

Neutral Metrics on Four-Dimensional Lie Groups

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Abstract. We consider four-dimensional Lie groups equipped with left-invariant metrics of signature $(2, 2)$. After describing the general structure of four-dimensional Lie algebras with a neutral inner product, we obtain a complete classification of left-invariant Einstein and Ricci-parallel metrics of neutral signature.

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1. Introduction

Four-dimensional homogeneous pseudo-Riemannian manifolds with nontrivial isotropy have been completely described at the local level in [12]. On the other hand, up to recently, a systematic study of left-invariant metrics on four-dimensional Lie groups only concerned the Riemannian case (see for example [1]). Curvature conditions on four-dimensional Lorentzian Lie groups have been studied in [7] and [16]. These results leave to consider left-invariant metrics of neutral signature on four-dimensional Lie groups (briefly, four-dimensional neutral Lie groups). In this paper, we shall undertake such a study and completely classify Einstein and Ricci-parallel left-invariant neutral metrics on four-dimensional Lie groups.

After discussing the general structure of a four-dimensional Lie algebra equipped with a neutral inner product in Section 2, a complete classification of four-dimensional simply connected neutral Einstein Lie groups will be given in Section 3. Note that Theorem 3.1, together with the results obtained in [12], leads to the complete local classification of four-dimensional homogeneous Einstein manifolds of neutral signature. Section 4 deals with the classification of the Ricci-parallel examples which are not Einstein. We shall determine which cases correspond to left-invariant Ricci solitons, the examples which are not locally symmetric and the ones admitting a parallel degenerate distribution. The software *Maple 16*[©] has been used to check all the needed calculations.

2. On four-dimensional Lie groups

Four-dimensional homogeneous Riemannian manifolds have been classified by Bérard-Bergery in [2]. This classification yields that a simply connected four-dimensional homogeneous Riemannian manifold is either symmetric, or isometric to a Lie group equipped with a left-invariant Riemannian metric. The classification of four-dimensional simply connected Riemannian Lie groups is resumed in the following result.

Proposition 2.1. [1] *A simply connected four-dimensional Riemannian Lie groups is:*

- (i) *either one of the unsolvable direct products $\mathbb{R} \times SU(2)$ and $\mathbb{R} \times \widetilde{SL}(2, \mathbb{R})$; or*
- (ii) *one of the following solvable Lie groups:*
 - (ii1) *the non-trivial semi-direct products $\mathbb{R} \ltimes E(2)$ and $\mathbb{R} \ltimes E(1, 1)$;*
 - (ii2) *the non-nilpotent semi-direct products $\mathbb{R} \ltimes H$, where H denotes the Heisenberg group;*
 - (ii3) *the semi-direct products $\mathbb{R} \ltimes \mathbb{R}^3$.*

Consider now an n -dimensional simply connected Lie group G and the corresponding Lie algebra \mathfrak{g} . A left-invariant metric on G , of prescribed signature, corresponds to an inner product of the same signature on \mathfrak{g} . Conversely, any inner product g on \mathfrak{g} induces a left-invariant metric on G , of the same signature. Thus, we can work at the Lie algebra level henceforth.

It is easily seen that G admits left-invariant metrics of any signature $(p, n - p)$. In fact, it suffices to choose a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ of \mathfrak{g} and consider the inner product g on \mathfrak{g} , uniquely determined by having \mathcal{B} as a pseudo-orthonormal basis, with e_1, \dots, e_p space-like and e_{p+1}, \dots, e_n time-like vectors.

In particular, suppose now that G is four-dimensional. Let \bar{g} be a positive definite inner product on \mathfrak{g} . By a well-known argument of linear algebra, it exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of \mathfrak{g} . Then, a corresponding left-invariant metric g of neutral signature on G is uniquely determined at the Lie algebra level by having $\{e_1, e_2, e_3, e_4\}$ as a pseudo-orthonormal basis of \mathfrak{g} , with e_1, e_2 space-like and e_3, e_4 time-like. Conversely, if g is an inner product of neutral signature $(2, 2)$ on \mathfrak{g} , it suffices to consider a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for g , with e_1, e_2 space-like and e_3, e_4 time-like, and we have a corresponding left-invariant Riemannian metric \bar{g} on G , described at the Lie algebra level by having $\{e_1, e_2, e_3, e_4\}$ as an orthonormal basis. Thus, G is necessarily one of Lie groups listed in Proposition 2.1, and we proved the following result.

Proposition 2.2. *Every n -dimensional simply connected Lie group G admits left-invariant metrics of any prescribed signature $(p, n - p)$. In particular, if G is a four-dimensional simply connected Lie group, equipped with a left-invariant metric of neutral signature, then G is one of Lie groups listed in Proposition 2.1.*

Any simply connected four-dimensional Lie group is a semi-direct product of \mathbb{R} by a three-dimensional Lie group, also including in this description the case of direct products of \mathbb{R} by one of unsolvable Lie groups $SU(2)$ or $\widetilde{SL}(2, \mathbb{R})$. Correspondingly, the Lie algebra of G is given by $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, that is, \mathfrak{g} is the direct sum of \mathfrak{r} and \mathfrak{g}_3 , where \mathfrak{g}_3 is a three-dimensional Lie algebra, and the generator of the one-dimensional Lie algebra \mathfrak{r} acts as a derivation on \mathfrak{g}_3 .

Although by Proposition 2.2 pseudo-Riemannian and Riemannian Lie groups coincide in any dimension, the study of left-invariant pseudo-Riemannian metrics on Lie groups cannot use the same techniques of the Riemannian case. In fact, if g is a positive definite inner product over $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, then its restriction to \mathfrak{g}_3 is still positive definite. Hence, one can use the description of three-dimensional Riemannian Lie groups given in [13] and then choose a basis of the Lie algebra adapted to the inner product. An example of this technique is given by the study of curvature properties of four-dimensional Riemannian Lie groups made in [1].

Suppose now that g is of neutral signature. Then, its restriction to \mathfrak{g}_3 is

- (a) either of signature $(2, 1)$, (a') of signature $(1, 2)$, or (b) *degenerate*.

We referred to the first two cases as “(a)” and “(a’)” because they are indeed equivalent to one another, up to reversing the metric. In fact, in case (a’), we have a neutral inner product g over a four-dimensional Lie algebra $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, where a space-like vector e_4 (spanning \mathfrak{r}) acts as a derivation over a three-dimensional Lie algebra \mathfrak{g}_3 , on which g has signature $(1, 2)$. It now suffices to *reverse the metric* [14], to find the same Lie algebra \mathfrak{g} , equipped with the neutral inner product $-g$, for which a time-like vector e_4 acts as a derivation over the three-dimensional Lorentzian Lie algebra \mathfrak{g}_3 of signature $(2, 1)$ (case (a)). We also recall that being homothetic, the metrics g and $-g$ share the same curvature properties, like being Einstein, Ricci-parallel and so on.

In case (b), since the restriction of g to \mathfrak{g}_3 is degenerate, the approach used in the Riemannian case to study curvature properties fails. We emphasize the fact that such cases explicitly occurred, for example, in the classification of left-invariant conformally flat neutral metrics on four-dimensional Lie groups [8].

In order to study neutral inner products over an arbitrary four-dimensional Lie algebra $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, we shall first discuss the standard forms of such inner products with respect to the semi-direct product structure of \mathfrak{g} . Then, we shall impose the required curvature condition, like being Einstein or Ricci-parallel, together with the Jacobi identity. Once we determine explicitly the Lie algebras satisfying the required curvature property, we identify *a posteriori* the corresponding simply connected Lie groups.

Using the well known facts that (1) any symmetric bilinear form admits an orthogonal basis, and (2) if g is nondegenerate of signature (p, q) , then $r = \min(p, q)$ is the maximal dimension of a vector subspace W such that $g|_W = 0$, we now prove the following result, which will be crucial to describe four-dimensional Lorentzian Lie algebras equipped with neutral inner products.

Lemma 2.3. *Let \mathfrak{g} denote any four-dimensional Lie algebra and g an inner product on \mathfrak{g} , of signature $(2, 2)$. Then, there exists a basis $\{e_1, e_2, e_3, e_4\}$ of \mathfrak{g} ,*

such that

- $\mathfrak{g}_3 = \text{Span}(e_1, e_2, e_3)$ is a three-dimensional Lie algebra and e_4 acts as a derivation on \mathfrak{g}_3 (that is, $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, where $\mathfrak{r} = \text{Span}(e_4)$), and
- with respect to $\{e_1, e_2, e_3, e_4\}$, the neutral inner product g takes one of the following forms:

$$(a) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (b) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proof. Consider a semi-direct product $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{k}$ of two Lie algebras \mathfrak{r} and \mathfrak{k} , with $\mathfrak{r} = \text{Span}(v)$ one-dimensional. For any vector $w \in \mathfrak{k}$, put $\mathfrak{r}' = \text{Span}(v + w)$. Then, we still have $\mathfrak{g} = \mathfrak{r}' \ltimes \mathfrak{k}$. In fact, since \mathfrak{r} is one-dimensional, $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{k}$ means that $[\mathfrak{r}, \mathfrak{r}] = 0$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{r}, \mathfrak{k}] \subset \mathfrak{k}$. It is then obvious from these equations and the definition of \mathfrak{r}' that the same conditions are satisfied when we replace \mathfrak{r} by \mathfrak{r}' . Hence, $\mathfrak{g} = \mathfrak{r}' \ltimes \mathfrak{k}$.

Next, let g denote a left-invariant neutral inner product on a four-dimensional Lie algebra \mathfrak{g} . Because of Propositions 2.1 and 2.2, $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$ is a semi-direct product, where $\mathfrak{r} = \text{Span}(v)$ is one-dimensional. Since g has signature $(2, 2)$, a subspace of \mathfrak{g} (and so, of \mathfrak{g}_3) on which g vanishes has dimension at most two. So, the nullity index of $g|_{\mathfrak{g}_3}$ is less than or equal to 2. Up to reversing the metric, the possible cases in terms of the signature of $g|_{\mathfrak{g}_3}$ are then the following:

- (I) $\text{sgn}(g|_{\mathfrak{g}_3}) = (2, 1, 0)$ (and so, $g|_{\mathfrak{g}_3}$ is nondegenerate);
- (II) $\text{sgn}(g|_{\mathfrak{g}_3}) = (2, 0, 1)$;
- (III) $\text{sgn}(g|_{\mathfrak{g}_3}) = (1, 1, 1)$;
- (IV) $\text{sgn}(g|_{\mathfrak{g}_3}) = (1, 0, 2)$.

We now study separately the above cases.

Case (I). Since $\text{sgn}(g|_{\mathfrak{g}_3}) = (2, 1, 0)$, there exists an orthogonal basis $\{e_1, e_2, e_3\}$ for $g|_{\mathfrak{g}_3}$, with e_1, e_2 unit space-like vectors and e_3 a time-like vector.

If $\mathfrak{r} = \text{Span}(v)$, we now consider the orthogonal projection w of v on \mathfrak{g}_3 , that is, $w := \sum_{i=1}^3 \varepsilon_i g(v, e_i) e_i$, where $\varepsilon_i = g(e_i, e_i) = \pm 1$. Next, we put $v' := v - w$ and $\mathfrak{r}' := \text{Span}(v')$. By the above remark, we still have $\mathfrak{g} = \mathfrak{r}' \ltimes \mathfrak{g}_3$.

Moreover, v' is orthogonal to e_1, e_2, e_3 and so, $\mathfrak{r}' = \mathfrak{g}_3^\perp$. Since $g|_{\mathfrak{g}_3}$ is nondegenerate, the index of g is the sum of the indices of $g|_{\mathfrak{g}_3}$ and $g|_{\mathfrak{g}_3^\perp}$. Hence, v' is necessarily time-like, and g takes the form (a) with respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of \mathfrak{g} , where $e_4 = v' / \sqrt{-g(v', v')}$.

Case (II). We shall prove that this case cannot occur for a neutral metric g . Since $\text{sgn}(g|_{\mathfrak{g}_3}) = (2, 0, 1)$, there exists an orthogonal basis $\{e_1, e_2, e_3\}$ for $g|_{\mathfrak{g}_3}$, with e_1, e_2 unit space-like vectors and e_3 a light-like vector.

We put $v' := v - \sum_{i=1}^2 g(v, e_i)e_i$ and $\mathfrak{r}' := \text{Span}(v')$, where $\mathfrak{r} = \text{Span}(v)$. Then, v' orthogonal to e_1, e_2 and $\mathfrak{g} = \mathfrak{r}' \times \mathfrak{g}_3$.

Note that necessarily $g(v', e_3) \neq 0$. In fact, if $g(v', e_3) = 0$, then e_3 is orthogonal to e_1, e_2, e_3 and v' and so, $e_3 \in \mathfrak{g}^\perp$, against the fact that g is nondegenerate.

It is now easily seen that there exists $\lambda_0 \in \mathbb{R}$, such that $v' + \lambda_0 e_3$ is light-like. In fact, one has

$$0 = g(v' + \lambda e_3, v' + \lambda e_3) = g(v', v') + 2\lambda g(v', e_3),$$

whose solution $\lambda_0 = -g(v', v')/2g(v', e_3)$ is uniquely determined for any value of $g(v', v')$ and $g(v', e_3) \neq 0$.

Since e_3 is light-like, we have $k = g(v' + \lambda_0 e_3, e_3) = g(v', e_3) \neq 0$. Putting $e_4 = \frac{1}{k}(v' + \lambda_0 e_3)$, we have that $e_4 (\neq 0)$ acts as a derivation on \mathfrak{g}_3 , and with respect to the basis $\{e_1, e_2, e_3, e_4\}$, g takes the form

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

But then, g would be Lorentzian and so, this case does not occur.

Case (III). This case is similar to the previous one but leads to case (b). Since $\text{sgn}(g|_{\mathfrak{g}_3}) = (1, 1, 1)$, there exists an orthogonal basis $\{e_1, e_2, e_3\}$ for $g|_{\mathfrak{g}_3}$, with e_1 a unit space-like vector, e_2 a unit time-like vector and e_3 a light-like vector.

If $\mathfrak{r} = \text{Span}(v)$, we consider $v' := v - \sum_{i=1}^2 \varepsilon_i g(v, e_i)e_i$ and obtain $\mathfrak{g} = \mathfrak{r}' \times \mathfrak{g}_3$, with $\mathfrak{r}' := \text{Span}(v')$ and v' orthogonal to e_1, e_2 . Moreover, because of the nondegeneracy of g , necessarily $g(v', e_3) \neq 0$.

Next, there exists a unique $\lambda_0 \in \mathbb{R}$, such that $v' + \lambda_0 e_3$ is light-like, given by $\lambda_0 = -g(v', v')/2g(v', e_3)$. Putting $k = g(v' + \lambda_0 e_3, e_3) = g(v', e_3) \neq 0$ and $e_4 = \frac{1}{k}(v' + \lambda_0 e_3)$, we get that e_4 acts as a derivation on \mathfrak{g}_3 , and g takes the form (b) with respect to the basis $\{e_1, e_2, e_3, e_4\}$.

Case (IV). This case is not compatible with the nondegeneracy of g . Starting from $\text{sgn}(g|_{\mathfrak{g}_3}) = (1, 0, 2)$, there exists an orthogonal basis $\{e_1, e_2, e_3\}$ for $g|_{\mathfrak{g}_3}$, where e_1 is a unit space-like vector and e_2, e_3 are light-like vectors.

We consider $v' := v - g(v, e_1)e_1$, where $\mathfrak{r} = \text{Span}(v)$. Then, $\mathfrak{g} = \mathfrak{r}' \times \mathfrak{g}_3$, with $\mathfrak{r}' := \text{Span}(v')$ and v' orthogonal to e_1 .

If $g(v', e_2) = 0$, then $e_2 \in \mathfrak{g}^\perp$, against the fact that g is nondegenerate. So, $k := g(v', e_2) \neq 0$. For the same reason, $\mu := g(v', e_3) \neq 0$.

As $g(v', e_2) \neq 0$, we can find $\lambda_0 \in \mathbb{R}$, such that $v' + \lambda_0 e_2$ is light-like: it suffices that $g(v', v') + 2g(v', e_2)\lambda_0 = 0$. We now put $e_4 = \frac{1}{k}(v' + \lambda_0 e_2)$. As $g(e_3, e_4) = \frac{1}{k}g(e_3, v') = \frac{\mu}{k}$, with respect to the basis $\{e_1, e_2, e_3, e_4\}$, the metric g takes the form

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \mu/k \\ 0 & 1 & \mu/k & 0 \end{pmatrix},$$

which is degenerate. Hence, this case does not occur. \square

Remark 2.4. By an argument very similar to the one used in the proof of the above Lemma 2.3, one can prove its Lorentzian analogue (see Proposition 2.3 in [7]): *Let (\mathfrak{g}, g) be an arbitrary four-dimensional Lorentzian Lie algebra. Then, there exists a basis $\{e_1, e_2, e_3, e_4\}$ of \mathfrak{g} , such that*

- $\mathfrak{g}_3 = \text{Span}(e_1, e_2, e_3)$ is a three-dimensional Lie algebra and e_4 acts as a derivation on \mathfrak{g}_3 (that is, $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, where $\mathfrak{r} = \text{Span}(e_4)$), and
- with respect to $\{e_1, e_2, e_3, e_4\}$, the Lorentzian inner product takes one of the following forms:

$$(\tilde{a}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (\tilde{b}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\tilde{c}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In the Lorentzian case, the maximal dimension of a subspace on which g vanishes is equal to one. Consequently, the nullity index of $g|_{\mathfrak{g}_3}$ is either 0 or 1. Up to reversing the metric, the possible cases in terms of the signature of $g|_{\mathfrak{g}_3}$ are then the following:

- (I) $\text{sgn}(g|_{\mathfrak{g}_3}) = (3, 0, 0)$, which leads to case (\tilde{a}) ;
- (II) $\text{sgn}(g|_{\mathfrak{g}_3}) = (2, 1, 0)$, which yields case (\tilde{b}) ;
- (III) $\text{sgn}(g|_{\mathfrak{g}_3}) = (1, 1, 1)$, which is incompatible with the fact that g is Lorentzian;
- (IV) $\text{sgn}(g|_{\mathfrak{g}_3}) = (1, 0, 2)$, which leads to case (\tilde{c}) .

The above argument corrects and replaces the proof of Proposition 2.3 in [7].

3. Einstein examples

We now give the following classification of left-invariant neutral Einstein metrics on four-dimensional Lie groups.

Theorem 3.1. *Let G be a four-dimensional simply connected Lie group. If g is a left-invariant neutral Einstein metric on G , then the Lie algebra \mathfrak{g} of G is isometric to $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, where $\mathfrak{g}_3 = \text{Span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \text{Span}\{e_4\}$, and one of the following cases occurs.*

(a) $\{e_1, e_2, e_3, e_4\}$ is a pseudo-orthonormal basis, with e_3, e_4 time-like. In this case, G is isometric to one of the following semi-direct products $\mathbb{R} \ltimes G_3$:

a1) $\mathbb{R} \ltimes H$, where \mathfrak{g} is described by one of the following sets of conditions:

$$1) \begin{aligned} [e_1, e_2] &= \frac{\varepsilon\sqrt{A^2-B^2}}{2} e_1, & [e_1, e_3] &= \frac{\varepsilon\delta\sqrt{A^2-B^2}}{2} e_1, & [e_1, e_4] &= \left(\frac{A}{2} - \delta\frac{B}{2}\right) e_1, \\ [e_2, e_4] &= Ae_2 - \delta Ae_3, & [e_3, e_4] &= Be_2 - \delta Be_3, & & (A \neq \pm B), \end{aligned}$$

$$2) [e_1, e_2] = \frac{\sqrt{A^2 - B^2}B}{A}e_1, \quad [e_1, e_3] = \sqrt{A^2 - B^2}e_1, \quad [e_2, e_4] = Ae_2 - Be_3, \\ [e_3, e_4] = Be_2 - \frac{B^2}{A}e_3, \quad (A \neq \pm B),$$

a2) $\mathbb{R} \times \mathbb{R}^3$ and \mathfrak{g} is described by one of the following sets of conditions:

$$3) [e_1, e_4] = Ae_1, \quad [e_2, e_4] = Ae_2 + Be_3, \quad [e_3, e_4] = Be_2 + Ae_3,$$

$$4) [e_1, e_4] = -\frac{A+B}{3}e_1, \quad [e_2, e_4] = -\frac{5A-B}{6}e_2 + Ae_3, \quad [e_3, e_4] = Be_2 - \frac{5B-A}{6}e_3,$$

$$5) [e_1, e_4] = \frac{2\sqrt{2}}{3}Ae_1 - Ae_3, \quad [e_2, e_4] = -\frac{\sqrt{2}}{3}Ae_2, \quad [e_3, e_4] = Ae_2 + \frac{\sqrt{2}}{6}Ae_3,$$

$$6) [e_1, e_4] = Ae_1 - \sqrt{C^2 - AB - A^2 - B^2}e_2, \\ [e_2, e_4] = -\sqrt{C^2 - AB - A^2 - B^2}e_1 - (A+B)e_2 - Ce_3, \quad [e_3, e_4] = Ce_2 + Be_3,$$

$$7) [e_1, e_4] = Ae_1 - Be_2, \quad [e_2, e_4] = Be_1 + Ae_2, \quad [e_3, e_4] = Ae_3,$$

$$8) [e_1, e_4] = -\frac{2A^2+5B^2}{6\sqrt{B^2+A^2}}e_1 - \frac{AB}{2\sqrt{B^2+A^2}}e_2 + Be_3, \\ [e_2, e_4] = -\frac{AB}{2\sqrt{B^2+A^2}}e_1 - \frac{5A^2+2B^2}{6\sqrt{B^2+A^2}}e_2 + Ae_3, \quad [e_3, e_4] = \frac{\sqrt{B^2+A^2}}{6}e_3,$$

$$9) [e_1, e_4] = \frac{A+B}{2}e_1 - \frac{\sqrt{6(B^2-A^2)}}{2}e_2 - \frac{\sqrt{6(B^2-A^2)}}{2}e_3, \\ [e_2, e_4] = Ae_2 + (B+2A)e_3, \quad [e_3, e_4] = (A+2B)e_2 + Be_3,$$

$$10) [e_1, e_4] = \frac{5\sqrt{A^2-B^2}}{6}e_1 + Be_2 + Ae_3, \quad [e_2, e_4] = \frac{B^2+2A^2}{6\sqrt{A^2-B^2}}e_2 + \frac{AB}{2\sqrt{A^2-B^2}}e_3, \\ [e_3, e_4] = -\frac{AB}{2\sqrt{A^2-B^2}}e_2 - \frac{2B^2+A^2}{6\sqrt{A^2-B^2}}e_3$$

b) $\{e_1, e_2, e_3, e_4\}$ is a basis, with the inner product g on \mathfrak{g} completely determined by $g(e_1, e_1) = -g(e_2, e_2) = g(e_3, e_4) = g(e_4, e_3) = 1$ and $g(e_i, e_j) = 0$ otherwise. In this case, G is isometric to one of the following semi-direct products $\mathbb{R} \times G_3$:

b1) $\mathbb{R} \times H$ and \mathfrak{g} is described by one of the following sets of conditions:

$$11) [e_1, e_2] = A(e_1 + e_2) + (2B - C - D)e_3, \\ [e_1, e_4] = D(e_1 + e_2) + \frac{3B(C+D) - 2B^2 - AE - (C+D)^2}{A}e_3, \\ [e_2, e_4] = C(e_1 + e_2) + Ee_3, \quad [e_3, e_4] = Be_3,$$

$$12) [e_1, e_2] = A(e_1 + e_2) - (C + D)e_3, \\ [e_1, e_4] = C(e_1 + e_2) - \frac{(C+D)^2 - B(C+D) + AE}{A}e_3, \\ [e_2, e_4] = D(e_1 + e_2) + Ee_3, \quad [e_3, e_4] = Be_3,$$

$$13) [e_1, e_2] = A(e_1 - e_2) + Be_3, \quad [e_1, e_4] = 2B(e_1 - e_2) + \frac{B^2+AD}{A}e_3, \\ [e_2, e_4] = B(e_1 - e_2) + De_3,$$

$$14) [e_1, e_2] = A(e_1 + \varepsilon e_2) - \frac{CA + \varepsilon AD}{A}e_3, \\ [e_1, e_4] = D(e_1 + \varepsilon e_2) - \frac{DC + \varepsilon D^2 - EC}{A}e_3, \\ [e_2, e_4] = C(e_1 + \varepsilon e_2) - \frac{C^2 + \varepsilon CD - ED}{A}e_3, \quad [e_3, e_4] = Ee_3,$$

$$15) \quad [e_1, e_2] = Ae_3, \quad [e_1, e_4] = \frac{A^2-(C-D)^2}{4B}e_1 + De_2 + Ee_3, \\ [e_2, e_4] = Ce_1 + Be_2 + Fe_3, \quad [e_3, e_4] = \frac{A^2-(C-D)^2+4B^2}{4B}e_3, \quad (A \neq 0),$$

$$16) \quad [e_1, e_2] = \varepsilon(B-A)e_3, \quad [e_1, e_4] = Ce_1 + Ae_2 + De_3, \\ [e_2, e_4] = Be_1 + Ee_3, \quad [e_3, e_4] = Ce_3, \quad (A \neq B),$$

b2) Either $\mathbb{R} \times E(2)$ or $\mathbb{R} \times E(1, 1)$, with \mathfrak{g} described by one of the following sets of conditions:

$$17) \quad [e_1, e_2] = -\frac{A}{2}e_1 + \frac{B}{2}e_2 - \frac{CB-AD}{A}e_3, \quad [e_1, e_3] = Be_3, \\ [e_1, e_4] = Ce_1 - \frac{CB}{A}e_2 - \frac{2D(-CA+EA+DB)}{A^2}e_3, \quad [e_2, e_3] = Ae_3, \\ [e_2, e_4] = De_1 - \frac{DB}{A}e_2 - \frac{2C(-CA+EA+DB)}{A^2}e_3, \quad [e_3, e_4] = Ee_3,$$

$$18) \quad [e_1, e_2] = -\frac{A}{2}(e_1 - \varepsilon e_2) + (B - \varepsilon C)e_3, \quad [e_1, e_3] = \varepsilon Ae_3, \\ [e_1, e_4] = C(e_1 - \varepsilon e_2) + De_3, \quad [e_2, e_3] = Ae_3, \\ [e_2, e_4] = B(e_1 - \varepsilon e_2) + \frac{2B^2+\varepsilon(-4CB+AD+2EB)-2EC+2C^2}{A}e_3, \quad [e_3, e_4] = Ee_3,$$

$$19) \quad [e_1, e_2] = \varepsilon Ae_2, \quad [e_1, e_3] = \varepsilon Ae_3, \quad [e_1, e_4] = \varepsilon Be_2 - \frac{\varepsilon(2CA+B^2-2DB)}{2A}e_3, \\ [e_2, e_4] = Be_2 + Ce_3, \quad [e_3, e_4] = Ae_2 + De_3$$

b3) $\mathbb{R} \times \mathbb{R}^3$ and \mathfrak{g} is described by one of the following sets of conditions:

$$20) \quad [e_1, e_4] = Ae_1 + Be_2 + Ce_3, \quad [e_2, e_4] = De_1 + Ee_2 + Fe_3, \\ [e_3, e_4] = \frac{-(B-D)^2+2A^2+2E^2}{2(A+E)}e_3,$$

$$21) \quad [e_1, e_4] = \frac{-(A-C)^2}{4B}e_1 + Ce_2 + De_3, \quad [e_2, e_4] = Ae_1 + Be_2 + Ee_3, \\ [e_3, e_4] = \frac{-(A-C)^2+4B^2}{4B}e_3,$$

$$22) \quad [e_1, e_4] = -Ae_1 + (B + 2\varepsilon A)e_2 + Ce_3, \quad [e_2, e_4] = Be_1 + Ae_2 + De_3, \\ [e_3, e_4] = Ee_3,$$

In the cases listed above, $\varepsilon = \pm 1$ and $\delta = \pm 1$.

Proof. Let G denote a four-dimensional Lie group and g a left-invariant neutral metric on it. Because of Lemma 2.3, the Lie algebra \mathfrak{g} of G is a semi-direct product $\mathfrak{r} \ltimes \mathfrak{g}_3$, where $\mathfrak{r} = \text{Span}(e_4)$ acts on $\mathfrak{g}_3 = \text{Span}(e_1, e_2, e_3)$. Since $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, we have

$$\begin{aligned} [e_1, e_2] &= a_1e_1 + a_2e_2 + a_3e_3, & [e_1, e_3] &= b_1e_1 + b_2e_2 + b_3e_3, \\ [e_1, e_4] &= c_1e_1 + c_2e_2 + c_3e_3, & [e_2, e_3] &= d_1e_1 + d_2e_2 + d_3e_3, \\ [e_2, e_4] &= p_1e_1 + p_2e_2 + p_3e_3, & [e_3, e_4] &= q_1e_1 + q_2e_2 + q_3e_3, \end{aligned} \quad (3.1)$$

for some real constants a_i, \dots, q_i . In terms of these coefficients, the Jacobi identity is equivalent to the following system of algebraic equations:

$$\begin{aligned}
 a_2d_1 + b_3d_1 - b_1d_3 - a_1d_2 &= 0, \\
 a_2d_3 + a_1b_3 - b_1a_3 - d_2a_3 &= 0, \\
 d_2b_3 - b_1a_2 + a_1b_2 - d_3b_2 &= 0, \\
 b_1p_3 - c_3d_1 - q_1a_3 + a_1p_2 - a_2p_1 &= 0, \\
 b_1q_3 - b_3q_1 + a_1q_2 + d_1c_2 - b_2p_1 &= 0, \\
 b_2p_1 - d_3q_2 - d_1c_2 + d_2q_3 - a_2q_1 &= 0, \\
 b_2p_3 + b_1c_3 - d_3c_2 - c_1b_3 - a_3q_2 &= 0, \\
 d_2c_3 - c_1a_2 + a_3q_2 + a_1c_2 - b_2p_3 &= 0, \\
 p_2d_3 + b_3p_1 - d_2p_3 - c_3d_1 - q_1a_3 &= 0, \\
 b_3q_2 + p_2b_2 + b_1c_2 - d_2c_2 - c_1b_2 - q_3b_2 - a_2q_2 &= 0, \\
 d_3c_3 - c_1a_3 + a_3q_3 + a_2p_3 + a_1c_3 - b_3p_3 - p_2a_3 &= 0, \\
 q_3d_1 - d_3q_1 + p_2d_1 - d_2p_1 - a_1q_1 + b_1p_1 - c_1d_1 &= 0.
 \end{aligned} \tag{3.2}$$

Again by Lemma 2.3, g is described at the algebraic level by conditions (a) or (b). We shall now treat these cases separately.

Case (a). Starting from the above description of the Lie algebra, we apply the well known *Koszul formula* and describe the Levi-Civita connection ∇ of g . With respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ with e_3 and e_4 time-like, we set $\Lambda_i = \nabla_{e_i}$ for all indices $i = 1, \dots, 4$ and we get

$$\begin{aligned}
 \Lambda_1 &= \begin{pmatrix} 0 & a_1 & b_1 & c_1 \\ -a_1 & 0 & \frac{b_2+d_1+a_3}{2} & \frac{c_2+p_1}{2} \\ b_1 & \frac{b_2+d_1+a_3}{2} & 0 & \frac{c_3-q_1}{2} \\ c_1 & \frac{c_2+p_1}{2} & \frac{q_1-c_3}{2} & 0 \end{pmatrix}, \\
 \Lambda_2 &= \begin{pmatrix} 0 & a_2 & \frac{d_1+b_2-a_3}{2} & \frac{c_2+p_1}{2} \\ -a_2 & 0 & d_2 & p_2 \\ \frac{d_1+b_2-a_3}{2} & d_2 & 0 & \frac{p_3-q_2}{2} \\ \frac{c_2+p_1}{2} & p_2 & \frac{q_2-p_3}{2} & 0 \end{pmatrix}, \\
 \Lambda_3 &= \begin{pmatrix} 0 & \frac{b_2-d_1-a_3}{2} & -b_3 & \frac{q_1-c_3}{2} \\ \frac{a_3-b_2+d_1}{2} & 0 & -d_3 & \frac{q_2-p_3}{2} \\ -b_3 & -d_3 & 0 & q_3 \\ \frac{q_1-c_3}{2} & \frac{-p_3+q_2}{2} & -q_3 & 0 \end{pmatrix}, \\
 \Lambda_4 &= \begin{pmatrix} 0 & \frac{c_2-p_1}{2} & -\frac{q_1+c_3}{2} & 0 \\ \frac{p_1-c_2}{2} & 0 & -\frac{q_2+p_3}{2} & 0 \\ -\frac{q_1+c_3}{2} & -\frac{q_2+p_3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
 \end{aligned}$$

Then, we determine the curvature tensor, calculating $R(e_i, e_j) = \Lambda_i\Lambda_j - \Lambda_j\Lambda_i - \Lambda_{[e_i, e_j]}$ for all indices i, j and the curvature components $R_{ijkh} = g(R(e_i, e_j)e_k, e_h)$. We find

$$\begin{aligned}
R_{1212} &= b_1d_2 + c_1p_2 - a_1^2 - a_2^2 + \frac{1}{2}(d_1a_3 - b_2a_3 - c_2p_1 - b_2d_1) \\
&\quad - \frac{1}{4}(p_1^2 + b_2^2 + c_2^2 + d_1^2) + \frac{3}{4}a_3^2, \\
R_{1213} &= b_3d_1 + a_3b_3 - b_1d_3 - a_1b_1 - a_2b_2 + \frac{1}{2}(c_1q_2 - c_1p_3) \\
&\quad + \frac{1}{4}(c_2c_3 - c_2q_1 + p_1c_3 - p_1q_1), \\
R_{1214} &= -a_1c_1 - a_2c_2 - \frac{1}{2}(b_1q_2 + b_1p_3) + \frac{1}{4}(b_2q_1 - b_2c_3 + d_1q_1 + a_3q_1) \\
&\quad + \frac{3}{4}(d_1c_3 + a_3c_3), \\
R_{1223} &= d_2b_3 - d_3b_2 + a_3d_3 - a_1d_1 - a_2d_2 + \frac{1}{2}(p_2c_3 - p_2q_1) \\
&\quad + \frac{1}{4}(p_1q_2 - c_2p_3 + c_2q_2 - p_1p_3), \\
R_{1224} &= -a_1p_1 - a_2p_2 + \frac{1}{2}(d_2q_1 + d_2c_3) + \frac{1}{4}(a_3q_2 - d_1q_2 + d_1p_3 - b_2q_2) \\
&\quad + \frac{3}{4}(a_3p_3 - b_2p_3), \\
R_{1234} &= -a_1q_1 - a_2q_2 + \frac{1}{2}(b_3q_2 + b_3p_3 - d_3q_1 - d_3c_3 + q_3d_1) \\
&\quad + a_3q_3 - q_3b_2), \\
R_{1313} &= b_3^2 - b_1^2 - a_1d_3 - c_1q_3 + \frac{1}{4}(d_1^2 + a_3^2 - c_3^2 - q_1^2) \\
&\quad + \frac{1}{2}(d_1a_3 + c_3q_1 - b_2d_1 + b_2a_3) - \frac{3}{4}b_2^2, \\
R_{1314} &= b_3c_3 - b_1c_1 - \frac{1}{2}(a_1q_2 + a_1p_3) + \frac{1}{4}(b_2p_1 + d_1p_1 + a_3p_1 + a_3c_2) \\
&\quad - \frac{3}{4}(b_2c_2 + d_1c_2), \\
R_{1323} &= b_3d_3 - b_1d_1 - a_2d_3 + d_2a_3 - b_2d_2 - \frac{1}{2}(q_3c_2 + q_3p_1) \\
&\quad + \frac{1}{4}(q_1p_3 + c_3q_2 - c_3p_3 - q_1q_2), \\
R_{1324} &= b_3p_3 - b_1p_1 + \frac{1}{2}(d_2p_1 + p_2a_3 - a_2q_2 - a_2p_3 - d_2c_2) \\
&\quad - p_2d_1 - p_2b_2), \\
R_{1334} &= b_3q_3 - b_1q_1 + \frac{1}{2}(d_3c_2 - d_3p_1) + \frac{1}{4}(d_1p_3 + a_3p_3 - d_1q_2 - b_2p_3) \\
&\quad + \frac{3}{4}(a_3q_2 - b_2q_2), \\
R_{1414} &= -c_1^2 - \frac{1}{2}(c_2p_1 + c_3q_1) + \frac{1}{4}(p_1^2 - q_1^2) + \frac{3}{4}(-c_2^2 + c_3^2), \\
R_{1423} &= -c_1d_1 + \frac{1}{2}(a_2q_2 - a_2p_3 + q_3d_1 + q_3b_2 - a_3q_3 + p_2d_1 + p_2a_3) \\
&\quad - p_2b_2 - b_3q_2 + b_3p_3 - d_2c_2 - d_2p_1 + d_3c_3 - d_3q_1), \\
R_{1424} &= -c_1p_1 - c_2p_2 - \frac{1}{4}(q_1q_2 + c_3q_2 + q_1p_3) + \frac{3}{4}c_3p_3, \\
R_{1434} &= q_3c_3 - c_1q_1 + \frac{1}{4}(c_2p_3 - p_1q_2 - p_1p_3) - \frac{3}{4}c_2q_2, \\
R_{2323} &= a_2b_3 - p_2q_3 - d_2^2 + d_3^2 + \frac{1}{4}(b_2^2 + a_3^2 - p_3^2 - q_2^2) \\
&\quad + \frac{1}{2}(p_3q_2 - b_2a_3 - b_2d_1 - d_1a_3) - \frac{3}{4}d_1^2, \\
R_{2324} &= d_3p_3 - d_2p_2 + \frac{1}{2}(a_2q_1 + a_2c_3) + \frac{1}{4}(d_1c_2 + b_2c_2 - a_3c_2 - a_3p_1) \\
&\quad - \frac{3}{4}(d_1p_1 + b_2p_1), \\
R_{2334} &= d_3q_3 - d_2q_2 + \frac{1}{2}(b_3p_1 - b_3c_2) + \frac{1}{4}(b_2c_3 - b_2q_1 - a_3c_3 - d_1c_3) \\
&\quad - \frac{3}{4}(a_3q_1 + d_1q_1), \\
R_{2424} &= -p_2^2 - \frac{1}{2}(c_2p_1 + p_3q_2) + \frac{1}{4}(c_2^2 - q_2^2) + \frac{3}{4}(p_3^2 - p_1^2), \\
R_{2434} &= q_3p_3 - p_2q_2 + \frac{1}{4}(p_1c_3 - c_2q_1 - c_2c_3) - \frac{3}{4}p_1q_1, \\
R_{3434} &= q_3^2 + \frac{1}{2}(c_3q_1 + p_3q_2) + \frac{1}{4}(c_3^2 + p_3^2) - \frac{3}{4}(q_1^2 + q_2^2).
\end{aligned}$$

By contraction we then get the components for the Ricci tensor with respect to the basis $\{e_1, e_2, e_3, e_4\}$:

$$\begin{aligned}
 \varrho_{11} &= b_1^2 + c_1^2 - a_2^2 - a_1^2 - b_3^2 - b_2a_3 + a_1d_3 + c_1q_3 + b_1d_2 + c_1p_2 + \frac{1}{2}(c_2^2 + a_3^2 + q_1^2 + b_2^2 - c_3^2 - d_1^2 - p_1^2), \\
 \varrho_{12} &= b_2d_2 + a_2d_3 + b_1d_1 + c_1p_1 + c_2p_2 - d_2a_3 - b_3d_3 + \frac{1}{2}(q_3c_2 + q_3p_1 - c_3p_3 + q_1q_2), \\
 \varrho_{13} &= a_1d_1 + a_2d_2 + c_1q_1 + d_3b_2 - q_3c_3 - d_2b_3 - a_3d_3 + \frac{1}{2}(c_2q_2 + p_1p_3 - p_2c_3 + p_2q_1), \\
 \varrho_{14} &= a_1p_1 + a_2p_2 - b_1q_1 + b_3q_3 + \frac{1}{2}(d_3c_2 + b_2p_3 + a_3q_2 - d_2q_1 - d_2c_3 - b_2q_2 - a_3p_3 - d_3p_1), \\
 \varrho_{22} &= d_2^2 + p_2^2 - a_1^2 - d_3^2 - a_2^2 + p_2q_3 + d_1a_3 + b_1d_2 + c_1p_2 - a_2b_3 + \frac{1}{2}(a_3^2 + p_1^2 + q_2^2 + d_1^2 - p_3^2 - b_2^2 - c_2^2), \\
 \varrho_{23} &= b_3d_1 + a_3b_3 + p_2q_2 - a_1b_1 - a_2b_2 - q_3p_3 - b_1d_3 + \frac{1}{2}(c_1q_2 + c_2c_3 + p_1q_1 - c_1p_3), \\
 \varrho_{24} &= d_3q_3 - a_1c_1 - a_2c_2 - d_2q_2 + \frac{1}{2}(d_1c_3 + a_3c_3 + b_3p_1 - b_1q_2 - b_1p_3 - d_1q_1 - a_3q_1 - b_3c_2), \\
 \varrho_{33} &= d_3^2 + b_3^2 - d_2^2 - b_1^2 - q_3^2 + a_2b_3 - p_2q_3 - b_2d_1 - a_1d_3 - c_1q_3 + \frac{1}{2}(q_1^2 + a_3^2 - p_3^2 + q_2^2 - c_3^2 - b_2^2 - d_1^2), \\
 \varrho_{34} &= b_3c_3 + d_3p_3 - b_1c_1 - d_2p_2 + \frac{1}{2}(a_2q_1 + a_2c_3 - a_1q_2 - a_1p_3 - b_2p_1 - d_1p_1 - b_2c_2 - d_1c_2), \\
 \varrho_{44} &= -c_1^2 - p_2^2 - q_3^2 - c_2p_1 - c_3q_1 - p_3q_2 + \frac{1}{2}(q_1^2 + c_3^2 + q_2^2 + p_3^2 - p_1^2 - c_2^2).
 \end{aligned}$$

The Einstein equation is equivalent to the algebraic system of the equations:

$$\varrho_{ij} = \lambda g_{ij}, \quad \text{for all indices } i, j, \tag{3.3}$$

where ϱ_{ij} are given by the equations above. Solving the algebraic systems (3.2) and (3.3), we determined cases **1)-10)** of the Lie algebras listed in the case *a)* of the statement of the theorem, which list all the generally non-isomorphic solutions. In cases **1)** and **2)** we find that $[\mathfrak{g}_3, \mathfrak{g}_3]$ is one-dimensional. By the classification of three-dimensional Lorentzian Lie algebras given in [15], we conclude that \mathfrak{g}_3 is the Lie algebra of the Heisenberg group. For the cases **3)-10)**, the Lie algebra corresponds to a semi-direct product $\mathbb{R} \ltimes \mathbb{R}^3$, since $[\mathfrak{g}_3, \mathfrak{g}_3] = 0$.

Case (b) By the same argument of the previous case, we consider the Lie algebra $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$ spanned by $\{e_1, e_2, e_3, e_4\}$, where the inner product is described as in case (b) of the Lemma 2.3. Starting from the Lie algebra (3.1), with respect to the basis $\{e_1, e_2, e_3, e_4\}$, we now find

$$\Lambda_1 = \begin{pmatrix} 0 & a_1 & b_1 & c_1 \\ a_1 & 0 & \frac{b_2-d_1}{2} & \frac{a_3-p_1+c_2}{2} \\ -c_1 & \frac{a_3-p_1+c_2}{2} & \frac{b_3-q_1}{2} & 0 \\ -b_1 & \frac{b_2-d_1}{2} & 0 & \frac{q_1-b_3}{2} \end{pmatrix},$$

$$\Lambda_2 = \begin{pmatrix} 0 & -a_2 & \frac{d_1-b_2}{2} & \frac{p_1-c_2+a_3}{2} \\ -a_2 & 0 & d_2 & p_2 \\ \frac{c_2-a_3-p_1}{2} & p_2 & \frac{d_3+q_2}{2} & 0 \\ \frac{b_2-d_1}{2} & d_2 & 0 & -\frac{q_2+d_3}{2} \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} 0 & -\frac{d_1+b_2}{2} & 0 & \frac{q_1+b_3}{2} \\ -\frac{d_1+b_2}{2} & 0 & 0 & \frac{q_2-d_3}{2} \\ -\frac{b_3+q_1}{2} & \frac{q_2-d_3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_4 = \begin{pmatrix} 0 & \frac{a_3-p_1-c_2}{2} & \frac{b_3-q_1}{2} & c_3 \\ \frac{a_3-p_1-c_2}{2} & 0 & -\frac{q_2+d_3}{2} & -p_3 \\ -c_3 & -p_3 & -q_3 & 0 \\ \frac{q_1-b_3}{2} & -\frac{q_2+d_3}{2} & 0 & q_3 \end{pmatrix}.$$

A long but straightforward calculation yields that the components of the Ricci tensor with respect to the basis $\{e_1, e_2, e_3, e_4\}$ are given by

$$\begin{aligned} \varrho_{11} &= a_1^2 - a_2^2 + b_2c_2 - b_1q_3 - a_1d_3 - b_2a_3 - d_1p_1 - 2b_1c_1 - b_1p_2 \\ &\quad - c_1d_2 - \frac{1}{2}(q_1^2 + b_3^2), \\ \varrho_{12} &= d_2c_2 + a_2d_3 + p_2b_2 - b_1p_1 - c_1d_1 - d_2a_3 \\ &\quad + \frac{1}{2}(q_2q_1 + q_3b_2 - d_3b_3 - q_3d_1), \\ \varrho_{13} &= b_2d_3 + a_2d_2 - a_1d_1 - b_1q_1 + \frac{1}{2}(b_2q_2 + d_1d_3 - d_2q_1 - d_2b_3), \\ \varrho_{14} &= a_2p_2 + c_1q_1 - d_2c_3 - a_1p_1 + \frac{1}{2}(p_3b_2 + c_2d_3 + a_3q_2 + p_2q_1 - c_2q_2 \\ &\quad - a_3d_3 - p_2b_3 - p_3d_1), \\ \varrho_{22} &= a_2^2 - a_1^2 + 2d_2p_2 + a_2b_3 + b_1p_2 + c_1d_2 + b_2c_2 + d_1a_3 + d_2q_3 \\ &\quad - d_1p_1 - \frac{1}{2}(d_3^2 + q_2^2), \\ \varrho_{23} &= d_1b_3 + a_2b_2 + d_2q_2 - a_1b_1 + \frac{1}{2}(b_2b_3 + b_1q_2 - b_1d_3 - d_1q_1), \\ \varrho_{24} &= a_2c_2 - a_1c_1 - p_2q_2 - b_1p_3 + \frac{1}{2}(p_1q_1 + c_3d_1 + p_1b_3 - c_1q_2 - c_1d_3 \\ &\quad - a_3b_3 - a_3q_1 - c_3b_2), \\ \varrho_{33} &= -b_1^2 - d_2^2 - d_1b_2 + \frac{1}{2}(d_1^2 + b_2^2), \\ \varrho_{34} &= -b_1c_1 - b_1q_3 - d_2p_2 - d_2q_3 + \frac{1}{2}(q_1^2 + d_3^2 - q_2^2 - b_3^2 + b_2c_2 + d_1p_1 \\ &\quad + a_2q_1 - a_1d_3 - a_1q_2 - b_2p_1 - d_1c_2 - a_2b_3), \\ \varrho_{44} &= -c_1^2 - p_2^2 + p_2q_3 - c_3q_1 - p_1c_2 - a_2c_3 - p_3q_2 - a_1p_3 + c_1q_3 \\ &\quad + \frac{1}{2}(p_1^2 + c_2^2 - a_3^2). \end{aligned}$$

We then determined all the sets of solutions of (3.2) and (3.3). The generally non-isomorphic solutions are the cases **11)**-**22)** of the statement of the theorem. Note that $[\mathfrak{g}_3, \mathfrak{g}_3]$ is one-dimensional for the cases **11)**-**16)**, two-dimensional for

the cases **17)**-**19)** and trivial for the cases **20)**-**22)**. Thus, the corresponding Lie algebra \mathfrak{g} is $\mathbb{R} \times H$, either $\mathbb{R} \times E(2)$ or $\mathbb{R} \times E(1, 1)$, and $\mathbb{R} \times \mathbb{R}^3$ respectively. This completes the proof. ■

We can now study curvature properties of left-invariant neutral Einstein metrics on four-dimensional Lie groups. For each of the examples listed in the above Theorem 3.1, considering the tensor components with respect to the basis $\{e_1, e_2, e_3, e_4\}$ used to describe the Lie algebra \mathfrak{g} , we recognize the flat and Ricci-flat cases, as solutions of $R_{ijkh} = 0$ for all $i, j, k, h = 1, \dots, 4$ and $\varrho_{ij} = 0$ for all $i, j = 1, \dots, 4$, respectively. Conformal flatness translates into the following system of algebraic equations:

$$\begin{aligned} W_{ijkh} &= R_{ijkh} - \frac{1}{2}(g_{ik}\varrho_{jh} + g_{jh}\varrho_{ik} - g_{ih}\varrho_{jk} - g_{jk}\varrho_{ih}) + \frac{r}{6}(g_{ik}g_{jh} - g_{ih}g_{jk}) \\ &= 0 \quad \text{for all indices } i, j, k, h = 1, \dots, 4, \end{aligned}$$

where W denotes the *Weyl tensor* and r the scalar curvature. It is well known that an Einstein manifold is conformally flat if and only if it is of constant sectional curvature.

With respect to the basis $\{e_1, e_2, e_3, e_4\}$, the local symmetry condition $\nabla R = 0$ is equivalent to the system of algebraic equations:

$$\begin{aligned} \nabla_s R_{ijkh} &= -R(\nabla_{e_s} e_i, e_j, e_k, e_h) - R(e_i, \nabla_{e_s} e_j, e_k, e_h) - R(e_i, e_j, \nabla_{e_s} e_k, e_h) \quad (3.4) \\ &\quad - R(e_i, e_j, e_k, \nabla_{e_s} e_h) = 0, \quad \text{for all indices } s, i, j, k, h = 1, \dots, 4. \end{aligned}$$

We checked the above curvature conditions for cases **1)**-**22)** listed in Theorem 3.1, proving the following result.

Theorem 3.2. *Consider four-dimensional neutral Einstein Lie groups, as classified in Theorem 3.1 up to isometries. Then, the locally symmetric, conformally flat, flat and Ricci-flat examples are listed in the following Table I, where the checkmark “✓” (respectively, “✗”) means that the corresponding condition holds for all Lie algebras of that form (respectively, never holds for Lie algebras of that form).*

(G, g)	Locally symmetric	Constant curvature	Flat	Ricci-flat
1)	✓	✓	✗	✗
2)	✓	✗	✗	✗
3)	✓	✓	$A = 0$	$A = 0$
4)	$B = \pm A$	$B = \pm A$	$B = -A$	$B = -A$
5)	✗	✗	✗	✗
6)	$B = C = 0$ or $A = B \pm C = 0$	$B = C = 0$ or $A = B \pm C = 0$	$B = C = 0$ or $A = B \pm C = 0$	✓
7)	✓	✓	$A = 0$	$A = 0$
8)	✗	✗	✗	✗
9)	$B = \pm A$	$B = \pm A$	$B = -A$	$B = -A$
10)	✗	✗	✗	$B = \pm A$
11)	$E = -\frac{(C-B)D+(B-C)^2}{A}$	$E = -\frac{(C-B)D+(B-C)^2}{A}$ or $B - C - D = E = 0$ or $B - C = E = 0$	$E = -\frac{(C-B)D+(B-C)^2}{A}$ or $B - C - D = E = 0$ or $B - C = E = 0$	✓
12)	$E = \frac{BC-CD-D^2}{A}$	$E = \frac{BC-CD-D^2}{A}$	$E = \frac{BC-CD-D^2}{A}$	✓
13)	$D = \frac{B^2}{A}$	$D = \frac{B^2}{A}$	$D = \frac{B^2}{A}$	✓
14)	✓	✓	✓	✓
15)	$A = C + D$ or $A = \pm 2B$ or $(A, B) = (-C - D, \pm D)$ or $(A, B) = \frac{1}{2}(C + D, \pm(C - 2D))$	$A = C + D$ or $A = \pm 2B$	$A = C + D$ or $A = \pm 2B$	✓
16)	$\varepsilon = 1, A = 0$ or $\varepsilon = -1, B = 0$ or $\varepsilon = -1, (A, B) = (0, \pm C)$ or $\varepsilon = 1, (B, C) = (2A, \pm A)$	$\varepsilon = 1, A = 0,$ or $\varepsilon = -1, B = 0$	$\varepsilon = 1, A = 0$ or $\varepsilon = -1, B = 0$	✓
17)	✓	✓	$B = \pm A$	$B = \pm A$
18)	$D = \frac{2B(C-\varepsilon B-E)}{A}$	$D = \frac{2B(C-\varepsilon B-E)}{A}$	$D = \frac{2B(C-\varepsilon B-E)}{A}$	✓
19)	✓	✗	✗	✗
20)	$(D, E) = (B, A)$ or $(D, E) = (-B, -\frac{B^2}{A})$ or $A = B = D = 0$ or $A = \frac{\varepsilon}{2}(3B + D)$ and $E = \frac{\varepsilon}{2}(3D + B)$	$(D, E) = (B, A)$ or $(D, E) = (-B, -\frac{B^2}{A})$ or $A = B = D = 0$ or $A = \frac{\varepsilon}{2}(3B + D)$ and $E = \frac{\varepsilon}{2}(3D + B)$	$(D, E) = (B, A)$ or $(D, E) = (-B, -\frac{B^2}{A})$ or $A = B = D = 0$ or $A = \frac{\varepsilon}{2}(3B + D)$ and $E = \frac{\varepsilon}{2}(3D + B)$	✓
21)	$C = -A$	$C = -A$	$C = -A$	✓
22)	$A = 0$ or $E = A + \varepsilon B$ or $E = 2(A + \varepsilon B)$	$A = 0$ or $E = 2(A + \varepsilon B)$	$A = 0$ or $E = 2(A + \varepsilon B)$	✓

Table I: Geometry of 4D Einstein neutral Lie groups

Four-dimensional simply connected homogeneous Riemannian Einstein manifolds are symmetric [11]. However, the results listed in the above Table I show the existence of left-invariant Einstein neutral metrics on four-dimensional Lie groups, which are not even locally symmetric.

4. Ricci-parallel examples

By an argument similar to the one used to prove Theorem 3.1, we obtain the classification of Ricci-parallel left-invariant neutral metrics on four-dimensional Lie groups, proving the following result.

Theorem 4.1. *Let G be a four-dimensional simply connected Lie group. If g is a left-invariant neutral Ricci-parallel (not Einstein) metric on G , then the Lie algebra \mathfrak{g} of G is isometric to $\mathfrak{g} = \mathfrak{r} \times \mathfrak{g}_3$, where $\mathfrak{g}_3 = \text{Span}\{e_1, e_2, e_3\}$ and*

$\mathfrak{r} = \text{Span}\{e_4\}$, one of the following cases occurs:

a) $\{e_1, e_2, e_3, e_4\}$ is a pseudo-orthonormal basis, with e_3 time-like. In this case, G is isometric to one of the following semi-direct products $\mathbb{R} \ltimes G_3$:

a1) $\mathbb{R} \ltimes H$, where \mathfrak{g} is described by one of the following sets of conditions:

- 1) $[e_1, e_2] = Ae_1, [e_1, e_3] = Be_1, [e_1, e_4] = Ce_1, (A^2 + B^2 \neq 0),$
- 2) $[e_1, e_2] = Ae_1, [e_1, e_3] = \varepsilon Ae_1, [e_2, e_4] = B(\varepsilon e_2 - e_3),$
 $[e_3, e_4] = B(e_2 - \varepsilon e_3), (A \neq 0),$
- 3) $[e_1, e_2] = \frac{BC}{A}e_1, [e_1, e_3] = Be_1, [e_2, e_4] = Ae_2 - Ce_3,$
 $[e_3, e_4] = Ce_2 - \frac{C^2}{A}e_3, (B \pm \sqrt{A^2 - C^2} \neq 0, (B \pm A)^2 + C^2 \neq 0, B \neq 0),$
- 4) $[e_1, e_2] = Ae_1 + Be_2, [e_3, e_4] = Ce_3, (A^2 + B^2 \neq 0),$

a2) $\mathbb{R} \ltimes \mathbb{R}^3$ and \mathfrak{g} is described by one of the following sets of conditions:

- 5) $[e_2, e_4] = Ae_2 + Be_3, [e_3, e_4] = Be_2 + Ae_3, (A \neq 0),$
- 6) $[e_1, e_4] = Ae_1 - Be_2, [e_2, e_4] = Be_1 + Ae_2, (A \neq 0),$
- 7) $[e_1, e_4] = \frac{A^2 - B^2}{A}e_1, [e_2, e_4] = -\frac{B^2}{A}e_2 - Be_3, [e_3, e_4] = Be_2 + Ae_3, (B \neq \pm A),$
- 8) $[e_1, e_4] = Ae_1, [e_2, e_4] = Be_3, [e_3, e_4] = Be_2, (A \neq 0),$
- 9) $[e_1, e_4] = \frac{B^2}{A}e_1 + Be_2, [e_2, e_4] = Be_1 + Ae_2, [e_3, e_4] = \frac{B^2 + A^2}{A}e_3,$
- 10) $[e_1, e_4] = A(\varepsilon e_2 + e_3), [e_2, e_4] = B(\varepsilon e_2 + e_3), [e_3, e_4] = C(\varepsilon e_2 + e_3),$
 $(A^2 + (B \pm C)^2 \neq 0),$
- 11) $[e_1, e_4] = \frac{B^2}{4A}e_1 + B(\varepsilon e_2 + e_3), [e_2, e_4] = Ae_2 - \frac{\varepsilon(B^2 - 4A^2)}{4A}e_3,$
 $[e_3, e_4] = -\frac{\varepsilon(B^2 + 4A^2)}{4A}e_2 - Ae_3, (B \neq 0),$
- 12) $[e_1, e_4] = Ae_2 + Be_3, [e_2, e_4] = \frac{\varepsilon BA}{\sqrt{A^2 - B^2}}e_2 + \frac{\varepsilon B^2}{\sqrt{A^2 - B^2}}e_3,$
 $[e_3, e_4] = -\frac{\varepsilon A^2}{\sqrt{A^2 - B^2}}e_2 - \frac{\varepsilon BA}{\sqrt{A^2 - B^2}}e_3,$
- 13) $[e_1, e_4] = -Ae_2, [e_2, e_4] = Ae_1, [e_3, e_4] = Be_3, (B \neq 0),$

b) $\{e_1, e_2, e_3, e_4\}$ is a basis, with the inner product g on \mathfrak{g} completely determined by $g(e_1, e_1) = -g(e_2, e_2) = g(e_3, e_4) = g(e_4, e_3) = 1$ and $g(e_i, e_j) = 0$ otherwise. In this case, G is isometric to one of the following semi-direct products $\mathbb{R} \ltimes G_3$:

b1) $\mathbb{R} \ltimes H$ and \mathfrak{g} is described by one of the following sets of conditions:

- 14) $[e_1, e_2] = Ae_3, [e_1, e_3] = Be_3, [e_1, e_4] = C(e_1 - \varepsilon e_2) - \varepsilon \frac{AE - BD}{B}e_3,$
 $[e_2, e_3] = \varepsilon Be_3, [e_2, e_4] = C(\varepsilon e_1 - e_2) + De_3, [e_3, e_4] = Ee_3,$

- 15) $[e_1, e_2] = Ae_3, [e_1, e_3] = Be_3,$
 $[e_1, e_4] = C(e_1 - \varepsilon e_2) + \frac{\varepsilon(AC+BE)+A^2-AD}{B}e_3, [e_2, e_3] = \varepsilon Be_3,$
 $[e_2, e_4] = D(e_1 - \varepsilon e_2) + Ee_3, [e_3, e_4] = -\varepsilon Ae_3,$
- 16) $[e_1, e_2] = A(\varepsilon e_1 + e_2) + Be_3, [e_1, e_4] = -\varepsilon Ce_3, [e_2, e_4] = Ce_3, (B \neq 0),$
- 17) $[e_1, e_2] = A(e_1 - e_2) + Be_3, [e_1, e_4] = C(e_1 - e_2) + \frac{BC+AD-B^2}{A}e_3,$
 $[e_2, e_4] = B(e_1 - e_2) + De_3, (C \neq 2B),$
- 18) $[e_1, e_2] = Ae_1 + Be_2 - \frac{CA+BD}{A}e_3,$
 $[e_1, e_4] = De_1 + \frac{BD}{A}e_2 - \frac{AC(D-E)+BD^2}{A^2}e_3,$
 $[e_2, e_4] = Ce_1 + \frac{CB}{A}e_2 - \frac{AC^2+BCD-ADE}{A^2}e_3, [e_3, e_4] = Ee_3, (A \neq \pm B),$
- 19) $[e_1, e_2] = Ae_3, [e_1, e_4] = -Be_1 + Ce_2 + De_3, [e_2, e_4] = Ee_1 + Be_2 + Fe_3,$
 $(A \neq 0),$
- 20) $[e_1, e_2] = \frac{BC}{A}e_3, [e_1, e_3] = Ce_3, [e_1, e_4] = De_3, [e_2, e_4] = Be_3,$
 $[e_3, e_4] = Ae_3, (C \neq 0),$
- 21) $[e_1, e_2] = Ae_3, [e_1, e_3] = Be_3, [e_1, e_4] = Ce_3, (A^2 + B^2 \neq 0),$
- 22) $[e_1, e_2] = Ae_2 - Be_3, [e_1, e_4] = Be_2 - \frac{B^2-CD}{A}e_3, [e_2, e_4] = Ce_2 - \frac{B(C-D)}{A}e_3,$
 $[e_3, e_4] = De_3,$

b2) $\mathbb{R} \times \mathbb{R}^3$, where \mathfrak{g} is described by one of the following sets of conditions:

- 23) $[e_1, e_4] = Ae_1 + Be_2 + Ce_3, [e_2, e_4] = De_1 + Ee_2 + Fe_3,$
 $(A^2 + E^2 + (B - D)^2 \neq 0),$
- 24) $[e_1, e_4] = A(e_1 + \varepsilon e_2) + \frac{AD+\varepsilon(D^2-BC)}{B}e_3, [e_2, e_4] = D(e_1 + \varepsilon e_2) + Ce_3,$
 $[e_3, e_4] = B(e_1 + \varepsilon e_2) + De_3.$

In the cases listed above, $\varepsilon = \pm 1$.

Proof. Equations (3.1) and (3.2), and the description of the components of the Ricci tensor listed in cases (a) and (b) of the proof of Theorem 3.1, are valid for any four-dimensional semi-direct product Lie algebra $\mathfrak{g} = \mathfrak{r} \times \mathfrak{g}_3$ equipped with a neutral inner product. We now apply to them the equations expressing the fact that the left-invariant metric g is Ricci-parallel. With respect to a basis $\{e_1, e_2, e_3, e_4\}$, with $\mathfrak{r} = \text{Span}(e_4)$ and $\mathfrak{g}_3 = \text{Span}(e_1, e_2, e_3)$, the Ricci-parallel condition $\nabla \varrho = 0$ translates as

$$\nabla_i \varrho_{jk} = -\varrho(\nabla_{e_i} e_j, e_k) - \varrho(e_j, \nabla_{e_i} e_k) = 0, \quad \text{for all indices } i, j, k = 1, \dots, 4. \quad (4.5)$$

Proceeding as in the proof of Theorem 3.1, we treat separately the cases (a) and (b) and find the solutions of (3.2) and (4.5). We exclude the Einstein cases and keep only the generally non-isomorphic examples, obtaining the cases 1)-24) listed above. ■

A *Walker manifold* is a pseudo-Riemannian manifold (M, g) , admitting a null (that is, degenerate) parallel distribution. Walker structures do not exist in Riemannian settings and often provide an interesting source of examples in pseudo-Riemannian geometry. We may refer to [3] for a recent survey on Walker geometry.

With regard to Ricci-parallel neutral metrics on four-dimensional Lie groups, we proved the following.

Theorem 4.2. *Among four-dimensional Ricci-parallel neutral Lie groups, as classified in Theorem 4.1 up to isometries, the left-invariant Walker examples occur in the cases listed in the following Table II, where the checkmark “✓” means that all examples of that form admit a parallel degenerate line field, and the checkmark “✗” means that do not exist any left-invariant null parallel plane fields. In all the different cases, we listed the generators of parallel degenerate line and plane fields.*

case	Walker	line field generator	plane field generators
1)	✓	$\sqrt{B^2 + C^2 - A^2}e_1 - Ae_2 + Be_3 + Ce_4$	$e_1 + e_4, e_2 - e_3$ for $A = B$
2)	✓	$e_2 - \varepsilon e_3$	$e_2 - \varepsilon e_3, e_1 - e_4$
3)	✓	$\sqrt{A^2 - C^2}e_1 + Ce_2 - Ae_3$	✗
8)	✓	$e_1 + e_4$	$e_1 + e_4, e_2 - e_3$
10)	$B = -\varepsilon C$	$e_2 + \varepsilon e_3$	$e_2 + \varepsilon e_3, e_1 - e_4$
11)	✓	$e_2 + \varepsilon e_3$	$e_2 + \varepsilon e_3, e_1 + e_4$
12)	✓	$\varepsilon\sqrt{A^2 - B^2}e_1 + Be_2 + Ae_3$	✗
14)	✓	$e_1 - \varepsilon e_2 + \frac{A\varepsilon}{B}e_3$	$e_1 - \varepsilon e_2, e_3$
15)	✓	$e_1 - \varepsilon e_2 + \frac{(A-D)\varepsilon+C}{B}e_3$	$e_1 - \varepsilon e_2, e_3$
16)	✓	e_3	$e_1 - \varepsilon e_2, e_3$
17)	✓	e_3	$e_1 - e_2, e_3$
18)	✓	e_3	$e_1 + e_2, e_3$
19)	✓	e_3	$e_1 + e_2, e_3$
21)	$B = 0$	e_3	$e_1 + e_2, e_3$
22)	✓	e_3	$e_1 - e_2, e_3$
23)	✓	e_3	$e_1 + e_2, e_3$
24)	✓	$e_1 + \varepsilon e_2 - \frac{A+\varepsilon D}{B}e_3$	$e_1 + \varepsilon e_2, e_3$

Table II: Left-invariant Walker structures on 4D Ricci-parallel neutral Lie groups

Proof. We prove this theorem considering case by case the Ricci parallel examples we listed in Theorem 4.1. We report the full details of the case 1), which admits a left-invariant null parallel line (and also a null parallel plane field when $A = B$), and of the case 4), which does not admit any left-invariant Walker structure. The other cases are obtained by similar calculations.

For the case 1), by using the *Koszul formula* and setting $\Lambda_i = \nabla_{e_i}$, the only possibly non-vanishing components of the Levi-Civita connection are given by

$$\Lambda_1 = \begin{pmatrix} 0 & A & B & C \\ -A & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ C & 0 & 0 & 0 \end{pmatrix}.$$

Now, if set $v = \sqrt{B^2 + C^2 - A^2}e_1 - Ae_2 + Be_3 + Ce_4$, then we get $g(v, v) = 0$ and $\nabla_{e_1}v = \sqrt{B^2 + C^2 - A^2}v$ (while $\nabla_{e_i}v = 0$ for $i = 2, 3, 4$). Thus, $\mathcal{D} = \text{Span}\{v\}$ is a left-invariant null parallel line field. Moreover, if $A \neq B$, then the above description of Λ_1 yields that there are not null parallel plane fields. On the other hand, if $A = B$, setting $u = e_1 + e_4$ and $v = e_2 - e_3$ we find $g(u, u) = g(v, v) = g(u, v) = 0$ and the only non-zero covariant derivative of u, v is $\nabla_{e_1}u = Cu - Av$. So, $\bar{\mathcal{D}} = \text{Span}\{u, v\}$ is a left-invariant parallel degenerate plane field.

For the case 4), the non-zero components of the Levi-Civita connection are

$$\Lambda_1 = \begin{pmatrix} 0 & A & 0 & 0 \\ -A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} 0 & B & 0 & 0 \\ -B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Lambda_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & -C & 0 \end{pmatrix}.$$

Let $\mathcal{D} = \text{Span}\{v\}$ denote a left-invariant null parallel line field, where $v = \sum_{i=1}^4 v_i e_i$. Thus, $v_1^2 + v_2^2 - v_3^2 - v_4^2 = 0$ and $\nabla_{e_i}v = c_i v$, $i = 1, \dots, 4$, where $\{c_i\}_{i=1}^4$ are real constants. This leads to the following system of algebraic equations:

$$\begin{aligned} v_1^2 + v_2^2 - v_3^2 - v_4^2 &= 0, \\ c_1 v_3 &= 0, \quad c_1 v_4 = 0, \quad c_1 v_1 - Av_2 = 0, \quad c_1 v_2 + Av_1 = 0, \\ c_2 v_3 &= 0, \quad c_2 v_4 = 0, \quad c_2 v_1 - Bv_2 = 0, \quad c_2 v_2 + Bv_1 = 0, \\ c_3 v_1 &= 0, \quad c_3 v_2 = 0, \quad c_3 v_3 - Cv_4 = 0, \quad c_3 v_4 + Cv_3 = 0, \\ c_4 v_1 &= 0, \quad c_4 v_2 = 0, \quad c_4 v_3 = 0, \quad c_4 v_4 = 0, \end{aligned}$$

which is satisfied only when $v = 0$. So, no left-invariant null parallel line fields exist for this case.

By a similar argument, we set $u = \sum_{i=1}^4 u_i e_i$ and $v = \sum_{i=1}^4 v_i e_i$ and suppose that $\bar{\mathcal{D}} = \text{Span}\{u, v\}$ is a left-invariant null parallel plane field. Since $\bar{\mathcal{D}}$ is a null plane field we must have $g(u, u) = g(v, v) = g(u, v) = 0$, that is,

$$\begin{cases} u_1^2 + u_2^2 - u_3^2 - u_4^2 = 0, \\ v_1^2 + v_2^2 - v_3^2 - v_4^2 = 0, \\ v_1 u_1 + v_2 u_2 - v_3 u_3 - v_4 u_4 = 0. \end{cases} \tag{4.6}$$

On the other hand, since $\bar{\mathcal{D}}$ is parallel, we get

$$\begin{aligned} \nabla_{e_1}u &= a_1 u + b_1 v, & \nabla_{e_2}u &= a_2 u + b_2 v, & \nabla_{e_3}u &= a_3 u + b_3 v, & \nabla_{e_4}u &= a_4 u + b_4 v, \\ \nabla_{e_1}v &= c_1 u + d_1 v, & \nabla_{e_2}v &= c_2 u + d_2 v, & \nabla_{e_3}v &= c_3 u + d_3 v, & \nabla_{e_4}v &= c_4 u + d_4 v, \end{aligned}$$

for some real constants $\{a_i, b_i, c_i, d_i\}_{i=1}^4$. By (4.6), the last equations are equivalent to satisfying the following algebraic system of equations:

$$\begin{aligned}
 b_1v_3 + a_1u_3 = 0, & & b_1v_4 + a_1u_4 = 0, & & b_1v_1 + a_1u_1 - Au_2 = 0, \\
 b_1v_2 + a_1u_2 + Au_1 = 0, & & b_2v_3 + a_2u_3 = 0, & & b_2v_4 + a_2u_4 = 0, \\
 b_2v_1 + a_2u_1 - Bu_2 = 0, & & b_2v_2 + a_2u_2 + Bu_1 = 0, & & b_3v_1 + a_3u_1 = 0, \\
 b_3v_2 + a_3u_2 = 0, & & b_3v_3 + a_3u_3 - Cu_4 = 0, & & b_3v_4 + a_3u_4 + Cu_3 = 0, \\
 b_4v_1 + a_4u_1 = 0, & & b_4v_2 + a_4u_2 = 0, & & b_4v_3 + a_4u_3 = 0, \\
 b_4v_4 + a_4u_4 = 0, & & d_1v_3 + c_1u_3 = 0, & & d_1v_4 + c_1u_4 = 0, \\
 d_1v_1 + c_1u_1 - Av_2 = 0, & & d_1v_2 + c_1u_2 + Av_1 = 0, & & d_2v_3 + c_2u_3 = 0, \\
 d_2v_4 + c_2u_4 = 0, & & d_2v_1 + c_2u_1 - Bv_2 = 0, & & d_2v_2 + c_2u_2 + Bv_1 = 0, \\
 d_3v_1 + c_3u_1 = 0, & & d_3v_2 + c_3u_2 = 0, & & d_3v_3 + c_3u_3 - Cv_4 = 0, \\
 d_3v_4 + c_3u_4 + Cv_3 = 0, & & d_4v_1 + c_4u_1 = 0, & & d_4v_2 + c_4u_2 = 0, \\
 d_4v_3 + c_4u_3 = 0, & & d_4v_4 + c_4u_4 = 0. & &
 \end{aligned} \tag{4.7}$$

We then solve the algebraic systems (4.6) and (4.7) together. As $A^2 + B^2 \neq 0$, these systems do not admit any solution. Thus, no left-invariant null parallel plane field exist in this case. ■

In the Riemannian case, any four-dimensional Ricci-parallel homogeneous manifold is locally symmetric. On the other hand, the existence of Walker structures is also related to the existence of Ricci-parallel examples which are not locally symmetric. With regard to local symmetry condition (3.4), we proved the following.

Theorem 4.3. *Among four-dimensional Ricci-parallel neutral Lie groups, as classified in Theorem 4.1 up to isometries, the not locally symmetric examples are the ones listed in Table III below, where the checkmark “✓” means that no examples of that form are locally symmetric.*

(G, g)	conditions for being not locally symmetric
11)	✓
12)	✓
14)	$C(E + \varepsilon A) \neq 0$
16)	$AC \neq 0$
17)	$AD \neq B^2$
19)	$A \neq C + E$ and $(B, E) \neq (0, C)$
23)	$B \neq -D, (B, E) \neq (D, A)$ and $(A, E) \neq \pm(B, D)$
24)	$BC \neq D^2$ and $(C, D) \neq (0, 0)$

Table III: Ricci-parallel, not locally symmetric 4D neutral Lie groups

Next, in order to determine conformally flat Ricci-parallel neutral Lie groups, we check when the Weyl tensor vanishes and we prove the following.

Theorem 4.4. *Among four-dimensional Ricci-parallel neutral Lie groups, as classified in Theorem 4.1 up to isometries, the conformally flat examples are the ones listed in the following Table IV, where the checkmark “✓” means that all examples of that form are conformally flat.*

(G, g)	condition for conformal flatness
3)	$B = \pm\sqrt{C^2 - A^2}$
4)	$A = \pm\sqrt{C^2 - B^2}$
5)	✓
6)	✓
7)	✓
9)	✓
10)	✓
14)	$C(E + \varepsilon A) = 0$
15)	✓
16)	$AC = 0$
17)	$AD = B^2$
19)	$A = C + E$ or $B = C - E = 0$
20)	✓
21)	✓
23)	$(D, E) = \pm(B, A)$ or $(B, D) = \pm(A, E)$
24)	$BC = D^2$

Table IV: Ricci-parallel conformally flat 4D neutral Lie groups

We now report some basic information on Ricci solitons, referring to [10] for a survey and further references. A *Ricci soliton* is a pseudo-Riemannian manifold (M, g) admitting a smooth vector field V , such that

$$\mathcal{L}_V g + \varrho = \lambda g, \quad (4.8)$$

where \mathcal{L} denotes the Lie derivative and λ a real constant. A Ricci soliton is said to be *shrinking*, *steady* or *expanding* depending on whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Ricci solitons are the self-similar solutions of the *Ricci flow*.

Originally introduced in the Riemannian case, Ricci solitons have been intensively studied in pseudo-Riemannian settings in recent years. The Ricci soliton equation is also a special case of Einstein field equations.

An *invariant Ricci soliton* is a homogeneous space $(M = G/H, g)$, such that equation (4.8) holds for some invariant vector field V . The natural problem to determine which homogeneous manifolds G/H admit invariant Ricci soliton shows different behaviours depending on the signature of the metric: while do not exist left-invariant Ricci solitons on three-dimensional Lie groups, there exist three-dimensional left-invariant Lorentzian Ricci solitons [4]. Invariant Ricci solitons on four-dimensional homogeneous pseudo-Riemannian manifolds with nontrivial isotropy were recently investigated in [5],[6],[9]. Checking equation (4.8) for the Ricci-parallel examples listed in Theorem 4.1, we found several examples of left-invariant Ricci solitons, proving the following result.

Theorem 4.5. *Among four-dimensional Ricci-parallel neutral Lie groups, as classified in Theorem 4.1 up to isometries, the left-invariant Ricci solitons are the following:*

2) always, with $X = \varepsilon k e_2 - k e_3 + \varepsilon \frac{A^2}{2B} e_4$ and $\lambda = 0$.

3) when $A = \varepsilon C$, with $X = \varepsilon k e_2 - k e_3 + \varepsilon \frac{B^2}{2C} e_4$ and $\lambda = 0$.

14) when $E = -\varepsilon A$, with

$$X = \frac{-\varepsilon(4kC+AB)}{4C} e_1 + k e_2 - \frac{B(A^2+\varepsilon(BD-AC))+4kAC}{4BC} e_3 + \frac{B^2}{4C} e_4 \text{ and } \lambda = 0.$$

15) when $E = \frac{\varepsilon(C^2-(A-D)^2)}{2B}$, with $X = \frac{B(C-\varepsilon D)-2k(D+\varepsilon C)+\varepsilon AB}{2(C+\varepsilon D)} e_1 - k e_2$
 $+ \frac{\varepsilon((2BC-4kD)(A-D)-4C^2k)+B(A^2+C^2+D^2)-4kAC-2ABD}{4B(C+\varepsilon D)} e_3 - \frac{B^2}{2(C+\varepsilon D)} e_4$ and $\lambda = \frac{B^2(C-\varepsilon D)}{2(C+\varepsilon D)}$.

18) always, with $X = \frac{A^2-B^2}{AE} \left(-C e_1 + D e_2 + \frac{C^2-D^2}{2A} e_3 - A e_4 \right)$ and $\lambda = A^2 - B^2$.

19) when $A = \pm\sqrt{(C - E)^2 - 4B^2}$, with $X = e_3$ and $\lambda = 0$.

22) always, with $X = \frac{1}{D} \left(A C e_1 - A B e_2 + \frac{B^2-C^2}{2} e_3 + A^2 e_4 \right)$ and $\lambda = -A^2$.

23) when $E = \frac{BD}{A}$, with

$$X = \frac{D(A^2(B-D)^2-2A^4-2D^2B^2)}{4A^2(AF-CD)} e_1 + \frac{2A^4-A^2(B-D)^2+2D^2B^2}{4A(AF-CD)} e_2 + k e_3 \text{ and } \lambda = 0.$$

In the cases listed above, k denotes an arbitrary real constant and $\varepsilon = \pm 1$.

In particular, comparison between Theorems 4.3 and 4.5 shows that cases **19)** and **23)** yield some Ricci-parallel, not locally symmetric Ricci solitons.

References

[1] Arias-Marco, T., and O. Kowalski, *Classification of 4-dimensional homogeneous D'Atri spaces*, Czechoslovak Math. J. **58** (2008), 203–239.

[2] Bérard-Bergery, L., “Homogeneous Riemannian spaces of dimension four,” Seminar A. Besse, *Four-dimensional Riemannian geometry*, 1985.

[3] Brozos-Vázquez, M., E. García-Río, P. Gilkey, S. Nikčević, and R. Vázquez-Lorenzo, “The geometry of Walker manifolds,” *Synthesis Lect. on Math. and Stat.*, Morgan and Claypool Publishers, 2009.

[4] Brozos-Vázquez, M., G. Calvaruso, E. García-Río, and S. Gavino-Fernández, *Three-dimensional Lorentzian homogeneous Ricci solitons*, Israel J. Math. **188** (2012), 385–403.

[5] Calvaruso, G., and A. Fino, *Ricci solitons and geometry of four-dimensional non-reductive homogeneous spaces*, Canad. J. Math. **64** (2012), 778–804.

[6] —, *Four-dimensional pseudo-Riemannian homogeneous Ricci solitons*, Int. J. Geom. Meth. Mod. Phys., to appear.

[7] Calvaruso, G., and A. Zaeim, *Four-dimensional Lorentzian Lie groups*, Diff. Geom. Appl. **31** (2013), 496–509.

[8] —, *Conformally flat homogeneous pseudo-Riemannian four-manifolds*, Tôhoku Math. J. **66** (2014), 31–54.

- [9] —, *A complete classification of Ricci and Yamabe solitons of non-reductive homogeneous 4-spaces*, J. Geom. Phys. **80** (2014), 15–25.
- [10] Cao, H.-D., *Recent progress on Ricci solitons*, Recent Advances in Geometric Analysis, 1–38, in: Adv. Lect. Math. (ALM) **11**, Int. Press, Somerville, MA, 2010.
- [11] Jensen, G. R., *Homogeneous Einstein spaces of dimension four*, J. Diff. Geom. **3** (1969), 309–349.
- [12] Komrakov Jr., B., *Einstein-Maxwell equation on four-dimensional homogeneous spaces*, Lobachevskii J. Math. **8** (2001), 33–165.
- [13] Milnor, J., *Curvatures of left-invariant metrics on Lie groups*, Adv. Math. **21** (1976), 293–329.
- [14] O’Neill, B., “Semi-Riemannian Geometry,” Academic Press, New York, 1983.
- [15] Rahmani, S., *Métriques de Lorentz sur les groupes de Lie unimodulaires de dimension trois*, J. Geom. Phys. **9** (1992), 295–302.
- [16] Zaeim, A., *Einstein-like four-dimensional Lorentzian Lie groups*, submitted.

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