

Isomorphism Classes of Involutions of $\mathrm{SP}(2n, k)$, $n > 2$

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Abstract. A first characterization of the isomorphism classes of k -involutions for any reductive algebraic groups defined over a perfect field was given in *On the Classification of k -involutions I* using 3 invariants. In *Involutions of $\mathrm{SL}(n, k)$* , ($n > 2$), a classification of all involutions on $\mathrm{SL}(n, k)$ for k algebraically closed, the real numbers, the p -adic numbers or a finite field was provided. In this paper, we build on these results to develop a detailed characterization of the involutions of $\mathrm{SP}(2n, k)$. We use these results to classify the isomorphism classes of involutions of $\mathrm{SP}(2n, k)$ where k is any field not of characteristic 2.

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1. Introduction

Let G be a connected reductive algebraic group defined over a field k of characteristic not 2, θ an involution of G defined over k , H a k -open subgroup of the fixed point group of θ and G_k (resp. H_k) the set of k -rational points of G (resp. H). The variety G_k/H_k is called a symmetric k -variety. For $k = \mathbb{R}$ these symmetric k -varieties are also called real reductive symmetric spaces. These varieties occur in many problems in representation theory, geometry and singularity theory. To study these symmetric k -varieties one needs first a classification of the related k -involutions. A characterization of the isomorphism classes of the k -involutions was given in [Helm2000] essentially using the following 3 invariants:

1. classification of admissible (Γ, θ) -indices.
2. classification of the G_k -isomorphism classes of k -involutions of the k -isotropic kernel of G .
3. classification of the G_k -isomorphism classes of k -inner elements of G .

For more details, see [Helm2000]. The admissible (Γ, θ) -indices determine most of the fine structure of the symmetric k -varieties and a classification of these was included in [Helm2000] as well. For k algebraically closed or k the real

numbers the full classification can be found in [Hel88]. For other fields a full classification of the remaining two invariants is still lacking. In particular the case of symmetric k -varieties over the p -adic numbers is of interest. We note that the above characterization was only proven for k a perfect field.

In [HWD04] a full characterization of the isomorphism classes of k -involutions was given in the case that $G = \mathrm{SL}(n, k)$ which does not depend on any of the results in [Helm2000]. It was also shown how one may construct an outer-involution from a given non-degenerate symmetric or skew-symmetric bilinear form β of k^n . Using this characterization the possible isomorphism classes for k algebraically closed, the real numbers, the p -adic numbers and finite fields were classified.

In this paper we build upon the results of [HWD04] to give a characterization of involutions of $\mathrm{SP}(2n, k)$, the symplectic group.

We first show that if an automorphism $\theta = \mathrm{Inn}_A$ where $A \in \mathrm{GL}(2n, \bar{k})$ leaves $\mathrm{SP}(2n, k)$ invariant, then we can assume A in $\mathrm{SP}(n, k[\sqrt{\alpha}])$ where $k[\sqrt{\alpha}]$ is a quadratic extension of k . Thus, to classify the involutions of $\mathrm{SP}(2n, k)$ it suffices to determine which $A \in \mathrm{SP}(2n, k[\sqrt{\alpha}])$ induce involutions of $\mathrm{SP}(2n, k)$, and to then determine the isomorphism classes of these involutions over $\mathrm{SP}(2n, k)$. Using these results, we give a full classification of involutions of $\mathrm{SP}(2n, k)$ for k algebraically closed, the real numbers, or a finite field.

2. Preliminaries

Our basic reference for reductive groups will be the papers of Borel and Tits [BT65], [BT72] and also the books of Borel [Bor91], Humphreys [Hum75] and Springer [Spr81]. We shall follow their notations and terminology. All algebraic groups and algebraic varieties are taken over an arbitrary field k (of characteristic $\neq 2$) and all algebraic groups considered are linear algebraic groups.

Our main reference for results regarding involutions of $\mathrm{SL}(n, k)$ will be [HWD04]. Let k be a field of characteristic not 2, \bar{k} the algebraic closure of k ,

$$\mathrm{M}(n, k) = \{n \times n\text{-matrices with entries in } k\},$$

$$\mathrm{GL}(n, k) = \{A \in \mathrm{M}(n, k) \mid \det(A) \neq 0\}$$

and

$$\mathrm{SL}(n, k) = \{A \in \mathrm{M}(n, k) \mid \det(A) = 1\}.$$

Let k^* denote the product group of all the nonzero field elements, $(k^*)^2 = \{a^2 \mid a \in k^*\}$ and $I_n \in \mathrm{M}(n, k)$ denote the identity matrix. We will sometimes use I instead of I_n when the dimension of the identity matrix is clear.

We recall some important definitions and theorems from [HWD04].

Definition 2.1. Let G be an algebraic group defined over a field k , and let G_k be the set of k -rational points. Let $\mathrm{Aut}(G_k)$ denote the set of all automorphisms of G_k . For $A \in \mathrm{GL}(n, k)$ let Inn_A denote the inner automorphism defined by $\mathrm{Inn}_A(X) = A^{-1}XA$ for all $X \in \mathrm{GL}(n, k)$. Let $\mathrm{Inn}_k(G_k) = \{\mathrm{Inn}_A \mid A \in G_k\}$ denote the set of all *inner* automorphisms of G_k and let $\mathrm{Inn}(G_k)$ denote the set of automorphisms Inn_A of G_k with $A \in G$ such that $\mathrm{Inn}_A(G_k) = G_k$. If Inn_A is

order 2, that is Inn_A^2 is the identity but Inn_A is not, then we call Inn_A an *inner involution* of G_k . We say that θ and τ in $\text{Aut}(G_k)$ are *Inn(G_k)-isomorphic* if there is a ϕ in $\text{Inn}(G_k)$ such that $\tau = \phi^{-1}\theta\phi$. Equivalently, we say that τ and θ are in the same *isomorphism class*.

In [HWD04], the isomorphism classes of the inner-involutions of $\text{SL}(n, k)$ were classified, and they are as follows:

Theorem 2.2. *Suppose the involution $\theta \in \text{Aut}(\text{SL}(n, k))$ is of inner type. Then up to isomorphism θ is one of the following:*

1. $\text{Inn}_Y|_G$, where $Y = I_{n-i,i} \in \text{GL}(n, k)$ where $i \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ where

$$I_{n-i,i} = \begin{pmatrix} I_{n-i} & 0 \\ 0 & -I_i \end{pmatrix}$$

2. $\text{Inn}_Y|_G$, where $Y = L_{\frac{n}{2},x} \in \text{GL}(n, k)$ where $x \in k^*/k^{*2}$, $x \not\equiv 1 \pmod{k^{*2}}$ and

$$L_{n,x} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ x & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & x & 0 \end{pmatrix}.$$

Note that (ii) can only occur when n is even.

For the purposes of this paper, we will use matrices of the form $\begin{pmatrix} 0 & I_{\frac{n}{2}} \\ xI_{\frac{n}{2}} & 0 \end{pmatrix}$ (and there multiples) rather than $L_{\frac{n}{2},x}$. Either of these serves as a member of the isomorphism class listed in the previous theorem. We will eventually see that all of the isomorphism classes of $\text{SP}(2n, k)$ are just isomorphism classes of $\text{SL}(n, k)$ that have been divided into multiple isomorphism classes.

We now begin to define the notion of a symplectic group. To do this, we must first define orthogonal groups. Let M be the matrix of a non-degenerate bilinear form β over k^n with respect to a basis $\{e_1, \dots, e_n\}$ of V . We will say that M is the matrix of β if the basis $\{e_1, \dots, e_n\}$ is the standard basis of k^n .

The typical notation for the orthogonal group is $\text{O}(n, k)$, which is the group

$$\text{O}(n, k) = \{A \in \text{M}(n, k) \mid (Ax)^T(Ay) = x^T y\}.$$

This group consists of the matrices which fix the standard dot product. This can be generalized to any non-degenerate bilinear β , which will yield the group

$$\text{O}(n, k, \beta) = \{A \in \text{M}(n, k) \mid \beta(Ax, Ay) = \beta(x, y)\}.$$

If M is the matrix of β with respect to the standard basis, then we can equivalently say

$$\text{O}(n, k, \beta) = \{A \in \text{M}(n, k) \mid A^T M A = M\}.$$

It is clear from this definition that all matrices in $O(n, k, \beta)$ have determinant 1 or -1 . We are interested in the case where M is a skew-symmetric matrix.

We note a couple of important facts, the first of which will be used repeatedly throughout this paper.

1. Skew-symmetric matrices of even dimension are congruent to the matrix $J = J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
2. If β_1 and β_2 correspond to M_1 and M_2 , then $O(n, k, \beta_1)$ and $O(n, k, \beta_2)$ are isomorphic via

$$\Phi : O(n, k, \beta_1) \rightarrow O(n, k, \beta_2) : X \rightarrow Q^{-1}XQ$$

for some $Q \in GL(n, k)$ if $Q^T M_1 Q = M_2$ (M_1 and M_2 are congruent via Q).

So, we will assume that β is such that we can replace M with the matrix J . When we do this, then we write $SP(2n, k) = O(n, k, \beta)$, and we call this the *Symplectic Group*. It can be shown that all matrices in $SP(2n, k)$ have determinant 1, so in fact $SP(2n, k)$ is a subgroup of $SL(2n, k)$. Lastly, note that to classify the involutions of an orthogonal group where M is skew-symmetric, one can apply the classification that will follow by simply using the isomorphism given above.

3. Automorphisms of $SP(2n, k)$

It follows from a proposition on page 191 of [Bor91] that

$$\text{Aut}(SP(2n, \bar{k})) / \text{Inn}(SP(2n, \bar{k}))$$

must be a subgroup of the diagram automorphisms of the Dynkin diagram C_n . Since C_n only has the trivial diagram automorphism, then we have that

$$\text{Aut}(SP(2n, \bar{k})) = \text{Inn}(SP(2n, \bar{k})).$$

When k is not algebraically closed, then all automorphisms of $SP(2n, k)$ will still be of the form Inn_A for some $A \in SP(n, \bar{k})$ since all automorphisms of $SP(2n, k)$ must also be an automorphism of $SP(n, \bar{k})$. Thus, the classifications and characterizations that follow in this paper consider all automorphisms and involutions of $SP(2n, k)$.

We now examine which automorphisms will act as the identity on $SP(2n, k)$. This will prove to be useful when we classify matrix representatives for automorphisms.

Theorem 3.1. *Let $G = SP(2n, k)$. If $\text{Inn}_A|_G = \text{Id}$ for some $A \in GL(2n, \bar{k})$ then $A = pI$ for some $p \in \bar{k}$.*

Proof. Suppose $\text{Inn}_A|_G = \text{Id}$ for some $A \in GL(2n, \bar{k})$. Then for all $X \in G$ we have $\text{Inn}_A(X) = A^{-1}XA = X$ which means that $AX = XA$ for all $X \in G$. Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

and consider the matrix

$$W_1 = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}.$$

Since $W_1 \in G$, $AW_1 = W_1A$ which implies

$$\begin{aligned} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} &= \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \\ \begin{pmatrix} A_1 & A_1 + A_2 \\ A_3 & A_3 + A_4 \end{pmatrix} &= \begin{pmatrix} A_1 + A_3 & A_2 + A_4 \\ A_3 & A_4 \end{pmatrix}. \end{aligned}$$

Hence, $A_3 = 0$ and $A_1 = A_4$. With this information in hand we are now able to rewrite A as $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_1 \end{pmatrix}$. We now consider the matrix $W_2 = \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix}$. Now W_2 is also in G and thus $AW_2 = W_2A$ and thus

$$\begin{aligned} \begin{pmatrix} A_1 & A_2 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} &= \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ 0 & A_1 \end{pmatrix} \\ \begin{pmatrix} A_1 + A_2 & A_2 \\ A_1 & A_1 \end{pmatrix} &= \begin{pmatrix} A_1 & A_2 \\ A_1 & A_2 + A_1 \end{pmatrix}. \end{aligned}$$

Which implies that $A_2 = 0$ and thus $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}$.

Let

$$\bar{X}_k = \begin{pmatrix} X_k & 0 \\ 0 & X_k \end{pmatrix}$$

where

$$X_k := \begin{pmatrix} I_{n-k-1} & \dots & 0 \\ \vdots & -1 & \vdots \\ 0 & \dots & I_k \end{pmatrix}$$

and $k = 0, 1, \dots, n - 1$. Then $\bar{X}_k \in G$ and hence we may utilize the fact that $A\bar{X}_k = \bar{X}_kA$, to conclude that

$$\begin{pmatrix} A_1X_k & 0 \\ 0 & A_1X_k \end{pmatrix} = \begin{pmatrix} X_kA_1 & 0 \\ 0 & X_kA_1 \end{pmatrix}.$$

From the above equality we see that $A_1X_k = X_kA_1$. Define $A_1 = (a_{i,j})$ for $i, j = 1, 2, \dots, n$. Then $A_1X_k = X_kA_1$ implies

$$\begin{pmatrix} a_{11} & a_{12} & \dots & -a_{1,n-k} & \dots & a_{1,n} \\ a_{21} & a_{22} & \dots & -a_{2,n-k} & \dots & a_{2,n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ a_{n-k,1} & a_{n-k,2} & \dots & -a_{n-k,n-k} & \dots & a_{n-k,n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & -a_{n,n-k} & \dots & a_{n,n} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,n-k} & \dots & a_{1,n} \\ a_{21} & a_{22} & \dots & a_{2,n-k} & \dots & a_{2,n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ -a_{n-k,1} & -a_{n-k,2} & \dots & -a_{n-k,n-k} & \dots & -a_{n-k,n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-k} & \dots & a_{n,n} \end{pmatrix}.$$

Hence, it follows that $a_{n-k,j} = a_{j,n-k} = 0$ for $j \neq n-k$ and $k = 0, 1, \dots, n-1$, $j = 1, 2, \dots, n$. Therefore we now obtain the fact that A is a diagonal matrix say,

$$A = \begin{pmatrix} A_d & 0 \\ 0 & A_d \end{pmatrix} \text{ with } A_d = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & a_{n,n} \end{pmatrix}.$$

Let

$$\bar{Y}_l = \begin{pmatrix} Y_l & 0 \\ 0 & Y_l \end{pmatrix} \text{ where } Y_l = \begin{pmatrix} I_l & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & & I_{n-l-2 \times n-l-2} \\ 0 & 0 & & & \end{pmatrix}$$

and $l = 0, 1, \dots, n-2$. Then $\bar{Y}_l \in \text{SP}(2n, k)$ and again $A\bar{Y}_l = \bar{Y}_l A$ which implies $A_d Y_l = Y_l A_d$. Therefore, we obtain the following equality

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & a_{ll} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{l+1,l+1} & 0 & 0 & \\ 0 & 0 & 0 & 0 & a_{l+2,l+2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{l+3,l+3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{n,n} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & a_{ll} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{l+2,l+2} & 0 & 0 & \\ 0 & 0 & 0 & 0 & a_{l+1,l+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{l+3,l+3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{n,n} \end{pmatrix}$$

Hence $a_{l+1,l+1} = a_{l+2,l+2}$ for $l = 0, 1, \dots, n-2$. That is $A = p \text{Id}$ for some $p \in \bar{k}$. ■

The following is a list of notation which will be used in the proof of Theorem 3.2.

Let $X_{r,s}$ be the $n \times n$ diagonal matrix with a -1 in the (r, r) and (s, s) entries and 1 's everywhere else.

Let X_r be the $n \times n$ diagonal matrix with a -1 in the (r, r) position and 1 's everywhere else.

Let $E_{r,s}$ be the $n \times n$ matrix with a 1 in the (r, s) entry and 0 's everywhere else.

Let T_c be the $c \times c$ antidiagonal matrix with 1 's on the antidiagonal and 0 's everywhere else.

Let I_c be the $c \times c$ identity matrix. If the size of the identity matrix is understood from the context then I may be used to represent I_c .

$$J_{2n} = J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{1}$$

$$Y_{r,s} = \begin{pmatrix} T_{r+s-1} & 0 \\ 0 & I_{n-(r+s-1)} \end{pmatrix} \tag{2}$$

$$Z_{r,s} = \begin{pmatrix} Y_{r,s} & 0 \\ E_{r,s} & Y_{r,s} \end{pmatrix} \tag{3}$$

$$\bar{Z}_{r,s} = \begin{pmatrix} -Y_{r,s} & 0 \\ E_{r,s} & -Y_{r,s} \end{pmatrix} \tag{4}$$

$$Z'_{r,s} = \begin{pmatrix} Y_{r-n,s-n} & E_{r-n,s-n} \\ 0 & Y_{r-n,s-n} \end{pmatrix} \tag{5}$$

$$\bar{Z}'_{r,s} = \begin{pmatrix} Y_{r-n,s-n} & E_{r-n,s-n} \\ 0 & Y_{r-n,s-n} \end{pmatrix} \tag{6}$$

$$M_{r,s} = \begin{pmatrix} E_{s-n,r} & Y_{s-n,r} \\ -Y_{s-n,r} & 0 \end{pmatrix} \tag{7}$$

$$\bar{M}_{r,s} = \begin{pmatrix} E_{s-n,r} & -Y_{s-n,r} \\ Y_{s-n,r} & 0 \end{pmatrix} \tag{8}$$

$$U_{r,s} = \begin{pmatrix} I_{-n+(r+s-1)} & 0 \\ 0 & T_{2n-(r+s-1)} \end{pmatrix} \tag{9}$$

$$V_{r,s} = \begin{pmatrix} U_{r,s} & 0 \\ E_{r,s} & U_{r,s} \end{pmatrix} \tag{10}$$

$$\bar{V}_{r,s} = \begin{pmatrix} -U_{r,s} & 0 \\ E_{r,s} & -U_{r,s} \end{pmatrix} \tag{11}$$

$$V'_{r,s} = \begin{pmatrix} U_{r-n,s-n} & E_{r-n,s-n} \\ 0 & U_{r-n,s-n} \end{pmatrix} \tag{12}$$

$$\bar{V}'_{r,s} = \begin{pmatrix} -U_{r-n,s-n} & E_{r-n,s-n} \\ 0 & -U_{r-n,s-n} \end{pmatrix} \tag{13}$$

$$N_{r,s} = \begin{pmatrix} E_{s-n,r} & U_{s-n,r} \\ -U_{s-n,r} & 0 \end{pmatrix} \tag{14}$$

$$\bar{N}_{r,s} = \begin{pmatrix} E_{s-n,r} & -U_{s-n,r} \\ U_{s-n,r} & 0 \end{pmatrix} \quad (15)$$

$$W_{r,s} = \begin{pmatrix} T_n & 0 \\ E_{r,s} & T_n \end{pmatrix} \quad (16)$$

$$\bar{W}_{r,s} = \begin{pmatrix} -T_n & 0 \\ E_{r,s} & -T_n \end{pmatrix} \quad (17)$$

$$W'_{r,s} = \begin{pmatrix} T_n & E_{r-n,s-n} \\ 0 & T_n \end{pmatrix} \quad (18)$$

$$\bar{W}'_{r,s} = \begin{pmatrix} -T_n & E_{r-n,s-n} \\ 0 & -T_n \end{pmatrix} \quad (19)$$

$$F_{r,s} = \begin{pmatrix} E_{s-n,r} & T_n \\ T_n & 0 \end{pmatrix} \quad (20)$$

$$\bar{F}_{r,s} = \begin{pmatrix} E_{s-n,r} & -T_n \\ -T_n & 0 \end{pmatrix} \quad (21)$$

We now have the following result that characterizes inner-automorphisms of $\mathrm{SP}(2n, k)$. We will see that for Inn_A to be an inner-involution of $\mathrm{SP}(2n, k)$, that we can not only assume that A is symplectic, but for the entries of A , we do not need the algebraic closure of the field k , but either the field itself or a quadratic extension of k .

Theorem 3.2. *Suppose $A \in \mathrm{GL}(2n, \bar{k})$, $\bar{G} = \mathrm{SP}(2n, \bar{k})$ and $G = \mathrm{SP}(2n, k)$.*

1. *The inner automorphism Inn_A keeps $\mathrm{SP}(2n, \bar{k})$ invariant if and only if $A = pM$ for some $p \in \bar{k}$ and $M \in \mathrm{SP}(2n, \bar{k})$.*
2. *If $A \in \mathrm{SP}(2n, \bar{k})$, then Inn_A keeps $\mathrm{SP}(2n, k)$ invariant if and only if we can show $A \in \mathrm{SP}(2n, k(\sqrt{\alpha}))$ where each entry of A is a k -multiple of $\sqrt{\alpha}$, for some $\alpha \in k$.*

Proof.

1. \Leftarrow Suppose $A = pM$ for some $p \in \bar{k}$ and $M \in \bar{G}$. Let $X \in \bar{G}$, then

$$\mathrm{Inn}_A(X) = \mathrm{Inn}_{pM}(X) = (pM)^{-1}X(pM) = M^{-1}XM$$

Since $M, M^{-1}, X \in \bar{G}$, $M^{-1}XM \in \bar{G}$ and thus Inn_A keeps \bar{G} invariant.

\Rightarrow Suppose Inn_A keeps \bar{G} invariant. Then for any $X \in \bar{G}$,

$B = \mathrm{Inn}_A(X) = A^{-1}XA \in \bar{G}$. Since $B \in \bar{G}$, by definition $B^T J B = J$ which implies that $B = J^{-1}(B^T)^{-1}J$. In addition, since $B = A^{-1}XA$, we have that $(B^T)^{-1} = A^T(X^T)^{-1}(A^T)^{-1}$. Thus the following is true

$$A^{-1}XA = B$$

implies

$$A^{-1}XA = J^{-1}(B^T)^{-1}J$$

which implies

$$A^{-1}XA = J^{-1}(A^T(X^T)^{-1}(A^T)^{-1})J$$

hence

$$X = AJ^{-1}A^T(X^T)^{-1}(A^T)^{-1}JA^{-1}.$$

Now since $X \in \bar{G}$, we know $(X^T)^{-1} = JXJ^{-1}$ which means

$$X = AJ^{-1}A^T(JXJ^{-1})(A^T)^{-1}JA^{-1}$$

that is

$$X = (AJ^{-1}A^TJ)X(AJ^{-1}A^TJ)^{-1}$$

$$\text{i.e. } \text{Inn}_{AJ^{-1}A^TJ}(X) = X.$$

Therefore by Lemma 3.1 $AJ^{-1}A^TJ = q\text{Id}$ for some $q \in \bar{k}^*$ which implies $q^{-1}AJ^{-1}A^TJ = \text{Id}$. Let $p \in \bar{k}^*$ such that $p^2 = q^{-1}$. Then for $M = pA$ we have

$$MJ^{-1}M^TJ = pAJ^{-1}pA^TJ = p^2AJ^{-1}A^TJ = q^{-1}AJ^{-1}A^TJ = I.$$

Therefore, $MJ^{-1}M^TJ = \text{Id}$ which implies $M^TJM = J$ ie. $M \in \bar{G}$.

2. \Leftarrow Suppose $A = pM$ for some $p \in \bar{k}$ and $M \in G$. Let $X \in G$, then

$$\text{Inn}_A(X) = \text{Inn}_{pM}(X) = p^{-1}M^{-1}XpM = M^{-1}XM$$

Since $M^{-1}, X, M \in G$ we know $\text{Inn}_A(X) = M^{-1}XM \in G$ and thus Inn_A keeps G invariant.

\Rightarrow Suppose $A = (a_{ij}) \in \bar{G}$ and Inn_A keeps G invariant.

We will first show that $a_{ri}a_{rj} + a_{si}a_{sj} \in G$.

CASE 1: Suppose $r, s \leq n$.

Subcase a: Suppose $i \leq n$.

The (i, j) entry of $\text{Inn}_A(J)$ is given by

$$a_{1,n+i}a_{1,j} + a_{2,n+i}a_{2,j} + \cdots + a_{2n,n+i}a_{2n,j} \in k$$

since $J \in G$ and Inn_A keeps G invariant. By the same argument the (i, j)

entry of $\text{Inn}_A \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ given by

$$a_{1j}a_{n+1,n+i} + a_{2j}a_{n+2,n+i} + \cdots + a_{nj}a_{2n,n+i} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \cdots + a_{2n,n+i}a_{2n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \cdots - a_{n,n+i}a_{2n,j} \in k$$

Hence the (i, j) position of $\text{Inn}_A(J) - \text{Inn}_A \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ given by

$$-a_{1j}a_{n+1,n+i} - a_{2j}a_{n+2,n+i} - \cdots - a_{nj}a_{2n,n+i} + a_{1,n+i}a_{1,j} + a_{2,n+i}a_{2,j} + \cdots +$$

$$a_{n,n+i}a_{n,j} + a_{1,n+i}a_{n+1,j} + a_{2,n+i}a_{n+2,j} + \dots + a_{n,n+i}a_{2n,j} \in k.$$

We know the matrix $\begin{pmatrix} I & 0 \\ X_{rs} & I \end{pmatrix}$ is in G and hence the (i, j) entry of

$\text{Inn}_A \begin{pmatrix} I & 0 \\ X_{rs} & I \end{pmatrix}$ is given by

$$a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \dots + a_{n,j}a_{2n,n+i} - a_{1,n+i}a_{1,j} - a_{2,n+i}a_{2,j} - \dots - (-a_{r,n+i}a_{r,j}) - a_{r+1,n+i}a_{r+1,j} - \dots - (-a_{s,n+i}a_{s,j}) - a_{s+1,n+i}a_{s+1,j} - \dots - a_{n,n+i}a_{n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \dots - a_{2n,n+i}a_{2n,j} \in k$$

Now, the (i, j) entry of $\text{Inn}_A(J) - \text{Inn}_A \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} + \text{Inn}_A \begin{pmatrix} I & 0 \\ X_{rs} & I \end{pmatrix}$ is given by $2a_{r,n+i}a_{rj} + 2a_{s,n+i}a_{s,j}$ and hence $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ for all $l > n$ and $j = 1, 2, \dots, 2n$.

Subcase b: Suppose $i > n$

For $i > n$ the (i, j) entry of $\text{Inn}_A(J)$ yields

$$-a_{1,i-n}a_{1,j} - a_{2,i-n}a_{2,j} - \dots - a_{2n,i-n}a_{2n,j}$$

and the (i, j) position of $\text{Inn}_A \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ for $i > n$ is

$$-a_{n+1,i-n}a_{1j} - a_{n+2,i-n}a_{2,j} - \dots - a_{2n,i-n}a_{n,j} - a_{n+1,i-n}a_{n+1,j} - a_{n+2,i-n}a_{n+2,j} - \dots - a_{2n,i-n}a_{2n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \dots + a_{n,i-n}a_{2n,j}.$$

Hence the (i, j) entry of $\text{Inn}_A(J) - \text{Inn}_A \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ is given by

$$a_{n+1,i-n}a_{1j} + a_{n+2,i-n}a_{2,j} + \dots + a_{2n,i-n}a_{n,j} - a_{1,i-n}a_{1,j} - a_{2,i-n}a_{2,j} - \dots - a_{n,i-n}a_{n,j} - a_{1,i-n}a_{n+1,j} - a_{2,i-n}a_{n+2,j} - \dots - a_{n,i-n}a_{2n,j}.$$

For $i > n$ the (i, j) entry of $\text{Inn}_A \begin{pmatrix} I & 0 \\ X_{rs} & I \end{pmatrix}$ is

$$-a_{n+1,i-n}a_{1,j} - a_{n+2,i-n}a_{2,j} - \dots - a_{2n,i-n}a_{n,j} + a_{1,i-n}a_{1,j} + a_{2,i-n}a_{2,j} + \dots + a_{n,i-n}a_{n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \dots + a_{n,i-n}a_{2n,j}.$$

Therefore the (i, j) entry of $\text{Inn}_A(J) - \text{Inn}_A \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} + \text{Inn}_A \begin{pmatrix} I & 0 \\ X_{rs} & I \end{pmatrix}$ yields $-2a_{r,i-n}a_{rj} - 2a_{s,i-n}a_{s,j}$ and since $i > n$ we have that $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ for all $l \leq n$ and $j = 1, 2, \dots, 2n$. Combining subcases a and b we have that $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ whenever $r, s \leq n$.

CASE 2: Suppose $r, s > n$. Without loss of generality assume $r < s$.

Subcase a: Suppose $i \leq n$. Now the matrix $\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ is in G and since

Inn_A keeps G invariant the (i, j) entry of $\text{Inn}_A \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ given by

$$a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \dots + a_{n,j}a_{2n,n+i} - a_{1,n+i}a_{1,j} - a_{2,n+i}a_{2,j} - \dots - a_{n,n+i}a_{n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \dots - a_{n,n+i}a_{2n,j} \in k$$

Now the (i, j) entry of $\text{Inn}_A(J)$ was given in case 1 subcase a, therefore the

(i, j) entry of $\text{Inn}_A(J) + \text{Inn}_A \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ is

$a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \dots + a_{n,j}a_{2n,n+i} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \dots + a_{2n,n+i}a_{2n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \dots - a_{n,n+i}a_{2n,j}$
 which must lie in k . We know the matrix $\begin{pmatrix} I & X_{r-n,s-n} \\ 0 & I \end{pmatrix} \in G$ and thus the

automorphism $\text{Inn}_A \begin{pmatrix} I & X_{r-n,s-n} \\ 0 & I \end{pmatrix} \in G$ and its (i, j) entry given by
 $a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \dots + a_{n,j}a_{2n,n+i} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \dots + (-a_{r,n+i}a_{r,j}) + a_{r+1,n+i}a_{r+1,j} + \dots + (-a_{s,n+i}a_{s,j}) + a_{s+1,n+i}a_{s+1,j} + \dots + a_{2n,n+i}a_{2n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \dots - a_n n + i a_{2n,j} \in k$.

Finally we observe that the (i, j) entry of $\text{Inn}_A(J) + \text{Inn}_A \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} - \text{Inn}_A \begin{pmatrix} I & X_{r-n,s-n} \\ 0 & I \end{pmatrix}$ is given by $2a_{r,n+i}a_{r,j} + 2a_{s,n+i}a_{s,j}$ and hence $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ for all $l > n$ and $j = 1, 2, \dots, 2n$.

Subcase b: Suppose $i > n$. The (i, j) entry of $\text{Inn}_A \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ is in k and is given by

$$-a_{n+i,i-n}a_{1,j} - a_{n+2,i-n}a_{2,j} - \dots - a_{2n,i-n}a_{n,j} + a_{1,i-n}a_{1,j} + a_{2,i-n}a_{2,j} + \dots + a_{n,i-n}a_{n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \dots + a_{n,i-n}a_{2n,j}.$$

Hence the (i, j) position of $\text{Inn}_A(J) + \text{Inn}_A \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ is
 $-a_{n+i,i-n}a_{1,j} - a_{n+2,i-n}a_{2,j} - \dots - a_{2n,i-n}a_{n,j} - a_{n+1,i-n}a_{n+1,j} - a_{n+2,i-n}a_{n+2,j} - \dots - a_{2n,i-n}a_{2n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \dots + a_{n,i-n}a_{2n,j}$

must reside in k . For $i > n$ the (i, j) entry of $\text{Inn}_A \begin{pmatrix} I & X_{r-n,s-n} \\ 0 & I \end{pmatrix}$ is given by

$$-a_{n+1,i-n}a_{1,j} - a_{n+2,i-n}a_{2,j} - \dots - a_{2n,i-n}a_{n,j} - a_{n+1,i-n}a_{n+1,j} - a_{n+2,i-n}a_{n+2,j} - \dots - (-a_{r,i-n}a_{r,j}) - a_{r+1,i-n}a_{r+1,j} - \dots - (-a_{s,i-n}a_{s,j}) - a_{s+1,i-n}a_{s+1,j} - \dots - a_{2n,i-n}a_{2n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \dots + a_{n,i-n}a_{2n,j}.$$

Therefore by considering the (i, j) entry of $\text{Inn}_A(J) + \text{Inn}_A \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} -$

$\text{Inn}_A \begin{pmatrix} I & X_{r-n,s-n} \\ 0 & I \end{pmatrix}$ we see that $-2a_{r,i-n}a_{r,j} - 2a_{s,i-n}a_{s,j}$ must be in k . Since we assumed $i > n$ we have that $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ for all $l \leq n$ and $j = 1, 2, \dots, 2n$. By combining subcases a and b we obtain $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ whenever $r, s > n$.

CASE 3: Suppose $r \leq n$ and $s > n$.

Subcase a: Suppose $i \leq n$. The matrix $\begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix} \in G$ and therefore

$\text{Inn}_A \begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix} \in G$. Specifically, the (i, j) entry of $\text{Inn}_A \begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix}$ given by
 $a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \dots + a_{n,j}a_{2n,n+i} - a_{1,n+i}a_{1,j} - a_{2,n+i}a_{2,j} - \dots - (-a_{r,n+i}a_{r,j}) - a_{r+1,n+i}a_{r+1,j} - \dots - a_{n,n+i}a_{n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \dots - a_{n,n+i}a_{2n,j}$

lies in k . Now the (i, j) entry of $\text{Inn}_A(J) + \text{Inn}_A \begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix}$, which must be in k , is

$$a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \cdots + a_{n,j}a_{2n,n+i} + 2a_{r,n+i}a_{r,j} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \cdots + a_{2n,n+i}a_{2n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \cdots - a_{n,n+i}a_{2n,j}.$$

If we now consider the automorphism Inn_A on the matrix $\begin{pmatrix} I & X_{s-n} \\ 0 & I \end{pmatrix} \in G$

then we see that the (i, j) entry of $\text{Inn}_A \begin{pmatrix} I & X_{s-n} \\ 0 & I \end{pmatrix}$ is given by

$$a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \cdots + a_{n,j}a_{2n,n+i} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \cdots + (-a_{s,n+i}a_{s,j}) + \cdots + a_{2n,n+i}a_{2n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \cdots - a_{n,n+i}a_{2n,j}.$$

Hence, the (i, j) entry of $\text{Inn}_A(J) + \text{Inn}_A \begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix} - \text{Inn}_A \begin{pmatrix} I & X_{s-n} \\ 0 & I \end{pmatrix}$ gives us $2a_{r,n+i}a_{r,j} + 2a_{s,n+i}a_{s,j}$ and more importantly since we assumed $i \leq n$ we have that $a_{r,l}a_{r,j} + a_{s,l}a_{s,j} \in k$ for all $l > n$ and $j = 1, 2, \dots, 2n$.

Subcase b: Suppose $i > n$. For $i > n$ the (i, j) entry of $\text{Inn}_A \begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix}$ yields

$$-a_{n+1,i-n}a_{i,j} - a_{n+2,i-n}a_{2,j} - \cdots - a_{2n,i-n}a_{n,j} + a_{1,i-n}a_{1,j} + a_{2,i-n}a_{2,j} + \cdots + a_{r,i-n}a_{r,j} + \cdots + a_{n,i-n}a_{n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \cdots + a_{n,i-n}a_{2n,j}.$$

Therefore the (i, j) entry of $\text{Inn}_A(J) + \text{Inn}_A \begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix}$ is

$$-a_{n+1,i-n}a_{i,j} - a_{n+2,i-n}a_{2,j} - \cdots - a_{2n,i-n}a_{n,j} - 2a_{r,i-n}a_{r,j} - a_{n+1,i-n}a_{n+1,j} - a_{n+2,i-n}a_{n+2,j} - \cdots - a_{2n,i-n}a_{2n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \cdots + a_{n,i-n}a_{2n,j}.$$

Lastly we consider the (i, j) entry of $\text{Inn}_A \begin{pmatrix} I & X_{s-n} \\ 0 & I \end{pmatrix}$ which is given by

$$-a_{n+1,i-n}a_{i,j} - a_{n+2,i-n}a_{2,j} - \cdots - a_{2n,i-n}a_{n,j} - a_{n+1,i-n}a_{n+1,j} - a_{n+2,i-n}a_{n+2,j} - \cdots - (-a_{s,i-n}a_{s,j}) - \cdots - a_{2n,i-n}a_{2n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \cdots + a_{n,i-n}a_{2n,j}.$$

So the (i, j) entry of $\text{Inn}_A(J) + \text{Inn}_A \begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix} - \text{Inn}_A \begin{pmatrix} I & X_{s-n} \\ 0 & I \end{pmatrix}$ gives us $-2a_{r,i-n}a_{r,j} - 2a_{s,i-n}a_{s,j}$ and since $i > n$ we have that $a_{r,l}a_{r,j} + a_{s,l}a_{s,j} \in k$ for all $l \leq n$ and $j = 1, 2, \dots, 2n$.

Combining subcases a and b we have that $a_{r,l}a_{r,j} + a_{s,l}a_{s,j} \in k$ whenever $r \leq n$ and $s > n$

In conclusion, by combining Cases 1,2,and 3 we can conclude that $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for all $i, j = 1, 2, \dots, 2n$ and $r \neq s$.

We are now able to use the fact that $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for all $i, j = 1, 2, \dots, 2n$ and $r \neq s$ to show that $a_{r,i}a_{r,j} \in k$ for all $i, j = 1, 2, \dots, 2n$. However, we must show this in two cases. We will first show that $a_{r,l}a_{r,j} \in k$ for all $l \leq n$ and then show that $a_{r,l}a_{r,j} \in k$ for all $l > n$. Without loss of generality it shall suffice to show $a_{1,l}a_{1,j} \in k$ for all l .

CASE 1: Assume $i > n$. The (i, j) entry of $\text{Inn}_A(J)$ is given by $-a_{1,i-n}a_{1,j} - a_{2,i-n}a_{2,j} - \dots - a_{2n,i-n}a_{2n,j}$ which is in k and implies that $a_{1,i-n}a_{1,j} + a_{2,i-n}a_{2,j} + \dots + a_{2n,i-n}a_{2n,j} \in k$. From our previous argument we know that $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for all $i, j = 1, 2, \dots, 2n$, so obviously $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for $i > n$. Making use of that fact the equality given by

$$a_{1,i-n}a_{1,j} =$$

$$(a_{1,i-n}a_{1,j} + a_{2,i-n}a_{2,j} + \dots + a_{2n,i-n}a_{2n,j}) - (1/2)(a_{2,i-n}a_{2,j} + a_{3,i-n}a_{3,j}) - (1/2)(a_{3,i-n}a_{3,j} + a_{4,i-n}a_{4,j}) - (1/2)(a_{4,i-n}a_{4,j} + a_{5,i-n}a_{5,j}) - \dots$$

$$\dots - (1/2)(a_{2n,i-n}a_{2n,j} + a_{2,i-n}a_{2,j})$$

must be in k , ie. $a_{1,i-n}a_{1,j} \in k$. Since we assumed that $i > n$ we have that $a_{1,l}a_{1,j} \in k$ for $l \leq n$. Furthermore, we can conclude that $a_{r,l}a_{r,j} \in k$ for $l \leq n$

CASE 2: Assume $i \leq n$. Then the (i, j) entry of $\text{Inn}_A(J)$, which is in k , is given by $a_{1,i+n}a_{1,j} + a_{2,i+n}a_{2,j} + \dots + a_{2n,i+n}a_{2n,j}$. We again make use of the fact that $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for $i = 1, 2, \dots, 2n$, and have an equality similar to the one in case 1 ($i - n$ is simply replaced by $i + n$)

$$a_{1,i+n}a_{1,j} =$$

$$(a_{1,i+n}a_{1,j} + a_{2,i+n}a_{2,j} + \dots + a_{2n,i+n}a_{2n,j}) - (1/2)(a_{2,i+n}a_{2,j} + a_{3,i+n}a_{3,j}) - (1/2)(a_{3,i+n}a_{3,j} + a_{4,i+n}a_{4,j}) - (1/2)(a_{4,i+n}a_{4,j} + a_{5,i+n}a_{5,j}) - \dots$$

$$\dots - (1/2)(a_{2n,i+n}a_{2n,j} + a_{2,i+n}a_{2,j})$$

which again must be in k . Since we assumed $i \leq n$ we have that $a_{1,l}a_{1,j} \in k$ for $l > n$ and furthermore, $a_{r,l}a_{r,j} \in k$ for $l > n$. Combining cases 1 and 2 shows that $a_{r,l}a_{r,j} \in k$ for $i, j = 1, 2, \dots, 2n$.

We will finally show that $a_{r,i}a_{s,j} \in k$ for $r \neq s$.

CASE I: Suppose $r, s \leq n$. Without loss of generality we will assume that $r < s$.//

1. **Subcase 1:** Suppose $r + s < n + 1$.

Let $Y_{r,s} = \begin{pmatrix} T_{s+r-1} & 0 \\ 0 & I_{n-(s+r-1)} \end{pmatrix}$ and

$$Z_{r,s} = \begin{pmatrix} Y_{r,s} & 0 \\ E_{r,s} & Y_{r,s} \end{pmatrix}$$

Now $Z_{r,s} \in G$ and hence Inn_A must keep $Z_{r,s}$ invariant and thus all the entries of $\text{Inn}_A(Z_{r,s})$ must lie in k .

- (a) Assume $i \leq n$. Then the (i, j) entry of $\text{Inn}_A(Z_{r,s})$ is given by
- $$\begin{aligned}
 & -a_{r,n+i}a_{s,j} + a_{n+s+r-1,n+i}a_{1,j} + a_{n+s+r-2,n+i}a_{2,j} + \cdots + \\
 & a_{n+2,n+i}a_{s+r-2,j} + a_{n+1,n+i}a_{s+r-1,j} - a_{1,n+i}a_{n+s+r-1,j} - a_{2,n+i}a_{n+s+r-2,j} - \\
 & \cdots - a_{s+r-2,n+i}a_{n+2,j} - a_{s+r-1,n+i}a_{n+1,j} + a_{n+r+s,n+i}a_{r+s,j} \\
 & + a_{n+r+s+1,n+i}a_{r+s+1,j} + \cdots + a_{2n,n+i}a_{n,j} - a_{r+s,n+i}a_{n+r+s,j} \\
 & - a_{r+s+1,n+i}a_{n+r+s+1,j} - \cdots - a_{n,n+i}a_{2n,j}.
 \end{aligned}$$

Let $\bar{Z}_{r,s} = \begin{pmatrix} -Y_{r,s} & 0 \\ E_{r,s} & -Y_{r,s} \end{pmatrix}$. Now $\bar{Z}_{r,s} \in G$ and thus $\text{Inn}_A(\bar{Z}_{r,s}) \in G$. In fact, the (i, j) entry of $\text{Inn}_A(\bar{Z}_{r,s})$ is the negative of the (i, j) entry of $\text{Inn}_A Z_{r,s}$ with the exception of $-a_{r,n+i}a_{s,j}$ which remains negative. Therefore, $\text{Inn}_A(Z_{r,s}) + \text{Inn}_A(\bar{Z}_{r,s})$ has an (i, j) entry of $-2a_{r,n+i}a_{s,j}$. Since both $\text{Inn}_A(Z_{r,s})$ and $\text{Inn}_A(\bar{Z}_{r,s})$ are both in G their sum is in G and hence $-2a_{r,n+i}a_{s,j} \in k$. Since we assumed $i \leq n$ we have $a_{r,l}a_{s,j} \in k$ for $l > n$.

- (b) Assume $i > n$. Then the (i, j) entry of $\text{Inn}_A(Z_{r,s})$ is given by
- $$\begin{aligned}
 & a_{r,i-n}a_{s,j} - a_{n+s+r-1,i-n}a_{1,j} - a_{n+s+r-2,i-n}a_{2,j} - \cdots - a_{n+2,i-n}a_{s+r-2,j} - \\
 & a_{n+1,i-n}a_{s+r-1,j} + a_{1,i-n}a_{n+s+r-1,j} - a_{2,i-n}a_{n+s+r-2,j} + \cdots + a_{s+r-2,i-n}a_{n+2,j} + \\
 & a_{s+r-1,i-n}a_{n+1,j} + a_{n+r+s,i-n}a_{r+s,j} - a_{n+r+s+1,i-n}a_{r+s+1,j} - \cdots - a_{2n,i-n}a_{n,j} + \\
 & a_{r+s,i-n}a_{n+r+s,j} + a_{r+s+1,i-n}a_{n+r+s+1,j} + \cdots + a_{n,i-n}a_{2n,j}
 \end{aligned}$$

Note that the (i, j) entry of $\text{Inn}_A(Z_{r,s})$ for $i > n$ is the negative of the (i, j) entry of $\text{Inn}_A(Z_{r,s})$ for $i \leq n$ with the simple change that $n + i$ becomes $i - n$. Again we have that the (i, j) entry of $\text{Inn}_A(\bar{Z}_{r,s})$ is the negative of the (i, j) entry of $\text{Inn}_A(Z_{r,s})$ with the exception of $a_{r,i-n}a_{s,j}$ which remains positive. Hence as in the previous case the (i, j) entry of $\text{Inn}_A Z_{r,s} + \text{Inn}_A(\bar{Z}_{r,s})$, gives us $2a_{r,i-n}a_{s,j} \in k$. Since we assumed that $i > n$ we can conclude that $a_{r,l}a_{s,j} \in k$ for $l < n$.

Combining a and b we have that $a_{r,i}a_{s,j} \in k$ for $r + s < n + 1$.

2. Subcase 2: Suppose $r + s > n + 1$.

Let

$$U_{r,s} = \begin{pmatrix} I_{-n+(r+s-1)} & 0 \\ 0 & (T_{2n-(r+s-1)}) \end{pmatrix}$$

and

$$V_{r,s} = \begin{pmatrix} U_{r,s} & 0 \\ E_{r,s} & U_{r,s} \end{pmatrix}.$$

$V_{r,s} \in G$ so $\text{Inn}_A(V_{r,s}) \in G$ since Inn_A keeps G invariant.

- (a) Suppose $i \leq n$. Then the (i, j) entry of $\text{Inn}_A(V_{r,s})$ is given by
- $$\begin{aligned}
 & -a_{r,n+i}a_{s,j} + a_{2n,n+i}a_{s+r-n,j} + a_{2n-1,n+i}a_{s+r-n+1,j} + \cdots + a_{s+r,n+i}a_{n,j} - \\
 & a_{2n,j}a_{s+r-n,n+i} - a_{2n-1,j}a_{s+r-n+1,n+i} - \cdots - a_{s+r,j}a_{n,n+i} + a_{n+1,n+i}a_{1,j} + \\
 & a_{n+2,n+i}a_{2,j} + \cdots + a_{r+s-1,n+i}a_{(r+s-1)-n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \\
 & \cdots - a_{(r+s-1)-n,n+i}a_{r+s-1,j}.
 \end{aligned}$$

Let $\bar{V}_{r,s} = \begin{pmatrix} -U_{r,s} & 0 \\ E_{r,s} & -U_{r,s} \end{pmatrix}$. Now $\bar{V}_{r,s} \in G$ which implies that $\text{Inn}_A(\bar{V}_{r,s}) \in G$. The (i, j) entry of $\text{Inn}_A(\bar{V}_{r,s})$ is the negative of the (i, j) entry of

$\text{Inn}_A(V_{r,s})$ with the exception of the term $-a_{r,n+i}a_{s,j}$ which remains negative. Hence the (i, j) entry of $\text{Inn}_A(V_{r,s}) + \text{Inn}_A(\bar{V}_{r,s})$, $-2a_{r,n+i}a_{s,j}$ is in k . Since we assumed $i \leq n$ we have $a_{r,l}a_{s,j} \in k$ for $l > n$.

- (b) Assume $i > n$. As in the previous case, the (i, j) entry of $\text{Inn}_A(V_{r,s})$ for $i > n$ is the negative of the (i, j) entry of $\text{Inn}_A(V_{r,s})$ for $i \leq n$ with the simple change that $n + i$ becomes $i - n$. Again we have that the (i, j) entry of $\text{Inn}_A(\bar{V}_{r,s})$ is the negative of the (i, j) entry of $\text{Inn}_A(V_{r,s})$ with the exception of $a_{r,i-n}a_{s,j}$ which remains positive. Hence as in the previous case the (i, j) entry of $\text{Inn}_A Z_{r,s} + \text{Inn}_A(\bar{Z}_{r,s})$, gives us $2a_{r,i-n}a_{s,j}$. Since we assumed that $i > n$ we can conclude that $a_{r,l}a_{s,j} \in k$ for $l < n$.

Combining a and b we have that $a_{r,i}a_{s,j} \in k$ for $r + s < n + 1$

- 3. **Subcase 3:** Suppose $r + s = n + 1$. Here we choose $W_{r,s} = \begin{pmatrix} T_n & 0 \\ E_{r,s} & T_n \end{pmatrix}$. Now $W_{r,s} \in G$ and hence, $\text{Inn}_A(W_{r,s}) \in G$ since Inn_A keeps G invariant.

- (a) Suppose $i \leq n$ Then the (i, j) entry of $\text{Inn}_A(W_{r,s})$ is given by $-a_{r,n+i}a_{s,j} + a_{2n,n+i}a_{1,j} + a_{2n-1,n+i}a_{2,j} + \dots + a_{n+1,n+i}a_{n,j} - a_{2n,j}a_{1,n+i} + a_{2n-1,j}a_{2,n+i} + \dots + a_{n+1,j}a_{n,n+i}$.
Let $\bar{W}_{r,s} = \begin{pmatrix} -T_n & 0 \\ E_{r,s} & -T_n \end{pmatrix}$. $\bar{W}_{r,s} \in G$ which means that $\text{Inn}_A(\bar{W}_{r,s}) \in G$. The (i, j) entry of $\text{Inn}_A(\bar{W}_{r,s})$ is the negative of the (i, j) entry of $\text{Inn}_A(W_{r,s})$ with the exception that the term $-a_{r,n+i}a_{s,j}$ which remains negative. Using the fact that $\text{Inn}_A(W_{r,s}) + \text{Inn}_A(\bar{W}_{r,s}) \in G$ we have that the term $-2a_{r,n+i}a_{s,j} \in k$. However, since we assumed that $i \leq n$ we have that $a_{r,l}a_{s,j} \in k$ for $l > n$.
- (b) The case where $i > n$ follows exactly as above by simply changing the signs of each term and replacing $n + i$ by $i - n$.

Combining Subcases 1,2, and 3 gives us $a_{r,i}a_{s,j} \in k$ for $r, s > n$.

CASE II: Suppose $r, s > n$. Without loss of generality assume $r < s$.

- 1. **Subcase 1:** Suppose $r + s - 2n < n + 1$.

Let $Z'_{r,s} = \begin{pmatrix} Y_{r-n,s-n} & E_{r-n,s-n} \\ 0 & Y_{r-n,s-n} \end{pmatrix}$

- (a) Suppose $i \leq n$, Since $Z'_{r,s} \in G$, $\text{Inn}_A(Z'_{r,s})$ must lie in G and hence its (i, j) entry of $a_{r,n+i}a_{s,j} + a_{n+s+r-1,n+i}a_{1,j} + a_{n+s+r-2,n+i}a_{2,j} + \dots + a_{n+2,n+i}a_{s+r-2,j} + a_{n+1,n+i}a_{s+r-1,j} - a_{1,n+i}a_{n+s+r-1,j} - a_{2,n+i}a_{n+s+r-2,j} - \dots - a_{s+r-2,n+i}a_{n+2,j} - a_{s+r-1,n+i}a_{n+1,j} + a_{n+r+s,n+i}a_{r+s,j} + a_{n+r+s+1,n+i}a_{r+s+1,j} + \dots + a_{2n,n+i}a_{n,j} - a_{r+s,n+i}a_{n+r+s,j} - a_{r+s+1,n+i}a_{n+r+s+1,j} - \dots - a_{n,n+i}a_{2n,j}$

is in k . Note that the (i, j) entry of $\text{Inn}_A(Z'_{r,s})$ is precisely the (i, j) entry of $\text{Inn}_A(Z_{r,s})$ given in part I with the exception of the first term. Let $\bar{Z}'_{r,s} = \begin{pmatrix} -Y_{r-n,s-n} & E_{r-n,s-n} \\ 0 & -Y_{r-n,s-n} \end{pmatrix}$. The (i, j) entry of $\text{Inn}_A(\bar{Z}'_{r,s})$ is the negative of the (i, j) entry of $\text{Inn}_A(Z'_{r,s})$ excluding the term $a_{r,n+i}a_{s,j}$ which remains positive. Hence the (i, j) entry of $\text{Inn}_A(Z'_{r,s}) + \text{Inn}_A(\bar{Z}'_{r,s})$, given by $2a_{r,n+i}a_{s,j}$ lies in k . Since we assumed $i \leq n$ we have $a_{r,l}a_{s,j} \in k$ for $l > n$.

- (b) As in the previous cases, for $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n + i$ by $i - n$. You will get that the (i, j) entry of $\text{Inn}_A(Z'_{r,s}) + \text{Inn}_A(\bar{Z}'_{r,s})$ yields $-2a_{r,i-n}a_{s,j} \in k$. Or more specifically, $a_{r,l}a_{s,j} \in k$ for $l \leq n$.

Combining a and b gives $a_{r,i}a_{s,j} \in k$ for $r + s - 2n < n + 1$

2. Subcase 2: Suppose $r + s - 2n > n + 1$

- (a) Let $V'_{r,s} = \begin{pmatrix} U_{r-n,s-n} & E_{r-n,s-n} \\ 0 & U_{r-n,s-n} \end{pmatrix}$. Now $\text{Inn}_A(V'_{r,s})$ must lie in G and hence its (i, j) entry of

$$a_{r,n+i}a_{s,j} + a_{2n,n+i}a_{s+r-n,j} + a_{2n-1,n+i}a_{s+r-n+1,j} + \dots + a_{s+r,n+i}a_{n,j} - a_{2n,j}a_{s+r-n,n+i} - a_{2n-1,j}a_{s+r-n+1,n+i} - \dots - a_{s+r,j}a_{n,n+i} + a_{n+1,n+i}a_{1,j} + a_{n+2,n+i}a_{2,j} + \dots + a_{r+s-1,n+i}a_{(r+s-1)-n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \dots - a_{(r+s-1)-n,n+i}a_{r+s-1,j}$$

must lie in k . If we define $\bar{V}'_{r,s} = \begin{pmatrix} -U_{r,s} & E_{r-n,s-n} \\ 0 & -U_{r,s} \end{pmatrix}$, which is in G , then we see that the (i, j) entry of $\text{Inn}_A(\bar{V}'_{r,s})$ is the negative of the (i, j) entry of $\text{Inn}_A(V'_{r,s})$ excluding the term $a_{r,n+i}a_{s,j}$ which remains positive. Hence the (i, j) entry of $\text{Inn}_A(V'_{r,s}) + \text{Inn}_A(\bar{V}'_{r,s})$, $2a_{r,n+i}a_{s,j}$ is in k . Since we assumed $i \leq n$ we have $a_{r,l}a_{s,j} \in k$ for $l > n$.

- (b) Again as in the previous cases, for $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n + i$ by $i - n$. You will get that the (i, j) entry of $\text{Inn}_A(V'_{r,s}) + \text{Inn}_A(\bar{V}'_{r,s})$ yields that the term $-2a_{r,i-n}a_{s,j}$ is in k . Or more specifically, $a_{r,l}a_{s,j} \in k$ for $l \leq n$.

Combining a and b gives $a_{r,i}a_{s,j} \in k$ for $r + s - 2n > n + 1$

3. Subcase 3: Suppose $r + s - 2n = n + 1$. Let $W'_{r,s} = \begin{pmatrix} T_n & E_{r-n,s-n} \\ 0 & T_n \end{pmatrix}$. Now $W'_{r,s} \in G$ and thus $\text{Inn}_A(W'_{r,s}) \in G$.

- (a) Suppose $i \leq n$, then the (i, j) entry of $\text{Inn}_A(W'_{r,s})$ is given by $a_{r,n+i}a_{s,j} + a_{2n,n+i}a_{1,j} + a_{2n-1,n+i}a_{2,j} + \dots + a_{n+1,n+i}a_{n,j} - a_{2n,j}a_{1,n+i} +$

$$a_{2n-1,j}a_{2,n+i} + \dots + a_{n+1,j}a_{n,n+i}.$$

If we let $\bar{W}'_{r,s} = \begin{pmatrix} -T_n & E_{r-n,s-n} \\ 0 & -T_n \end{pmatrix}$, then we see that the (i, j) entry is simply the negative of the (i, j) entry of $\text{Inn}_A(W'_{r,s})$ excluding the term $a_{r,n+i}a_{s,j}$ which remains positive. Hence the (i, j) entry of $\text{Inn}_A(W'_{r,s}) + \text{Inn}_A(\bar{W}'_{r,s})$, $2a_{r,n+i}a_{s,j}$ is in k . Since we assumed $i \leq n$ we have $a_{r,l}a_{s,j} \in k$ for $l > n$.

- (b) Again as in the previous cases, for $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n + i$ by $i - n$. You will get that the (i, j) entry of $\text{Inn}_A(W'_{r,s}) + \text{Inn}_A(\bar{W}'_{r,s})$ yields that the term $-2a_{r,i-n}a_{s,j}$ is in k . Or more specifically, $a_{r,l}a_{s,j} \in k$ for $l \leq n$.

Combining a and b gives $a_{r,i}a_{s,j} \in k$ for $r + s - 2n > n + 1$

CASE III: Suppose $r \leq n$ and $s > n$

- 1. **Subcase 1:** Suppose $r + s < 2n + 1$.

Let $M_{r,s} = \begin{pmatrix} E_{s-n,r} & Y_{s-n,r} \\ -Y_{s-n,r} & 0 \end{pmatrix}$. Now $M_{r,s} \in G$ and thus $\text{Inn}_A(M_{r,s}) \in G$ by assumption.

- (a) Suppose $i \leq n$. Now the (i, j) entry of $\text{Inn}_A(M_{r,s})$ is given by

$$a_{r,j}a_{s,n+i} + a_{s+r,n+i}a_{s+r,j} + a_{s+r+1,n+i}a_{s+r+1,j} + \dots + a_{2n,n+i}a_{2n,j} +$$

$$a_{s+r-n,n+i}a_{s+r-n,j} + a_{s+r-n+1,n+i}a_{s+r-n+1,j} + \dots + a_{n,n+i}a_{n,j} + a_{s+r-1,n+i}a_{n+1,j}$$

$$+ a_{s+r-2,n+i}a_{n+2,j} + \dots + a_{n+1,n+i}a_{s+r-1,j} + a_{s-n+r-1,n+i}a_{1,j}$$

$$+ a_{s-n+r-2,n+i}a_{2,j} + \dots + a_{1,n+i}a_{s-n+r-1,j}.$$

Now define $\bar{M}_{r,s} = \begin{pmatrix} E_{s-n,r} & -Y_{s-n,r} \\ Y_{s-n,r} & 0 \end{pmatrix}$, then $\bar{M}_{r,s} \in G$ and therefore $\text{Inn}_A(\bar{M}_{r,s}) \in G$. In addition the (i, j) entry of $\bar{M}_{r,s}$ is the negative of the (i, j) entry of $\text{Inn}_A(M_{r,s})$ with the exception of the term $a_{r,j}a_{s,i+n}$ which remains positive. The sum $\text{Inn}_A(M_{r,s}) + \text{Inn}_A(\bar{M}_{r,s}) \in G$ and thus its (i, j) entry of $2a_{r,j}a_{s,i+n} \in k$. Since we assumed that $i \leq n$ this gives us $a_{r,j}a_{s,l} \in k$ for $l > n$.

- (b) As in the previous cases, for $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n + i$ by $i - n$. You will get that the (i, j) entry of $\text{Inn}_A(M_{r,s}) + \text{Inn}_A(\bar{M}_{r,s})$ yields that the term $-2a_{r,j}a_{s,i-n}$ is in k . Or more specifically, $a_{r,j}a_{s,l} \in k$ for $l \leq n$.

Combining a and b gives $a_{r,j}a_{s,l} \in k$ for $r + s < 2n + 1$.

- 2. **Subcase 2:** Suppose $r + s > 2n + 1$.

With $N_{r,s} = \begin{pmatrix} E_{s-n,r} & U_{s-n,r} \\ -U_{s-n,r} & 0 \end{pmatrix}$ it is seen that $N_{r,s} \in G$ and thus by assumption $\text{Inn}_A(N_{r,s}) \in G$

- (a) Suppose $i \leq n$ then the (i, j) entry of $\text{Inn}_A(N_{r,s})$ is given by
- $$a_{r,j}a_{s,n+i} + a_{2n,n+i}a_{s+r-n,j} + a_{2n-1,n+i}a_{s+r-n+1,j} + \cdots + a_{s+r-n,n+i}a_{2n,j} + a_{n,n+i}a_{s+r-2n,j} + a_{n-1,n+i}a_{s+r-2n+1,j} + \cdots + a_{s+r-2n,n+i}a_{n,j} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \cdots + a_{-n+r+s-1,n+i}a_{-n+r+s-1,j} + a_{1,n+i}a_{1,j} + a_{2,n+i}a_{2,j} + \cdots + a_{-2n+r+s-1,n+i}a_{-2n+r+s-1,j}$$
- Define $\bar{N}_{r,s} = \begin{pmatrix} E_{s-n,r} & -U_{s-n,r} \\ U_{s-n,r} & 0 \end{pmatrix}$. We again can make use of the fact that $\bar{N}_{r,s} \in G$ implies that $\text{Inn}_A(\bar{N}_{r,s}) \in G$. Now the (i, j) entry of $\text{Inn}_A(\bar{N}_{r,s})$ is the negative of the (i, j) entry of $\text{Inn}_A(N_{r,s})$ with the exception of the the term $a_{r,j}a_{s,n+i}$ which remains positive. Hence the (i, j) entry of $\text{Inn}_A(\bar{N}_{r,s}) + \text{Inn}_A(N_{r,s})$ given by $2a_{r,j}a_{s,n+i}$ must lie in k . Furthermore, since we assumed that $i \leq n$ we can conclude that $a_{r,j}a_{s,l} \in k$ for $l > n$.
- (b) As in the previous cases, if $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n + i$ by $i - n$. You will get that the (i, j) entry of $\text{Inn}_A(N_{r,s}) + \text{Inn}_A(\bar{N}_{r,s})$ yields that the term $-2a_{r,j}a_{s,i-n}$ is in k . Or more specifically, $a_{r,j}a_{s,l} \in k$ for $l \leq n$.

Combining a and b gives $a_{r,j}a_{s,l} \in k$ for $r + s < 2n + 1$

3. Subcase 3: Suppose $r + s = 2n + 1$. Let $F_{r,s} = \begin{pmatrix} E_{s-n,r} & T_n \\ T_n & 0 \end{pmatrix}$.

Now $F_{r,s} \in G$ and therefore $\text{Inn}_A(F_{r,s}) \in G$ since Inn_A keeps G invariant.

- (a) Suppose $i \leq n$. Then the (i, j) entry of $\text{Inn}_A(F_{r,s})$ is given by
- $$a_{r,j}a_{s,n+i} + a_{1,n+i}a_{1,j} + a_{2,n+i}a_{2,j} + a_{2,n+i}a_{3,j} + \cdots + a_{2n,n+i}a_{2n,j}$$
- Let $\bar{F}_{r,s} = \begin{pmatrix} E_{s-n,r} & -T_n \\ -T_n & 0 \end{pmatrix}$. Then since $\bar{F}_{r,s} \in G$ we have that $\text{Inn}_A(\bar{F}_{r,s}) \in G$. More importantly, the (i, j) entry of $\text{Inn}_A(\bar{F}_{r,s})$ is the negative of the (i, j) entry of $\text{Inn}_A(F_{r,s})$ with the exception that the term $a_{r,j}a_{s,i+n}$ remains positive. Again using the fact that $\text{Inn}_A(\bar{F}_{r,s}) + \text{Inn}_A(F_{r,s}) \in G$ we have that its (i, j) entry of $2a_{r,j}a_{s,n+i} \in k$. Since we assumed $i \leq n$ we have that $a_{r,j}a_{s,l} \in k$ for $l > n$.
- (b) As in the previous cases, if $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n + i$ by $i - n$. You will get that the (i, j) entry of $\text{Inn}_A(\bar{F}_{r,s}) + \text{Inn}_A(F_{r,s}) \in G$ yields that the term $-2a_{r,j}a_{s,i-n}$ is in k . Or more specifically, $a_{r,j}a_{s,l} \in k$ for $l \leq n$

Combining subcases a and b gives us $a_{r,j}a_{s,l} \in k$ for $r + s = 2n + 1$.

Combining cases 1,2, and 3 gives us $a_{r,i}a_{s,j} \in k$ for $r \leq n$ and $s > n$.

Cases I, II and III show that $a_{r,i}a_{s,j} \in k$. From this it is clear that $k[a_{is}] = k[a_{jt}]$ for all i, j, s, t (assuming that a_{is} and a_{jt} are both nonzero). So, let $\alpha = a_{is}^2$ where a_{is} is a fixed nonzero entry of A . Then, we have shown that all

the entries of A are in $k[\sqrt{\alpha}]$. This means that $A \in \text{SP}(2n, k[\sqrt{\alpha}])$, and all of the entries of A are k -multiples of $\sqrt{\alpha}$, as desired. ■

4. Involutions of $\text{SP}(2n, k)$

We now begin to focus on involutions and the classification of their isomorphism classes. We will distinguish different types of involutions. First, we note that for some involutions, ϕ , there exists $A \in \text{SP}(2n, k)$ such that $\phi = \text{Inn}_A$, but not in all cases. Sometimes we must settle for $A \in \text{SP}(2n, k[\sqrt{\alpha}]) \setminus \text{SP}(2n, k)$.

This is not the only way in which we can distinguish between different types of involutions. If Inn_A is an involution, then $\text{Inn}_{A^2} = (\text{Inn}_A)^2$ is the identity map. We know from above that this means that $A^2 = \gamma I$ for some $\gamma \in \bar{k}$. But, we know that A is symplectic. So, A^2 is also symplectic. That means that $(A^2)^T J (A^2) = J$, which implies $(\gamma I)^T J (\gamma I) = J$, which means $\gamma^2 = 1$. So, $\gamma = \pm 1$. Thus, we can also distinguish between different types of involutions by seeing if $A^2 = I$ or $A^2 = -I$. This gives the four types of involutions, which are outlined in Table 1.

Table 1: The various possible types of involutions of $\text{SP}(2n, k)$

	$A \in \text{SP}(2n, k)$	$A \in \text{SP}(2n, k[\sqrt{\alpha}]) \setminus \text{SP}(n, k)$
$A^2 = I$	Type 1	Type 2
$A^2 = -I$	Type 3	Type 4

4.1. Type 1 Involutions.

We first characterize the matrices that induce Type 1 involutions in the following lemma.

Lemma 4.1. *Suppose θ is a Type 1 involution of $\text{SP}(2n, k)$. Then,*

$$A = X \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} X^{-1}$$

where $s + t = 2n$ and $X^T J X = J$. That is, $X \in \text{SP}(2n, k)$.

Proof. Since $\text{Inn}_A^2 = I$ and $A \in \text{SP}(2n, k)$, then it follows that $A^2 = I$. So, all eigenvalues of A are ± 1 . Since there are no repeated roots in the minimal polynomial of A , then we see that A is diagonalizable. Let $s = \dim(E(A, 1))$ and $t = \dim(E(A, -1))$, and observe that $s + t = 2n$ since A is diagonalizable. We will first show that both s and t must be even. To do this, we proceed by contradiction and assume that s and t are both odd. So, there exists some $Y \in \text{GL}(n, \bar{k})$ such

that $Y^{-1}AY = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}$. Since A is symplectic, then it follows that

$$\begin{aligned} J &= A^T J A \\ &= \left(Y \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Y^{-1} \right)^T J Y \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Y^{-1} \\ &= (Y^{-1})^T \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Y^T J Y \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Y^{-1}. \end{aligned}$$

This implies that

$$\begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Y^T J Y = (Y^T J Y) \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix},$$

where $Y^T J Y$ is an invertible skew-symmetric matrix. So, $Y^T J Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}$ for some invertible skew symmetric matrices Y_1 and Y_2 , which are $s \times s$ and $t \times t$, respectively. But odd dimensional skew symmetric matrices cannot be invertible, so this is a contradiction. Thus, s and t must be even.

We now wish to construct bases for $E(A, 1)$ and $E(A, -1)$ such that all the vectors lie in k^n . Let $\{z_1, \dots, z_n\}$ be a basis for k^n . For each i , let $u_i = (A + I)z_i$. Note that

$$A u_i = A(A + I)z_i = (A + I)z_i = u_i.$$

So, $\{u_1, \dots, u_n\}$ must span $E(A, 1)$. Thus, we can appropriately choose s of these vectors and form a basis for $E(A, 1)$. Label these basis vectors as $y_1, \dots, y_{\frac{s}{2}}, y_{n+1}, \dots, y_{n+\frac{s}{2}}$. We can similarly form a basis for $E(A, -1)$. We shall call these vectors $y_{\frac{s}{2}+1}, \dots, y_n, y_{n+\frac{s}{2}+1}, \dots, y_{2n}$. Let Y be the matrix with the vectors y_1, \dots, y_{2n} as its columns. Then, by construction,

$$Y^{-1}AY = \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix}.$$

We can rearrange to get

$$A = Y \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} Y^{-1}.$$

Recall that $A^T = JAJ^{-1}$, since $A \in \text{SP}(2n, k)$. So,

$$\left(Y \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} Y^{-1} \right)^T = J \left(Y \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} Y^{-1} \right) J^{-1}.$$

This implies

$$(Y^{-1})^T \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} Y^T = JY \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} (JY)^{-1},$$

which means

$$\begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} Y^T JY = Y^T JY \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix}.$$

So, $Y^T JY = \begin{pmatrix} Y_1 & 0 & Y_2 & 0 \\ 0 & Y_3 & 0 & Y_4 \\ -Y_2^T & 0 & Y_5 & 0 \\ 0 & -Y_4^T & 0 & Y_6 \end{pmatrix}$, where Y_1 and Y_5 are $\frac{s}{2} \times \frac{s}{2}$ skew-symmetric matrices, Y_3 and Y_6 are $\frac{t}{2} \times \frac{t}{2}$ skew-symmetric matrices, Y_2 is a $\frac{s}{2} \times \frac{s}{2}$ matrix and Y_4 is a $\frac{t}{2} \times \frac{t}{2}$ matrix.

We can choose a permutation matrix $Q \in O(2n, k)$ such that

$$A = YQ \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Q^{-1}Y^{-1}$$

and

$$Y^T JY = Q \begin{pmatrix} Y_1 & Y_2 & 0 & 0 \\ -Y_2^T & Y_5 & 0 & 0 \\ 0 & 0 & Y_3 & Y_4 \\ 0 & 0 & -Y_4^T & Y_6 \end{pmatrix} Q^{-1}.$$

Let $Y_7 = \begin{pmatrix} Y_1 & Y_2 \\ -Y_2^T & Y_5 \end{pmatrix}$ and $Y_8 = \begin{pmatrix} Y_3 & Y_4 \\ -Y_4^T & Y_6 \end{pmatrix}$. Note that both Y_7 and Y_8 are skew-symmetric. We can rearrange the above statement to be

$$Q^T Y^T JY Q = \begin{pmatrix} Y_7 & 0 \\ 0 & Y_8 \end{pmatrix}.$$

It follows that there exists $N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \in GL(n, k)$ such that

$$N^T Q^T Y^T JY Q N = \begin{pmatrix} 0 & I_{\frac{s}{2}} & 0 & 0 \\ -I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\frac{t}{2}} \\ 0 & 0 & -I_{\frac{t}{2}} & 0 \end{pmatrix}.$$

We see that we can again use the permutation matrix Q to get

$$QN^T Q^T Y^T JY Q N Q^T = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = J.$$

Let $X = YQNQ^T$. Then,

$$\begin{aligned} X \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} X^{-1} &= YQNQ^T \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} (YQNQ^T)^{-1} \\ &= YQN \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} N^{-1}Q^{-1}Y^{-1} \\ &= YQ \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} Q^{-1}Y^{-1} \\ &= Y \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} Y^{-1} \\ &= A, \end{aligned}$$

where $X^T J X = J$. From this last observation, it follows that $X \in \text{SP}(2n, k)$. ■

Using this characterization, we now find conditions on these involutions that are equivalent to isomorphy.

Theorem 4.2. *Suppose Inn_A and Inn_B both induce Type 1 involutions for $\text{SP}(2n, k)$ for A and $B \in \text{SP}(2n, k)$. Then, Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$ if and only if the dimension of $E(A, 1)$ equals the dimension of $E(B, 1)$ or $E(B, -1)$.*

Proof. We first prove that Inn_A is isomorphic to Inn_B over $\text{SP}(2n, k)$ is equivalent to A being conjugate to B or $-B$ over $\text{SP}(2n, k)$. Suppose A is conjugate to B over $\text{SP}(2n, k)$. Choose $Q \in \text{SP}(2n, k)$ such that $B = Q^{-1}AQ$. Then, for all $U \in \text{SP}(2n, k)$, we have

$$\begin{aligned} Q^{-1}A^{-1}QUQ^{-1}AQ &= (Q^{-1}AQ)^{-1}U(Q^{-1}AQ) \\ &= B^{-1}UB. \end{aligned}$$

So, $(\text{Inn}_Q)^{-1}\text{Inn}_A\text{Inn}_Q = \text{Inn}_B$. That is, Inn_A is isomorphic to Inn_B over $\text{SP}(2n, k)$. Likewise, if A is conjugate to $-B$, then we can show Inn_A is isomorphic to Inn_B over $\text{SP}(2n, k)$. This argument is easily reversible.

From this, it is clear that if Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$, then the dimension of $E(A, 1)$ equals the dimension of $E(B, 1)$ or $E(B, -1)$. We need only show the converse.

First, suppose that the dimension of $E(A, 1)$ equals the dimension of $E(B, 1)$. By the previous lemma, we can choose $X, Y \in \text{SP}(2n, k)$ such that

$$X^{-1}AX = \begin{pmatrix} I_{\frac{s}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{t}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{s}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{t}{2}} \end{pmatrix} = Y^{-1}BY.$$

Let $Q = XY^{-1}$. Note that $Q \in \text{SP}(2n, k)$. Then, we have $Q^{-1}AQ = B$, and we have already shown that this implies Inn_A is isomorphic to Inn_B over $\text{SP}(2n, k)$.

If the dimension of $E(A, 1)$ equals the dimension of $E(B, -1)$, then we can similarly show that there exists $Q \in \text{SP}(2n, k)$ such that $Q^{-1}AQ = -B$, which also implies Inn_A is isomorphic to Inn_B over $\text{SP}(2n, k)$. ■

From this theorem, the number of isomorphism classes of Type 1 involutions is clear. We note that this number is independent of the field k .

Corollary 4.3. $\text{SP}(2n, k)$ has $\frac{n}{2}$ or $\frac{n-1}{2}$ isomorphism classes of Type 1 involutions. (Whichever is an integer.)

4.2. Type 2 Involutions.

We have a similar characterization of the matrices and isomorphism classes in the Type 2 case. We first prove a result that characterizes the eigenvectors in the Type 2 case.

Lemma 4.4. Suppose $A \in \text{SP}(2n, k[\sqrt{\alpha}], \beta) \setminus \text{SP}(2n, k, \beta)$ induces a Type 2 involution of $\text{SP}(n, k, \beta)$ where $\sqrt{\alpha} \notin k$. Also suppose $x, y \in k^{2n}$ such that $x + \sqrt{\alpha}y \in E(A, -1)$. Then, $x - \sqrt{\alpha}y \in E(A, 1)$. Likewise, if $u, v \in k^{2n}$ such that $u + \sqrt{\alpha}v \in E(A, 1)$. Then, $u - \sqrt{\alpha}v \in E(A, -1)$. Further, $\dim(E(A, 1)) = \dim(E(A, -1))$.

Proof. First, we observe that “ $\sqrt{\alpha}$ -conjugation,” similar to the familiar complex conjugation (i -conjugation), preserves multiplication. That is,

$$(a + \sqrt{\alpha}b)(c + \sqrt{\alpha}d) = (ac + \alpha bd) + \sqrt{\alpha}(ad + bc)$$

and

$$(a - \sqrt{\alpha}b)(c - \sqrt{\alpha}d) = (ac + \alpha bd) - \sqrt{\alpha}(ad + bc).$$

So, “ $\sqrt{\alpha}$ -conjugation” will preserve multiplication on the matrix level as well. Because of this and since

$$A(x + \sqrt{\alpha}y) = -x - \sqrt{\alpha}y,$$

then it follows that

$$(-A)(x - \sqrt{\alpha}y) = -x + \sqrt{\alpha}y.$$

We can multiply both sides to see that

$$A(x - \sqrt{\alpha}y) = x - \sqrt{\alpha}y.$$

That is, $x - \sqrt{\alpha}y \in E(A, 1)$. This proves the first statement. An analogous argument proves the second.

To see that $\dim(E(A, 1)) = \dim(E(A, -1))$ is the case, note that the first statement tells us that $\dim(E(A, 1)) \leq \dim(E(A, -1))$, and that the second statement tells us that $\dim(E(A, 1)) \geq \dim(E(A, -1))$, since “ $\sqrt{\alpha}$ -conjugation” is an invertible operator on $k[\sqrt{\alpha}]^n$. ■

We are now able to characterize the Type 2 involutions. Note that this result combined with our results from the Type 1 case shows that if n is odd, then $\text{SP}(2n, k)$ will not have any Type 2 involutions.

Lemma 4.5. *Suppose θ is a Type 2 involution of $\text{SP}(2n, k)$. Let A be the symplectic matrix in $\text{SP}(2n, k[\sqrt{\alpha}])$ such that $\theta = \text{Inn}_A$. Then,*

$$A = \frac{\sqrt{\alpha}}{\alpha} X \begin{pmatrix} 0 & I_n \\ \alpha I_n & 0 \end{pmatrix} X^{-1}$$

where

$$X = (x_1 \ x_2 \ \cdots \ x_n \ y_1 \ y_2 \ \cdots \ y_n) \in \text{GL}(2n, k),$$

where for each i , we have that $x_i + \sqrt{\alpha}y_i \in E(A, 1)$ and $x_i - \sqrt{\alpha}y_i \in E(A, -1)$. Further,

$$X^T J X = \frac{1}{2} \begin{pmatrix} J & 0 \\ 0 & \frac{1}{\alpha} J \end{pmatrix}.$$

Proof. We wish to construct bases for $E(A, 1)$ and $E(A, -1)$ such that all the vectors lie in $k[\sqrt{\alpha}]^{2n}$. From the previous lemma, we know that $\dim(E(A, 1)) = \dim(E(A, -1)) = n$. Since Inn_A is a Type 1 involution of $\text{SP}(2n, k[\alpha])$, then we can apply Lemma 4.1 to find a basis $\{x_1 + \sqrt{\alpha}y_1, \dots, x_n + \sqrt{\alpha}y_n\}$ of $E(A, 1)$, where $x_1, \dots, x_n, y_1, \dots, y_n \in k^{2n}$. By the previous lemma, we know that $\{x_1 - \sqrt{\alpha}y_1, \dots, x_{\frac{n}{2}} - \sqrt{\alpha}y_{\frac{n}{2}}\}$ must be a basis for $E(A, -1)$. Further, based on Lemma 4.1, we can assume that these vectors are chosen so that if

$$Y = (x_1 + \sqrt{\alpha}y_1, \dots, x_{\frac{n}{2}} + \sqrt{\alpha}y_{\frac{n}{2}}, x_1 - \sqrt{\alpha}y_1, \dots, x_{\frac{n}{2}} - \sqrt{\alpha}y_{\frac{n}{2}}, x_{\frac{n}{2}+1} + \sqrt{\alpha}y_{\frac{n}{2}+1}, \dots, x_n + \sqrt{\alpha}y_n, x_{\frac{n}{2}+1} - \sqrt{\alpha}y_{\frac{n}{2}+1}, \dots, x_n - \sqrt{\alpha}y_n),$$

then we know that

$$A = Y \begin{pmatrix} I_{\frac{n}{2}} & 0 & 0 & 0 \\ 0 & -I_{\frac{n}{2}} & 0 & 0 \\ 0 & 0 & I_{\frac{n}{2}} & 0 \\ 0 & 0 & 0 & -I_{\frac{n}{2}} \end{pmatrix} Y^{-1}$$

where $Y^T J Y = J$.

Let $X = (x_1 \ x_2 \ \cdots \ x_{\frac{n}{2}} \ y_1 \ y_2 \ \cdots \ y_{\frac{n}{2}}) \in \text{GL}(n, k)$.

We now make a couple of observations. Suppose $u = x + \sqrt{\alpha}y$ is a 1-eigenvector of A such that $x, y \in k^n$. Then, we know $v = x - \sqrt{\alpha}y$ is a -1-eigenvector of A . Observe that

$$Ax = \frac{1}{2}A(u + v) = \frac{1}{2}(u - v) = \sqrt{\alpha}y.$$

It follows from this that

$$Ay = \frac{\sqrt{\alpha}}{\alpha}x.$$

Since $Ax = \sqrt{\alpha}y$ and $Ay = \frac{\sqrt{\alpha}}{\alpha}x$, then it follows that

$$X^{-1}AX = \begin{pmatrix} 0 & \frac{\sqrt{\alpha}}{\alpha}I_{\frac{n}{2}} \\ \sqrt{\alpha}I_{\frac{n}{2}} & 0 \end{pmatrix}.$$

Rearranging this, we see that

$$A = \frac{\sqrt{\alpha}}{\alpha}X \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \alpha I_{\frac{n}{2}} & 0 \end{pmatrix} X^{-1}.$$

Now, we need only prove the last statement to prove the Lemma. Since $Y^T J Y = J$, then we know that if $1 \leq i \leq \frac{n}{2}$ and $j \neq \frac{n}{2} + i$, then

$$0 = \beta(x_i + \sqrt{\alpha}y_i, x_j + \sqrt{\alpha}y_j) = (\beta(x_i, x_j) + \alpha\beta(y_i, y_j)) + \sqrt{\alpha}(\beta(x_i, y_j) + \beta(x_j, y_i))$$

and that

$$0 = \beta(x_i + \sqrt{\alpha}y_i, x_j - \sqrt{\alpha}y_j) = (\beta(x_i, x_j) - \alpha\beta(y_i, y_j)) + \sqrt{\alpha}(-\beta(x_i, y_j) + \beta(x_j, y_i)).$$

So, we have that $\beta(x_i, x_j) + \alpha\beta(y_i, y_j) = 0$, $\beta(x_i, y_j) + \beta(x_j, y_i) = 0$, $\beta(x_i, x_j) - \alpha\beta(y_i, y_j) = 0$, and $-\beta(x_i, y_j) + \beta(x_j, y_i) = 0$. It follows from this that when $1 \leq i \leq \frac{n}{2}$ and $j \neq \frac{n}{2} + i$, we have

$$\beta(x_i, x_j) = \beta(y_i, y_j) = \beta(x_i, y_j) = \beta(y_i, x_j) = 0.$$

Now suppose that $1 \leq i \leq \frac{n}{2}$ and $j = \frac{n}{2} + i$. Then, we have

$$1 = \beta(x_i + \sqrt{\alpha}y_i, x_j + \sqrt{\alpha}y_j) = (\beta(x_i, x_j) + \alpha\beta(y_i, y_j)) + \sqrt{\alpha}(\beta(x_i, y_j) + \beta(x_j, y_i))$$

and that

$$0 = \beta(x_i + \sqrt{\alpha}y_i, x_j - \sqrt{\alpha}y_j) = (\beta(x_i, x_j) - \alpha\beta(y_i, y_j)) + \sqrt{\alpha}(-\beta(x_i, y_j) + \beta(x_j, y_i)).$$

Similar to the first case, we have that $\beta(x_i, y_j) = 0 = \beta(y_i, x_j) = 0$, and we have

$$1 = \beta(x_i, x_j) + \alpha\beta(y_i, y_j)$$

and

$$0 = \beta(x_i, x_j) - \alpha\beta(y_i, y_j).$$

Thus, when $1 \leq i \leq \frac{n}{2}$ and $j = \frac{n}{2} + i$, we have that $\beta(x_i, x_j) = \frac{1}{2}$ and $\beta(y_i, y_j) = \frac{1}{2\alpha}$. So, we have that $X^T J X = \frac{1}{2} \begin{pmatrix} J & 0 \\ 0 & \frac{1}{\alpha} J \end{pmatrix}$. ■

We now consider a couple of examples of Type 2 involutions.

Example 4.6. Consider the matrix

$$A = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Inn_A is a Type 2 involution of $\text{SP}(4, \mathbb{Q})$ since $A^2 = I$ and each entry of A is a \mathbb{Q} -multiple of $\sqrt{2}$. A basis for $E(A, 1)$ that matches the conditions of Lemma 4.5 is formed by the vectors

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{8} \\ -\frac{1}{8} \end{pmatrix} + \sqrt{2} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{8} \\ 0 \end{pmatrix}$$

and

$$v_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 1 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

It can be shown that

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{8} \\ -\frac{1}{8} \end{pmatrix} - \sqrt{2} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{8} \\ 0 \end{pmatrix}$$

and

$$v_4 = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 1 \end{pmatrix} - \sqrt{2} \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

are basis vectors for $E(A, -1)$ that also match the conditions of Lemma 4.5.

Following the notation of the previous lemma, we have

$$X = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & -2 \\ -\frac{1}{8} & 1 & -\frac{1}{8} & 1 \\ -\frac{1}{8} & 1 & 0 & 0 \end{pmatrix},$$

where $X^T J X = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & 0 \end{pmatrix}$ and $A = \frac{\sqrt{2}}{2} X \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ 2I_{\frac{n}{2}} & 0 \end{pmatrix} X^{-1}$.

Example 4.7. Let k be any field that does not contain $i = \sqrt{-1}$. For example, k could be \mathbb{R} , or \mathbb{F}_p or \mathbb{Q}_p where p is congruent to 3 mod 4. Consider the matrix

$$A = i \begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Inn_A is a Type 2 involution of $\text{SP}(4, k)$ since $A^2 = I$ and each entry of A is a k -multiple of i . A basis for $E(A, 1)$ that matches the conditions of Lemma 4.5 is formed by the vectors

$$v_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} \frac{1}{2} \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

and

$$v_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \\ 0 \end{pmatrix}$$

It can be shown that

$$v_3 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \\ 1 \end{pmatrix} - i \begin{pmatrix} \frac{1}{2} \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

and

$$v_4 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \\ 0 \end{pmatrix}$$

are basis vectors for $E(A, -1)$ that also match the conditions of Lemma 4.5.

Following the notation of the previous lemma, we have

$$X = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

where $X^T J X = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $A = -iXJX^{-1}$.

Using our characterization of Type 2 involutions, we now find conditions on Type 2 involutions that are equivalent to isomorphy.

Theorem 4.8. *Suppose A and B both induce Type 2 involutions of $\text{SP}(2n, k)$ where we write*

$$A = \frac{\sqrt{\alpha}}{\alpha} X \begin{pmatrix} 0 & I_n \\ \alpha I_n & 0 \end{pmatrix} X^{-1}$$

and

$$B = \frac{\sqrt{\beta}}{\beta} Y \begin{pmatrix} 0 & I_n \\ \beta I_n & 0 \end{pmatrix} Y^{-1}$$

where

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n & y_1 & y_2 & \cdots & y_n \end{pmatrix} \in \text{GL}(2n, k)$$

and

$$Y = \begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_n & \tilde{y}_1 & \tilde{y}_2 & \cdots & \tilde{y}_n \end{pmatrix} \in \text{GL}(2n, k),$$

where for each i , we have that $x_i + \sqrt{\alpha}y_i \in E(A, 1)$, $x_i - \sqrt{\alpha}y_i \in E(A, -1)$, $\tilde{x}_i + \sqrt{\alpha}\tilde{y}_i \in E(B, 1)$, $\tilde{x}_i - \sqrt{\alpha}\tilde{y}_i \in E(B, -1)$, and we know that

$$X^T J X = \frac{1}{2} \begin{pmatrix} J & 0 \\ 0 & \frac{1}{\alpha} J \end{pmatrix}$$

and

$$Y^T J Y = \frac{1}{2} \begin{pmatrix} J & 0 \\ 0 & \frac{1}{\beta} J \end{pmatrix}.$$

Then, Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$ if and only α and β lie in the same square class of k .

Proof. First, we note that if there exists $Q \in \text{SP}(2n, k)$ such that $Q^{-1}AQ = B$, then Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$. Secondly, we note that this can be the case if and only if α and β are in the same square class. So, to prove this theorem, we can simply assume that $\alpha = \beta$ and we will show that there exists such a $Q \in \text{SP}(2n, k)$.

Let $Q = XY^{-1}$. First, we note that

$$Q^T J Q = (XY^{-1})^T J (XY^{-1}) = (Y^{-1})^T (X^T J X) Y^{-1} = (Y^{-1})^T (Y^T J Y) Y^{-1} = J,$$

so we see that $Q \in \text{SP}(2n, k)$.

Lastly, we see that

$$\begin{aligned} Q^{-1}AQ &= (XY^{-1})^{-1}A(XY^{-1}) \\ &= Y(X^{-1}AX)Y^{-1} \\ &= \frac{\sqrt{\alpha}}{\alpha} Y \begin{pmatrix} 0 & I_n \\ \alpha I_n & 0 \end{pmatrix} Y^{-1} \\ &= B. \end{aligned} \quad \blacksquare$$

From here, it is clear that the number of Type 2 involution isomorphism classes is dependent on n and on the number of square classes of the field k .

Corollary 4.9. *If n is even, then $\text{SP}(2n, k)$ has at most $|k^*/(k^*)^2| - 1$ isomorphism classes of Type 2 involutions. If n is odd, then $\text{SP}(2n, k)$ has no Type 2 involutions.*

4.3. Type 3 Involutions.

We now examine the Type 3 case. Recall that ϕ is a Type 3 involution if $\phi = \text{Inn}_A$, where $A \in \text{SP}(2n, k)$ and $A^2 = -I$. Such matrices have eigenvalues $\pm i$, and are diagonalizable because the minimal polynomial has no repeated roots. We begin by proving a couple of results about the eigenvectors of such matrices.

Lemma 4.10. *Suppose $A \in \text{SP}(2n, k)$ induces a Type 3 involution of $\text{SP}(2n, k)$. Also suppose $x, y \in k^n$ such that $x + iy \in E(A, -i)$. Then, $x - iy \in E(A, i)$. Likewise, if $u, v \in k^n$ such that $u + iv \in E(A, i)$. Then, $u - iv \in E(A, -i)$. Further, $\dim(E(A, i)) = \dim(E(A, -i))$.*

Proof. Recall that complex conjugation preserves multiplication. This applies at the matrix level as well as at the scalar level. Because of this and since

$$A(x + iy) = -i(x - iy) = y - ix,$$

then it follows that

$$A(x - iy) = y + ix = i(x - iy).$$

That is, $x - iy \in E(A, -i)$. This proves the first statement. An analogous argument proves the second.

To see that $\dim(E(A, i)) = \dim(E(A, -i))$ is the case, note that the first statement tells us that $\dim(E(A, i)) \leq \dim(E(A, -i))$, and that the second statement tells us that $\dim(E(A, i)) \geq \dim(E(A, -i))$. ■

Lemma 4.11. *Suppose $A \in \text{SP}(2n, k)$ induces a Type 3 involution of $\text{SP}(2n, k)$. Then, there exists $x_1, \dots, x_n, y_1, \dots, y_n \in k^{2n}$ such that the $x_j + iy_j$ form a basis for $E(A, i)$ and the $x_j - iy_j$ form a basis for $E(A, -i)$.*

Proof. Since Inn_A is Type 3, then we are assuming that $A \in \text{SP}(2n, k)$ and $A^2 = -I$. It follows that all eigenvalues of A are $\pm i$. Since there are no repeated roots in the minimal polynomial of A , then we see that A is diagonalizable. We wish to construct bases for $E(A, i)$ and $E(A, -i)$ such that all the vectors lie in $k[i]^{2n}$. Let $\{z_1, \dots, z_{2n}\}$ be a basis for k^{2n} . For each j , let $u_j = (A + iI)z_j$. Note that

$$Au_j = A(A + iI)z_j = (A^2 + iA)z_j = (-I + iA)z_j = i(A + iI)z_j = iu_j.$$

So, $\{u_1, \dots, u_{2n}\}$ must span $E(A, i)$. Thus, we can appropriately choose n of these vectors and form a basis for $E(A, i)$. We can reorder, and assume that the n chosen vectors are u_1, \dots, u_n . Let $x_j = Ax_j$ and $y_j = z_j$. Then, these eigenvectors are of the form $x_j + iy_j$. By the previous lemma, we know that $x_j - iy_j \in E(A, -i)$. This proves the statement. ■

We are now able to prove results that characterize the matrices that induce Type 3 involutions, and then use these characterizations to find conditions on these involutions that are equivalent to isomorphy. We will have to prove this by looking at separate cases depending on whether or not $i = \sqrt{-1}$ lies in k . We begin by assuming that $i \in k$.

Lemma 4.12. *Assume $i \in k$ and suppose $\theta = \text{Inn}_A$ is a Type 3 involution of $\text{SP}(2n, k)$, where $A \in \text{SP}(2n, k)$. Then, $A = X \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix} X^{-1}$ for some $X \in \text{GL}(n, k)$, where $X^T J X = \begin{pmatrix} 0 & X_1 \\ -X_1 & 0 \end{pmatrix}$ where X_1 is diagonal.*

Proof. We know from Lemma 4.11 that we have bases for $E(A, i)$ and $E(A, -i)$ that lie in k^{2n} . We will show that we can in fact choose bases a_1, \dots, a_n for $E(A, i) \cap k^{2n}$ and b_1, \dots, b_n for $E(A, -i) \cap k^{2n}$ such that $\beta(a_j, a_l) = 0 = \beta(b_j, b_l)$ and $\beta(a_j, b_l)$ is nonzero if and only if $j = l$. We will build these bases recursively.

First, we know that we can choose some nonzero $a_1 \in E(A, i) \cap k^{2n}$. Then, since β is non degenerate, we can choose a vector t such that $\beta(a_1, t) \neq 0$. We note that $E(A, i) \oplus E(A, -i) = k^{2n}$, so we can choose $t_i \in E(A, i) \cap k^{2n}$ and $t_{-i} \in E(A, -i) \cap k^{2n}$ such that $t = t_i + t_{-i}$. Since $\beta(a_1, t_i) = 0$, then it follows that $\beta(a_1, t_{-i}) \in k$ is nonzero. Let $b_1 = t_{-i}$.

Let $E_1 = \text{Span}_k(a_1, b_1)$ and let F_1 be the orthogonal complement of E_1 in k^{2n} . Since the system of linear equations

$$\beta(a_1, x) = 0$$

$$\beta(b_1, x) = 0$$

has $2n - 2$ free variables, then we see that F_1 has dimension $2n - 2$.

We now wish to find $a_2 \in F_1 \cap E(A, i)$. Similar to the construction in the previous lemma, we can choose $x \in F_1$, and let $a_2 = Ax + ix$. Now we want $b_2 \in F_2 \cap E(A, -i)$ such that $\beta(a_2, b_2)$ is nonzero. Since $\beta|_{F_1}$ is non degenerate, then there exists some $y \in F_2$ such that $\beta(a_2, y) \neq 0$. Similar to the construction of b_1 , we see that this implies the existence of a vector b_2 that fits our criteria.

Now, we let $E_2 = \text{Span}_k(a_1, a_2, b_1, b_2)$ and let F_2 be the orthogonal complement of E_2 in k^n . We continue this same argument n times, until we have the bases that we wanted to find. Let

$$X = (a_1, \dots, a_n, b_1, \dots, b_n).$$

Then, the result follows. ■

We can now use this characterization to show that all such involutions must be isomorphic.

Theorem 4.13. *Assume that $i \in k$. Then, if Inn_A and Inn_B are both Type 3 involutions of $\text{SP}(2n, k)$, then Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$.*

Proof. Suppose we have two such involutions of $\text{SP}(2n, k)$. Let them be represented by matrices $A, B \in \text{SP}(2n, k)$. By the previous Lemma, we can choose $X, Y \in \text{GL}(n, k)$ such that

$$X^{-1}AX = \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix} = Y^{-1}BY,$$

$$X^T J X = \begin{pmatrix} 0 & X_1 \\ -X_1 & 0 \end{pmatrix}$$

and

$$Y^T J Y = \begin{pmatrix} 0 & Y_1 \\ -Y_1 & 0 \end{pmatrix}$$

where X_1 and Y_1 are diagonal.

Since X_1 and Y_1 are both invertible diagonal matrices, then we can choose R_1 and $R_2 \in \text{GL}(\frac{n}{2}, k)$ such that $Y_1 = R_1^T X_1 R_2$. Let $R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$ and $Q = XRY^{-1}$. We will show that $Q \in \text{SP}(2n, k)$ and $Q^{-1}AQ = B$. This will then prove that Inn_A and Inn_B lie in the same isomorphism class.

First we show that $Q \in \text{SP}(2n, k)$. Note that

$$\begin{aligned} Q^T J Q &= (XRY^{-1})^T J (XRY^{-1}) \\ &= (Y^{-1})^T R^T (X^T J X) R Y^{-1} \\ &= (Y^{-1})^T (Y^T J Y) Y^{-1} \\ &= J, \end{aligned}$$

which proves this claim.

Lastly, we show that $Q^{-1}AQ = B$. We first note that R and $\begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}$ commute. Then, we see that

$$\begin{aligned} Q^{-1}AQ &= (XRY^{-1})^{-1} A (XRY^{-1}) \\ &= YR^{-1}(X^{-1}AX)RY^{-1} \\ &= YR^{-1} \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} RY^{-1} \\ &= YR^{-1}R \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} Y^{-1} \\ &= Y \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} Y^{-1} \\ &= B. \end{aligned}$$

We have shown what was needed. ■

We now examine the case where $i \notin k$, beginning with a characterization of the matrices that induce these involutions.

Lemma 4.14. *Assume $i \notin k$. Suppose $\theta = \text{Inn}_A$ is a Type 3 involution of $\text{SP}(2n, k)$. Then, $A = U \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} U^{-1} = UJU^{-1}$ for*

$$U = \begin{pmatrix} a_1 & a_2 & \cdots & a_n & b_1 & b_2 & \cdots & b_n \end{pmatrix} \in \text{GL}(2n, k),$$

where the $a_j + ib_j$ are a basis for $E(A, i)$, the $a_j - ib_j$ are a basis for $E(A, -i)$, and $U^T J U = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix}$, where U_1 is diagonal.

Proof. We know from Lemma 4.11 that we have bases for $E(A, i)$ and $E(A, -i)$ that lie in $k[i]^{2n}$. We will show that we can in fact choose bases $a_1 + ib_1, \dots, a_n + ib_n$ for $E(A, i) \cap k[i]^{2n}$ and $a_1 - ib_1, \dots, a_n - ib_n$ for $E(A, -i) \cap k[i]^{2n}$ such that $\beta(a_j + ib_j, a_l - ib_l)$ is nonzero if and only if $j = l$. From this, we will be able to show that $\beta(a_j, a_l) = 0 = \beta(b_j, b_l)$ when $j \neq l$ and $\beta(a_j, b_l) = 0$ for all j and l . We will build these bases recursively.

Recall that given any vector $x \in k^{2n}$, we know that $Ax + ix \in E(A, i)$. We want to choose $x \in k^{2n}$ such that $x^T A^T J x \neq 0$. (The reasons for this will become apparent.) If $e_j^T A^T J e_j \neq 0$, we can let $x = e_j$. Suppose that this doesn't occur for any j .

Since $A^T J$ is invertible, we know that for more than $2n$ pairs of j and l we have $e_j^T A^T J e_l \neq 0$. Also, we see that since A is symplectic and $A^T J A = J$, then we have that

$$A^T J = J A^{-1} = J A^3 = -J A$$

and that

$$(A^T J)^T = J^T A = -J A = A^T J.$$

That is, $A^T J$ is symmetric. So, $e_j^T A^T J e_j = e_l^T A^T J e_l$. Then, we can let $x = e_j + e_l$. Then, we have

$$x^T A^T J x = e_j A^T J e_l + e_l A^T J e_j = 2e_j A^T J e_l \neq 0.$$

In either case, we have many choices for x .

Let $x \in k^{2n}$ be a vector from above. We have $Ax + ix \in E(A, i)$. Let $a_1 = Ax$ and $b_1 = x$. So, $a_1 + ib_1 \in E(A, i)$ and $a_1 - ib_1 \in E(A, -i)$. From this, it follows that

$$\begin{aligned} \beta(a_1 + ib_1, a_1 - ib_1) &= (\beta(a_1, a_1) + \beta(b_1, b_1)) + i(-\beta(a_1, b_1) + \beta(b_1, a_2)) \\ &= 0 + i(-\beta(Ax, x) + \beta(x, Ax)) \\ &= -2i\beta(Ax, x) \\ &= -2i(x^T A^T J x) \\ &\neq 0. \end{aligned}$$

Let $E_1 = \text{Span}_{k[i]}(a_1 + ib_1, a_1 - ib_1) = \text{Span}_{k[i]}(a_1, b_1)$, and let F_1 be the orthogonal complement of E_1 over $k[i]$. F_1 has dimension $2n - 2$, and $\beta|_{F_1}$ is nondegenerate. So, we can find a nonzero vector $x \in F_1 \cap k^{2n}$ such that $\beta|_{F_1}(Ax, x) \neq 0$. So, as in the last case, let $a_2 = Ax$ and $b_2 = x$. Similar to before, we have $\beta(a_2 + ib_2, a_2 - ib_2) \neq 0$.

Let $E_2 = \text{Span}_{k[i]}(a_1, a_2, b_1, b_2)$, and let F_2 be the orthogonal complement of E_2 over $k[i]$. In this manner, we can create the bases that we noted in the opening paragraph of this proof.

Note that we always have

$$0 = \beta(a_j + ib_j, a_l + ib_l) = (\beta(a_j, a_l) - \beta(b_j, b_l)) + i(\beta(a_j, b_l) + \beta(b_j, a_l)),$$

and when $j \neq l$ we have

$$0 = \beta(a_j + ib_j, a_l - ib_l) = (\beta(a_j, a_l) + \beta(b_j, b_l)) + i(-\beta(a_j, b_l) + \beta(b_j, a_l)).$$

This tells us that when $j \neq l$ that

$$\beta(a_j, b_l) = \beta(a_j, a_l) = \beta(b_j, b_l) = 0.$$

When $j = l$, we know that $\beta(b_j, b_j) = 0 = \beta(a_j, a_j)$. Lastly, we see that $\beta(a_j, b_j) = -\beta(b_j, a_j)$.

Let

$$U = (a_1, \dots, a_n, b_1, \dots, b_n).$$

Then, it follows that $U^T J U = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix}$ where X_1 is a diagonal $n \times n$ matrix.

Lastly, since $Ab_j = a_j$, then it follows that $Aa_j = -b_j$. So, we have that

$$A = U \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} U^{-1}. \quad \blacksquare$$

We now show that if $i \notin k$, then we also have that there is only one isomorphism class of Type 3 involutions.

Theorem 4.15. *Assume $i \notin k$. Then, if Inn_A and Inn_B are both Type 3 involutions of $\text{SP}(2n, k)$, then Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$.*

Proof. By the previous Lemma, we can choose a matrix $U \in \text{GL}(n, k)$ such that

$$A = U \begin{pmatrix} 0 & -I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0 \end{pmatrix} U^{-1}$$

for

$$U = \begin{pmatrix} a_1 & a_2 & \cdots & a_{\frac{n}{2}} & b_1 & b_2 & \cdots & b_{\frac{n}{2}} \end{pmatrix} \in \text{GL}(n, k),$$

where the $a_j + ib_j$ are a basis for $E(A, i)$, the $a_j - ib_j$ are a basis for $E(A, -i)$, and $U^T J U = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix}$ for diagonal matrix U_1 .

Let

$$X = (a_1 + ib_1, \dots, a_{\frac{n}{2}} + ib_{\frac{n}{2}}, a_1 - ib_1, \dots, a_{\frac{n}{2}} - ib_{\frac{n}{2}}),$$

and consider Inn_A and Inn_B as involutions of $\text{SP}(2n, k[i])$. By construction, we see that X is a matrix that satisfies the conditions of Lemma 4.12 for the group $\text{SP}(2n, k[i])$. We note that $X_1 = -2iU_1$. We also know by the previous Theorem that Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k[i])$. So, we can choose $Q_i \in \text{SP}(2n, k[i])$ such that $Q_i^{-1} A Q_i = B$. Let $Y = Q_i^{-1} X$. We now show a couple of facts about Y .

First, we note that since Y was obtained from X via row operations, then for $1 \leq j \leq \frac{n}{2}$, the j th and $\frac{n}{2} + j$ th columns are i -conjugates of one another.

Also, note that

$$\begin{aligned} Y^{-1} B Y &= (Q_i^{-1} X)^{-1} B (Q_i^{-1} X) \\ &= X^{-1} Q_i B Q_i^{-1} X \\ &= X^{-1} A X \\ &= \begin{pmatrix} -iI_{\frac{n}{2}} & 0 \\ 0 & iI_{\frac{n}{2}} \end{pmatrix}. \end{aligned}$$

Lastly, we see that

$$\begin{aligned}
 Y^T J Y &= (Q_i^{-1} X)^T J (Q_i^{-1} X) \\
 &= X^T ((Q_i^{-1})^T J Q_i) X \\
 &= X^T J X \\
 &= \begin{pmatrix} 0 & X_1 \\ -X_1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & -2iU_1 \\ 2iU_1 & 0 \end{pmatrix}.
 \end{aligned}$$

Write

$$Y = (c_1 + id_1, \dots, c_{\frac{n}{2}} + id_{\frac{n}{2}}, c_1 - id_1, \dots, c_{\frac{n}{2}} - id_{\frac{n}{2}}),$$

and let

$$V = (c_1, \dots, c_{\frac{n}{2}}, d_1, \dots, d_{\frac{n}{2}}).$$

It follows from what we have shown that

$$B = V \begin{pmatrix} 0 & -I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0 \end{pmatrix} V^{-1} \text{ where } V^T J V = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix} = U^T M U.$$

Now, let $Q = UV^{-1}$. We will show that $Q^{-1} A Q = B$ and $Q \in \text{SP}(2n, k)$. This will prove that Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$.

We first show that $Q \in \text{SP}(2n, k)$.

$$\begin{aligned}
 Q^T J Q &= (UV^{-1})^T J UV^{-1} \\
 &= (V^{-1})^T (U^T J U) V^{-1} \\
 &= (V^{-1})^T (V^T J V) V^{-1} \\
 &= J.
 \end{aligned}$$

Lastly, we show that $Q^{-1} A Q = B$.

$$\begin{aligned}
 Q^{-1} A Q &= (UV^{-1})^{-1} A (UV^{-1}) \\
 &= VU^{-1} A UV^{-1} \\
 &= V \begin{pmatrix} 0 & -I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0 \end{pmatrix} V^{-1} \\
 &= B.
 \end{aligned}$$

We have shown what was needed. ■

Combining the results from this section, we get the following corollary.

Corollary 4.16. *If Inn_A and Inn_B are both Type 3 involutions of $\text{SP}(2n, k)$, then Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$. That is, $\text{SP}(2n, k)$ has exactly one isomorphism class of Type 3 involutions. Further, the matrix J is a representative matrix for this isomorphism class.*

4.4. Type 4 Involutions.

We now move on to a similar classification in the Type 4 case. First, we characterize the eigenvectors of the matrices that induce these involutions. Recall that we can choose $A \in \text{SP}(2n, k[\sqrt{\alpha}])$ such that each entry of A is a k -multiple of $\sqrt{\alpha}$, and that we know $A^2 = -I$. We begin by proving a couple of lemmas about the eigenspaces of these matrices.

Lemma 4.17. *Suppose $A \in \text{SP}(2n, k[\sqrt{\alpha}])$ induces a Type 4 involution of $\text{SP}(2n, k)$. Also suppose $x, y \in k^{2n}$ such that $x + \sqrt{-\alpha}y \in E(A, i)$. Then, $x - \sqrt{-\alpha}y \in E(A, -i)$. Likewise, if $u, v \in k^{2n}$ such that $u + \sqrt{-\alpha}v \in E(A, -i)$. Then, $u - \sqrt{-\alpha}v \in E(A, i)$. Further, $\dim(E(A, i)) = \dim(E(A, -i))$.*

Proof. Suppose $x, y \in k^n$ such that $x + \sqrt{-\alpha}y \in E(A, -i)$. Then,

$$A(x + \sqrt{-\alpha}y) = -i(x + \sqrt{-\alpha}y)$$

which implies

$$Ax + \sqrt{-\alpha}Ay = \sqrt{\alpha}y - ix.$$

Then, complex conjugation tells us that

$$Ax - \sqrt{-\alpha}Ay = \sqrt{\alpha}y + ix,$$

which tells us that

$$A(x - \sqrt{-\alpha}y) = i(x - \sqrt{-\alpha}y).$$

A similar argument shows that if $u, v \in k^n$ such that $u + \sqrt{-\alpha}v \in E(A, i)$. Then, $u - \sqrt{-\alpha}v \in E(A, -i)$.

Since $x + \sqrt{-\alpha}y \in E(A, -i)$ implies $x - \sqrt{-\alpha}y \in E(A, i)$ and vice versa, then we see that $\dim(E(A, i)) = \dim(E(A, -i))$. ■

Lemma 4.18. *Suppose $\theta = \text{Inn}_A$ is a Type 4 involution of $\text{SP}(2n, k)$ where $A \in \text{SP}(2n, k[\sqrt{\alpha}])$. Then, we can find $x_1, \dots, x_n, y_1, \dots, y_n \in k^n$ such that the $x + \sqrt{-\alpha}y$ are a basis for $E(A, i)$ and the $x - \sqrt{-\alpha}y$ are a basis for $E(A, -i)$.*

Proof. Since Inn_A is Type 4, then we are assuming that $A \in \text{SP}(2n, k[\sqrt{\alpha}])$ and $A^2 = -I$. It follows that all eigenvalues of A are $\pm i$. Since there are no repeated roots in the minimal polynomial of A , then we see that A is diagonalizable. We wish to construct bases for $E(A, i)$ and $E(A, -i)$ such that all the vectors lie in $k[i]^{2n}$. Let $\{z_1, \dots, z_{2n}\}$ be a basis for k^{2n} . For each j , let $u_j = (\sqrt{\alpha}A + \sqrt{-\alpha}I)z_j$. Note that

$$Au_j = A(\sqrt{\alpha}A + \sqrt{-\alpha}I)z_j = (\sqrt{\alpha}A^2 + \sqrt{-\alpha}A)z_j = i(\sqrt{\alpha}A + \sqrt{-\alpha}I)z_j = iu_j.$$

So, $\{u_1, \dots, u_n\}$ must span $E(A, i)$. Thus, we can appropriately choose n of these vectors and form a basis for $E(A, i)$. Note that each of these vectors lies in $k[i]^{2n}$. Label these basis vectors as v_1, \dots, v_n . We can write each of these vectors as $v_j = x_j + \sqrt{-\alpha}y_j$. By the previous lemma, we know that $x_j - \sqrt{-\alpha}y_j \in E(A, -i)$, and that these vectors form a basis for $E(A, -i)$. ■

We are now able to prove results that characterize the matrices that induce Type 4 involutions, and then use these characterizations to find conditions on these involutions that are equivalent to isomorphy. We will have separate cases, depending on whether or not $\sqrt{-\alpha}$ lies in k . We begin by assuming that $\sqrt{-\alpha} \in k$. Since we are also assuming that $\sqrt{\alpha} \notin k$, then it follows from these two assumptions that α and -1 lie in the same square class of k . Thus, we can assume in this case that $\alpha = -1$, which means $\sqrt{-\alpha} = 1$.

Lemma 4.19. *Assume $\sqrt{-\alpha} \in k$ and suppose θ is a Type 4 involution of $\text{SP}(2n, k)$. Then, $A = X \begin{pmatrix} iI_{\frac{n}{2}} & 0 \\ 0 & -iI_{\frac{n}{2}} \end{pmatrix} X^{-1}$ for some $X \in \text{GL}(2n, k)$, where $X^T J X = \begin{pmatrix} 0 & X_1 \\ -X_1 & 0 \end{pmatrix}$ and X_1 is diagonal.*

Proof. We know from Lemma 4.18 that we have bases for $E(A, i)$ and $E(A, -i)$ that lie in k^{2n} . We will show that we can in fact choose bases a_1, \dots, a_n for $E(A, i) \cap k^{2n}$ and b_1, \dots, b_n for $E(A, -i) \cap k^{2n}$ such that $\beta(a_j, a_l) = 0 = \beta(b_j, b_l)$ and $\beta(a_j, b_l)$ is nonzero if and only if $j = l$. We will build these bases recursively.

First, we know that we can choose some nonzero $a_1 \in E(A, i) \cap k^{2n}$. Then, since β is non degenerate, we can choose a vector t such that $\beta(a_1, t) \neq 0$. We note that $E(A, i) \oplus E(A, -i) = k^{2n}$, so we can choose $t_i \in E(A, i) \cap k^{2n}$ and $t_{-i} \in E(A, -i) \cap k^{2n}$ such that $t = t_i + t_{-i}$. Since $\beta(a_1, t_i) = 0$, then it follows that $\beta(a_1, t_{-i}) \in k$ is nonzero. Let $b_1 = t_{-i}$.

Let $E_1 = \text{Span}_k(a_1, b_1)$ and let F_1 be the orthogonal complement of E_1 in k^{2n} . Since the system of linear equations

$$\beta(a_1, x) = 0$$

$$\beta(b_1, x) = 0$$

has $2n - 2$ free variables, then we see that F_1 has dimension $2n - 2$.

We now wish to find $a_2 \in F_2 \cap E(A, i)$. Similar to the construction in the previous lemma, we can choose $x \in F_1$, and let $a_2 = \sqrt{\alpha}Ax + \sqrt{-\alpha}x$. Now we want $b_2 \in F_2 \cap E(A, -i)$ such that $\beta(a_2, b_2)$ is nonzero. Since $\beta|_{F_1}$ is non degenerate, then there exists some $y \in F_2$ such that $\beta(a_2, y) \neq 0$. Similar to the construction of b_1 , we see that this implies the existence a vector b_2 that fits our criteria.

Now, we let $E_2 = \text{Span}_k(a_1, a_2, b_1, b_2)$ and let F_2 be the orthogonal complement of E_2 in k^n . We continue this same argument n times, until we have the bases that we wanted to find. Let

$$X = (a_1, \dots, a_n, b_1, \dots, b_n).$$

Then, the result follows. ■

Here is an example of a Type 4 involution when $\sqrt{-\alpha} \in k$.

Example 4.20. Let k be \mathbb{R} . So, $\alpha = -1$. Notice that $\sqrt{-\alpha} = 1 \in \mathbb{R}$.

Consider the matrix

$$A = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Inn_A is a Type 4 involution of $\text{SP}(4, k)$ since $A^2 = -I$ and each entry of A is a k -multiple of i . A basis for $E(A, 1)$ that matches the conditions of Lemma 4.19 is formed by the vectors $v_1 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $v_2 = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$. It can also be shown that $v_3 = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $v_4 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ are basis vectors for $E(A, -1)$ that also match the conditions of Lemma 4.19.

Following the notation of the previous lemma, we have

$$X = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

where $X^T J X = J$ and $A = X \begin{pmatrix} iI_{\frac{n}{2}} & 0 \\ 0 & -iI_{\frac{n}{2}} \end{pmatrix} X^{-1}$.

Now we characterize the isomorphism classes of Type 4 involutions in the case where $\sqrt{-\alpha} \in k$.

Theorem 4.21. *Assume that $\sqrt{-\alpha} \in k$. Then, if Inn_A and Inn_B are both Type 4 involutions of $\text{SP}(2n, k)$ such that $A, B \in \text{SP}(2n, k[\sqrt{\alpha}])$, then Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$.*

Proof. Suppose we have two such involutions of $\text{SP}(2n, k)$. Let them be represented by matrices $A, B \in \text{SP}(2n, k)$. By the previous Lemma, we can choose $X, Y \in \text{GL}(n, k)$ such that

$$X^{-1} A X = \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix} = Y^{-1} B Y,$$

$$X^T J X = \begin{pmatrix} 0 & X_1 \\ -X_1 & 0 \end{pmatrix},$$

and

$$Y^T J Y = \begin{pmatrix} 0 & Y_1 \\ -Y_1 & 0 \end{pmatrix},$$

where X_1 and Y_1 are diagonal.

Since X_1 and Y_1 are both invertible diagonal matrices, then we can choose R_1 and $R_2 \in \text{GL}(\frac{n}{2}, k)$ such that $Y_1 = R_1^T X_1 R_1$. Let $R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$ and $Q = X R Y^{-1}$. It follows from this that $R^T X^T J X R = Y^T J Y$. We will show that $Q \in \text{SP}(2n, k)$ and $Q^{-1} A Q = B$. This will then prove that Inn_A and Inn_B lie in the same isomorphism class.

First we show that $Q \in \text{SP}(2n, k)$. Note that

$$\begin{aligned} Q^T J Q &= (X R Y^{-1})^T J (X R Y^{-1}) = (Y^{-1})^T R^T (X^T J X) R Y^{-1} \\ &= (Y^{-1})^T (Y^T J Y) Y^{-1} = J, \end{aligned}$$

which proves this claim.

Lastly, we show that $Q^{-1} A Q = B$. We first note that R and $\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}$ commute. Then, we see that

$$\begin{aligned} Q^{-1} A Q &= (X R Y^{-1})^{-1} A (X R Y^{-1}) \\ &= Y R^{-1} (X^{-1} A X) R Y^{-1} \\ &= Y R^{-1} \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} R Y^{-1} \\ &= Y R^{-1} R \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} Y^{-1} \\ &= Y \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} Y^{-1} \\ &= B. \end{aligned} \quad \blacksquare$$

We now examine the case where $\sqrt{-\alpha} \notin k$. We begin with a characterization of the matrices that induce Type 4 involutions in this case.

Lemma 4.22. *Assume $\sqrt{-\alpha} \notin k$. Suppose $\theta = \text{Inn}_A$ is a Type 4 involution of $\text{SP}(2n, k)$. Then, $A = \frac{\sqrt{\alpha}}{\alpha} U \begin{pmatrix} 0 & I_n \\ -\alpha I_n & 0 \end{pmatrix} U^{-1}$ for*

$$U = \begin{pmatrix} a_1 & a_2 & \cdots & a_n & b_1 & b_2 & \cdots & b_n \end{pmatrix} \in \text{GL}(2n, k),$$

where the $a_j + \sqrt{-\alpha} b_j$ are a basis for $E(A, i)$, the $a_j - \sqrt{-\alpha} b_j$ are a basis for $E(A, -i)$, and $U^T J U = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix}$, where U_1 is diagonal.

Proof. We know from Lemma 4.18 that we have bases for $E(A, i)$ and $E(A, -i)$ that lie in $k[\sqrt{-\alpha}]^{2n}$. We will show that we can in fact choose bases $a_1 + \sqrt{-\alpha} b_1, \dots, a_n + \sqrt{-\alpha} b_n$ for $E(A, i) \cap k[\sqrt{-\alpha}]^{2n}$ and $a_1 - \sqrt{-\alpha} b_1, \dots, a_n - \sqrt{-\alpha} b_n$ for $E(A, -i) \cap k[\sqrt{-\alpha}]^{2n}$ such that $\beta(a_j + i b_j, a_l - i b_l)$ is nonzero if and only if $j = l$. From this, we will be able to show that $\beta(a_j, a_l) = 0 = \beta(b_j, b_l)$ when $j \neq l$ and $\beta(a_j, b_l) = 0$ for all j and l . We will build these bases recursively.

Recall that given any vector $x \in k^{2n}$, we know that $\sqrt{\alpha} A x + \sqrt{-\alpha} x \in E(A, i)$. We want to choose $x \in k^{2n}$ such that $x^T A^T J x \neq 0$. That is, such that $\beta(Ax, x) \neq 0$. (The reasons for this will become apparent.) If $e_j^T A^T J e_j \neq 0$, we can let $x = e_j$. Suppose that this doesn't occur for any j .

Since $A^T J$ is invertible, we know that for more than $2n$ pairs of j and l we have $e_j^T A^T J e_l \neq 0$. Also, we see that since A is symplectic and $A^T J A = J$, then we have that

$$A^T J = J A^{-1} = J A^3 = -J A$$

and that

$$(A^T J)^T = J^T A = -J A = A^T J.$$

That is, $A^T J$ is symmetric. So, $e_j^T A^T J e_j = e_l^T A^T J e_l$. Then, we can let $x = e_j + e_l$. Then, we have

$$x^T A^T J x = e_j A^T J e_l + e_l A^T J e_j = 2e_j A^T J e_l \neq 0.$$

In either case, we have many choices for x .

Let $x \in k^{2n}$ be a vector from above. We have $\sqrt{\alpha}Ax + \sqrt{-\alpha}x \in E(A, i)$. Let $a_1 = \sqrt{\alpha}Ax$ and $b_1 = x$. So, $a_1 + \sqrt{-\alpha}b_1 \in E(A, i)$ and $a_1 - \sqrt{-\alpha}b_1 \in E(A, -i)$. From this, it follows that

$$\begin{aligned} \beta(a_1 + \sqrt{-\alpha}b_1, a_1 - \sqrt{-\alpha}b_1) &= (\beta(\sqrt{\alpha}Ax, \sqrt{\alpha}Ax) + \alpha\beta(x, x)) \\ &\quad + \sqrt{-\alpha}(-\beta(\sqrt{\alpha}Ax, x) + \beta(x, \sqrt{\alpha}Ax)) \\ &= 2\alpha i \beta(x, Ax) \\ &\neq 0. \end{aligned}$$

Let $E_1 = \text{Span}_{k[\sqrt{-\alpha}]}(a_1 + \sqrt{-\alpha}b_1, a_1 - \sqrt{-\alpha}b_1) = \text{Span}_{k[\sqrt{-\alpha}]}(a_1, b_1)$, and let F_1 be the orthogonal complement of E_1 over $k[\sqrt{-\alpha}]$. F_1 has dimension $2n - 2$, and $\beta|_{F_1}$ is nondegenerate. So, we can find a nonzero vector $x \in F_1 \cap k^{2n}$ such that $\beta|_{F_1}(x, -Ax) \neq 0$. So, as in the last case, let $a_2 = \sqrt{\alpha}Ax$ and $b_2 = x$. As before, we have $\beta(a_2 + \sqrt{-\alpha}b_2, a_2 - \sqrt{-\alpha}b_2) \neq 0$.

Let $E_2 = \text{Span}_{k[\sqrt{-\alpha}]}(a_1, a_2, b_1, b_2)$, and let F_2 be the orthogonal complement of E_2 over $k[\sqrt{-\alpha}]$. In this manner, we can create the bases that we noted in the opening paragraph of this proof.

Note that we always have

$$0 = \beta(a_j + \sqrt{-\alpha}b_j, a_l + \sqrt{-\alpha}b_l) = (\beta(a_j, a_l) - \alpha\beta(b_j, b_l)) + \sqrt{-\alpha}(\beta(a_j, b_l) + \beta(b_j, a_l)),$$

and when $j \neq l$ we have

$$0 = \beta(a_j + \sqrt{-\alpha}b_j, a_l - \sqrt{-\alpha}b_l) = (\beta(a_j, a_l) + \alpha\beta(b_j, b_l)) + \sqrt{-\alpha}(-\beta(a_j, b_l) + \beta(b_j, a_l)).$$

This tells us that when $j \neq l$ that

$$\beta(a_j, b_l) = \beta(a_j, a_l) = \beta(b_j, b_l) = 0.$$

When $j = l$, we know that $\beta(b_j, b_j) = 0 = \beta(a_j, a_j)$. Lastly, we see that $\beta(a_j, b_j) = -\beta(b_j, a_j)$.

Let

$$U = (a_1, \dots, a_n, b_1, \dots, b_n).$$

Then, it follows that $U^T J U = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix}$ where U_1 is a diagonal $n \times n$ matrix.

Since $Aa_j = -\sqrt{\alpha}b_j$ and $Ab_j = \frac{\sqrt{\alpha}}{\alpha}a_j$, then we have $A = \frac{\sqrt{\alpha}}{\alpha}U \begin{pmatrix} 0 & I_n \\ -\alpha I_n & 0 \end{pmatrix} U^{-1}$.

We have shown what was needed. ■

The following is an example of a Type 4 involution where $\sqrt{-\alpha} \notin k$.

Example 4.23. Let $k = \mathbb{F}_5$ and consider $\alpha = 2$. Note that $\sqrt{-\alpha} = \sqrt{3} \notin k$.

Consider the matrix

$$A = \sqrt{2} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}.$$

Inn_A is a Type 4 involution of $\text{SP}(4, k)$ since $A^2 = -I$ and each entry of A is a k -multiple of $\sqrt{2}$. A basis for $E(A, 1)$ that matches the conditions of Lemma 4.22 is formed by the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 4 \\ 0 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

It can be shown that

$$v_3 = \begin{pmatrix} 1 \\ 0 \\ 4 \\ 0 \end{pmatrix} - \sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \end{pmatrix} - \sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

are basis vectors for $E(A, -1)$ that also match the conditions of Lemma 4.22.

Following the notation of the Lemma 4.22, we have

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{pmatrix},$$

where $U^T J U = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix}$ for $U_1 = 2I$ and $A = \frac{\sqrt{2}}{2} U \begin{pmatrix} 0 & I \\ -2I & 0 \end{pmatrix} U^{-1}$.

We now find conditions on Type 4 involutions where $\sqrt{-\alpha} \notin k$ that are equivalent to isomorphy.

Theorem 4.24. *Assume $\sqrt{-\alpha} \notin k$. Then, if Inn_A and Inn_B are both Type 4 involutions of $\text{SP}(2n, k)$ where the entries of A and B are k -multiples of $\sqrt{\alpha}$, then Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$.*

Proof. By Lemma 4.22, we can choose a matrix $U \in \text{GL}(n, k)$ such that

$$A = \frac{\sqrt{\alpha}}{\alpha} U \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -\alpha I_{\frac{n}{2}} & 0 \end{pmatrix} U^{-1}$$

for

$$U = \left(\begin{array}{cccccc} a_1 & a_2 & \cdots & a_{\frac{n}{2}} & b_1 & b_2 & \cdots & b_{\frac{n}{2}} \end{array} \right),$$

where the $a_j + \sqrt{-\alpha}b_j$ are a basis for $E(A, i)$, the $a_j - \sqrt{-\alpha}b_j$ are a basis for $E(A, -i)$, and $U^T J U = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix}$ for diagonal U_1 .

Consider Inn_A and Inn_B as involutions of $\text{SP}(2n, k[\sqrt{-\alpha}])$. If $k[\sqrt{-\alpha}] = k[\sqrt{\alpha}]$, then these are Type 3 involutions of $\text{SP}(2n, k[\sqrt{-\alpha}])$, since A and B would have entries in the field, and $i \in k[\sqrt{-\alpha}]$. Otherwise, if $k[\sqrt{-\alpha}] \neq k[\sqrt{\alpha}]$, then these are Type 4 involutions where $\sqrt{-\alpha} \in k[\sqrt{-\alpha}]$.

Let

$$X = (a_1 + \sqrt{-\alpha}b_1, \dots, a_{\frac{n}{2}} + \sqrt{-\alpha}b_{\frac{n}{2}}, a_1 - \sqrt{-\alpha}b_1, \dots, a_{\frac{n}{2}} - \sqrt{-\alpha}b_{\frac{n}{2}}).$$

By construction, we see that X is a matrix that satisfies the conditions of Lemma 4.14 or Lemma 4.19 for the group $\text{SP}(2n, k[\sqrt{\alpha}])$. We note that $X_1 = -2iU_1$. We also know by Corollary 4.16 or Theorem 4.21 that Inn_A and Inn_B are isomorphic (when viewed as involutions of $\text{SP}(2n, k[\sqrt{-\alpha}])$) over $\text{SP}(2n, k[\sqrt{-\alpha}])$. So, we can choose $Q_\alpha \in \text{SP}(2n, k[\sqrt{-\alpha}])$ such that $Q_\alpha^{-1} A Q_\alpha = B$. Let $Y = Q_\alpha^{-1} X$. Since Y is constructed by doing row operations on X , then we can write

$$Y = (c_1 + \sqrt{-\alpha}d_1, \dots, c_{\frac{n}{2}} + \sqrt{-\alpha}d_{\frac{n}{2}}, c_1 - \sqrt{-\alpha}d_1, \dots, c_{\frac{n}{2}} - \sqrt{-\alpha}d_{\frac{n}{2}}),$$

where $c_j, d_j \in k^n$. We now show a couple of facts about Y .

First, we note that since Y was obtained from X via row operations, then for $1 \leq j \leq \frac{n}{2}$, the j th and $\frac{n}{2} + j$ th columns are i -conjugates of one another.

Next, we observe that

$$\begin{aligned} Y^{-1} B Y &= (Q_\alpha^{-1} X)^{-1} B (Q_\alpha^{-1} X) \\ &= X^{-1} Q_\alpha B Q_\alpha^{-1} X \\ &= X^{-1} A X \\ &= \begin{pmatrix} iI_{\frac{n}{2}} & 0 \\ 0 & -iI_{\frac{n}{2}} \end{pmatrix}. \end{aligned}$$

Lastly, we see that

$$\begin{aligned} Y^T J Y &= (Q_\alpha^{-1} X)^T J (Q_\alpha^{-1} X) \\ &= X^T ((Q_\alpha^{-1})^T J Q_\alpha) X \\ &= X^T J X \\ &= \begin{pmatrix} 0 & X_1 \\ -X_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2iU_1 \\ 2iU_1 & 0 \end{pmatrix}. \end{aligned}$$

Let

$$V = (c_1, \dots, c_{\frac{n}{2}}, d_1, \dots, d_{\frac{n}{2}}) \in \text{GL}(n, k).$$

It follows from what we have shown that $B = \frac{\sqrt{\alpha}}{\alpha} V \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -\alpha I_{\frac{n}{2}} & 0 \end{pmatrix} V^{-1}$ where $V^T J V = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix} = U^T J U$.

Now, let $Q = UV^{-1}$. We will show that $Q^{-1} A Q = B$ and $Q \in \text{SP}(2n, k)$. This will prove that Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$.

We first show that $Q \in \text{SP}(2n, k)$.

$$\begin{aligned} Q^T J Q &= (UV^{-1})^T J UV^{-1} \\ &= (V^{-1})^T (U^T J U) V^{-1} \\ &= (V^{-1})^T (V^T J V) V^{-1} \\ &= J. \end{aligned}$$

Lastly, we show that $Q^{-1} A Q = B$.

$$\begin{aligned} Q^{-1} A Q &= (UV^{-1})^{-1} A (UV^{-1}) \\ &= VU^{-1} A UV^{-1} \\ &= \frac{\sqrt{\alpha}}{\alpha} V \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -\alpha I_{\frac{n}{2}} & 0 \end{pmatrix} V^{-1} \\ &= B. \end{aligned}$$

We have shown what was needed. ■

Combining the results from this section, we get the following corollary.

Corollary 4.25. *If Inn_A and Inn_B are both Type 4 involutions of $\text{SP}(2n, k)$, then Inn_A and Inn_B are isomorphic over $\text{SP}(2n, k)$ if and only if A and B have entries lying in the same field extension of k . That is, $\text{SP}(2n, k)$ has at most $|k^*/(k^*)^2| - 1$ isomorphism classes of Type 4 involutions.*

5. Maximal Number of Isomorphism classes

From the work we have done, it follows that the maximum number of isomorphism classes of $\text{SP}(2n, k)$ is a function of the number of square classes of k and n . We first define the following formulas.

Definition 5.1. Let $C_1(2n, k)$, $C_2(2n, k)$, $C_3(2n, k)$ and $C_4(2n, k)$ be the number of isomorphism classes of involutions of $\text{SP}(2n, k)$ of types 1, 2, 3, and 4, respectively.

From our previous work, we have the following:

Corollary 5.2. *1. If n is odd, then $C_1(2n, k) = \frac{n-1}{2}$. If n is even, then $C_1(2n, k) = \frac{n}{2}$.*

- 2. If n is odd, then $C_2(2n, k) = 0$. If n is even, then $C_2(2n, k) \leq |k^*/(k^*)^2| - 1$.
- 3. $C_3(2n, k) = 1$.
- 4. $C_4(2n, k) \leq |k^*/(k^*)^2| - 1$.

6. Explicit Examples

We have shown that the number of isomorphism classes of Type 1 and Type 3 involutions depends only on n , and not the field k . Since Type 2 and Type 4 involutions do not occur when k is algebraically closed, then the previous corollary tells us the number of isomorphism classes in this case. In addition to this example, we will also consider the cases where $k = \mathbb{R}$ and $k = \mathbb{F}_p$.

6.1. Type 2 Examples.

We first consider the Type 2 case. So, we may assume that n is even. First, let us suppose that k is \mathbb{R} or \mathbb{F}_q where -1 is not a square in \mathbb{F}_q . Without loss of generality, assume $\alpha = -1$. Let A_1 be an $n \times n$ block diagonal matrix where each block is the 2×2 matrix $i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, let $A = \begin{pmatrix} A_1 & 0 \\ 0 & (A_1^{-1})^T \end{pmatrix}$. This matrix induces a Type 2 involution on $SP(2n, k)$.

Now, let us suppose that $k = \mathbb{F}_q$ where -1 is a square. Let $\alpha \in k^*$ be a non-square. Then, we can choose $a, b \in k$ such that $a^2 + b^2 = \frac{1}{\alpha}$. Let A_1 be an $n \times n$ block diagonal matrix where each block is the 2×2 matrix $\sqrt{\alpha} \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. Then, let $A = \begin{pmatrix} A_1 & 0 \\ 0 & (A_1^{-1})^T \end{pmatrix}$. This matrix induces a Type 2 involution on $SP(2n, k)$. So if k is finite or real, then $SP(2n, k)$ has the maximal number of Type 2 isomorphism classes.

6.2. Type 4 Examples.

Now we consider the Type 4 case. So, n may be even or odd. Let us again begin by supposing that k is \mathbb{R} or \mathbb{F}_q where -1 is not a square in \mathbb{F}_q . Then, the matrix $\begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix}$ induces a Type 4 involution, and $SP(2n, k)$ has the maximal number of isomorphism classes in this case, regardless of if n is odd or even.

Now, let us suppose that $k = \mathbb{F}_q$ where -1 is a square. Let $\alpha \in k^*$ be a non-square and choose $a, b \in k$ such that $a^2 + b^2 = \alpha$. If we let $U = \begin{pmatrix} cI_n & dI_n \\ -dI_n & cI_n \end{pmatrix}$ and then let

$$\begin{aligned} A &= \frac{\sqrt{\alpha}}{\alpha} U \begin{pmatrix} 0 & I_n \\ -\alpha I_n & 0 \end{pmatrix} U^{-1} \\ &= \frac{\sqrt{\alpha}}{\alpha^2} \begin{pmatrix} (1 - \alpha)cdI_n & (c^2 + \alpha d^2)I_n \\ -(c^2 + \alpha d^2)I_n & -(1 - \alpha)cdI_n \end{pmatrix}. \end{aligned}$$

A induces a Type 4 involution on $SP(2n, k)$. We have shown that if k is finite or real, then $SP(2n, k)$ has the maximal number of Type 4 isomorphism classes. Thus, if k is real or finite it has the maximal number of all types of isomorphism classes.

While we have been unable to prove that this is the case for any field k , we believe that this is the case. That is, we have the following conjecture:

Conjecture 6.1. 1. If n is odd, then $C_1(2n, k) = \frac{n-1}{2}$. If n is even, then $C_1(2n, k) = \frac{n}{2}$.

2. If n is odd, then $C_2(2n, k) = 0$. If n is even, then $C_2(2n, k) = |k^*/(k^*)^2| - 1$.
3. $C_3(2n, k) = 1$.
4. $C_4(2n, k) = |k^*/(k^*)^2| - 1$.

We have classified the involutions for symplectic groups over algebraically closed fields, the real numbers, and for a finite field of characteristic not 2. We also have constructed the tools to classify the involutions of other symplectic groups. In addition to proving (or disproving) the above conjecture, further areas of research in this area to be completed are to classify the (θ, k) -split tori for given involutions θ , classify the k -inner elements, and to study the fixed point groups, which would give rise to a symmetric space.

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