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On a Question of Ross and Stromberg

Hatem Hamrouni

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Abstract. A topological group G is said an [AFG]-group if G contains an increasing sequence of finite subgroups having dense union. In this paper it is proved that the identity component G_0 of a locally compact [AFG]-group G is a pro-torus. This partially answers an old open question posed by Ross and Stromberg in Pacific J. Math., **20** (1967), 135–147. Other results included in this paper give a necessary and sufficient condition for an almost connected Lie group to be an [AFG]-group.
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1. Introduction and main results

In 1967, K. Ross and K. Stromberg ([13, page 145]) posed the following open question.

Question (K. Ross and K. Stromberg, 1967). When does a locally compact group contain an increasing sequence of finite subgroups with dense union?

We call a group of the form $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k = (\mathbb{R}/\mathbb{Z})^k$ a *torus.* A *pro-torus* is a compact connected abelian group. In this paper, we shall prove the following theorem, which will provide a partial answer to Question 1.

Theorem A. *Let G be a locally compact group. If G contains an increasing sequence of finite subgroups with dense union, then the identity component G_0 of G is a pro-torus and G is a union of compact subgroups.*

As an immediate consequence, we have the following.

Corollary. *If G is a connected locally compact group which contains an increasing sequence of finite subgroups with dense union, then G is a pro-torus.*

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Also included in this paper is the following theorem which gives a complete answer to Question 1 for Lie groups with finitely many connected components.

Theorem B. *Let G be a Lie group of dimension k with finitely many connected components. Then the following two conditions are equivalent.*

(1) *The group G contains an increasing sequence of finite subgroups with dense union;*

(2) *there is a finite subgroup F of G such that $G = G_0F$ and $G_0 \cong \mathbb{T}^k$ for some k.*

2. Background

This section is devoted to setting up some notation, primarily from [7], and to establish a few more or less well-known results that we shall need.

2.1. Periodic elements.

Let G be a topological group. An element x of G is called periodic (or compact) if x is contained in a compact subgroup of G. We denote by $\mathrm{comp}(G)$ the set of all periodic elements of G.

Definition 2.1 (Periodic group). A topological group G is called *periodic* if $\mathrm{comp}(G) = G$.

A *clopen set* in a topological space is a set which is both open and closed.

Theorem 2.2. *Let G be a locally compact totally disconnected group. Then* $\mathrm{comp}(G)$ *is a clopen subset of G.*

Proof. By Theorem 2 of [19], $\mathrm{comp}(G)$ is a closed subset of G. On the other hand, as every compact subgroup of a totally disconnected locally compact group is contained in an open compact subgroup ([18, Lemma 7]), $\mathrm{comp}(G)$ is open in G. ■

Corollary 2.3. *Let G be a locally compact group such that the identity component G_0 is compact. Then* $\mathrm{comp}(G)$ *is a clopen subset of G.*

Proof. Since G_0 is compact, the canonical projection $\pi : G \longrightarrow G/G_0$ is proper ([15, Lemma 32.8]); that is, for each compact subset C of G/G_0 the inverse image $\pi^{-1}(C)$ is compact. Then it is easy to verify that

$$\mathrm{comp}(G) = \pi^{-1}\left(\mathrm{comp}(G/G_0)\right).$$

As the quotient group G/G_0 is totally disconnected, the desired result follows from the continuity of π and Theorem 2.2. ■

Remark 2.4. We recall that the identity component G_0 of a locally compact group G is compact if and only if G contains a compact open subgroup.

2.2. Pro-Lie groups.

The following definitions are introduced in [7, page 148-149].

Definition 2.5 (Co-Lie group). A subgroup N of a topological group G is called a *co-Lie subgroup* if it is normal and G/N is a Lie group.

For a topological group G, let $\mathcal{N}(G)$ be the set of all co-Lie subgroups. We say that a topological group G *has arbitrarily small co-Lie subgroups if every identity neighborhood of* G *contains a member of* $\mathcal{N}(G)$.

Definition 2.6 (Pro-Lie group). A topological group G is called *a pro-Lie group* if it is complete and has arbitrarily small co-Lie subgroups.

A topological group G is said to be *almost connected* if the factor group G/G_0 is compact. The following theorem is proved in [20].

Theorem 2.7 (Yamabe's Theorem). *Every almost connected locally compact topological group is a pro-Lie group.*

It is often unsful to express this in a seemingly stronger form:

Theorem 2.8 (Gleason-Yamabe Theorem, stronger version). *Let* G *be a locally compact group. Then there exists an open sub-group* H *of* G *such that, for any open neighborhood* U *of the identity in* H, *there exists a compact normal subgroup* K *of* H *in* U *such that* H/K *is isomorphic to a Lie group.*

2.3. Chabauty topology.

Let G be a locally compact group and $\mathcal{SUB}(G)$ the hyperspace of all closed subgroups of G. The *Chabauty topology* on $\mathcal{SUB}(G)$ has the sets

$$\mathcal{O}_1(K) = \{H \in \mathcal{SUB}(G) \mid H \cap K = \emptyset\},$$
$$\mathcal{O}_2(V) = \{H \in \mathcal{SUB}(G) \mid H \cap V \neq \emptyset\},$$

as an open subbase, where V and K run, respectively, over all open and compact subsets of G.

The hyperspace $\mathcal{SUB}(G)$ endowed with the Chabauty topology is called the *Chabauty space* of the group G.

Proposition 2.9. *In* $\mathcal{SUB}(G)$, *a basis of neighborhoods of a closed subgroup* H_0 *is given by*

$$\mathcal{V}[K, U; H_0] = \{H \in \mathcal{SUB}(G) \mid H \cap K \subset H_0 U, \quad H_0 \cap K \subset HU\},$$

where U *and* K *run, respectively, over all open neighborhoods of* e *and compact subsets of* G.

The following well-known result can be found in [2, Théorème 1, page 181] (see also [1, Lemma E.1.1]).

Proposition 2.10. *The Chabauty space of a locally compact group is compact.*

2.4. Discrete approximation.

Recall the following definition, which first appeared in [16, page 36].

Definition 2.11. A locally compact group G is said to be *approximated by discrete subgroups*, if there is a sequence of discrete subgroups $(H_n)_{n\in\mathbb{N}}$ of G satisfying the condition that for any nonempty open set O of G, there exists an integer k such that $O \cap H_n \neq \varnothing$, for every $n \geq k$.

The following theorem was proved by H. Tôyama in the case when the group G is a compact connected Lie group ([16, Theorem 2]), and extended independently by M. Kuranishi ([11, Corollary, page 64]) and S. P. Wang ([17, Theorem 2]) for the case of Lie groups.

Theorem 2.12 (The Tôyama-Kuranishi-Wang Theorem). *Let G be a Lie group. If G is approximated by discrete subgroups, then the identity component G_0 is nilpotent.*

The following generalization of the Tôyama-Kuranishi-Wang theorem has recently been presented in [5].

Theorem 2.13 (Generalized Tôyama-Kuranishi-Wang Theorem). *Let G be a pro-Lie group. If G is approximated by discrete subgroups, then the identity component G_0 is nilpotent.*

Further results on discrete approximation can be found in [16, 11, 17, 4, 5].

3. Local systems

The following definition is adapted from [10, page 8].

Definition 3.1 (Local system). Let G be a topological group. An increasing sequence $(H_n)_{n\in\mathbb{N}}$ of finite subgroups of G such that $G = \overline{\bigcup_{n\geq 0} H_n}$ is called a *local system of* G.

Let us define the class [AFG] as the class of topological groups G admitting a local system. A topological group belonging to the class [AFG] will often be called [AFG]- *group*. The class [AFG] contains for instance the class of discretely topologized finite groups.

The terminology AFG ("approximable by finite subgroups") is justified by the following proposition.

Proposition 3.2. *Let $(H_n)_{n\in\mathbb{N}}$ be an increasing sequence of closed subgroups of a locally compact group G. Then the following assertions are equivalent:*

(1) $G = \overline{\bigcup_{n\geq 0} H_n}$;

(2) *the sequence* $(H_n)_{n\in\mathbb{N}}$ *converges to* G *in the Chabauty topology;*

(3) *the group* G *is approximated by the subgroups* H_n.

The proof is straightforward with the use of the definition of the Chabauty topology and Definition 2.11.

Example 3.3. The torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has a local system

$$\frac{1}{2}\mathbb{Z}/\mathbb{Z} \leq \cdots \leq \frac{1}{2^n}\mathbb{Z}/\mathbb{Z} \leq \cdots .$$

Example 3.4. The group $O_2(\mathbb{R})$ of orthogonal transformations of the plane has a local system $(H_n)_{n\in\mathbb{N}}$, where H_n is the dihedral group D_{2^n}.

We collect some simple properties of the class $[\mathsf{AFG}]$.

Lemma 3.5. *If* $f : G_1 \longrightarrow G_2$ *is a morphism of locally compact groups, and if* G_1 *has a local system* $(H_n)_{n\in\mathbb{N}}$, *then* $(f(H_n))_{n\in\mathbb{N}}$ *is a local system of* $f(G_1)$.

The proof is immediate from the definitions.

Lemma 3.6. *Let* $(G_i)_{i\in\mathbb{N}}$ *be a countable family of* $[\mathsf{AFG}]$*-groups. For each* $i \in \mathbb{N}$, *let* $(A_n^{(i)})_{n\in\mathbb{N}}$ *be a local system of* G_i. *Then* $(H_n)_{n\in\mathbb{N}}$ *is a local system of* $\prod_{i\in\mathbb{N}} G_i$, *where*

$$H_n = \prod_{i=0}^{n} A_n^{(i)} \times \prod_{i\geq n+1} \{e\}.$$

Proof. To prove the lemma, it suffices to show that the sequence $(H_n)_{n\in\mathbb{N}}$ has dense union in $\prod_{i\in\mathbb{N}} G_i$. Let $O = \prod_{i\in\mathbb{N}} O_i$ be a non-empty elementary set. We have $O_i = G_i$, except for indices i belonging to a finite subset J of $\mathbb{N}$. For each $i \in J$, as $(A_n^{(i)})_{n\in\mathbb{N}}$ has dense union in G_i, there exists $n_i \in \mathbb{N}$ such that $O_i \cap A_{n_i}^{(i)} \neq \emptyset$. Let

$$m = \max\{\max(J), n_i \mid i \in J\}.$$

It is clear that $H_m \cap O \neq \emptyset$. ■

Lemma 3.7. *Let* $(H_n)_{n\geq 0}$ *be a local system of a locally compact group* G *and* A *an open subgroup of* G. *Then* $(H_n \cap A)_{n\geq 0}$ *is a local system of* A.

The proof is clear.

We summarize the above results as follows:

Proposition 3.8. *The class* $[\mathsf{AFG}]$ *is closed under forming open subgroups, continuous homomorphic images and countable direct products.*

The following result will be used in Section 5.

Proposition 3.9. *Let G be a locally compact topological group such that $G = G_1G_2$ for two closed subgroups G_1 and G_2. If $(A_n)_{n\in\mathbb{N}}$ (resp. $(B_n)_{n\in\mathbb{N}}$) is a local system of G_1 (resp. of G_2) and if, for every $n \in \mathbb{N}$, A_nB_n is a subgroup of G then $(A_nB_n)_{n\in\mathbb{N}}$ is a local system of G.*

Proof. Since the sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are increasing then we have

$$\bigcup_{n\in\mathbb{N}} A_nB_n = \left(\bigcup_{n\in\mathbb{N}} A_n\right)\left(\bigcup_{n\in\mathbb{N}} B_n\right).$$

On the other hand, as

$$\left(\overline{\bigcup_{n\in\mathbb{N}} A_n}\right)\left(\overline{\bigcup_{n\in\mathbb{N}} B_n}\right) \subset \overline{\left(\bigcup_{n\in\mathbb{N}} A_n\right)\left(\bigcup_{n\in\mathbb{N}} B_n\right)}$$

then

$$\overline{\left(\bigcup_{n\in\mathbb{N}} A_n\right)\left(\bigcup_{n\in\mathbb{N}} B_n\right)} = G$$

and therefore

$$\overline{\bigcup_{n\in\mathbb{N}} A_nB_n} = G,$$

which is the desired conclusion. ■

Recall the following definition ([10]).

Definition 3.10 (Locally finite group)**.** A group G is said to be *locally finite* if every finite subset of G generates a finite subgroup.

Proposition 3.11. *A discrete group G belongs to [AFG] if and only if it is countable locally finite group.*

Proof. See [10, Lemma 1.A.9]. ■

Remark 3.12. Let us note that a locally compact group is an [AFG]-group if and only if it contains a dense countable locally finite subgroup.

From the given circumstances, the following result clear.

Lemma 3.13. *For every [AFG]-group G, the set $\mathrm{comp}(G)$ is dense in G.*

4. Locally compact abelian groups

The torsion subgroup of an abelian group G is denoted by $\mathrm{tor}(G)$.

Proposition 4.1. *Let G be a locally compact abelian group with dense torsion subgroup. Then the following holds*

(1) $\mathrm{comp}(G) = G$.

(2) *G contains a compact open subgroup U with the following properties:*

(*a*) *G/U is a torsion group.*
(*b*) $U = \overline{\mathrm{tor}(U)}$.

For a proof of the next result see [8], Corollary 8.9 (ii).

Proposition 4.2. *A compact abelian group has a dense torsion group if and only if its discrete character group has no divisible elements.*

Proposition 4.3. *Let G be a compact connected abelian group. Then the following are equivalent:*

(1) $\overline{\mathrm{tor}(G)} = G$;

(2) *$\widehat{G}$ is a torsion free group not having a direct summand $\mathbb{Q}$;*

(3) *G does not have a direct factor isomorphic to $\widehat{\mathbb{Q}}$;*

(3) *G does not have a torsionfree direct factor.*

For a topological group G we shall write G_d for the underlying group endowed with the discrete topology. We note that

(A) $G = \overline{\mathrm{tor}(G)}$

is equivalent to

(B) In the category of locally compact abelian groups there is an injective morphism $f : \mathrm{tor}(G)_d \longrightarrow G$ with dense image; that is, f is a monic epic morphism.

Dually, this means that

(C) We have a monic epic morphism $\widehat{f} : \widehat{G} \longrightarrow (\mathrm{tor}(G)_d)\widehat{}$.

Proposition 4.4. *Let G be a locally compact abelian group.*

(1) *The group G is profinite if and only if its character group $\widehat{G}$ is a discrete torsion group.*

(2) *The group G is metric profinite if and only if its character group $\widehat{G}$ is a countable torsion group.*

Proof. (1) See [8], Corollary 8.5 $(a) \Longleftrightarrow (c)$.
(2) See [8, Theorem 7.76 (ii)]; see also [8, Theorem A4.16]. ∎

Proposition 4.5. *A locally compact abelian group G is an $[\mathsf{AFG}]$-group if its dual $\widehat{G}$ can be densely injected into a metric profinite abelian group M.*

The converse of Proposition 4.5 is not true in general as seen in the following example. Let $\mathbb{Z}(2)$ be the two elements group. The group $G = \mathbb{Z}(2)^{\mathfrak{c}}$ ($\mathfrak{c}$ is the cardinality of the continuum) is an $[\mathsf{AFG}]$-group but its character group is not metric.

Let A be any LCA group. Its universal compactification is $\alpha(A) = \widehat{\widehat{A}_d}$, and its universal zero-dimensional compactification is

$$\alpha_0(A) = \alpha(A)/\alpha(A)_0 \cong \mathrm{tor}(\widehat{A}_d)\widehat{}.$$

This we apply with $A = \widehat{G}$, and so $G = \widehat{A}$ by duality. Therefore

$$\alpha_0(\widehat{G}) \cong \mathrm{tor}(G_d)\widehat{}.$$

This profinite group is metric if and only if $\mathrm{tor}(G_d) = \mathrm{tor}(G)_d$ is countable. Thus we summarize

> *Any discrete torsion group T is densely injected into the profinite group $\alpha_0(T)$. The weight of $\alpha_0(T)$ is $\mathfrak{c}$, the cardinality of the continuum, and $\alpha_0(T)$ is, therefore, not metric.*

Example 4.6. The set of all primes is denoted by $\mathbb{P}$. For a prime p we denote by $\mathbb{Z}_p$ the group of p-adic integers. The profinite group

$$\mathbb{Z}^* = \alpha_0(\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})\widehat{} \cong \prod_{p\in\mathbb{P}} \mathbb{Z}_p$$

is metric and its underlying abelian group has torsion free rank $\mathfrak{c}$. Thus, if $A = (\mathbb{Z}^*)_d$ then $G = \widehat{A}$ is a compact connected $[\mathsf{AFG}]$-group whose weight has continuum cardinality $\mathfrak{c}$.

Example 4.7. Since the rank of the torsion free group A of Example 4.6 is $\mathfrak{c}$, it contains a copy of the free abelian group $\mathbb{Z}^{(\mathfrak{c})}$ of continuum rank. The dual of this group is the torus $\mathbb{T}^{\mathfrak{c}}$ of dimension continuum. Its character group is densely injected into a metric profinite group and therefore

$$\mathbb{T}^{\mathfrak{c}} \text{ is an } [\mathsf{AFG}]\text{-group.}$$

The following example shows that the class $[\mathsf{AFG}]$ is not closed under forming closed subgroups.

Example 4.8. The character group H of the discrete group $\mathbb{Q}$ is a compact connected torsion free group (see [8, Corollary 8.5]) and thus is guaranteed to fail to be an $[\mathsf{AFG}]$-group. On the other hand, in the category of abelian groups there is a quotient morphism $q: \mathbb{Z}^{(\mathbb{N})} \longrightarrow \mathbb{Q}$ from the free abelian group in countably many generators $e_1, e_2, \ldots$ to $\mathbb{Q}$ defined by $q(e_n) = \frac{1}{n}$ (cf. [8, Proposition A1.8]).

Let G denote the character group of $\mathbb{Z}^{(\mathbb{N})}$ which is isomorphic to $\mathbb{T}^{\mathbb{N}}$. Then by duality, $\widehat{q}: \ H \longrightarrow G$ is an embedding. So the [AFG]-group $\mathbb{T}^{\mathbb{N}}$ has a closed subgroup which is not an [AFG]-group.

We close this section with the following proposition.

Proposition 4.9. *The class [AFG] contains the class of compact abelian Lie groups.*

Proof. This follows from the fact that a compact abelian Lie group is of the form $\mathbb{T}^n \times F$ with a finite abelian group F and Lemma 3.6. ∎

5. Main results

We begin by recording the following result, due to S. P. Wang ([17, Lemma 5.7]), wich will be crucial to our argument.

Proposition 5.1. *Let G be a compactly generated locally compact group, and $(H_n)_{n\in\mathbb{N}}$ a sequence of subgroups of G converging to a closed subgroup H of G with G/H compact. Then there is a compact subset K such that $G = KH_n$ for a large n.*

This has the following immediate consequence.

Proposition 5.2. *Every compactly generated locally compact [AFG]-group is compact.*

Proof. Let G be a compactly generated locally compact [AFG]-group and $(H_n)_{n\in\mathbb{N}}$ a local system of G. As the sequence $(H_n)_{n\in\mathbb{N}}$ converges to G in the Chabauty topology, then by Proposition 5.1 there is a compact subset K of G such that $G = KH_n$ for large n. ∎

Since every connected locally compactis compactly generated, we have the following consequence:

Corollary 5.3. *Every connected locally compact [AFG]-group is compact.*

A topological space is called σ-compact if it is a countable union of compact subsets. It is known that every locally compact σ-compact group G contains an increasing sequence $A_0 \leq A_1 \leq \cdots \leq A_n \leq A_{n+1} \leq \cdots$ of compactly generated open subgroups of G such that $\bigcup_{n\geq 0} A_n = G$. Thus, we obtain:

Corollary 5.4. *Let G be a locally compact σ-compact [AFG]-group. Then there exists an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of compact open subgroups of G such that*

$$\bigcup_{n\geq 0} A_n = G.$$

Theorem 5.5. *Every connected locally compact* [AFG]*-group is a pro-torus.*

Proof. Let G be a connected locally compact [AFG]-group. By hypothesis there exists an increasing sequence $(H_n)_{n\geq 0}$ of finite subgroups of G such that $G = \overline{\bigcup_{n\geq 0} H_n}$. It is clear that the group G is approximable by the discrete subgroups $(H_n)_{n\geq 0}$ (Proposition 3.2). Then, by Theorem 2.13 the group G is nilpotent. By Corollary 5.3, G is compact. But a connected compact nilpotent group is abelian ([8], Proposition 9.4, page 450). ■

This theorem has several direct and important consequences.

Corollary 5.6. *Let G be a Lie group. If G is a* [AFG]*-group then the identity component G_0 is a torus.*

Proof. As G_0 is open in G then G_0 is a [AFG]-group (Lemma 3.7). Using Theorem 5.5 we deduce the desired result. ■

Corollary 5.7. *Let G be a Lie group. If G is a* [AFG]*-group then G is a (torus)-by-(countable locally finite).*

Proof. By Corollary 5.5 we have G_0 is a torus. On the other hand, as every Lie group is discrete modulo its identity component, then the result follows from Lemma 3.5 and Proposition 3.11. ■

For Lie groups with finitely many connected components, we have the following characterization.

Theorem 5.8. *Let G be an almost connected Lie group of dimension k. The following two conditions are equivalent.*

(1) *The group G is a* [AFG]*-group;*

(2) *there is a finite subgroup F of G such that $G = G_0F$, $G_0 \cong \mathbb{T}^k$.*

Proof. Let $(H_n)_{n\in\mathbb{N}}$ be a local system of G and let $p : G \longrightarrow G/G_0$ be the canonical projection. We have

$$G/G_0 = \bigcup_{n\in\mathbb{N}} p(H_n).$$

As G/G_0 is finite then $G/G_0 = p(H_m)$ for some $m \in \mathbb{N}$ and therefore $G = G_0H_m$. On the other hand, by Corollary 5.7 we have $G_0 \cong \mathbb{T}^k$. This show (1) $\Longrightarrow$ (2). To prove the converse, we first observe that $G_0 = \mathbb{T}^k$. For $n \in \mathbb{N}$, let $A_n = \frac{1}{2^n}\mathbb{Z}/\mathbb{Z}$ and $H_n = A_n^k = A_n \times \cdots \times A_n$. Then by Lemma 3.6, $(H_n)_{n\in\mathbb{N}}$ is a local system of G_0. On the other hand, it is easy to see that for every $n \in \mathbb{N}$ H_n is a characteristic subgroup of G_0, because $\mathrm{Aut}(\mathbb{T}^k) = \mathrm{GL}(k,\mathbb{Z})$. Then, for every $n \in \mathbb{N}$, H_n is normal in G. Finally, by Proposition 3.9, we have $(H_nF)_{n\in\mathbb{N}}$ is a local system of G, which completes the proof. ■

Proposition 5.9. *Let G be a locally compact pro-Lie group. If G is a $[\mathsf{AFG}]$-group then the identity component G_0 is a pro-torus.*

Proof. Using Proposition 2.4 of [6] (see also [7]), every locally compact pro-Lie group G gives rise to a strong projective system

$$\{p_{NM} : G/M \longrightarrow G/N \mid (M, N) \in \mathcal{N}_c(G) \times \mathcal{N}_c(G),\ N \leq M\}$$

whose projective limit is isomorphic to G, where

$$\mathcal{N}_c(G) = \{N \in \mathcal{N}(G) \mid N \text{ is a compact subgroup of } G\}.$$

Let

$$G = \varprojlim_{N \in \mathcal{N}_c(G)} G/N.$$

For every $N \in \mathcal{N}_c(G)$, let $(G/N)_0$ be the identity component of the group G/N. By Proposition 4 of [3, page 292], we have

$$G_0 = \varprojlim_{N \in \mathcal{N}_c(G)} (G/N)_0.$$

On the other hand, by Proposition 3.8, the Lie group G/N is a $[\mathsf{AFG}]$-group. By Corollary 5.7, the identity component $(G/N)_0$ is a torus. By Theorem 6.1.20 of [14] the group G_0 is a pro-torus. ∎

The following is the main result of this paper.

Theorem 5.10. *Let G be a locally compact $[\mathsf{AFG}]$-group. Then the identity component G_0 is a pro-torus. Moroever, G is periodic.*

Proof. By Theorem 2.8, the group G contains an open subgroup H such that H is a pro-Lie group. By Proposition 3.8, H is a $[\mathsf{AFG}]$-group and therefore the identity component H_0 is a pro-torus. On the other hand, it is clear that $G_0 = H_0$. Finally, the periodicity of the group G follows Lemma 3.13 and Corollary 2.3. ∎

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Hatem Hamrouni
Faculty of Sciences at Sfax
Department of Mathematics
Sfax University
B.P. 1171. 3000 Sfax, Tunisia
hatemhhamrouni@voila.fr

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