Journal of Lie Theory
Volume **25** (2015) 875–888

Split Regular Hom-Lie Algebras

M. J. Aragón Periñán and A. J. Calderón Martín*

Communicated by A. Pasquale

Abstract. We introduce the class of split regular Hom-Lie algebras as the natural extension of the one of split Lie algebras. We study its structure by showing that an arbitrary split regular Hom-Lie algebra $\mathfrak{L}$ is of the form $\mathfrak{L} = U + \sum_j I_j$, where U is a certain linear subspace of a maximal abelian subalgebra of $\mathfrak{L}$ and the I_j are well described (split) ideals of $\mathfrak{L}$ satisfying $[I_j, I_k] = 0$ if $j \neq k$. Under certain conditions, the simplicity of $\mathfrak{L}$ is characterized and it is shown that $\mathfrak{L}$ is the direct sum of the family of its simple ideals.
Mathematics Subject Classification 2010: 17A30, 17A60, 17B65, 17B22.
Key Words and Phrases: Hom-Lie algebra, roots, root space, structure theory.

1. Introduction and first definitions

A Hom-algebra is an algebra in which the multiplication is twisted by a linear homomorphism. In the Lie case, the twisting of the Jacobi identity leads to the so-called Hom-Lie algebras. This class of algebras appeared in the study of quasi-deformations of Lie algebras of vector fields, in particular quasi-deformations of Witt and Virasoro algebras, [9]. The pioneering works in this subject are [9, 11, 12, 13], where we can find, in particular, the relations of Hom-Lie algebras with discrete and deformed vector fields and differential calculus. Since then, many authors have been interested in the study of Hom-Lie algebras, motivated in part for their applications in physics. See for instance the recent works of Ammar, Arnlind, Fregier, Gohr, Jin, Li, Makhlouf, Sheng, Silvestrov and Yau among other authors, [1, 2, 8, 10, 16, 18, 19]. We also note that other categories of Hom-algebras, such as the Hom-associative algebras, Hom-alternative algebras, Hom-Leibniz algebras and Hom-Lie superalgebras, have been considered in the literature (see [1, 7, 12, 14, 15, 17, 20]).

In the present paper we introduce the class of split regular Hom-Lie algebras as a natural extension of that of split Lie algebras, and study its structure. In §2 we develop the technique of connections of roots, which will become the main tool

* Supported by the PCI of the UCA 'Teoría de Lie y Teoría de Espacios de Banach', by the PAI with project numbers FQM298, FQM7156 and by the project of the Spanish Ministerio de Educación y Ciencia MTM2010-15223.

in our study. In §3 we apply all the machinery introduced in the previous section to show that a split Lie algebra $\mathfrak{L}$ can be written in the form $\mathfrak{L} = U + \sum_j I_j$, where U a certain linear subspace of a maximal abelian subalgebra H of $\mathfrak{L}$ and the I_j are well described ideals of $\mathfrak{L}$ satisfying $[I_j, I_k] = 0$ if $j \neq k$. Finally, in §4, under certain conditions, we characterize the simplicity of $\mathfrak{L}$ and we show that $\mathfrak{L}$ is the direct sum of the family of its simple ideals.

Definition 1.1. A *Hom-Lie algebra* $\mathfrak{L}$ is a vector space over a base field $\mathbb{K}$ endowed with a bilinear product

$$[\cdot,\cdot] : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$$

and with a linear map $\phi : \mathfrak{L} \longrightarrow \mathfrak{L}$ such that

1. $[x, y] = -[y, x]$,
2. $[[x, y], \phi(z)] + [[y, z], \phi(x)] + [[z, x], \phi(y)] = 0$ (Hom-Jacobi identity),

for any $x, y, z \in \mathfrak{L}$. When ϕ furthermore is an algebra automorphism it is said that $\mathfrak{L}$ is a *regular Hom-Lie algebra*.

Throughout this paper we will consider regular Hom-Lie algebras $\mathfrak{L}$. We will denote by $\mathbb{N}$ the set of all non-negative integers and by $\mathbb{Z}$ the set of all integers. Finally, we would like to note that we consider $\mathfrak{L}$ of arbitrary dimension and over an arbitrary base field $\mathbb{K}$.

A *subalgebra* A of $\mathfrak{L}$ is a linear subspace such that $[A, A] \subset A$ and $\phi(A) = A$. A linear subspace I of $\mathfrak{L}$ is called an *ideal* if $[I, \mathfrak{L}] \subset I$ and $\phi(I) = I$. A Hom-Lie algebra $\mathfrak{L}$ will be called *simple* if $[\mathfrak{L}, \mathfrak{L}] \neq 0$ and its only ideals are $\{0\}$ and $\mathfrak{L}$.

Let us introduce the class of split algebras in the framework of regular Hom-Lie algebras $\mathfrak{L}$. Denote by H a maximal abelian subalgebra of $\mathfrak{L}$. For a linear functional

$$\alpha : H \longrightarrow \mathbb{K},$$

we define the *root space* of $\mathfrak{L}$ (with respect to H) associated to α as the subspace

$$\mathfrak{L}_\alpha = \{v_\alpha \in \mathfrak{L} : [h, v_\alpha] = \alpha(h)\phi(v_\alpha) \text{ for any } h \in H\}.$$

The elements $\alpha : H \longrightarrow \mathbb{K}$ satisfying $\mathfrak{L}_\alpha \neq 0$ are called *roots* of $\mathfrak{L}$ with respect to H. We set $\Lambda := \{\alpha \in H^* \setminus \{0\} : \mathfrak{L}_\alpha \neq 0\}$.

Definition 1.2. We say that $\mathfrak{L}$ is a *split regular Hom-Lie algebra*, with respect to H, if

$$\mathfrak{L} = H \oplus (\bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha).$$

We also say that Λ is the *roots system* of $\mathfrak{L}$.

Note that by taking $\phi = Id$, the split Lie algebras become examples of split regular Hom-Lie algebras. Hence, the present paper extends the results in [3].

For an easier notation, the mappings $\phi|_H, \phi|_H^{-1} : H \to H$ will be denoted by ϕ and ϕ^{-1} respectively, when there is no possible confusion.

Lemma 1.3. *For any $\alpha, \beta \in \Lambda \cup \{0\}$ the following assertions hold.*

1. $\phi(\mathfrak{L}_\alpha) \subset \mathfrak{L}_{\alpha\phi^{-1}}$ *and* $\phi^{-1}(\mathfrak{L}_\alpha) \subset \mathfrak{L}_{\alpha\phi}$.
2. $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \subset \mathfrak{L}_{\alpha\phi^{-1}+\beta\phi^{-1}}$.

Proof. 1. For $h \in H$ write $h' = \phi(h)$. Then for all $h \in H$ and $v_\alpha \in \mathfrak{L}_\alpha$, since $[h, v_\alpha] = \alpha(h)\phi(v_\alpha)$, we have

$$[h', \phi(v_\alpha)] = \phi([h, v_\alpha]) = \alpha(h)\phi(\phi(v_\alpha)) = \alpha\phi^{-1}(h')\phi(\phi(v_\alpha)).$$

That is, $\phi(v_\alpha) \in \mathfrak{L}_{\alpha\phi^{-1}}$ and so $\phi(\mathfrak{L}_\alpha) \subset \mathfrak{L}_{\alpha\phi^{-1}}$. In a similar way, we can verify $\phi^{-1}(\mathfrak{L}_\alpha) \subset \mathfrak{L}_{\alpha\phi}$.

2. For any $h \in H$, $v_\alpha \in \mathfrak{L}_\alpha$ and $v_\beta \in \mathfrak{L}_\beta$, by denoting $h' = \phi(h)$, we have that $[h', [v_\alpha, v_\beta]] = -[[h, v_\beta], \phi(v_\alpha)] + [[h, v_\alpha], \phi(v_\beta)] = (\alpha+\beta)(h)\phi([v_\alpha, v_\beta]) = (\alpha+\beta)\phi^{-1}(h')([v_\alpha, v_\beta])$. That is, $[v_\alpha, v_\beta] \in \mathfrak{L}_{\alpha\phi^{-1}+\beta\phi^{-1}}$. ■

Lemma 1.4. *The following assertions hold.*

1. *If $\alpha \in \Lambda$ then $\alpha\phi^{-z} \in \Lambda$ for any $z \in \mathbb{Z}$.*
2. $\mathfrak{L}_0 = H$.

Proof. 1. Consequence of Lemma 1.3-1.

2. The inclusion $H \subset \mathfrak{L}_0$ is a direct consequence of the fact that H is abelian. Let us show that $\mathfrak{L}_0 \subset H$. For any $0 \neq x \in \mathfrak{L}_0$ we can write $x = h \oplus (\bigoplus_{i=1}^{m} v_{\alpha_i})$, where $h \in H$ and $v_{\alpha_i} \in \mathfrak{L}_{\alpha_i}$ with $\alpha_i \neq \alpha_j$ when $i \neq j$. Since for any $h' \in H$ we have $[h', v] = 0$, then $0 = \bigoplus_{i=1}^{m} \alpha_i(h')\phi(v_{\alpha_i})$. Hence, Lemma 1.3-1 together with the fact $\alpha_i \neq 0$ gives us all $v_{\alpha_i} = 0$. So $x = h \in H$. ■

2. Connections of roots

In the following, $\mathfrak{L}$ denotes a split regular Hom-Lie algebra and

$$\mathfrak{L} = \mathfrak{L}_0 \oplus (\bigoplus_{\alpha \in \Lambda} \mathfrak{L}_\alpha)$$

is the corresponding root space decomposition. Given a linear functional $\alpha : H \to \mathbb{K}$, we denote by $-\alpha : H \to \mathbb{K}$ the element in H^* defined by $(-\alpha)(h) := -\alpha(h)$ for all $h \in H$. We also indicate

$$-\Lambda = \{-\alpha : \alpha \in \Lambda\}.$$

Definition 2.1. Let α and β be two elements in Λ. We shall say that α is *connected* to β if there exists $\{\alpha_1, \alpha_2, ..., \alpha_k\} \subset \pm\Lambda$ such that:

If $k = 1$, then

1. $\alpha_1 \in \{\alpha\phi^{-n} : n \in \mathbb{N}\} \cap \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$.

If $k \geq 2$, then

1. $\alpha_1 \in \{\alpha\phi^{-n} : n \in \mathbb{N}\}$.
2. $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \pm\Lambda$,
$\alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1} \in \pm\Lambda$,
$\alpha_1\phi^{-3} + \alpha_2\phi^{-3} + \alpha_3\phi^{-2} + \alpha_4\phi^{-1} \in \pm\Lambda$,
....
$\alpha_1\phi^{-i} + \alpha_2\phi^{-i} + \alpha_3\phi^{-i+1} + \cdots + \alpha_{i+1}\phi^{-1} \in \pm\Lambda$,
....
$\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+2} + \alpha_3\phi^{-k+3} + \cdots + \alpha_i\phi^{-k+i} + \cdots + \alpha_{k-1}\phi^{-1} \in \pm\Lambda$.
3. $\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1} \in \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$.

We shall also say that $\{\alpha_1, ..., \alpha_k\}$ is a *connection* from α to β.

Observe that the case $k = 1$ in Definition 2.1 is equivalent to the fact $\beta = \epsilon\alpha\phi^z$ for some $z \in \mathbb{Z}$ and $\epsilon \in \{\pm 1\}$.

Our next goal is to show that the connection is an equivalence relation on Λ. We first need to prove two lemmas.

Lemma 2.2. *For any $\alpha \in \Lambda$, we have that $\alpha\phi^{z_1}$ is connected to $\alpha\phi^{z_2}$ for every $z_1, z_2 \in \mathbb{Z}$. We also have that $\alpha\phi^{z_1}$ is connected to $-\alpha\phi^{z_2}$ in case $-\alpha\phi^{z_2} \in \Lambda$.*

Proof. By Lemma 1.4-1 we have that $\alpha\phi^{z_1}, \alpha\phi^{z_2} \in \Lambda$. Set $z = \min\{z_1, z_2\}$. Then $\{\alpha\phi^z\}$ is a connection from $\alpha\phi^{z_1}$ to $\alpha\phi^{z_2}$ and to $-\alpha\phi^{z_2}$ in case $-\alpha\phi^{z_2} \in \Lambda$. ■

Lemma 2.3. *Let $\{\alpha_1, ..., \alpha_k\}$ be a connection from α to β. Then the following assertions hold.*

1. *Suppose $\alpha_1 = \alpha\phi^{-n}$, $n \in \mathbb{N}$. Then for any $r \in \mathbb{N}$ such that $r \geq n$, there exists a connection $\{\bar{\alpha}_1, ..., \bar{\alpha}_k\}$ from α to β such that $\bar{\alpha}_1 = \alpha\phi^{-r}$.*

2. *Suppose that $\alpha_1 = \epsilon\beta\phi^{-m}$ in case $k = 1$ or*

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_k\phi^{-1} = \epsilon\beta\phi^{-m}$$

in case $k \geq 2$, with $m \in \mathbb{N}$ and $\epsilon \in \{\pm 1\}$. Then for any $r \in \mathbb{N}$ such that $r \geq m$, there exists a connection $\{\bar{\alpha}_1, ..., \bar{\alpha}_k\}$ from α to β such that $\bar{\alpha}_1 = \epsilon\beta\phi^{-r}$ in case $k = 1$ or

$$\bar{\alpha}_1\phi^{-k+1} + \bar{\alpha}_2\phi^{-k+1} + \bar{\alpha}_3\phi^{-k+2} + \cdots + \bar{\alpha}_k\phi^{-1} = \epsilon\beta\phi^{-r}$$

in case $k \geq 2$.

Proof. 1. By Lemma 1.4-1 we have $\{\alpha_1\phi^{n-r}, ..., \alpha_k\phi^{n-r}\} \subset \pm\Lambda$. Define $\bar{\alpha}_i := \alpha_i\phi^{n-r}$, $i = 1, ..., k$. Then Lemma 1.4-1 allows us to verify that $\{\bar{\alpha}_1, ..., \bar{\alpha}_k\}$ is a connection from α to β. This connection clearly satisfies $\bar{\alpha}_1 = (\alpha\phi^{-n})\phi^{n-r} = \alpha\phi^{-r}$.

2. Lemma 1.4-1 allows us to assert that $\{\alpha_1\phi^{m-r}, ..., \alpha_k\phi^{m-r}\} \subset \pm\Lambda$. Define now $\bar{\alpha}_i := \alpha_i\phi^{m-r}$, $i = 1, ..., k$. Then Lemma 1.4-1 gives us that $\{\bar{\alpha}_1, ..., \bar{\alpha}_k\}$ is a connection from α to β. It is clear that $\bar{\alpha}_1 = \epsilon\beta\phi^{-r}$ in case $k = 1$, and also

$$\begin{aligned}
&\bar{\alpha}_1\phi^{-k+1} + \bar{\alpha}_2\phi^{-k+1} + \bar{\alpha}_3\phi^{-k+2} + \cdots + \bar{\alpha}_k\phi^{-1} \\
&\quad = (\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_k\phi^{-1})\phi^{m-r} \\
&\quad = \epsilon\beta\phi^{-r}
\end{aligned}$$

in case $k \geq 2$. ■

Proposition 2.4. *The relation $\sim$ in Λ defined by*

$$\alpha \sim \beta \text{ if and only if } \alpha \text{ is connected to } \beta$$

is an equivalence relation.

Proof. Lemma 2.2 gives us $\alpha \sim \alpha$ for any $\alpha \in \Lambda$. That is, the relation $\sim$ is reflexive.

To verify the symmetric character of $\sim$, suppose $\alpha \sim \beta$. Then there exists a connection

$$\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{k-1}, \alpha_k\} \subset \pm\Lambda \tag{1}$$

from α to β.

If $k = 1$, then $\alpha_1 = \alpha\phi^{-n}$ and $\alpha_1 = \epsilon\beta\phi^{-m}$ with $n, m \in \mathbb{N}$ and $\epsilon \in \{\pm 1\}$. Hence, $\{\epsilon\alpha_1\}$ is a connection from β to α.

If $k \geq 2$, observe that condition 3 in Definition 2.1 let us distinguish two possibilities. In the first one

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_{k-i+1}\phi^{-i} + \cdots + \alpha_k\phi^{-1} = \beta\phi^{-m}, \tag{2}$$

while in the second one

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_{k-i+1}\phi^{-i} + \cdots + \alpha_k\phi^{-1} = -\beta\phi^{-m} \tag{3}$$

for some $m \in \mathbb{N}$.

Suppose first that (2) holds. Then Lemma 1.4-1 allows us to take the set

$$\{\beta\phi^{-m}, -\alpha_k\phi^{-1}, -\alpha_{k-1}\phi^{-3}, -\alpha_{k-2}\phi^{-5}, ..., -\alpha_{k-i}\phi^{-2i-1}, ..., -\alpha_2\phi^{-2k+3}\} \pm \Lambda.$$

We are going to show that this set is a connection from β to α. It is clear that it satisfies condition 1 of Definition 2.1, so let us check that it also satisfies condition 2. We have

$$\begin{aligned}
&(\beta\phi^{-m})\phi^{-1} - (\alpha_k\phi^{-1})\phi^{-1} = (\beta\phi^{-m} - \alpha_k\phi^{-1})\phi^{-1} \\
&\quad = (\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_{k-1}\phi^{-2})\phi^{-1},
\end{aligned}$$

the last equality being a consequence of equation (2). So

$$(\beta\phi^{-m})\phi^{-1} - (\alpha_k\phi^{-1})\phi^{-1} = \\ (\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+2} + \alpha_3\phi^{-k+3} + \cdots + \alpha_i\phi^{-k+i} + \cdots + \alpha_{k-1}\phi^{-1})\phi^{-2}.$$

Since

$$\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+2} + \alpha_3\phi^{-k+3} + \cdots + \alpha_i\phi^{-k+i} + \cdots + \alpha_{k-1}\phi^{-1} \in \pm\Lambda$$

by condition 2 of Definition 2.1 applied to the connection (1), Lemma 1.4-1 let us assert $(\beta\phi^{-m})\phi^{-1} - (\alpha_k\phi^{-1})\phi^{-1} \in \pm\Lambda$.

For any $1 \leq i \leq k-2$ we also have that

$$\begin{aligned}(\beta\phi^{-m})\phi^{-i} - (\alpha_k\phi^{-1})\phi^{-i} - (\alpha_{k-1}\phi^{-3})\phi^{-i+1} - \cdots - (\alpha_{k-(i-1)}\phi^{-2i+1})\phi^{-1} \\ = (\beta\phi^{-m} - \alpha_k\phi^{-1} - \alpha_{k-1}\phi^{-2} - \cdots - \alpha_{k-(i-1)}\phi^{-i})\phi^{-i} \\ = (\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \cdots + \alpha_{k-i}\phi^{-i-1})\phi^{-i},\end{aligned}$$

the last equality being a consequence of equation (2). Hence,

$$(\beta\phi^{-m})\phi^{-i} - (\alpha_k\phi^{-1})\phi^{-i} - (\alpha_{k-1}\phi^{-3})\phi^{-i+1} - \cdots - (\alpha_{k-(i-1)}\phi^{-2i+1})\phi^{-1} = \\ (\alpha_1\phi^{-k+i+1} + \alpha_2\phi^{-k+i+1} + \cdots + \alpha_{k-i}\phi^{-1})\phi^{-2i}.$$

Taking now into account that, by condition 2 of Definition 2.1 applied to (1), $\alpha_1\phi^{-k+1+i} + \alpha_2\phi^{-k+1+i} + \cdots + \alpha_{k-i}\phi^{-1} \in \pm\Lambda$, we get as consequence of Lemma 1.4-1 that

$$(\beta\phi^{-m})\phi^{-i} - (\alpha_k\phi^{-1})\phi^{-i} - (\alpha_{k-1}\phi^{-3})\phi^{-i+1} - \cdots - (\alpha_{k-(i-1)}\phi^{-2i+1})\phi^{-1} \in \pm\Lambda.$$

We have showed that our set satisfies condition 2 of Definition 2.1. It just remains to prove that this set also satisfies condition 3 of this definition. We have, as above,

$$\begin{aligned}(\beta\phi^{-m})\phi^{-k+1} - (\alpha_k\phi^{-1})\phi^{-k+1} - (\alpha_{k-1}\phi^{-3})\phi^{-k+2} - \cdots - (\alpha_2\phi^{-2k+3})\phi^{-1} \\ = (\beta\phi^{-m} - \alpha_k\phi^{-1} - \alpha_{k-1}\phi^{-2} - \cdots - \alpha_2\phi^{-k+1})\phi^{-k+1} \\ = (\alpha_1\phi^{-k+1})\phi^{-k+1}.\end{aligned}$$

Condition 1 of Definition 2.1 applied to the connection (1) gives us now that $\alpha_1 = \alpha\phi^{-n}$ for some $n \in \mathbb{N}$ and so

$$\begin{aligned}(\beta\phi^{-m})\phi^{-k+1} - (\alpha_k\phi^{-1})\phi^{-k+1} - (\alpha_{k-1}\phi^{-3})\phi^{-k+2} - \cdots - (\alpha_2\phi^{-2k+3})\phi^{-1} \\ = \alpha\phi^{-2k-n+2} \in \{\alpha\phi^{-h} : h \in \mathbb{N}\}.\end{aligned}$$

We have therefore showed that our set is actually a connection from β to α.

Suppose now we are in the second possibility, given by equation (3). Then we can prove, as in the first possibility given by equation (2), that

$$\{\beta\phi^{-m}, \alpha_k\phi^{-1}, \alpha_{k-1}\phi^{-3}, \alpha_{k-2}\phi^{-5}, ..., \alpha_{k-i}\phi^{-2i-1}, ..., \alpha_2\phi^{-2k+3}\}$$

is a connection from β to α. We conclude $\beta \sim \alpha$, and thus the relation $\sim$ is symmetric.

Finally, let us verify that $\sim$ is transitive. Suppose $\alpha \sim \beta$ and $\beta \sim \gamma$, and write $\{\alpha_1, ..., \alpha_k\}$ for a connection from α to β which satisfies

$$\alpha_1 = \epsilon\beta\phi^{-m} \quad \text{if } k = 1 \tag{4}$$

or

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_k\phi^{-1} = \epsilon\beta\phi^{-m} \quad \text{if } k \geq 2, \tag{5}$$

for some $m \in \mathbb{N}$, $\epsilon \in \{\pm 1\}$; and $\{h_1, ..., h_p\}$ for a connection from β to γ, being then

$$h_1 = \beta\phi^{-r} \tag{6}$$

for some $r \in \mathbb{N}$. Note that Lemma 2.3 let us suppose $m = r$.

If $p = 1$, then $h_1 = \tau\gamma\phi^{-t}$ with $t \in \mathbb{N}$ and $\tau \in \{\pm 1\}$. Since $m = r$, we have $\alpha_1 = \epsilon\beta\phi^{-m} = \epsilon h_1 = \epsilon\tau\gamma\phi^{-t}$ if $k = 1$, and

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_k\phi^{-1} = \epsilon\beta\phi^{-m} = \epsilon h_1 = \epsilon\tau\gamma\phi^{-t}$$

if $k \geq 2$. Hence, we get that $\{\alpha_1, ..., \alpha_k\}$ is also a connection from α to γ.

If $p \geq 2$, then, taking into account the equations (4), (5) and (6) and that $m = r$, it is easy to show that a connection from α to γ is $\{\alpha_1, ..., \alpha_k, h_2, ..., h_p\}$ if $\epsilon = 1$, and $\{\alpha_1, ..., \alpha_k, -h_2, ..., -h_p\}$ if $\epsilon = -1$. Summarizing, the connection relation is also transitive and so it is an equivalence relation. ■

3. Decompositions

By Proposition 2.4 the connection relation is an equivalence relation in Λ. We can therefore consider the quotient set

$$\Lambda/\sim = \{[\alpha] : \alpha \in \Lambda\},$$

where $[\alpha]$ denotes the set of nonzero roots of $\mathfrak{L}$ which are connected to α.

Our next goal in this section is to associate an (adequate) ideal $I_{[\alpha]}$ to any $[\alpha]$. For a fixed $\alpha \in \Lambda$, we start by defining the set $I_{0,[\alpha]} \subset \mathfrak{L}_0$ as follows:

$$I_{0,[\alpha]} := \operatorname{span}_{\mathbb{K}}\{[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] : \beta, \gamma \in [\alpha] \cup \{0\}\} \cap \mathfrak{L}_0.$$

Applying Lemma 1.3-2, we obtain

$$I_{0,[\alpha]} = \operatorname{span}_{\mathbb{K}}\{[\mathfrak{L}_\beta, \mathfrak{L}_{-\beta}] : \beta \in [\alpha]\}.$$

Next, we define

$$V_{[\alpha]} := \bigoplus_{\beta \in [\alpha]} \mathfrak{L}_\beta.$$

Finally, we denote by $I_{[\alpha]}$ the direct sum of the two subspaces above, that is,

$$I_{[\alpha]} := I_{0,[\alpha]} \oplus V_{[\alpha]}.$$

Proposition 3.1. *For any $\alpha \in \Lambda$, the linear subspace $I_{[\alpha]}$ is a subalgebra of $\mathfrak{L}$.*

Proof. First, we have to check that $I_{[\alpha]}$ satisfies $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$. Since $I_{0,[\alpha]} \subset \mathfrak{L}_0 = H$, then $[I_{0,[\alpha]}, I_{0,[\alpha]}] = 0$ and we have

$$[I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\alpha]} \oplus V_{[\alpha]}] \subset [I_{0,[\alpha]}, V_{[\alpha]}] + [V_{[\alpha]}, V_{[\alpha]}]. \tag{7}$$

Let us consider the first summand in equation (7). Given $\beta \in [\alpha]$ we have $[I_{0,[\alpha]}, \mathfrak{L}_\beta] \subset \mathfrak{L}_{\beta\phi^{-1}}$, being $\beta\phi^{-1} \in [\alpha]$ by Lemma 2.2. Hence $[I_{0,[\alpha]}, \mathfrak{L}_\beta] \subset V_{[\alpha]}$. Consider now the second summand in equation (7), that is, $[V_{[\alpha]}, V_{[\alpha]}]$. Given $\beta, \gamma \in [\alpha]$ such that $[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] \neq 0$, if $\gamma = -\beta$ then clearly $[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] = [\mathfrak{L}_\beta, \mathfrak{L}_{-\beta}] \subset I_{0,[\alpha]}$. Suppose then $\gamma \neq -\beta$. Since $[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] \neq 0$ together with Lemma 1.3-2 ensures that $\beta\phi^{-1} + \gamma\phi^{-1} \in \Lambda$, we have that $\{\beta, \gamma\}$ is a connection from β to $\beta\phi^{-1} + \gamma\phi^{-1}$. The transitivity of $\sim$ gives now that $\beta\phi^{-1} + \gamma\phi^{-1} \in [\alpha]$ and so $[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] \subset \mathfrak{L}_{\beta\phi^{-1}+\gamma\phi^{-1}} \subset V_{[\alpha]}$. Hence $[\bigoplus_{\beta\in[\alpha]} \mathfrak{L}_\beta, \bigoplus_{\beta\in[\alpha]} \mathfrak{L}_\beta] \subset I_{0,[\alpha]} \oplus V_{[\alpha]}$. That is,

$$[V_{[\alpha]}, V_{[\alpha]}] \subset I_{[\alpha]}. \tag{8}$$

From equations (7) and (8) we get $[I_{[\alpha]}, I_{[\alpha]}] = [I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\alpha]} \oplus V_{[\alpha]}] \subset I_{[\alpha]}$.

Second, we have to verify that $\phi(I_{[\alpha]}) = I_{[\alpha]}$. But this is a direct consequence of Lemma 1.3-1 and Lemma 2.2. ■

Proposition 3.2. *For any $[\alpha] \neq [\beta]$ we have $[I_{[\alpha]}, I_{[\beta]}] = 0$.*

Proof. We have

$$[I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\beta]} \oplus V_{[\beta]}] \subset [I_{0,[\alpha]} V_{[\beta]}] + [V_{[\alpha]}, I_{0,[\beta]}] + [V_{[\alpha]}, V_{[\beta]}]. \tag{9}$$

Consider the third summand $[V_{[\alpha]}, V_{[\beta]}]$ and suppose there exist $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$ such that $[\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{\alpha_2}] \neq 0$. As necessarily $\alpha_1 \neq -\alpha_2$, then $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$. So $\{\alpha_1, \alpha_2, -\alpha_1\phi^{-1}\}$ is a connection between α_1 and α_2. By the transitivity of the connection relation we have $\alpha \in [\beta]$, a contradiction. Hence $[\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{\alpha_2}] = 0$ and so

$$[V_{[\alpha]}, V_{[\beta]}] = 0. \tag{10}$$

Consider now the first summand $[I_{0,[\alpha]} V_{[\beta]}]$ in equation (9). Let us take $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$ and show that

$$\alpha_2([\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{-\alpha_1}]) = 0.$$

Indeed, by the Hom-Jacobi identity and equation (10), we have

$$[[\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{-\alpha_1}], \phi(\mathfrak{L}_{\alpha_2})] = 0$$

. So, taking into account that ϕ is injective (see Lemma 1.4), we can assert that $\alpha_2\phi^{-1}([\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{-\alpha_1}]) = 0$ for any $\alpha_1 \in [\alpha]$. Since

$$\phi([\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{-\alpha_1}]) \subset [\mathfrak{L}_{\alpha_1\phi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-1}}]$$

we get $[\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{-\alpha_1}] \subset \phi^{-1}([\mathfrak{L}_{\alpha_1\phi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-1}}])$ and by the above $\alpha_2([\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{-\alpha_1}]) = 0$. From here $[[\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{-\alpha_1}], \mathfrak{L}_{\alpha_2}] \subset \alpha_2([\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{-\alpha_1}])\phi(\mathfrak{L}_{\alpha_2}) = 0$. We have showed $[I_{0,[\alpha]} V_{[\beta]}] = 0$. In a similar way, we get $[V_{[\alpha]}, I_{0,[\beta]}] = 0$ and we conclude, together with equations (9) and (10), that $[I_{[\alpha]}, I_{[\beta]}] = 0$. ■

Theorem 3.3. *The following assertions hold.*

1. *For any $\alpha \in \Lambda$, the subalgebra*

$$I_{[\alpha]} = I_{0,[\alpha]} \oplus V_{[\alpha]}$$

of $\mathfrak{L}$ associated to $[\alpha]$ is an ideal of $\mathfrak{L}$.

2. *If $\mathfrak{L}$ is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$ and $H = \sum\limits_{\alpha \in \Lambda} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$.*

Proof. 1. Since $[I_{[\alpha]}, H] \subset I_{[\alpha]}$ we have by Propositions 3.1 and 3.2 that

$$[I_{[\alpha]}, \mathfrak{L}] = [I_{[\alpha]}, H \oplus (\bigoplus_{\beta \in [\alpha]} \mathfrak{L}_\beta) \oplus (\bigoplus_{\gamma \notin [\alpha]} \mathfrak{L}_\gamma)] \subset I_{[\alpha]}.$$

As we also have by Proposition 3.1 that $\phi(I_{[\alpha]}) = I_{[\alpha]}$, we conclude that $I_{[\alpha]}$ is an ideal of I.

2. The simplicity of $\mathfrak{L}$ implies $I_{[\alpha]} = \mathfrak{L}$. From here, it is clear that $[\alpha] = \Lambda$ and $H = \sum\limits_{\alpha \in \Lambda} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$. ■

Theorem 3.4. *We have*

$$\mathfrak{L} = U + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]},$$

where U is a linear complement in H of $\mathrm{span}_{\mathbb{K}}\{[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}] : \alpha \in \Lambda\}$ and any $I_{[\alpha]}$ is one of the ideals of $\mathfrak{L}$ described in Theorem 3.3-1, satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.

Proof. Each $I_{[\alpha]}$ is well defined and, by Theorem 3.3-1, an ideal of $\mathfrak{L}$. It is clear that

$$\mathfrak{L} = H \oplus (\bigoplus_{\alpha \in \Lambda} \mathfrak{L}_\alpha) = U + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}.$$

Finally Proposition 3.2 gives us $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. ■

Let us denote by $\mathcal{Z}(\mathfrak{L}) = \{v \in \mathfrak{L} : [v, \mathfrak{L}] = 0\}$ the *center* of $\mathfrak{L}$.

Corollary 3.5. *If $\mathcal{Z}(\mathfrak{L}) = 0$ and $H = \sum\limits_{\alpha \in \Lambda} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$. Then $\mathfrak{L}$ is the direct sum of the ideals given in Theorem 3.3:*

$$\mathfrak{L} = \bigoplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]},$$

being $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.

Proof. Since $H = \sum\limits_{\alpha \in \Lambda} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$ we get $\mathfrak{L} = \sum\limits_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$. Finally, the sum is direct because $\mathcal{Z}(\mathfrak{L}) = 0$ and $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. ■

4. The simple components

In this section we are going to study under which conditions $\mathfrak{L}$ decomposes as the direct sum of the family of its simple ideals. We recall that a roots system Λ of a split regular Hom-Lie algebra $\mathfrak{L}$ is called *symmetric* if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$. From now on we will suppose Λ is symmetric.

Lemma 4.1. *If I is an ideal of $\mathfrak{L}$ such that $I \subset H$, then $I \subset \mathcal{Z}(\mathfrak{L})$.*

Proof. Consequence of $[I, H] \subset [H, H] = 0$ and $[I, \bigoplus_{\alpha \in \Lambda} \mathfrak{L}_\alpha] \subset (\bigoplus_{\alpha \in \Lambda} \mathfrak{L}_\alpha) \cap H = 0$. ∎

Lemma 4.2. *For any $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ there exists $h_0 \in H$ such that $\alpha(h_0) \neq 0$ and $\alpha(h_0) \neq \beta(h_0)$.*

Proof. As $\alpha \neq \beta$, there exists $h \in H$ such that $\alpha(h) \neq \beta(h)$. If $\alpha(h) \neq 0$ we have finished. So let us suppose $\alpha(h) = 0$, which implies $\beta(h) \neq 0$. Since $\alpha \neq 0$, we can fix some $h' \in H$ such that $\alpha(h') \neq 0$. We can distinguish two cases, in the first one $\alpha(h') \neq \beta(h')$ and in the second one $\alpha(h') = \beta(h')$. Then, by taking $h_0 := h'$ in the first case and $h_0 := h + h'$ in the second, we complete the proof. ∎

Lemma 4.3. *Suppose that I is an ideal of $\mathfrak{L}$ and $x = h + \sum_{j=1}^{n} v_{\alpha_j} \in I$, with $h \in H$, $v_{\alpha_j} \in \mathfrak{L}_{\alpha_j}$ and $\alpha_j \neq \alpha_k$ if $j \neq k$. Then any $v_{\alpha_j} \in I$.*

Proof. If $n = 1$ we have $x = h + v_{\alpha_1} \in I$. By taking $h' \in H$ such that $\alpha_1(h') \neq 0$ we have $[h', x] = \alpha_1(h')\phi(v_{\alpha_1}) \in I$ and so $\phi(v_{\alpha_1}) \in I$. Thus $\phi^{-1}(\phi(v_{\alpha_1})) = v_{\alpha_1} \in I$.

Suppose now $n > 1$ and consider α_1 and α_2. By Lemma 4.2 there exists $h_0 \in H$ such that $\alpha_1(h_0) \neq 0$ and $\alpha_1(h_0) \neq \alpha_2(h_0)$. Then we have

$$I \ni [h_0, x] = \alpha_1(h_0)\phi(v_{\alpha_1}) + \alpha_2(h_0)\phi(v_{\alpha_2}) + \cdots + \alpha_n(h_0)\phi(v_{\alpha_n}) \tag{11}$$

and

$$I \ni \phi(x) = \phi(h) + \phi(v_{\alpha_1}) + \phi(v_{\alpha_2}) + \cdots + \phi(v_{\alpha_n}). \tag{12}$$

By multiplying (12) by $\alpha_2(h_0)$ and subtracting (11), we get

$$\alpha_2(h_0)\phi(h) + (\alpha_2(h_0) - \alpha_1(h_0))\phi(v_{\alpha_1}) \\ + (\alpha_2(h_0) - \alpha_3(h_0))\phi(v_{\alpha_3}) + \cdots + (\alpha_2(h_0) - \alpha_n(h_0))\phi(v_{\alpha_n}) \in I.$$

By setting $\tilde{h} := \alpha_2(h_0)\phi(h) \in H$ and $v_{\alpha_i\phi^{-1}} := (\alpha_2(h_0) - \alpha_i(h_0))\phi(v_{\alpha_i}) \in \mathfrak{L}_{\alpha_i\phi^{-1}}$, we can write

$$\tilde{h} + v_{\alpha_1\phi^{-1}} + v_{\alpha_3\phi^{-1}} + \cdots + v_{\alpha_n\phi^{-1}} \in I. \tag{13}$$

Now we can argue as above, with equation (13), to get

$$\tilde{\tilde{h}} + v_{\alpha_1\phi^{-2}} + v_{\alpha_4\phi^{-2}} + \cdots + v_{\alpha_n\phi^{-2}} \in I$$

for $\tilde{\tilde{h}} \in H$ and $v_{\alpha_1\phi^{-2}} \in \mathfrak{L}_{\alpha_i\phi^{-2}}$. By iterating this process, we obtain

$$\bar{h} + v_{\alpha_1\phi^{-n+1}} \in I$$

with $\bar{h} \in H$ and $v_{\alpha_1\phi^{-n+1}} \in \mathfrak{L}_{\alpha_i\phi^{-n+1}}$. As in the above case $n = 1$, we conclude $v_{\alpha_1\phi^{-n+1}} \in I$ and consequently $v_{\alpha_1} = \phi^{-n+1}(v_{\alpha_1\phi^{-n+1}}) \in I$.

In a similar way we can prove that $v_{\alpha_i} \in I$ for $i \in \{2, ..., n\}$, and the proof is complete. ■

Let us introduce the concepts of root-multiplicativity and maximal length in the framework of split Hom-Lie algebras, in a similar way to the ones for split Lie algebras, split Lie color algebras, split Leibniz algebras and split Poisson algebras (see [3, 4, 5, 6] for these notions and examples).

Definition 4.4. We say that a split regular Hom-Lie algebra $\mathfrak{L}$ is *root-multiplicative* if given $\alpha, \beta \in \Lambda$ such that $\alpha\phi^{-1} + \beta\phi^{-1} \in \Lambda$, then $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \neq 0$.

Definition 4.5. A split regular Hom-Lie algebra $\mathfrak{L}$ is said to be of *maximal length* if $\dim \mathfrak{L}_\alpha = 1$ for any $\alpha \in \Lambda$.

Theorem 4.6. *Let $\mathfrak{L}$ be a split regular Hom-Lie algebra of maximal length and root-multiplicative. Then $\mathfrak{L}$ is simple if and only if $\mathcal{Z}(\mathfrak{L}) = 0$, $H = \sum_{\alpha \in \Lambda} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$ and Λ has all of its elements connected.*

Proof. Suppose $\mathfrak{L}$ is simple. Since $\mathcal{Z}(\mathfrak{L})$ is an ideal of $\mathfrak{L}$ then $\mathcal{Z}(\mathfrak{L}) = 0$. From here, Theorem 3.3-2 completes the proof of the first implication. To prove the converse, consider I a nonzero ideal of $\mathfrak{L}$. By Lemma 4.3 we can write $I = (I \cap H) \oplus (\bigoplus_{\alpha \in \Lambda} I_\alpha)$, where $I_\alpha = I \cap \mathfrak{L}_\alpha$. By the maximal length of $\mathfrak{L}$, if we set $\Lambda_I := \{\alpha \in \Lambda : I_\alpha \neq 0\}$, we can write $I = (I \cap H) \oplus (\bigoplus_{\alpha \in \Lambda_I} \mathfrak{L}_\alpha)$, where $\Lambda_I \neq \emptyset$ as consequence of Lemma 4.1. Let us fix some $\alpha_0 \in \Lambda_I$ so that $0 \neq \mathfrak{L}_{\alpha_0} \subset I$. The fact $\phi(I) = I$ together with Lemma 1.3-1 allows us to assert that

$$\text{if } \alpha \in \Lambda_I \quad \text{then} \quad \{\alpha\phi^z : z \in \mathbb{Z}\} \subset \Lambda_I, \tag{14}$$

that is

$$\{\mathfrak{L}_{\alpha_0\phi^z} : z \in \mathbb{Z}\} \subset I. \tag{15}$$

Now, let us take any $\beta \in \Lambda$ satisfying $\beta \notin \{\pm\alpha_0\phi^z : z \in \mathbb{Z}\}$. Since α_0 and β are connected, we have a connection $\{\alpha_1, \alpha_2, ..., \alpha_k\}$, $k \geq 2$, from α_0 to β satisfying:

$\alpha_1 = \alpha_0\phi^{-n}$ for some $n \in \mathbb{N}$, and

$\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$,

$\alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1} \in \Lambda$,

....

$\alpha_1\phi^{-i} + \alpha_2\phi^{-i} + \alpha_3\phi^{-i+1} + \cdots + \alpha_{i+1}\phi^{-1} \in \Lambda$,

....

$$\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+2} + \alpha_3\phi^{-k+3} + \cdots + \alpha_i\phi^{-k+i} + \cdots + \alpha_{k-1}\phi^{-1} \in \Lambda,$$
$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1} = \epsilon\beta\phi^{-m}$$
for some $m \in \mathbb{N}$ and $\epsilon \in \{\pm 1\}$.

Taking into account that $\alpha_1, \alpha_2 \in \Lambda$ and $\alpha_1\phi^{-1}+\alpha_2\phi^{-1} \in \Lambda$, the root-multiplicativity and maximal length of $\mathfrak{L}$ allow us to assert $0 \neq [\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{\alpha_2}] = \mathfrak{L}_{\alpha_1\phi^{-1}+\alpha_2\phi^{-1}}$. Since $0 \neq \mathfrak{L}_{\alpha_1} \subset I$ as consequence of Equation (15) we get

$$0 \neq \mathfrak{L}_{\alpha_1\phi^{-1}+\alpha_2\phi^{-1}} \subset I.$$

A similar argument applied to $\alpha_1\phi^{-1} + \alpha_2\phi^{-1}$, α_3 and

$$(\alpha_1\phi^{-1} + \alpha_2\phi^{-1})\phi^{-1} + \alpha_3\phi^{-1} = \alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1}$$

gives us $0 \neq \mathfrak{L}_{\alpha_1\phi^{-2}+\alpha_2\phi^{-2}+\alpha_3\phi^{-1}} \subset I$. We can follow this process with the connection $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ to get

$$0 \neq \mathfrak{L}_{\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+1}+\alpha_3\phi^{-k+2}+\cdots+\alpha_i\phi^{-k+i-1}+\cdots+\alpha_k\phi^{-1}} \subset I$$

and then

$$\text{either} \quad \mathfrak{L}_{\beta\phi^{-m}} \subset I \quad \text{or} \quad \mathfrak{L}_{-\beta\phi^{-m}} \subset I.$$

From equations (14) and (15), we now get

$$\text{either} \quad \{\mathfrak{L}_{\alpha\phi^{-z}} : z \in \mathbb{Z}\} \subset I \quad \text{or} \quad \{\mathfrak{L}_{-\alpha\phi^{-z}} : z \in \mathbb{Z}\} \subset I \quad \text{for any } \alpha \in \Lambda. \tag{16}$$

Equation (16) can be reformulated by asserting that given any $\alpha \in \Lambda$ either $\{\alpha\phi^{-z} : z \in \mathbb{Z}\}$ or $\{-\alpha\phi^{-z} : z \in \mathbb{Z}\}$ is contained in Λ_I. Taking now into account $H = \sum_{\alpha\in\Lambda} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$, we obtain

$$H \subset I. \tag{17}$$

If we consider now any $\alpha \in \Lambda$, since $\mathfrak{L}_\alpha = [H, \mathfrak{L}_\alpha]$ by the maximal length of $\mathfrak{L}$, the inclusion (17) gives us $\mathfrak{L}_\alpha \subset I$ and so $I = \mathfrak{L}$. That is, $\mathfrak{L}$ is simple. ■

Theorem 4.7. *Let $\mathfrak{L}$ be a split regular Hom-Lie algebra of maximal length, root-multiplicative, with $\mathcal{Z}(\mathfrak{L}) = 0$, and satisfying $H = \sum_{\alpha\in\Lambda} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$. Then*

$$\mathfrak{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is a simple (split) ideal having its roots system, $\Lambda_{I_{[\alpha]}}$, with all of its elements $\Lambda_{I_{[\alpha]}}$-connected.

Proof. Taking into account Corollary 3.5 we can write $\mathfrak{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ as the direct sum of the family of ideals

$$I_{[\alpha]} = I_{0,[\alpha]} \oplus V_{[\alpha]} = \operatorname{span}_{\mathbb{K}}\{[\mathfrak{L}_\beta, \mathfrak{L}_{-\beta}] : \beta \in [\alpha]\} \oplus \bigoplus_{\beta\in[\alpha]} \mathfrak{L}_\beta,$$

where each $I_{[\alpha]}$ is a split regular Hom-Lie algebra having as roots system $\Lambda_{I_{[\alpha]}} = [\alpha]$. To apply Theorem 4.6 to each $I_{[\alpha]}$, we have to observe that the root-multiplicativity of $\mathfrak{L}$ and Proposition 3.2 show that $\Lambda_{I_{[\alpha]}}$ has all of its elements $\Lambda_{I_{[\alpha]}}$-connected, that is, connected through connections contained in $\Lambda_{I_{[\alpha]}}$. We also get that any of the $I_{[\alpha]}$ is root-multiplicative as consequence of the root-multiplicativity of $\mathfrak{L}$. Clearly $I_{[\alpha]}$ is of maximal length, and finally $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]}) = 0$, (where $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]})$ denotes de center of $I_{[\alpha]}$ in $I_{[\alpha]}$), as consequence of $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$, (Theorem 3.4), and $\mathcal{Z}(\mathfrak{L}) = 0$. We can therefore apply Theorem 4.6 to any $I_{[\alpha]}$ so as to conclude $I_{[\alpha]}$ is simple. It is clear that the decomposition $\mathfrak{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ satisfies the assertions of the theorem. ■

Remark 4.8. We finally note that the results in the present paper also hold for a class of non-regular Hom-Lie algebras $\mathfrak{L}$. Consider the class formed for those Hom-Lie algebras $(\mathfrak{L}, [\cdot,\cdot], \phi)$ such that ϕ is a (non-necessarily bijective) algebra homomorphism. Let us introduce for this class of Hom-Lie algebras the family of the split ones by asking for H not only the condition of being a maximal abelian subalgebra of $\mathfrak{L}$, but also that $\phi|_H$ is a linear bijection from H onto H. Then we can argue as in the case of split regular Hom-Lie algebras, to get that the results obtained in the present paper also hold for this family of split Hom-Lie algebras.

Acknowledgment. We would like to thank the referee and the editor for the detailed reading of this work and for the suggestions which have improved its final version.

References

[1] Ammar, F., and A. Makhlouf, *Hom-Lie superalgebras and Hom-Lie admissible superalgebras*, J. Algebra **324** (2007), 1513–1528.

[2] Arnlind, J., A. Makhlouf, and S. Silvestrov, *Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras*, J. Math. Phys. **51** (2010), 043515, 11 pp.

[3] Calderón, A. J., *On split Lie algebras with symmetric root systems*, Proc. Indian. Acad. Sci. Math. **118** (2008), 351–356.

[4] —, *On the structure of split non-commutative Poisson algebras*, Linear and Multilinear Algebra **60** (2012), 775–785.

[5] Calderón, A. J., and J. M. Sánchez, *On the structure of split Lie color algebras*, Linear Algebra Appl. **436** (2012), 307–315.

[6] —, *On split Leibniz algebras*, Linear Algebra Appl. **436** (2012), 1648–1660.

[7] Cheng, Y. S., and Y. C. Su, *(Co)Homology and universal central extension of Hom-Leibniz algebras*, Acta Math. Sin. (Engl. Ser.) **27** (2011), 813–830.

[8] Fregier, Y., A. Gohr, and S. Silvestrov, *Unital algebras of Hom-Lie type and surjective or injective twistings*, Gen. Lie Theory Appl. **3** (2009), 285–295.

[9] Hartwig, J., S. Larsson, and S. Silvestrov, *Deformations of Lie algebras using δ-derivations*, J. Algebra, **295** (2006), 314–361.

[10] Jin, Q., and X. Li, *Hom-Lie algebra structures on semi-simple Lie algebras*, J. Algebra **319** (2008), 1398–1408.

[11] Larsson, D., and S. Silvestrov, *Quasi-Hom-Lie algebras, central extensions and 2-cocycle-like identities*, J. Algebra **288** (2005), 321–344.

[12] —, *Quasi-Lie algebras*. In "Non-commutative Geometry and Representation Theory in Mathematical Physics", Contemp. Math. **391**, 241–248.

[13] —, *Quasi-deformations of $sl_2(\mathbb{F})$ using twisted derivations*, Comm. in Algebra **35** (2007), 4303–4318.

[14] Makhlouf, A., *Hom-alternative algebras and Hom-Jordan algebras*, Int. Electron. J. Algebra **8** (2010), 177–190.

[15] Makhlouf, A., and S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2008), 51–64.

[16] —, *Notes on 1-parameter formal deformations of Hom-Lie and Hom-Lie algebras*, Forum Math. **22** (2010), 715–739.

[17] —, *Hom-algebras and Hom-coalgebras*, J. Algebra Appl. **9** (2010), 553–589.

[18] Sheng, Y., *Representations of Hom-Lie algebras*, Algebr. Represent. Theor. **12** (2012), 1081–1098.

[19] Yau, D., *The Hom-Yang-Baxter equation and Hom-Lie algebras*, J. Math. Phys. **52** (2011), 053502, 19 pp.

[20] —, *Hom-Novikov algebras*, J. Phys. A **44** (2011), 085202, 20 pp.

M. J. Aragón Periñán
Department of Mathematics
University of Cádiz
11510 Puerto Real, Cádiz, Spain
mariajesus.aragonperin@alum.uca.es

A. J. Calderón Martín
Department of Mathematics
University of Cádiz
11510 Puerto Real, Cádiz, Spain
ajesus.calderon@uca.es

Received June 5, 2014
and in final form December 23, 2014