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Characterization of 9-Dimensional Anosov Lie Algebras

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Abstract. The classification of all real and rational Anosov Lie algebras up to dimension 8 is given by Lauret and Will. In this paper we study 9-dimensional Anosov Lie algebras by using the properties of very special algebraic numbers and Lie algebra classification tools. We prove that there exists a unique, up to isomorphism, complex 3-step Anosov Lie algebra of dimension 9. In the 2-step case, we prove that a 2-step 9-dimensional Anosov Lie algebra with no abelian factor must have a 3-dimensional derived algebra and we characterize these Lie algebras in terms of their Pfaffian forms. Among these Lie algebras, we exhibit a family of infinitely many complex non-isomorphic Anosov Lie algebras.
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1. Introduction

A diffeomorphism f of a compact differentiable manifold M is called *Anosov* if the tangent bundle $\mathrm{T}M$ admits a continuous invariant splitting $\mathrm{T}M = E^+ \oplus E^-$ such that $\mathrm{d}f$ expands E^+ and contracts E^- exponentially. A well-known class of Anosov diffeomorphisms arises as follows. Let N be a simply connected nilpotent Lie group and let Γ be a discrete subgroup of N such that N/Γ is compact. In this case, N/Γ is called a *nilmanifold.* If f is a hyperbolic automorphism of N (i.e. no eigenvalue of the differential df is of absolute value 1) such that $f(\Gamma) = \Gamma$, then the induced diffeomorphism $\overline{f}$ on N/Γ, defined by $\overline{f}(x\Gamma) = f(x)\Gamma$ for all $x \in N$, is an Anosov diffeomorphism of the nilmanifold N/Γ. An Anosov diffeomorphism of a nilmanifold N/Γ arising in this way is called an *Anosov automorphism* of N/Γ. More generally, one can get examples of Anosov diffeomorphisms on manifolds which are finitely covered by nilmanifolds in the following way. Let K be a finite group of automorphisms of a simply connected nilpotent Lie group N and let Γ be a torsion free discrete subgroup of $K \ltimes N$ such that the quotient N/Γ is

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compact. Here the action of Γ on N is given by $x(\tau, y) = y\tau(x)$ for $x \in N$ and $(\tau, y) \in \Gamma$. We call the quotient space N/Γ an *infranilmanifold.* If g is a hyperbolic automorphism of N such that g normalizes the subgroup K in the group of automorphisms of N and $g(\Gamma) = \Gamma$, then the induced diffeomorphism on N/Γ is called an *Anosov automorphism of the infranilmanifold* N/Γ.

In [20], S. Smale raised the problem of classifying the compact manifolds admitting Anosov diffeomorphisms, and up to now the only known examples of Anosov diffeomorphisms are the Anosov automorphisms of infranilmanifolds described above. J. Franks [8] and A. Manning [15] proved that an Anosov diffeomorphism of a nilmanifold N/Γ is topologically conjugate to an Anosov automorphism of N/Γ. This enhances the interest in the problem of classifying all nilmanifolds which admit Anosov automorphisms.

The first example of a non-toral nilmanifold admitting an Anosov automorphism was described by S. Smale ([20]). For many years only relatively few examples appeared in the literature, but in recent years families of nilmanifolds with Anosov automorphisms have been constructed showing that a complete classification does not seem to be possible (see [11, 2, 4, 5, 6, 7, 13, 14, 16, 18, 21]).

Any Anosov automorphism of a nilmanifold N/Γ gives rise to a hyperbolic automorphism τ of $\mathfrak{n}$, the Lie algebra of N, such that τ stabilizes a $\mathbb{Z}$-subalgebra Λ of $\mathfrak{n}$. Hence the matrix of τ with respect to a $\mathbb{Z}$-basis of Λ (i.e. with structure constants in $\mathbb{Z}$) lies in $GL_n(\mathbb{Z})$ where $n = \dim \mathfrak{n}$.

Definition 1.1. (See [11]) A rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ (i.e. with structure constants in $\mathbb{Q}$) of dimension n is said to be *Anosov* if it admits a hyperbolic automorphism τ which is *unimodular*, i.e. $[\tau]_\beta \in GL_n(\mathbb{Z})$ for some basis β of $\mathfrak{n}^{\mathbb{Q}}$, where $[\tau]_\beta$ denotes the matrix of τ with respect to the basis β. We will call a real or complex Lie algebra *Anosov* if it admits a rational form which is Anosov.

Therefore, if a nilmanifold N/Γ admits an Anosov automorphism then the Lie algebra of N is *Anosov.* It can be seen that real Anosov Lie algebras give rise to nilmanifolds admitting Anosov automorphisms. Hence the problem of classifying nilmanifolds admitting Anosov automorphisms reduces to the classification of real Anosov Lie algebras. So far, in [14], all real and rational Anosov Lie algebras of dimension ≤ 8 are classified up to isomorphism and we note that curiously enough, there are quite a few of them.

In this paper we consider the problem of classifying 9-dimensional complex Anosov Lie algebras. We note that in order to classify real Anosov Lie algebras, for each complex Anosov Lie algebras, one has to find all the real forms and study which of them are Anosov. Therefore, to study complex Anosov Lie algebras is the first step towards the classification.

In our approach we use two main tools, some algebraic number theory and Lie algebra classification results. More precisely, the eigenvalues the hyperbolic automorphism τ of an Anosov Lie algebra $\mathfrak{n}$ whose matrix with respect to a $\mathbb{Z}$-basis is in $GL(n, \mathbb{Z})$, are algebraic units (algebraic integers whose reciprocals are also algebraic integers) of absolute value different from 1. The properties of these type of algebraic units, studied in [17], can be used to give examples or to prove non-existence of Anosov Lie algebras. This approach was introduced in [11] and have also been used in [13],[14] and [18]. Using this, we prove that every 9-dimensional real Anosov Lie algebra, without an abelian factor, is either of type $(6,3)$ or $(3,3,3)$. Recall that

Definition 1.2. Let $\mathfrak{n}$ be an r-step nilpotent Lie algebra, i.e., the lower central series $\{C^i(\mathfrak{n})\}$ (defined by $C^0(\mathfrak{n}) = \mathfrak{n}$ and $C^i(\mathfrak{n}) = [\mathfrak{n}, C^{i-1}(\mathfrak{n})]$ for $i \geq 1$) satisfies $C^{r-1}(\mathfrak{n}) \neq 0$ and $C^r(\mathfrak{n}) = 0$. Then the *type* of $\mathfrak{n}$ is the r-tuple of positive integers $(n_1, \cdots, n_r)$, where $n_i = \dim C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n})$.

Definition 1.3. Let $\mathfrak{n}$ be a Lie algebra. An *abelian factor* of $\mathfrak{n}$ is an abelian (Lie) ideal $\mathfrak{a}$ of $\mathfrak{n}$ such that $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{a}$ for some ideal $\mathfrak{m}$ of $\mathfrak{n}$.

We then show that there exist a complex Anosov Lie algebra of type $(3,3,3)$ that is unique up to Lie algebra isomorphism. It can be seen that if $\mathfrak{n}$ is an Anosov Lie algebra of type $(3, n_2, n_3)$, then $n_2 = 3$ and $n_3 \geq 3$ (see [13, Proposition 2.3]). The type $(3,3,3)$ example mentioned above is the first example, to our knowledge, of an Anosov Lie algebra of type $(3,3,*)$ showing that the condition $n_3 \geq 3$ is in fact attained. To show that this algebra is actually an Anosov Lie algebra, we use some special algebraic units constructed from the roots of unity in a way that seems to be generalizable.

The $(6,3)$ case is more involved. By using the classification given by Galitzki and Timashev in [9], we show that if a Lie algebra of type $(6,3)$ is Anosov then its Pfaffian form is projectively equivalent to xyz or 0 (see [11]) then we give a list of possible candidates for $(6,3)$ type complex Anosov Lie algebras (Theorem 3.12). We note that among these algebras there is an infinite family of complex non-isomorphic Anosov Lie algebras (see Proposition 3.7 and Remark 3.8). It was proved in [14] that up to isomorphism there are only finitely many real Anosov Lie algebras up to dimension 8 (see [14, Table 3]). This shows that the smallest dimension in which there are infinitely many complex non-isomorphic Anosov Lie algebras is 9.

2. Preliminaries

In this section, in order to introduce the framework in which we are going to work, we begin by recalling the following proposition, proved in [13].

Proposition 2.1. *Let $\mathfrak{n}$ be a real r-step nilpotent Lie algebra. We define $C^i(\mathfrak{n})$ inductively by $C^i(\mathfrak{n}) = [\mathfrak{n}, C^{i-1}(\mathfrak{n})]$ for $i \geq 1$, where $C^0(\mathfrak{n}) = \mathfrak{n}$. If $\mathfrak{n}$ is an Anosov Lie algebra then there exist a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_r$ satisfying $C^i(\mathfrak{n}) = \mathfrak{n}_{i+1} \oplus ... \oplus \mathfrak{n}_r$, $i = 0, ..., r$, and a hyperbolic $\tau \in \mathrm{Aut}(\mathfrak{n})$ such that*

(i) $\tau(\mathfrak{n}_i) = \mathfrak{n}_i$ for all $i = 1, ..., r$.

(ii) τ is semisimple (in particular τ is diagonalizable over $\mathbb{C}$).

(iii) For each i, there exists a basis β_i of $\mathfrak{n}_i$ such that $[\tau_i]_{\beta_i} \in \mathrm{SL}_{n_i}(\mathbb{Z})$, where $n_i = \dim \mathfrak{n}_i$ and $\tau_i = \tau|_{\mathfrak{n}_i}$.

Let $\mathfrak{n}$ and τ be as in Proposition 2.1, i.e. τ is a hyperbolic automorphism of $\mathfrak{n}$ and there exists a $\mathbb{Z}$-basis of $\mathfrak{n}$, β, such that $[\tau]_\beta \in \mathrm{SL}_n(\mathbb{Z})$ where $n = \dim \mathfrak{n}$. We note that the eigenvalues of τ are algebraic units, that is each eigenvalue of τ satisfies a monic polynomial equation with integer coefficients and constant term 1. This follows from the existence of a basis of $\mathfrak{n}$ with respect to which the matrix of τ is in $\mathrm{SL}_n(\mathbb{Z})$.

For an algebraic number $\lambda \in \mathbb{C}$, we denote by $\deg(\lambda)$ the degree of $m_\lambda(x)$, the irreducible monic polynomial over $\mathbb{Q}$ annihilated by λ and by the *conjugate of* λ the other roots of $m_\lambda(x)$. We then have the following lemma:

Lemma 2.2. *Let $\mathfrak{n}$ be a real r-step Anosov Lie algebra, and let τ and $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_r$ be as in* Proposition 2.1. *Let $n_i = \dim \mathfrak{n}_i$ and $\tau_i = \tau|_{\mathfrak{n}_i}$ for $1 \le i \le r$. Then every eigenvalue λ_i of τ_i is an algebraic unit and $1 < deg(\lambda_i) \le n_i$ for all i.*

The following definition will be used in the following sections.

Definition 2.3. Let V be a real vector space of dimension n. Let σ be a linear automorphism of V whose characteristic polynomial has integer coefficients. We say that σ has a *splitting* $[k_1; \ldots; k_m]$, where $k_i \in \mathbb{N}, k_i \ge k_{i+1}$, if the characteristic polynomial of σ can be written as a product of m irreducible polynomials (over $\mathbb{Z}$) $f_1, f_2, \ldots f_m$ such that $\deg f_i = k_i$ for all i.

For example if $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear automorphism given by $\sigma(x, y, z) = (2x + y, x + y, 3z)$ for all $x, y, z \in \mathbb{R}$, then the characteristic polynomial of σ is $X^3 - 6X^2 + 10X - 3$ with irreducible factors over $\mathbb{Z}$, $X^2 - 3X + 1$ and $X - 3$. Hence the splitting of σ is $[2; 1]$.

In our setup, if $\mathfrak{n}, \tau, \mathfrak{n}_i$ and τ_i are as in Proposition 2.1, we are going to look at the possible splittings of τ_i's. Note that for a fixed i, the characteristic polynomial of τ_i has integer coefficients (Proposition 2.1 (iii)). If τ_i has a splitting $[k_1; \ldots; k_m]$, then $k_1 + \cdots + k_m = n_i$ where $n_i = \dim \mathfrak{n}_i$. Since τ_i is hyperbolic and $[\tau_i]_{\beta_i} \in \mathrm{SL}_{n_i}(\mathbb{Z})$, for some basis β_i of $\mathfrak{n}_i$, we also have that $k_j \neq 1$ for all j (see also [13] Appendix).

We note that if $i > 1$ each eigenvalue of τ_i is a product of certain eigenvalues of τ_j with $j < i$ and moreover, it is an algebraic unit.

We will investigate how the product of such special algebraic numbers can behave. It is proved in [17] that this behavior can not be too wild. In fact, one has the following lemma that can be deduced from [17].

Lemma 2.4. *Let τ be an Anosov automorphism of a nilpotent Lie algebra $\mathfrak{n}$. Let α and β be two eigenvalues of τ. Then the following hold:*

1. *If $deg(\alpha)$ and $deg(\beta)$ are relatively prime then $deg(\alpha\beta)$ cannot be a prime.*

2. *If $\alpha\beta$ is also an eigenvalue of τ and deg(α) $\leq$ deg(β) then*

$$g.c.d.(deg(\beta), deg(\alpha\beta)) \neq 1.$$

Proof. If deg(α) and deg(β) are relatively prime and deg($\alpha\beta$) is prime, then it follows from Corollary 2 of [17] that either $|\alpha| = 1$ or $|\beta| = 1$. This contradicts our assumption that both α and β are eigenvalues of a hyperbolic automorphism τ.

Suppose that $\alpha\beta$ is an eigenvalue of τ, deg(α) = deg(β) and the g.c.d. of deg(β) and deg($\alpha\beta$)) is 1. Then $|\alpha\beta| = 1$ by Corollary 1 of [17] which is a contradiction. Also if deg(α) < deg(β), then by Lemma 2 of [17] we have g.c.d.$(\deg(\beta), \deg(\alpha\beta)) \neq 1$. ■

We recall [13, Theorem 3.1]: Let $\mathfrak{n}$ be a rational Lie algebra and let $\mathfrak{n} = \widetilde{\mathfrak{n}} \oplus \mathfrak{m}$ be a Lie direct sum, where $\mathfrak{m}$ is a maximal abelian factor of $\mathfrak{n}$. Then $\mathfrak{n}$ is Anosov if and only if $\widetilde{\mathfrak{n}}$ is Anosov and dim $\mathfrak{m} \geq 2$. In view of this, we are interested in studying Anosov Lie algebras without an abelian factor (see Definition 1.3).

For an Anosov Lie algebra $\mathfrak{n}$ without an abelian factor, we will study the properties of eigenvalues of an Anosov automorphism of $\mathfrak{n}$ and by using these properties we will deduce the Lie bracket structure on $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n} \otimes \mathbb{C}$. Moreover, we also note that if $\mathfrak{n}_{\mathbb{C}}$ has an abelian factor, then $\mathfrak{n}$ must also carry an abelian factor. In fact, $\mathfrak{z}(\mathfrak{n}_{\mathbb{C}}) \cap [\mathfrak{n}_{\mathbb{C}}, \mathfrak{n}_{\mathbb{C}}] = (\mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{n}, \mathfrak{n}])_{\mathbb{C}}$ and hence if $\mathfrak{z}(\mathfrak{n}_{\mathbb{C}}) \cap [\mathfrak{n}_{\mathbb{C}}, \mathfrak{n}_{\mathbb{C}}] \neq \mathfrak{z}(\mathfrak{n}_{\mathbb{C}})$ then $(\mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{n}, \mathfrak{n}]) \neq \mathfrak{z}(\mathfrak{n})$ where $\mathfrak{z}(\mathfrak{n})$ denotes the center of $\mathfrak{n}$.

Finally, we state the following lemma that can be proved by using Proposition 2.1.

Lemma 2.5. *Let $\mathfrak{n}$ be an Anosov Lie algebra of type $(n_1, \ldots, n_r)$ and let $\mathfrak{n} = \mathfrak{n}_1 \oplus ... \oplus \mathfrak{n}_r$ be the decomposition of $\mathfrak{n}$ given in Proposition 2.1. Then $\widetilde{\mathfrak{n}} = \mathfrak{n}/\mathfrak{n}_r$ is Anosov.*

Proof. If $\mathfrak{n}$ is an Anosov Lie algebra, τ and $\mathfrak{n} = \mathfrak{n}_1 \oplus ... \oplus \mathfrak{n}_r$ are as in Proposition 2.1 then, since $\tau\mathfrak{n}_i = \mathfrak{n}_i$ for all $i = 1, ..., r$, it is easy to see that it induces an automorphism of $\widetilde{\mathfrak{n}} = \mathfrak{n}/\mathfrak{n}_r \simeq \mathfrak{n}_1 \oplus ... \oplus \mathfrak{n}_{r-1}$. Also this automorphism is hyperbolic and to see that it is unimodular, recall that for each i, there exists a basis β_i of $\mathfrak{n}_i$ such that $[\tau_i]_{\beta_i} \in \mathrm{S}L_{n_i}(\mathbb{Z})$, where $n_i = \dim \mathfrak{n}_i$ and $\tau_i = \tau|_{\mathfrak{n}_i}$. ■

Note that this argument is valid not only at the real or complex level but also in the rational case. We also note that if $\mathfrak{n}$ is a three step nilpotent Anosov Lie algebra, then $\widetilde{\mathfrak{n}}$ is 2-step and the decomposition $\widetilde{\mathfrak{n}} = \mathfrak{n}/\mathfrak{n}_3 \simeq \mathfrak{n}_1 \oplus \mathfrak{n}_2$ gives the type of $\widetilde{\mathfrak{n}}$.

3. Dimension 9

In this section we will study Anosov Lie algebras of dimension 9. We will prove that an Anosov Lie algebra without an abelian factor must be of type $(6, 3)$ or

$(3,3,3)$. Moreover, we will prove that there is only one complex Anosov Lie algebra (up to a Lie algebra isomorphism) without an abelian factor of type $(3,3,3)$. The case $(6,3)$ is the hardest.

Using [13, Proposition 2.3] we can see that the possible types for a 9-dimensional Anosov Lie algebra are

$(7,2)$, $(6,3)$, $(5,4)$, $(4,5)$, $(5,2,2)$, $(4,3,2)$, $(4,2,3)$ and $(3,3,3)$.

As a corollary of Lemma 2.5 and the fact that there is no non-toral 7-dimensional Anosov Lie algebra (see [14]), we get that there are no Anosov Lie algebras of type $(5,2,2)$ and $(4,3,2)$. Finally, it is not hard to see by using Lemma 2.4, that there are no Anosov Lie algebras with no abelian factor of type $(7,2)$, $(5,4)$, $(4,5)$ and $(4,2,3)$. We will illustrate by looking at the case $(5,4)$.

In the rest of this section, we will use the following notation:

Notation 3.1. If $\mathfrak{n}$ is an Anosov Lie algebra and τ, τ_i and $\mathfrak{n}_i$ are as in Proposition 2.1, we will denote the eigenvalues of τ_1 by λ_i's, the eigenvalues of τ_2 by μ_j's and the eigenvalues of τ_3 by ν_k's, and the corresponding eigenvectors by X_i's, Y_j's and Z_k's (in $\mathfrak{n}_{\mathbb{C}}$) respectively.

Proposition 3.2. *There is no Anosov Lie algebra of type* $(5,4)$.

Proof. Suppose that there exists an Anosov Lie algebra $\mathfrak{n}$ of type $(5,4)$, and let τ be an Anosov automorphism as in Proposition 2.1. We note that the possible splittings (see Definition 2.3) of τ_1 are [5] and [3; 2]. In the first case, for each nonzero bracket among the eigenvectors of τ_1 we get an eigenvector of τ_2 (of degree 2 or 4). That is if $[X_i, X_j] \neq 0$ for some $1 \leq i, j \leq 5$, we then have that $\lambda_i\lambda_j = \mu_k$ has degree 2 or 4. Either way this contradicts Lemma 2.4 since $g.c.d(5,4) = g.c.d(5,2) = 1$. On the other hand, if the splitting is [3; 2] by the same argument, we get that $[X_i, X_j] = 0$ for $i, j \in \{1,2,3\}$, and if $\mu_k = \lambda_i\lambda_j$ with $i \leq 3$ and $j = 4$ or 5, then $\text{g.c.d.}(\deg \lambda_i, \deg \mu_k) = 1$, contradicting again Lemma 2.4. This means that $[X_i, X_j] = 0$ for all i, j since $\lambda_4 = \lambda_5^{-1}$ and therefore $[X_4, X_5] = 0$. This is a contradiction because $\mathfrak{n}$ is of type $(5,4)$. ■

Proposition 3.3. *There exists only one complex Anosov Lie algebra of type* $(3,3,3)$ *up to Lie algebra isomorphism.*

Proof. Let $\mathfrak{n}$ be a nilpotent Lie algebra of type $(3,3,3)$ that admits an Anosov automorphism τ as in Proposition 2.1. In this case we note that the characteristic polynomials of τ_1, τ_2 and τ_3 are all irreducible degree 3 polynomials over $\mathbb{Q}$. Following Notation 3.1, we can assume with no loss of generality, that in $\mathfrak{n}_{\mathbb{C}}$ (complexification of $\mathfrak{n}$):

$$[X_1, X_2] = Y_1, \quad [X_2, X_3] = Y_2, \quad [X_1, X_3] = Y_3,$$

and since $\lambda_1\lambda_2\lambda_3 = \pm 1$ we also have that

$$[X_3, Y_1] = 0, \quad [X_1, Y_2] = 0, \quad [X_2, Y_3] = 0. \tag{1}$$

Let $\mathfrak{n}(a,b,c)$ denotes the Lie algebra given by

$$\begin{aligned} &[X_1,X_2]=Y_1, \quad [X_2,X_3]=Y_2, \quad [X_1,X_3]=Y_3,\\ &[X_1,Y_1]=a\,Z_1, \quad [X_2,Y_2]=b\,Z_2, \quad [X_3,Y_3]=c\,Z_3, \end{aligned} \tag{2}$$

for $a,b,c\in\mathbb{C}$. If $abc\neq 0$, it can be seen that $\mathfrak{n}(a,b,c)$ is isomorphic to $\mathfrak{n}(1,1,1)$ by changing the basis in the center. To see that $\mathfrak{n}_{\mathbb{C}}$ is isomorphic to $\mathfrak{n}(1,1,1)$, note that otherwise we have (in $\mathfrak{n}_{\mathbb{C}}$)

$$[X_i,Y_j]=a\,Z_k, \qquad \text{and} \qquad [X_i,Y_l]=b\,Z_r \tag{3}$$

for some non zero $a,b\in\mathbb{C}$ and some i that we can assume to be 1. It is clear from (1) that $j,l\neq 2$ so we may assume that $j=1$ and $l=3$. Hence, if $k=r$, (3) implies that $\lambda_1^2\lambda_2=\lambda_1^2\lambda_3$ and then $\lambda_2=\lambda_3$. This is a contradiction because λ_i's are roots of the characteristic polynomial of τ_1 and they have to be distinct because the characteristic polynomial of τ_1 is irreducible over $\mathbb{Q}$. On the other hand, if $k\neq r$, with no loss of generality we can assume that

$$[X_1,Y_1]=a\,Z_1, \quad [X_1,Y_3]=b\,Z_2.$$

Since Z_3 is an eigenvector of τ_3 corresponding to ν_3 (see Notation 3.1), $Z_3=[X_i,Y_j]$ for some i,j. Using (1) and that ν_k's are all distinct, the possibilities for (i,j) are $(2,1)$, $(2,2)$, $(3,2)$, and $(3,3)$ but all of them lead us to the same kind of contradiction by checking that the product of the eigenvalues in the center equals ν_1^2, ν_1, ν_2, ν_2^2 respectively. Indeed, if for example we consider case $(2,2)$, we obtain

$$[X_1,Y_1]=a\,Z_1, \quad [X_1,Y_3]=b\,Z_2, \quad [X_2,Y_2]=\,Z_3,$$

and therefore $1=\lambda_1^2\lambda_2.\lambda_1^2\lambda_3.\lambda_2^2\lambda_3=\lambda_1^2\lambda_2=\nu_1$, contradicting the fact that τ_3 is hyperbolic. This shows that $\mathfrak{n}_{\mathbb{C}}$ is isomorphic to $\mathfrak{n}(1,1,1)$.

In the following we will show that $\mathfrak{n}(1,1,1)$ is Anosov by constructing a hyperbolic automorphism σ and a $\mathbb{Z}$-basis of $\mathfrak{n}(1,1,1)$ preserved by σ such that the matrix of σ in that basis has integer entries.

Consider the polynomial in $\mathbb{Z}[X]$ given by

$$f(X)=X^3-3X+1. \tag{4}$$

Then its roots are given by

$$\lambda_1=\xi+\xi^8, \quad \lambda_2=\xi^2+\xi^7, \quad \lambda_3=\xi^4+\xi^5, \tag{5}$$

where $\xi=e^{2i\pi/9}$ a ninth root of unity. In this case we have that the extension $\mathbb{Q}(\lambda_1)$ is a cyclic extension of degree 3 over $\mathbb{Q}$ (see [1, p. 543]) and moreover, a straightforward calculation shows that

$$\lambda_1=\lambda_3^2-2 \quad \lambda_2=\lambda_1^2-2, \quad \lambda_3=\lambda_2^2-2. \tag{6}$$

Let

$$\mu_1=\lambda_1\lambda_2, \quad \mu_2=\lambda_2\lambda_3, \quad \mu_3=\lambda_1\lambda_3,$$

$$\nu_1=\lambda_1\mu_1, \quad \nu_2=\lambda_2\mu_2, \quad \nu_3=\lambda_3\mu_3.$$

We consider now the automorphism σ as above, corresponding to these λ_i's, that is, defined by $\sigma(X_i) = \lambda_i X_i$, $\sigma(Y_i) = \mu_i Y_i$ and $\sigma(Z_i) = \nu_i Z_i$ for $i = 1, 2, 3$.

We note that $\mu_i, \nu_i \notin \mathbb{Q}$ for all $1 \leq i \leq 3$. For, if $\mu_1 = \lambda_1\lambda_2 \in \mathbb{Q}$, then $\mu_1 = \pm 1$ because μ_1 is an algebraic unit. Then we get a contradiction that $\lambda_3 = \pm 1$ since $\lambda_1\lambda_2\lambda_3 = 1$. Similarly if $\nu_1 \in \mathbb{Q}$, then $\nu = \pm 1$ and $\lambda_1 = \pm\lambda_3$ which is a contradiction. Now since $\mu_i, \nu_i \in \mathbb{Q}(\lambda_1)$ and $\mathbb{Q}(\lambda_1)$ is an extension of degree 3 over $\mathbb{Q}$, $\deg(\mu_i) = 3 = \deg(\nu_i)$ for all $1 \leq i \leq 3$.

Concerning the new basis, for $i = 1, 2, 3$ let us denote by

$$\mathcal{X}_i = \sum_{j=1}^{3} \lambda_j^{i-1} X_j,$$

$$\mathcal{Y}_i = \lambda_3^{1-i}(\lambda_2 - \lambda_1)Y_1 + \lambda_1^{1-i}(\lambda_3 - \lambda_2)Y_2 + \lambda_2^{1-i}(\lambda_3 - \lambda_1)Y_3,$$

$$\mathcal{Z}_i = (\lambda_1)^{i-1}(\lambda_2 - \lambda_1)Z_1 + (\lambda_2)^{i-1}(\lambda_3 - \lambda_2)Z_2 + (\lambda_3)^{i-1}(\lambda_3 - \lambda_1)Z_3.$$

Let $\beta_1 = \{\mathcal{X}_i, 1 \leq i \leq 3\}$, $\beta_2 = \{\mathcal{Y}_i, 1 \leq i \leq 3\}$ and $\beta_3 = \{\mathcal{Z}_i, 1 \leq i \leq 3\}$. It is easy to see that β_i's are linearly independent sets. Moreover, we can see that $\sigma(\mathcal{X}_i) = \mathcal{X}_{i+1}$, $\sigma(\mathcal{Y}_i) = \mathcal{Y}_{i+1}$ for $i = 1, 2$, $\sigma(\mathcal{X}_3) = 3\mathcal{X}_2 - \mathcal{X}_1$ and $\sigma(\mathcal{Y}_3) = 3\mathcal{Y}_3 - \mathcal{Y}_1$ by using that $f(\lambda_i) = 0$ and therefore $\lambda_i^{-1} = 3 - \lambda_i^2$. Also, by using (6) one can see that $\sigma(\mathcal{Z}_i)$ is an integer linear combination of the $\mathcal{Z}_i's$ for all i. In fact, for example

$$\begin{aligned}
\sigma(\mathcal{Z}_1) &= \lambda_1^2\lambda_2(\lambda_2 - \lambda_1)Z_1 + \lambda_2^2\lambda_3(\lambda_3 - \lambda_2)Z_2 + \lambda_3^2\lambda_1(\lambda_3 - \lambda_1)Z_3 \\
&= \lambda_1^2(\lambda_1^2 - 2)(\lambda_2 - \lambda_1)Z_1 + \lambda_2^2(\lambda_2^2 - 2)(\lambda_3 - \lambda_2)Z_2 + \lambda_3^2(\lambda_3^2 - 2)(\lambda_3 - \lambda_1)Z_3 \\
&= (\lambda_1^2 - \lambda_1)(\lambda_2 - \lambda_1)Z_1 + (\lambda_2^2 - \lambda_2)(\lambda_3 - \lambda_2)Z_2 + (\lambda_3^2 - \lambda_3)(\lambda_3 - \lambda_1)Z_3 \\
&= \mathcal{Z}_3 - \mathcal{Z}_2.
\end{aligned}$$

Therefore, $\beta = \{\mathcal{X}_i, \mathcal{Y}_j, \mathcal{Z}_k : i, j, k = 1, 2, 3\}$ is a basis of $\mathfrak{n}(1, 1, 1)$ preserved by σ and moreover $[\sigma]_\beta \in GL(9, \mathbb{Z})$. To conclude, it remains to show that β is also a $\mathbb{Z}$-basis. We first note that since $\lambda_1\lambda_2\lambda_3 = 1$, $\lambda_1 + \lambda_2\lambda_3 = 0$ and $\lambda_i^{-2} = 3\lambda_i^{-1} - \lambda_i$, we have that

$$[\mathcal{X}_1, \mathcal{X}_2] = \mathcal{Y}_1, \quad [\mathcal{X}_1, \mathcal{X}_3] = \mathcal{Y}_3 - 3\mathcal{Y}_2. \tag{7}$$

Moreover, by using (6) as above, we have

$$[\mathcal{X}_1, \mathcal{Y}_1] = \mathcal{Z}_1, \quad [\mathcal{X}_1, \mathcal{Y}_2] = \mathcal{Z}_2 - \mathcal{Z}_1, \quad [\mathcal{X}_1, \mathcal{Y}_3] = \mathcal{Z}_3 - 2\mathcal{Z}_2 + \mathcal{Z}_1.$$

Note that since σ in an automorphism of $\mathfrak{n}$ and by our construction of the basis, these are the only Lie brackets one needs to check. Hence β is a $\mathbb{Z}$-basis of $\mathfrak{n}(1, 1, 1)$ preserved by the hyperbolic automorphism σ, such that $[\sigma]_\beta \in GL(9, \mathbb{Z})$, and therefore $\mathfrak{n}(1, 1, 1)$ is an Anosov Lie algebra. ■

Remark 3.4. We will use the above proof as a model proof to show that a given Lie algebra is Anosov in some of the following cases where we will just exhibit the automorphism and the $\mathbb{Z}$-basis.

Remark 3.5. Note that with this result we have shown that there is only one k-step nilpotent complex Anosov Lie algebra of dimension 9 with $k > 2$ (see the beginning of this section).

Type $(6,3)$. To study Anosov Lie algebras of type $(6,3)$, we will use the classification of complex nilpotent Lie algebras of type $(6,3)$ by Galitzki and Timashev given in [9]. We will begin by dividing the study according to the possible splittings of the Anosov automorphism to get some condition on the bracket structure. Then, using classification from [9], we give a list of non-isomorphic Lie algebras of type $(6,3)$ as possible candidates for Anosov algebras. In most cases we will give a rational basis to show that they are indeed Anosov Lie algebras.

Notation 3.6. To refer to the classification given in [9] we will introduce here notation for some Lie algebras from [9, Section 4] which will be used in this section. For s_1, s_2, $s_3 \in \mathbb{C}$, we will denote by $s_1\,\mathcal{U}_1 + s_2\,\mathcal{U}_2 + s_3\,\mathcal{U}_3$ a 2-step nilpotent Lie algebra of type $(6,3)$ with a basis $\{v_i : 1 \le i \le 6\} \cup \{w_j : 1 \le j \le 3\}$ and nonzero Lie brackets given by

$$\begin{array}{lll} [v_1, v_2] = s_1\, w_1 & [v_3, v_4] = s_1\, w_2 & [v_5, v_6] = s_1\, w_3 \\ [v_5, v_4] = s_2\, w_1 & [v_1, v_6] = s_2\, w_2 & [v_3, v_2] = s_2\, w_3 \\ [v_3, v_6] = s_3\, w_1 & [v_5, v_2] = s_3\, w_2 & [v_1, v_4] = s_3\, w_3. \end{array} \tag{8}$$

Here, $\mathcal{U}_i$ is the 9-dimensional nilpotent Lie algebra with Lie bracket given by the i^{th} row of (8) and with $s_i = 1$.

Let V be a 6-dimensional vector space and let W be a 3-dimensional vector space. The classification of nilpotent Lie algebras of type $(6,3)$ has been given in [9] by classifying the tensors in $\wedge^2 V \otimes W$ under the action of $SL(V) \times SL(W)$. Each tensor is decomposed into a semisimple and a nilpotent part. They obtained 7 families that are divided according to the semisimple part which is given in terms of $s_1\mathcal{U}_1 + s_2\mathcal{U}_2 + s_3\mathcal{U}_3$ for some s_1, s_2, s_3. The nilpotent parts are listed in tables which we are going to denote by $\mathfrak{m}_j$ where j is its number in the corresponding table. For example, by $\mathcal{U}_1 + \mathfrak{m}_{11}$ from Family 6 (and Table 7) from [9, Section 4], we denote the 9-dimensional Lie algebra with a basis $\{v_i, w_j : 1 \le i \le 6, 1 \le j \le 3\}$ and nonzero Lie brackets given by

$$\begin{array}{lll} [v_1, v_2] = w_1 & [v_3, v_4] = w_2 & [v_5, v_6] = w_3 \\ [v_1, v_5] = w_2 & [v_3, v_6] = w_1 & \end{array} \tag{9}$$

Note that since we are interested in Lie algebras of type $(6,3)$ up to isomorphism, we need orbits of tensors under the action of $GL(V) \times GL(W)$, and therefore to get this classification, the canonical form for the semisimple part of the tensors can be reduced by multiplying by a nonzero scalar. Then, for example we have that $s\mathcal{U}_2 + r\mathcal{U}_3 \simeq s'\mathcal{U}_2 + \mathcal{U}_3$ for any $r \neq 0$ and the semisimple part of Family 4 (and 5) are all isomorphic (see [9]).

Next, for completeness, we will list the $\mathfrak{m}_j$'s we are actually using in this section, by specifying the nonzero Lie brackets of their basis vectors

$$\{v_i, w_j : 1 \le i \le 6, 1 \le j \le 3\}.$$

From Family 6 and Table 7 :

$$[v_1, v_4] = w_3 \quad [v_1, v_6] = w_2 \quad [v_3, v_5] = w_1 \tag{$\mathfrak{m}_6$}$$

$$[v_1, v_3] = w_3 \quad [v_1, v_5] = w_2 \quad [v_3, v_6] = w_1 \tag{$\mathfrak{m}_7$}$$

$$[v_1, v_3] = w_3 \quad [v_1, v_5] = w_2 \quad [v_3, v_5] = w_1 \tag{$\mathfrak{m}_{10}$}$$

$$[v_1, v_5] = w_2 \quad [v_3, v_6] = w_1 \tag{$\mathfrak{m}_{11}$}$$

$$[v_1, v_3] = w_3 \quad [v_1, v_5] = w_2 \tag{$\mathfrak{m}_{12}$}$$

$$[v_1, v_3] = w_3 \tag{$\mathfrak{m}_{14}$}$$

From Family 4 and Table 5:

$$[v_5, v_3] = w_1 \quad [v_1, v_5] = w_2 \quad [v_3, v_1] = w_3 \tag{$\mathfrak{m}_3$}$$

Note that according to [9] the corresponding algebras are pairwise non-isomorphic, meaning that for example, $\mathcal{U}_1 + \mathfrak{m}_{11}$ and $\mathcal{U}_1 + \mathfrak{m}_{14}$ are non-isomorphic.

Anosov Lie algebras of type $(6, 3)$. Let $\mathfrak{n}$ be an Anosov Lie algebra of type $(6, 3)$ with no abelian factor, and let τ, τ_1, τ_2, be as in Proposition 2.1. As in the previous cases, we will denote by X_i and Y_j the eigenvectors of τ_1 and τ_2 respectively with corresponding eigenvalues λ_i and μ_j. Note that, by using Lemma 2.4, it is easy to see that the splitting of τ_1 is $[6]$ or $[3; 3]$ (see Definition 2.3). We are going to study now each one of these cases separately.

Case i: The splitting of τ_1 is $[6]$ or equivalently, the characteristic polynomial of τ_1 is irreducible over $\mathbb{Z}$. We note that, in particular, this implies that $\lambda_i \neq \lambda_j$ for all $i \neq j$. From this, it can be shown that one may assume that

$$[X_1, X_2] = Y_1, \qquad [X_3, X_4] = Y_2, \qquad [X_5, X_6] = Y_3. \tag{10}$$

In fact, if this is not the case we can reorder the basis so that

$$[X_1, X_2] = Y_1, \qquad [X_1, X_3] = Y_2, \tag{11}$$

and in this situation one would have to consider three possibilities for Y_3,

$$\textbf{(I)}\ [X_1, X_4] = Y_3, \qquad \textbf{(II)}\ [X_2, X_j] = Y_3, \qquad \textbf{(III)}\ [X_4, X_5] = Y_3. \tag{12}$$

Note that each one of these situations represents a few others that are totally equivalent to the one considered.

Case **(III)** is the simplest one because directly from (11) it follows that $\lambda_1^2 \lambda_2 \lambda_3 \lambda_4 \lambda_5 = 1$ and therefore we obtain the contradiction $\lambda_1 = \lambda_6$.

Cases **(I)** and **(II)** can be done in a very similar way. That is, by considering the possibilities for the other brackets among the X_i we get either a contradiction or (10). Therefore, we can assume (10):

$$[X_1, X_2] = Y_1, \qquad [X_3, X_4] = Y_2, \qquad [X_5, X_6] = Y_3.$$

If there are no other nontrivial Lie brackets but these, then $\mathfrak{n} \simeq \mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathfrak{h}_3 \simeq \mathcal{U}_1$ which is known to be Anosov (see [14]). If there are more nontrivial Lie brackets, without any loss of generality, we can assume that $[X_3, X_2] = c\, Y_3$ and moreover, using that there is no abelian factor, one can see that $\mathfrak{n}_{\mathbb{C}}$ is isomorphic to $\mathfrak{n}'(a, b, c)$, given by

$$\begin{array}{lll} [X_1, X_2] = Y_1, & [X_3, X_4] = Y_2 & [X_5, X_6] = Y_3, \\ [X_5, X_4] = a\, Y_1, & [X_1, X_6] = b\, Y_2 & [X_3, X_2] = c\, Y_3, \end{array} \tag{13}$$

for some $a, b, c \in \mathbb{C}$. It is easy to see that we can not add more nontrivial Lie brackets by using that τ is hyperbolic and $\prod_{i=1}^{6} \lambda_i = 1$. To know which non-isomorphic Lie algebras we obtain from (13), we start by noting that if $abc = 0$, from [9] we get that we have only two non-isomorphic Lie algebras:

$$\mathfrak{n}'(a, b, 0) \simeq \mathfrak{n}'(1, 1, 0) \simeq \mathcal{U}_1 + \mathfrak{m}_{11}, \qquad \text{and} \qquad \mathfrak{n}'(0, 0, c) \simeq \mathfrak{n}'(0, 0, 1) \simeq \mathcal{U}_1 + \mathfrak{m}_{14},$$

(see Notation 3.6).

On the other hand, if $abc \neq 0$, by changing the basis to

$$\beta' = \{X_1, aX_2, \tfrac{1}{a}X_3, \tfrac{1}{c}X_4, cX_5, X_6, aY_1, \tfrac{1}{ac}Y_2, cY_3\},$$

we have that $\mathfrak{n}'(a, b, c) \simeq \mathfrak{n}'(1, abc, 1)$, and then (if $abc \neq 0$)

$$\mathfrak{n}'(a, b, c) \simeq \mathfrak{n}'(s, s, s) \simeq \mathcal{U}_1 + s\, \mathcal{U}_2 \simeq s\, \mathcal{U}_2 + \mathcal{U}_3,$$

where $s^3 = abc \neq 0$. Hence, from [9] we get the following non-isomorphic Lie algebras

- $s\, \mathcal{U}_2 + \mathcal{U}_3$, corresponding to $s^3 \neq \pm 1$ (Family 2)
- $\mathcal{U}_2 + \mathcal{U}_3$, corresponding to $s^3 = 1$ (Family 4) (14)
- $-\mathcal{U}_2 + \mathcal{U}_3$, corresponding to $s^3 = -1$ (Family 5)

Here we are referring to the families from [9, Section 4].

We will consider now the ones corresponding to $s \in \mathbb{Q}$ which includes, in particular, the algebras $\mathcal{U}_2 + \mathcal{U}_3$ and $-\mathcal{U}_2 + \mathcal{U}_3$.

Proposition 3.7. *If $s \in \mathbb{Q}$, then a Lie algebra $s\, \mathcal{U}_2 + \mathcal{U}_3$ is Anosov.*

Proof. Let $s = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $\mathfrak{n} = s\, \mathcal{U}_2 + \mathcal{U}_3$. Then $\mathfrak{n} \simeq p\mathcal{U}_2 + q\mathcal{U}_3$. We recall that $p\mathcal{U}_2 + q\mathcal{U}_3$ is a 2-step nilpotent Lie algebra in which the nonzero Lie brackets on its basis vectors $\{X_i, Y_j : 1 \leq i \leq 6, 1 \leq j \leq 3\}$ are given by

$$\begin{array}{lll} [X_5, X_4] = p\, Y_1 & [X_1, X_6] = p\, Y_2 & [X_3, X_2] = p\, Y_3 \\ [X_3, X_6] = q\, Y_1 & [X_5, X_2] = q\, Y_2 & [X_1, X_4] = q\, Y_3. \end{array} \tag{15}$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the roots of the polynomial $x^3 - 3x + 1$. Note that we used these algebraic units in the proof of Proposition 3.3. Let σ be the automorphism of $\mathfrak{n}$ whose matrix with respect to a basis

$$\{X_1, \ldots, X_6, Y_1, Y_2, Y_3\}$$

is a diagonal matrix $D(\lambda_3, \lambda_3, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$.

Let

$$\begin{aligned} \mathcal{X}_l &= \lambda_3^{l-1} X_1 + \lambda_1^{l-1} X_3 + \lambda_2^{l-1} X_5 \qquad l = 1,2,3, \\ \mathcal{Y}_k &= \lambda_3^{k-1} X_2 + \lambda_1^{k-1} X_4 + \lambda_2^{k-1} X_6 \qquad k = 1,2,3, \\ \mathcal{Z}_r &= \lambda_1^{r} Y_1 + \lambda_2^{r} Y_2 + \lambda_3^{r} Y_3 \qquad r = 1,2,3. \end{aligned} \tag{16}$$

Hence,

$$\beta' = \{\mathcal{X}_l, \mathcal{Y}_k, \mathcal{Z}_r \mid l, k, r = 1,2,3\}. \tag{17}$$

is a basis of $\mathfrak{n}$. Using the properties of $\lambda_1, \lambda_2, \lambda_3$ it can be checked that $[\sigma]_{\beta'} \in GL(9, \mathbb{Z})$. For example, as $\lambda_1\lambda_2 = \lambda_1 - 1$ (see (5)) we get that $\sigma(\mathcal{Z}_1) = \mathcal{Z}_2 - \mathcal{Z}_1$.

To see that β' is a $\mathbb{Z}$-basis, we note the following:

$$\begin{aligned} &[\mathcal{X}_i, \mathcal{X}_j] = [\mathcal{Y}_i, \mathcal{Y}_j] = [\mathcal{Z}_i, \mathcal{Z}_j] = 0 \quad \forall\, 1 \le i, j \le 3, \\ &[\mathcal{X}_1, \mathcal{Y}_1] = (p+q)\mathcal{Z}_1, \qquad [\mathcal{X}_2, \mathcal{Y}_1] = -2p\mathcal{Z}_1 + q\mathcal{Z}_2 + p\mathcal{Z}_3, \\ &[\mathcal{X}_3, \mathcal{Y}_1] = 2p\mathcal{Z}_1 + p\mathcal{Z}_2 + q\mathcal{Z}_3. \end{aligned}$$

∎

Remark 3.8. Using the above proposition we get that $\{s\,\mathcal{U}_2 + \mathcal{U}_3 : s \in \mathbb{Q}\}$ is an infinite family of non-isomorphic Anosov Lie algebras of type $(6,3)$. We know that there are, up to isomorphism, only finitely many (real) Anosov Lie algebras up to dimension 8 (see [14, Table 3]) and hence this is the smallest dimension in which there are infinitely many non-isomorphic Anosov Lie algebras. We also note that since $\{s\,\mathcal{U}_2 + \mathcal{U}_3 : s \notin \mathbb{Q}, s^3 \neq \pm 1\}$ is an uncountable family of non-isomorphic Lie algebras, not all of them can be Anosov. Moreover, if $s^3 \notin \mathbb{Q}$ it is not even clear if $s\,\mathcal{U}_2 + \mathcal{U}_3$ admits a rational form.

Case ii: The splitting of τ_1 is $[3,3]$ (see Definition 2.3). Let $\lambda_1, \ldots, \lambda_6$ denote the eigenvalues of τ_1 such that $\lambda_1, \lambda_2, \lambda_3$ are conjugates over $\mathbb{Q}$ and $\lambda_4, \lambda_5, \lambda_6$ are conjugates over $\mathbb{Q}$. We will follow Notation 3.1.

In this case one may either have

(a) $[X_i, X_j] = Y_k$ for some $1 \le i, j \le 3$ (or equivalently $4 \le i, j \le 6$) or

(b) $[X_i, X_j] = 0$ for all $1 \le i, j \le 3$ and $[X_k, X_l] = 0$ for all $4 \le k, l \le 6$.

In **(a)**, with no loss of generality we may assume $[X_1, X_2] = Y_1$. At the eigenvalue level, this means that $\lambda_1\lambda_2 = \lambda_3^{-1}$, $\lambda_1\lambda_3 = \lambda_2^{-1}$ and $\lambda_2\lambda_3 = \lambda_1^{-1}$ must be the eigenvalues of τ_2 (τ on $[\mathfrak{n}, \mathfrak{n}]$). On the other hand, the absence of abelian factor implies that

$$[X_4, X_j] \neq 0 \quad \text{for some} \quad 1 \le j \le 6.$$

Hence $\lambda_4\lambda_j = \lambda_i^{-1}$ for some j and i, $1 \le j \le 6$, $1 \le i \le 3$. It is not hard to see that from here we can either have

$$\{\lambda_4, \lambda_5, \lambda_6\} = \{\lambda_1^{-2}, \lambda_2^{-2}, \lambda_3^{-2}\} \qquad \text{or} \qquad \{\lambda_4, \lambda_5, \lambda_6\} = \{\lambda_1, \lambda_2, \lambda_3\}.$$

If $\{\lambda_4, \lambda_5, \lambda_6\} = \{\lambda_1^{-2}, \lambda_2^{-2}, \lambda_3^{-2}\}$, we will prove that $\mathfrak{n}$ is Anosov. In this case, we may rearrange the basis so that $\lambda_4 = \lambda_1^{-2}, \lambda_5 = \lambda_2^{-2}$ and $\lambda_6 = \lambda_3^{-2}$. Hence we can assume that the nonzero Lie brackets in $\mathfrak{n}$ are given by

$$\begin{array}{lll} [X_1, X_2] = Y_1, & [X_2, X_3] = Y_2 & [X_1, X_3] = Y_3, \\ [X_3, X_6] = c\, Y_1, & [X_1, X_4] = a\, Y_2, & [X_2, X_5] = b\, Y_3 \end{array} \tag{18}$$

for some $a, b, c \in \mathbb{C}$. Here we need to use the special properties of λ_i's like $|\lambda_i| \neq 1$, λ_i's are all distinct for $1 \leq i \leq 3$ etc.

Let $\widetilde{\mathfrak{n}}(a, b, c)$ denote the Lie algebra defined by (18). Due to our assumption of no abelian factor we have that $abc \neq 0$ and then

$$\mathfrak{n} \simeq \widetilde{\mathfrak{n}}(a, b, c) \simeq \widetilde{\mathfrak{n}}(1, 1, 1) \simeq \mathcal{U}_1 + \mathfrak{m}_{10}$$

(see Notation 3.6 and ($\mathfrak{m}_{10}$)).

Proposition 3.9. *A Lie algebra $\widetilde{\mathfrak{n}}(1, 1, 1)$ defined by (18) with $a = b = c = 1$ is Anosov.*

Proof. We will use very similar arguments as used in the case of type $(3, 3, 3)$ (proof of Proposition 3.3). Let $\lambda_1, \lambda_2, \lambda_3$ be the roots of $X^3 - 3X + 1$. Let σ be the automorphism of $\widetilde{\mathfrak{n}}(1, 1, 1)$ whose matrix with respect to a basis

$$\{X_1, \ldots, X_6, Y_1, Y_2, Y_3\}$$

is a diagonal matrix

$$D(\lambda_1, \lambda_2, \lambda_3, \lambda_1^{-2}, \lambda_2^{-2}, \lambda_3^{-2}, \lambda_3^{-1}, \lambda_1^{-1}, \lambda_2^{-1}).$$

Let

$$\begin{aligned} \mathcal{X}_i &= \lambda_1^{i-1} X_1 + \lambda_2^{i-1} X_2 + \lambda_3^{i-1} X_3, \\ \mathcal{X}'_j &= \lambda_3^{1-j}(\lambda_2 - \lambda_1) X_6 + \lambda_1^{1-j}(\lambda_3 - \lambda_2) X_4 + \lambda_2^{1-j}(\lambda_3 - \lambda_1) X_5, \\ \mathcal{Y}_k &= \lambda_3^{1-k}(\lambda_2 - \lambda_1) Y_1 + \lambda_1^{1-k}(\lambda_3 - \lambda_2) Y_2 + \lambda_2^{1-k}(\lambda_3 - \lambda_1) Y_3. \end{aligned}$$

It can be shown that $\tilde{\beta} = \{\mathcal{X}_i, \mathcal{X}'_j, \mathcal{Y}_k \mid i, j, k = 1, 2, 3\}$ is a $\mathbb{Z}$-basis of $\widetilde{\mathfrak{n}}(1, 1, 1)$ such that $[\sigma]_{\tilde{\beta}} \in GL(9, \mathbb{Z})$. Note that due to the similarities with the $(3, 3, 3)$-case one only needs to check brackets that involve some $\mathcal{X}'_j$ (see [18]). ■

On the other hand, if $\{\lambda_4, \lambda_5, \lambda_6\} = \{\lambda_1, \lambda_2, \lambda_3\}$, then by rearranging the basis vectors we assume that $\lambda_4 = \lambda_1, \lambda_5 = \lambda_2, \lambda_6 = \lambda_3$ and the nonzero Lie brackets on $\mathfrak{n}$ are given by

$$\begin{array}{lll} [X_1, X_2] = a\, Y_1, & [X_2, X_3] = b\, Y_2 & [X_1, X_3] = c\, Y_3, \\ [X_4, X_5] = d\, Y_1, & [X_5, X_6] = e\, Y_2 & [X_4, X_6] = f\, Y_3, \\ [X_1, X_5] = g\, Y_1, & [X_2, X_6] = h\, Y_2 & [X_4, X_3] = i\, Y_3, \\ [X_4, X_2] = j\, Y_1, & [X_5, X_3] = k\, Y_2 & [X_1, X_6] = l\, Y_3, \end{array}$$

for some $a, b, c, d, e, f, g, h, i, j, k, l \in \mathbb{C}$. Note that the Pfaffian form of these Lie algebras (see [13]) is given by

$$Pf(x, y, z) = -xyz(afk - aei + bdl - bgf + cje - cdh + ihg - ljk). \tag{19}$$

Calculating the Pfaffian forms of the algebras listed in the classification given in [9, Section 4], one can see that $\mathfrak{n}$ should be isomorphic to one of the following (see Notation 3.6):

- $s\mathcal{U}_2 + \mathcal{U}_3$,
- $\mathcal{U}_2 + \mathcal{U}_3 + \mathfrak{m}_3$,
- $\mathcal{U}_1 + \mathfrak{m}_i$ for $i = 6, 7, 10, 11, 12, 14, 15$,

where $\mathfrak{m}_{15} = 0$.

Remark 3.10. Note that the algebras in [9, Section 4, Family 7] with 0 Pfaffian form have always an abelian factor and therefore are not included in our list.

The Lie algebras $s\mathcal{U}_2 + \mathcal{U}_3$ have been considered in case **i** (see (14), Proposition 3.7). Also $\mathcal{U}_1 + \mathfrak{m}_i$ for $i = 10, 11, 14, 15$ have been considered in case **i.** We don't know if $\mathcal{U}_1 + \mathfrak{m}_i$ is Anosov for $i = 6, 7, 12$. However we prove the following:

Proposition 3.11. *$\mathcal{U}_2 + \mathcal{U}_3 + \mathfrak{m}_3$ is an Anosov Lie algebra.*

Proof. Let $\mathfrak{n} = \mathcal{U}_2 + \mathcal{U}_3 + \mathfrak{m}_3$. We recall that the nonzero brackets in $\mathfrak{n}$ on the basis vectors are given by (see Notation 3.6):

$$\begin{array}{lll} [X_5, X_4] = Y_1, & [X_1, X_6] = Y_2 & [X_3, X_2] = Y_3, \\ [X_3, X_6] = Y_1, & [X_5, X_2] = Y_2 & [X_1, X_4] = Y_3, \\ [X_5, X_3] = Y_1, & [X_1, X_5] = Y_2 & [X_3, X_1] = Y_3. \end{array}$$

Once again we will take $\lambda_1, \lambda_2, \lambda_3$ to be the roots of $X^3 - 3X + 1$. Let σ be the automorphism of $\mathfrak{n}$ whose matrix with respect to the basis $\{X_i, Y_j : 1 \le i \le 6, 1 \le j \le 3\}$ is a diagonal matrix $D(\lambda_3, \lambda_3, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$. Then σ is a hyperbolic automorphism because of our choice of $\lambda_1, \lambda_2, \lambda_3$.

Let

$$\begin{array}{ll} \mathcal{X}_l = \lambda_3^{l-1} X_1 + \lambda_1^{l-1} X_3 + \lambda_2^{l-1} X_5 & l = 1, 2, 3, \\ \mathcal{Y}_k = \lambda_3^{k-1} X_2 + \lambda_1^{k-1} X_4 + \lambda_2^{k-1} X_6 & k = 1, 2, 3, \\ \mathcal{Z}_r = \lambda_3^{r-1} Y_1 + \lambda_1^{r-1} Y_2 + \lambda_2^{r-1} Y_3 & r = 1, 2, 3. \end{array} \tag{20}$$

We note that a similar construction of $\mathcal{X}_l$'s, $\mathcal{Y}_k$'s and σ have been used in the proof of Proposition 3.7. Let $\beta' = \{\mathcal{X}_l, \mathcal{Y}_k, \mathcal{Z}_r \mid l, k, r = 1, 2, 3\}$. It can seen that $[\sigma]_{\beta'} \in \mathrm{GL}(9, \mathbb{Z})$ To show that β' is a $\mathbb{Z}$-basis of $\mathfrak{n}$, we note that

$$\lambda_1 = \xi + \xi^8, \quad \lambda_2 = \xi^2 + \xi^7, \quad \lambda_3 = \xi^4 + \xi^5,$$

where $\xi = e^{2i\pi/9}$ a ninth root of unity.

Hence if $P(\lambda) = 2\lambda^2 + \lambda - 4$,

$$\lambda_2 - \lambda_3 = P(\lambda_1), \quad \lambda_3 - \lambda_1 = P(\lambda_2), \quad \lambda_1 - \lambda_2 = P(\lambda_3).$$

Moreover, $P(\lambda_i)\lambda_i = \lambda_i^2 + 2\lambda_i - 2$ for $1 \le i \le 3$. Bearing all this in mind, straightforward calculations show that

$$\begin{array}{ll} [\mathcal{Y}_j, \mathcal{Y}_k] = [\mathcal{Z}_j, \mathcal{Z}_k] = 0 \quad 1 \le j, k \le 3, & [\mathcal{X}_1, \mathcal{Y}_1] = 2\mathcal{Z}_1, \\ [\mathcal{X}_1, \mathcal{X}_2] = -4\mathcal{Z}_1 + \mathcal{Z}_2 + 2\mathcal{Z}_3, & [\mathcal{X}_1, \mathcal{Y}_2] = [\mathcal{X}_2, \mathcal{Y}_1] = -\mathcal{Z}_2, \\ [\mathcal{X}_1, \mathcal{X}_3] = 2\mathcal{Z}_1 - 2\mathcal{Z}_2 - \mathcal{Z}_3, & [\mathcal{X}_1, \mathcal{Y}_3] = [\mathcal{X}_3, \mathcal{Y}_1] = 6\mathcal{Z}_1 - \mathcal{Z}_2. \end{array}$$

Hence $\mathcal{U}_2 + \mathcal{U}_3 + \mathfrak{m}_3$ is an Anosov Lie algebra. ■

To conclude, let us study Case **(b)**. We recall that in this case, our assumption is

$$[X_i, X_j] = 0 = [X_k, X_l] \quad \text{for} \quad 1 \le i, j \le 3, \quad 4 \le k, l \le 6.$$

We will show that $\mathfrak{n}$ is isomorphic to one of the Lie algebras we came across in Case (a). In other words, we don't get any new Lie algebras in this case.

We note that we can assume

$$[X_1, X_4] = Y_1, \quad [X_2, X_5] = Y_2 \quad [X_3, X_6] = Y_3, \tag{21}$$

In fact, if this is not the case, rearranging a basis we can only assure that

$$[X_1, X_4] = Y_1, \quad [X_i, X_j] = Y_2 \quad [X_k, X_l] = Y_3,$$

$i, k \le 3$ $j, l \ge 4$ and i or $k = 1$. Let us say that $[X_1, X_5] = Y_2$ and then it is not hard to check that $k \ne 1$ so we may assume that $k = 2$. Also, since $l \ne 6$ ($l = 6$ implies the contradiction $\lambda_1 = \lambda_3$) , say $l = 5$ ($l = 4$ is similar). Then we get at least the following nonzero Lie brackets:

$$[X_1, X_4] = Y_1, \quad [X_1, X_5] = Y_2 \quad [X_2, X_5] = Y_3.$$

Therefore $\lambda_1^2 \lambda_2 \lambda_4 \lambda_5^2 = 1$ or equivalently, $\lambda_1 \lambda_5 = \lambda_3 \lambda_6$ and from this, since $\mathfrak{n}$ has no abelian factor, by considering all possibilities for $[X_r, X_6] = Y_k$ and $[X_3, X_s] = Y_l$ it is not hard to check that we should have $[X_3, X_6] = Y_2$, as desired, since any other possibility leads to a contradiction.

If $\mathfrak{n}$ has only the nonzero Lie brackets given in (21) then $\mathfrak{n} \simeq \mathcal{U}_1$ which is Anosov. If there are more nonzero Lie brackets, with no loss of generality, we may assume that $[X_3, X_5] = a\, Y_1$, and hence we have that

$$\mu_1 = \lambda_1 \lambda_4 = \lambda_3 \lambda_5.$$

Note that since λ_i's are distinct for $1 \le i \le 3$ and λ_j's are distinct for $4 \le j \le 6$, if $[X_i, X_j] = Y_k = [X_l, X_r]$ then $i \ne l$ and $j \ne r$. From this, it can be seen that the nonzero Lie brackets in $\mathfrak{n}$ are given by

$$\begin{aligned} &[X_1, X_4] = Y_1, \quad [X_2, X_5] = Y_2 \quad [X_3, X_6] = Y_3, \\ &[X_3, X_5] = a\, Y_1, \quad [X_1, X_6] = b\, Y_2 \quad [X_2, X_4] = c\, Y_3, \end{aligned} \tag{22}$$

for some constants $a, b, c \in \mathbb{C}$. By reordering the basis

$$\beta' = \{X_1, X_4, X_2, X_5, X_3, X_6, Y_1, Y_2, Y_3\}$$

one can see that this algebra is isomorphic to $\mathfrak{n}(a, b, c)$ given in (13) (from Case (a)) that has already been studied.

Finally, if $\mathfrak{n}$ is Anosov and has an abelian factor, and if $\mathfrak{n} = \tilde{\mathfrak{n}} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is a maximal abelian factor of $\mathfrak{n}$, according to [13, Theorem 3.1] one has that $\dim \mathfrak{m} \ge 2$. Moreover, $\tilde{\mathfrak{n}}$ is also an Anosov Lie algebra and $\dim \tilde{\mathfrak{n}} \le 7$. Since there are no 7 dimensional non-abelian Anosov Lie algebras, we then have that $\mathfrak{n}$ is isomorphic to one of the following (see [14]):

- $\mathbb{R}^9$
- $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^3$
- $\mathfrak{f}_3 \oplus \mathbb{R}^3$,

where $\mathfrak{h}_3$ is the 3-dimensional Heisenberg algebra, and $\mathfrak{f}_3$ is the free 2-step nilpotent Lie algebra on 3 generators.

Summarizing the results of this section we have the following Theorem.

Theorem 3.12. *If $\mathfrak{n}$ is a 9-dimensional Anosov Lie algebra then*

(i) *if it has no abelian factor, then one of the following holds*

- *it is a type $(3,3,3)$ nilpotent Lie algebra isomorphic to $\mathfrak{n}(1,1,1)$ given in (2) or*
- *it is a two step nilpotent Lie algebra of type $(6,3)$ and its Pfaffian form is projectively equivalent to xyz or 0. Moreover it is isomorphic to one of the following non-isomorphic Lie algebras: $\mathcal{U}_1$, $\mathcal{U}_1 + \mathfrak{m}_i$ for $i = 6, 7, 10, 11, 12, 14$, $s\mathcal{U}_2 + \mathcal{U}_3$ $s \in \mathbb{C}$ or $\mathcal{U}_2 + \mathcal{U}_3 + \mathfrak{m}_3$.*

(ii) *If $\mathfrak{n}$ has an abelian factor then it is isomorphic to $\mathbb{R}^9$, $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^3$ or $\mathfrak{f}_3 \oplus \mathbb{R}^3$.*

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