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The Torus-Equivariant Cohomology of Nilpotent Orbits

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Abstract. We consider aspects of the geometry and topology of nilpotent orbits in finite-dimensional complex simple Lie algebras. In particular, we give the equivariant cohomologies of the regular and minimal nilpotent orbits with respect to the action of a maximal compact torus of the overall group in question.
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1. Introduction

1.1. Generalities. Throughout, we let G be a connected, simply-connected complex simple linear algebraic group. Let $K \subseteq G$ be a maximal compact subgroup, and fix a maximal torus $T \subseteq K$. Set $H := T_{\mathbb{C}}$ and let $X^*(T)$ denote the weight lattice $\operatorname{Hom}(T, U(1)) = \operatorname{Hom}(H, \mathbb{C}^*)$. Denote by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{t}$, and $\mathfrak{h}$ the Lie algebras of G, K, T, and H, respectively. Let $W = N_K(T)/T = N_G(H)/H$ be the Weyl group. Also, let $\operatorname{Ad} : G \to \operatorname{GL}(\mathfrak{g})$ and $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ denote the adjoint representations of G and $\mathfrak{g}$, respectively. Let $\Delta \subseteq X^*(T)$ denote the set of roots. By fixing a Borel subgroup $B \subseteq G$ containing H, we specify collections $\Delta_+, \Delta_- \subseteq \Delta$ of positive and negative roots, respectively. Let $\Pi \subseteq \Delta_+$ denote the resulting collection of simple roots. Given a subset Λ of Π, we let P_Λ denote the corresponding standard parabolic subgroup of G, and we let W_Λ be the subgroup of W generated by the reflections $\{s_\beta : \beta \in \Lambda\}$.

Recall that a point $\xi \in \mathfrak{g}$ is called *nilpotent* if the vector space endomorphism $\operatorname{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. Recall also that the nilpotent cone is the closed subvariety $\mathcal{N}$ of $\mathfrak{g}$ consisting of the nilpotent elements. We call an adjoint G-orbit a *nilpotent orbit* if it is contained in $\mathcal{N}$. As an orbit of an algebraic G-action, any nilpotent orbit is a smooth locally closed subvariety of $\mathfrak{g}$.

It is well-known that there exist only finitely many nilpotent orbits of G. Indeed, if $G = \operatorname{SL}_n(\mathbb{C})$, then one can use Jordan canonical forms to give an explicit indexing of the nilpotent orbits by the partitions of n.

Furthermore, the nilpotent orbits constitute an algebraic stratification of $\mathcal{N}$ (see [6]). In other words, we have the partial order on the set of nilpotent

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orbits given by $\mathcal{O}_1 \leq \mathcal{O}_2$ if and only if $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$ (the Zariski-closure of $\mathcal{O}_2$ in $\mathcal{N}$). Hence,

$$\overline{\mathcal{O}} = \bigcup_{\Theta \leq \mathcal{O}} \Theta$$

for all nilpotent orbits $\mathcal{O}$.

It turns out that the set of nilpotent orbits has a unique maximal element, $\mathcal{O}_{\mathrm{reg}}$, and a unique minimal non-zero element, $\mathcal{O}_{\mathrm{min}}$. These orbits are called the *regular* and *minimal* nilpotent orbits, respectively. The former consists precisely of the regular nilpotent elements of $\mathfrak{g}$, while the latter is the orbit of a root vector for a long root.

1.2. Context. The study of nilpotent orbits lies at the interface of algebraic geometry, representation theory, and symplectic geometry. Indeed, one has the famous Springer resolution

$$\mu : T^*(G/B) \to \mathcal{N}$$

of the singular nilpotent cone (see [6]). The fibres of μ over a given nilpotent orbit $\mathcal{O}$ are isomorphic as complex varieties, and this isomorphism class is called the Springer fibre of $\mathcal{O}$. The Springer correspondence then gives a realization of the irreducible complex W-representations on the Borel-Moore homology groups of the Springer fibres (see [6]).

From the symplectic standpoint, we note that coadjoint G-orbits are canonically complex symplectic manifolds. Since the Killing form on $\mathfrak{g}$ provides an isomorphism between the adjoint and coadjoint representations of G, it follows that adjoint G-orbits (and in particular, nilpotent G-orbits) are naturally complex symplectic manifolds.

Some attention has also been given to the matter of computing topological invariants of nilpotent orbits. Indeed, the work of Springer, Steinberg, and others has led to a computation of the fundamental group of every nilpotent orbit in the classical Lie algebras (see [7]). Also, Juteau's paper [11] gives the integral cohomology groups of the minimal nilpotent orbit in each of the finite-dimensional complex simple Lie algebras. Additionally, Biswas and Chatterjee compute $H^2(\mathcal{O};\mathbb{R})$ for $\mathcal{O}$ any nilpotent orbit in a finite-dimensional complex simple Lie algebra (see their paper [2]).

Our contribution is a computation of the T-equivariant cohomology algebras of the G-orbits $\mathcal{O}_{\mathrm{reg}}$ and $\mathcal{O}_{\mathrm{min}}$. The following is our main result.

Theorem 1.1. *(i)* $H_T^*(\mathcal{O}_{reg};\mathbb{Q}) \cong H^*(G/B;\mathbb{Q})$

(ii) Let $\alpha \in \Delta_+$ *be the highest root, and let* $\Xi := \{\beta \in \Pi : \langle \alpha, \beta \rangle = 0\}$. *Then,* $H_T^*(\mathcal{O}_{min};\mathbb{Q})$ *is isomorphic to the quotient of*

$$\{f \in \mathrm{Map}(W/W_\Xi, \mathrm{Sym}(X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q})) : (w \cdot \beta) | (f([w]) - f([ws_\beta])) \\ \forall w \in W, \beta \in \Delta_-, \ \langle \alpha, \beta \rangle \neq 0\}$$

by the ideal generated by the map $W/W_\Xi \to \mathrm{Sym}(X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q})$, $[w] \mapsto w \cdot \alpha$.

1.3. Structure of the Article. We begin with Section 2, which briefly addresses some of our conventions concerning equivariant cohomology. Section 3 then provides a direct computation of the T-equivariant cohomology of $\mathcal{O}_{\text{reg}}$.

Section 4 treats the case of the minimal nilpotent orbit, but the approach differs considerably from that adopted when studying $\mathcal{O}_{\text{reg}}$. We begin by considering a natural $\mathbb{C}^*$-action on nilpotent orbits. Via this action, we introduce $\mathbb{P}(\mathcal{O}_{\text{min}})$, a smooth closed subvariety of $\mathbb{P}(\mathfrak{g})$. This variety has interesting properties beyond those materially relevant to computing $H_T^*(\mathcal{O}_{\text{min}};\mathbb{Q})$. In particular, $\mathbb{P}(\mathcal{O}_{\text{min}})$ is naturally a symplectic manifold, and the T-action on $\mathcal{O}_{\text{min}}$ descends to a Hamiltonian action on $\mathbb{P}(\mathcal{O}_{\text{min}})$. Accordingly, we give an explicit description of $\mathbb{P}(\mathcal{O}_{\text{min}})^T$ and use it to find the moment polytope of $\mathbb{P}(\mathcal{O}_{\text{min}})$.

We next recall a description of $H_T^*(G/P;\mathbb{Q})$ arising from GKM theory, where $P\subseteq G$ is a parabolic subgroup containing B. This is done in recognition of the fact that $\mathbb{P}(\mathcal{O}_{\text{min}})$ is G-equivariantly isomorphic to G/P_{Ξ}.

It then remains to relate the graded algebras $H_T^*(G/P_{\Xi};\mathbb{Q})$ and $H_T^*(\mathcal{O}_{\text{min}};\mathbb{Q})$. This is achieved via the Thom-Gysin sequence in T-equivariant cohomology, which allows us to exhibit $H_T^*(\mathcal{O}_{\text{min}};\mathbb{Q})$ as a quotient of $H_T^*(G/P_{\Xi};\mathbb{Q})$. Indeed, we take the quotient of $H_T^*(G/P_{\Xi};\mathbb{Q})$ by the ideal generated by the T-equivariant first Chern class of the associated line bundle $G\times_{P_{\Xi}}\mathfrak{g}_{\alpha}\to G/P_{\Xi}$.

2. Equivariant Cohomology Conventions

Let us briefly establish some of the conventions underlying the equivariant cohomology computations in this article. Accordingly, suppose that $\mathcal{G}$ is any Lie group. Let $E\mathcal{G}\to B\mathcal{G}$ be the universal principal $\mathcal{G}$-bundle, characterized by the condition that $E\mathcal{G}$ is a contractible space on which $\mathcal{G}$ acts freely. Given a $\mathcal{G}$-manifold X, let $\mathcal{G}$ act diagonally on the product $E\mathcal{G}\times X$. The (rational) $\mathcal{G}$-equivariant cohomology of X is then defined to be the ordinary (rational) cohomology of the quotient $(E\mathcal{G}\times X)/\mathcal{G}$, namely

$$H_{\mathcal{G}}^*(X;\mathbb{Q}):=H^*((E\mathcal{G}\times X)/\mathcal{G};\mathbb{Q}).$$

In this article, let us understand all cohomology (both equivariant and ordinary) to be over $\mathbb{Q}$. Henceforth, $H^*(X)$ will denote the ordinary cohomology of any manifold X over $\mathbb{Q}$. If X happens to be a $\mathcal{G}$-manifold, then $H_{\mathcal{G}}^*(X)$ will denote its $\mathcal{G}$-equivariant cohomology over $\mathbb{Q}$.

While we will consider equivariant cohomology for $\mathcal{G}=G$ in Section 3, we will be principally interested in the case $\mathcal{G}=T$. Furthermore, it will be advantageous to recall a particular description of the T-equivariant cohomology of a point, $H_T^*(\text{pt})$. Given a weight $\beta\in X^*(T)$, let $\mathbb{C}_{\beta}$ denote the one-dimensional complex T-representation of weight β. We may regard $\mathbb{C}_{\beta}$ as a T-equivariant complex line bundle over a point, allowing us to consider its T-equivariant first Chern class $c_1^T(\mathbb{C}_{\beta})\in H_T^2(\text{pt})$. One then has a degree-doubling $\mathbb{Q}$-algebra isomorphism

$$\varphi:\text{Sym}(X^*(T)\otimes_{\mathbb{Z}}\mathbb{Q})\xrightarrow{\cong}H_T^*(\text{pt}),\tag{1}$$

defined by the property that $\varphi(\beta)=c_1^T(\mathbb{C}_{\beta})$ for all $\beta\in X^*(T)$. From this point forward, we will freely identify $H_T^*(\text{pt})$ with $\text{Sym}(X^*(T)\otimes_{\mathbb{Z}}\mathbb{Q})$ via (1).

3. The Regular Nilpotent Orbit

Throughout this section, we may actually take G to be semisimple. Now, recall that an element $\xi \in \mathfrak{g}$ is called *regular* if the dimension of the Lie algebra centralizer $C_{\mathfrak{g}}(\xi) = \{X \in \mathfrak{g} : [X, \xi] = 0\}$ coincides with the rank of $\mathfrak{g}$. The regular nilpotent elements of $\mathfrak{g}$ constitute $\mathcal{O}_{\text{reg}}$.

Fix a point $\eta \in \mathcal{O}_{\text{reg}}$, and let $C_G(\eta) = \{g \in G : \text{Ad}_g(\eta) = \eta\}$ be the G-centralizer of η. This gives an isomorphism $\mathcal{O}_{\text{reg}} \cong G/C_G(\eta)$ of complex G-varieties, where the action of $C_G(\eta)$ on G is given by $x : g \mapsto gx^{-1}$, $x \in C_G(\eta)$, $g \in G$. Having realized $\mathcal{O}_{\text{reg}}$ in this way, we turn our attention to $C_G(\eta)$. Indeed, note that $\mathcal{O}_{\text{reg}}$ is a *distinguished* nilpotent orbit (see [7]), meaning that the unipotent radical of $C_G(\eta)$ coincides with the identity component $C_G(\eta)^0$. It follows that

$$C_G(\eta) = C_G(\eta)^0 \times Z(G)$$

is the Levi decomposition of $C_G(\eta)$, where $Z(G)$ is the (finite) centre of G. Hence,

$$EG/Z(G) \to EG/C_G(\eta)$$

is an affine bundle, so that it induces an isomorphism on ordinary cohomology. The ordinary cohomology of $EG/C_G(\eta)$ is precisely $H_G^*(G/C_G(\eta)) = H_G^*(\mathcal{O}_{\text{reg}})$, while that of $EG/Z(G)$ is isomorphic to $H_{Z(G)}^*(\text{pt})$. Therefore,

$$H_G^*(\mathcal{O}_{\text{reg}}) \cong H_{Z(G)}^*(\text{pt}).$$

However, since $Z(G)$ is a finite group, $H_{Z(G)}^*(\text{pt})$ is isomorphic to $\mathbb{Q}$ (by which we mean the graded $\mathbb{Q}$-algebra equal to $\mathbb{Q}$ in degree zero and vanishing in all other grading degrees). We thus have

$$H_G^*(\mathcal{O}_{\text{reg}}) \cong \mathbb{Q}. \tag{2}$$

Now, recall that the T-equivariant cohomology of a smooth K-manifold is obtained by tensoring its K-equivariant cohomology with $H_T^*(\text{pt})$ over $H_T^*(\text{pt})^W$ (see Proposition 1(c) of [5]). In our case, this gives

$$H_T^*(\mathcal{O}_{\text{reg}}) \cong H_T^*(\text{pt}) \otimes_{H_T^*(\text{pt})^W} H_K^*(\mathcal{O}_{\text{reg}}). \tag{3}$$

By (2), we may replace $H_K^*(\mathcal{O}_{\text{reg}}) = H_G^*(\mathcal{O}_{\text{reg}})$ with $H_K^*(K)$ in (3). Applying Proposition 1(c) of [5] again, the right-hand side of (3) is seen to be isomorphic to $H_T^*(K) \cong H^*(K/T) \cong H^*(G/B)$. In other words,

$$H_T^*(\mathcal{O}_{\text{reg}}) \cong H^*(G/B)$$

as claimed in the statement of Theorem 1.1.

Remark 3.1. Without modification, our arguments establish the slightly more general fact that the T-equivariant cohomology of a distinguished nilpotent orbit coincides with the ordinary cohomology of the flag variety.

4. The Minimal Nilpotent Orbit

We now address the matter of computing $H_T^*(\mathcal{O}_{\min})$. Note that we could try to proceed in analogy with Section 3 by fixing $\eta \in \mathcal{O}_{\min}$, taking L to be the reductive part in a Levi decomposition of $C_G(\eta)$, and so forth. While this approach is certainly legitimate, we will compute the T-equivariant cohomology of $\mathcal{O}_{\min}$ by first determining that of a closely related homogeneous G-variety $\mathbb{P}(\mathcal{O}_{\min})$. The latter variety is an example of a GKM manifold (see Definition 4.2), making its T-equivariant cohomology computable using theory not considered in Section 3.

4.1. A $\mathbb{C}^*$-Action on Nilpotent Orbits. Fix a non-zero nilpotent orbit $\mathcal{O} \subseteq \mathfrak{g}$ and a point $\xi \in \mathcal{O}$. By the Jacobson-Morozov Theorem, there exist a semisimple element $h \in \mathfrak{g}$ and a nilpotent element $f \in \mathfrak{g}$ for which (ξ, h, f) is an $\mathfrak{sl}_2(\mathbb{C})$-triple with nil-positive element ξ. We note that for all $\lambda \in \mathbb{C}$,

$$\mathrm{Ad}_{\exp(\lambda h)}(\xi) = e^{\mathrm{ad}_{\lambda h}}(\xi) = e^{2\lambda}\xi.$$

From this calculation, it follows that $\mathcal{O}$ is invariant under the scaling action of $\mathbb{C}^*$ on $\mathfrak{g}$. Accordingly, we introduce

$$\mathbb{P}(\mathcal{O}) := \mathcal{O}/\mathbb{C}^*,$$

a smooth quasi-projective subvariety of $\mathbb{P}(\mathfrak{g})$. Since the actions of G and $\mathbb{C}^*$ on $\mathfrak{g}$ commute, the G-action descends to the quotients $\mathbb{P}(\mathcal{O})$ and $\mathbb{P}(\mathfrak{g})$.

We remark that $\mathbb{P}(\mathcal{O})$ has a rich geometric structure. To see this, choose a K-invariant Hermitian inner product $\langle , \rangle : \mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g} \to \mathbb{C}$. This yields a K-invariant Kähler structure on $\mathbb{P}(\mathfrak{g})$. Since the usual action of $\mathrm{U}(n+1)$ on $\mathbb{P}^n$ is Hamiltonian, so too is the action of K on $\mathbb{P}(\mathfrak{g})$. Furthermore, one has the moment map $\Phi : \mathbb{P}(\mathfrak{g}) \to \mathfrak{k}^*$ defined by

$$\Phi([\xi])(X) = \frac{\mathrm{Im}(\langle [X,\xi], \xi\rangle)}{\langle \xi, \xi\rangle},$$

where $X \in \mathfrak{g} \setminus \{0\}$ and $\eta \in \mathfrak{k}$ (see [8] for a derivation of Φ). Note that the Kähler structure on $\mathbb{P}(\mathfrak{g})$ restricts to a K-invariant Kähler structure on the smooth subvariety $\mathbb{P}(\mathcal{O})$, and the action of K on $\mathbb{P}(\mathcal{O})$ is Hamiltonian.

It should be noted that $\mathbb{P}(\mathcal{O})$ is generally not projective. However, $\mathbb{P}(\mathcal{O}_{\min})$ is the G-orbit in $\mathbb{P}(\mathcal{N})$ of minimal dimension, meaning that it is a closed (hence projective) subvariety of $\mathbb{P}(\mathfrak{g})$. This will be crucial to our study of $\mathbb{P}(\mathcal{O}_{\min})$, and subsequently to our description of $\mathcal{O}_{\min}$ itself.

4.2. Description of the T-Fixed Points. Let us take a moment to examine the Hamiltonian action of T on $\mathbb{P}(\mathcal{O})$, where $\mathcal{O} \subseteq \mathfrak{g}$ is a non-zero nilpotent orbit. We have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta\in\Delta} \mathfrak{g}_\beta,$$

the weight space decomposition of the representation $\mathrm{Ad}|_T$. Note that a point in $\mathbb{P}(\mathfrak{g})$ is fixed by T if and only if it is a class of vectors in $\mathfrak{g} \setminus \{0\}$ with the property that T acts by scaling each vector. In other words,

$$\mathbb{P}(\mathfrak{g})^T = \mathbb{P}(\mathfrak{h}) \cup \{\mathfrak{g}_\beta : \beta \in \Delta\}.$$

With this description, we may determine $\mathbb{P}(\mathcal{O})^T$. Indeed, since $\mathfrak{h}$ consists of semisimple elements of $\mathfrak{g}$ while $\mathcal{O}$ consists of non-zero nilpotent elements, we find that $\mathfrak{h} \cap \mathcal{O} = \emptyset$. Hence,

$$\mathbb{P}(\mathcal{O})^T = \{\mathfrak{g}_\beta : \beta \in \Delta, \mathfrak{g}_\beta \cap \mathcal{O} \neq \emptyset\},$$

a finite set. In particular, $\mathbb{P}(\mathcal{O})^T$ is non-empty if and only if $\mathcal{O}$ is the orbit of a root vector.

Let us take a moment to provide a more refined description of $\mathbb{P}(\mathcal{O})^T$. To this end, we will require the following lemma.

Lemma 4.1. *Let $\beta, \gamma \in \Delta$ be roots. The root spaces $\mathfrak{g}_\beta$ and $\mathfrak{g}_\gamma$ are G-conjugate if and only if β and γ are conjugate under W.*

Proof. Suppose that $w \in W$ and that $\beta = w \cdot \gamma$. Choosing a representative $g \in N_G(H)$ of w, this means precisely that $\beta = \gamma \circ \varphi_{g^{-1}}|_H$, where $\varphi_{g^{-1}} : G \to G$ is conjugation by g^{-1}. Given $h \in H$ and $\xi \in \mathfrak{g}_\beta$, note that

$$\mathrm{Ad}_h(\mathrm{Ad}_g(\xi)) = \mathrm{Ad}_g(\mathrm{Ad}_{g^{-1}hg}(\xi)) = \mathrm{Ad}_g(\beta(g^{-1}hg)\xi) = \mathrm{Ad}_g(\gamma(h)\xi) = \gamma(h)(\mathrm{Ad}_g(\xi)).$$

It follows that $\mathfrak{g}_\gamma = \mathrm{Ad}_g(\mathfrak{g}_\beta)$.

Conversely, suppose that $g \in G$ and that $\mathfrak{g}_\gamma = \mathrm{Ad}_g(\mathfrak{g}_\beta)$. Consider the Zariski-closed subgroup

$$L := \{x \in G : \mathrm{Ad}_x(\mathfrak{g}_\gamma) = \mathfrak{g}_\gamma\},$$

noting that $H, gHg^{-1} \subseteq L$. Since H and gHg^{-1} are maximal tori of L, there exists $x \in L$ for which $xHx^{-1} = gHg^{-1}$. Hence, $x^{-1}g \in N_G(H)$ and $\mathrm{Ad}_{x^{-1}g}(\mathfrak{g}_\beta) = \mathfrak{g}_\gamma$. We may therefore assume that $g \in N_G(H)$. Now, let $w \in W$ denote the class of g. Given $h \in H$ and $\xi \in \mathfrak{g}_\beta$, we find that

$$(w \cdot \beta)(h)\xi = \beta(g^{-1}hg)\xi = \mathrm{Ad}_{g^{-1}hg}(\xi) = \mathrm{Ad}_{g^{-1}}(\gamma(h)\,\mathrm{Ad}_g(\xi)) = \gamma(h)\xi.$$

It follows that $\gamma = w \cdot \beta$. ■

Since $\mathfrak{g}$ is a simple Lie algebra, the root system associated with the pair $(\mathfrak{g}, \mathfrak{h})$ is irreducible. Hence, there are at most two distinct root lengths (namely, those of the long and short roots), and the roots of a given length constitute an orbit of W in Δ. By Lemma 4.1, there are at most two nilpotent G-orbits $\mathcal{O}$ for which $\mathbb{P}(\mathcal{O})^T$ is non-empty, the orbits of root vectors for the short and long roots. Furthermore, if $\mathcal{O}$ is the orbit of a root vector $e_\beta \in \mathfrak{g}_\beta \setminus \{0\}$, $\beta \in \Delta$, then $\mathbb{P}(\mathcal{O})^T$ is the union of the points $\mathfrak{g}_\gamma$ for all $\gamma \in \Delta$ with length equal to that of β. Since $\mathcal{O}_{\mathrm{min}}$ is the orbit of a long root vector, $\mathbb{P}(\mathcal{O}_{\mathrm{min}})^T = \{\mathfrak{g}_\gamma : \gamma \in \Delta_{\mathrm{long}}\}$, where $\Delta_{\mathrm{long}} \subseteq \Delta$ is the set of long roots.

4.3. The Moment Polytope of $\mathbb{P}(\mathcal{O}_{\mathrm{min}})$. Note that the moment map $\Phi : \mathbb{P}(\mathfrak{g}) \to \mathfrak{k}^*$ considered earlier can be modified to obtain a moment map for the Hamiltonian action of T on $\mathbb{P}(\mathcal{O}_{\mathrm{min}})$. Indeed, we denote by $\mu : \mathbb{P}(\mathcal{O}_{\mathrm{min}}) \to \mathfrak{t}^*$ the moment map given by the composition

$$\mathbb{P}(\mathcal{O}_{\mathrm{min}}) \hookrightarrow \mathbb{P}(\mathfrak{g}) \xrightarrow{\Phi} \mathfrak{k}^* \to \mathfrak{t}^*.$$

Recall that

$$\mathbb{P}(\mathcal{O}_{\min})^T = \{\mathfrak{g}_\beta : \beta \in \Delta_{\text{long}}\}.$$

Given $\beta \in \Delta_{\text{long}}$, choose a point $e_\beta \in \mathfrak{g}_\beta \setminus \{0\}$. Note that for $X \in \mathfrak{t}$,

$$\mu(\mathfrak{g}_\beta)(X) = \frac{\text{Im}(\langle [X, e_\beta], e_\beta \rangle)}{\langle e_\beta, e_\beta \rangle} = \frac{\text{Im}(d_e\beta(X)\langle e_\beta, e_\beta \rangle)}{\langle e_\beta, e_\beta \rangle} = \text{Im}(d_e\beta(X)),$$

where $d_e\beta : \mathfrak{t} \to i\mathbb{R}$ is the morphism of real Lie algebras induced by $\beta : T \to \text{U}(1)$. If one regards the weight lattice $X^*(T)$ as included into $\mathfrak{t}^*$ in the usual way, then our above calculation takes the form

$$\mu(\mathfrak{g}_\beta) = \beta.$$

The moment polytope $\mu(\mathbb{P}(\mathcal{O}_{\min}))$ is then the convex hull of Δ_{long} in $\mathfrak{t}^*$.

4.4. Partial Flag Varieties as GKM Manifolds. Let us consider the matter of computing $H_T^*(\mathbb{P}(\mathcal{O}_{\min}))$. To this end, choose a long root $\alpha \in \Delta_{\text{long}}$, so that $\mathfrak{g}_\alpha \in \mathbb{P}(\mathcal{O}_{\min})^T$. Let Q denote the G-centralizer of $\mathfrak{g}_\alpha$. Since $G/Q \cong \mathbb{P}(\mathcal{O}_{\min})$ is projective, Q is a parabolic subgroup of G, and G/Q is therefore a GKM (Goresky-Kottwitz-MacPherson) manifold (see [9]). Accordingly, the following section reviews those parts of GKM theory relevant to computing the T-equivariant cohomology of a partial flag variety.

Let us recall the definition of a GKM manifold.

Definition 4.2. A compact T-manifold X is called a GKM manifold if

(i) X^T is finite, and

(ii) for every codimension-one subtorus $S \subseteq T$, $\dim(X^S) \leq 2$.

Let us briefly address the significance of this notion in the context of computing T-equivariant cohomology. Suppose that X is a GKM manifold as in Definition 4.2. If $S \subseteq T$ is a subtorus of codimension one and Y is a connected component of X^S, then $Y \cap X^T \neq \emptyset$. In particular, Y is T-invariant. Furthermore, Y is either a point or is isomorphic as a T-manifold to S^2 on which T acts via some non-trivial character $\alpha_Y \in X^*(T)$. In the latter case, Y^T consists of two points, x_Y^+ and x_Y^-.

Let $\{Y_j\}_{j=1}^n$ be the collection of those two-spheres in X arising as connected components of fixed point submanifolds of codimension-one subtori (henceforth called special two-spheres). The inclusion $X^T \hookrightarrow X$ induces an injective graded algebra morphism $H_T^*(X) \hookrightarrow H_T^*(X^T) = \text{Map}(X^T, H_T^*(\text{pt}))$ with image

$$\{f \in \text{Map}(X^T, H_T^*(\text{pt})) : \forall j \in \{1, \ldots, n\}, \alpha_{Y_j} | (f(x_{Y_j}^+) - f(x_{Y_j}^-))\} \cong H_T^*(X).$$

Note that Definition 4.2 is precisely the definition of GKM manifold given in [10], where the authors exhibited certain homogeneous spaces of a compact connected simply-connected semisimple Lie group as GKM manifolds. Below is a statement of their result.

Theorem 4.3. *Let M be a compact connected simply-connected semisimple Lie group. Let $R \subseteq M$ be a maximal torus, and let U be a closed subgroup of M containing R. Assume that M/U is oriented. Then, the left-multiplicative action of R renders M/U a GKM manifold.*

For the duration of this section, let us fix a parabolic subgroup $P \subseteq G$ satisfying $B \subseteq P$. Note that P is then the standard parabolic subgroup P_Λ generated by B and the root subgroups $\{U_{-\beta} := \overline{\exp(\mathfrak{g}_{-\beta})} : \beta \in \Lambda\}$ for some unique subset Λ of Π.

Corollary 4.4. *The partial flag variety G/P is a GKM manifold for the left-multiplicative action of T.*

Proof. The Iwasawa decomposition of G tells us that $G = KB$. In particular, K acts transitively on G/P. Since the K-centralizer of the identity coset $[e] \in G/P$ is $K \cap P$, we have a K-manifold isomorphism $K/(K \cap P) \cong G/P$. It will therefore suffice to establish that $K/(K \cap P)$ is a GKM manifold for the left-multiplicative action of T. For this, we will invoke Theorem 4.3. We need only note that K is connected, simply-connected, and semisimple (since G is), that $T \subseteq K \cap P$, and that $K/(K \cap P)$ is oriented (as G/P is). ■

It thus remains to determine the fixed points $(G/P)^T$ and the special two-spheres. Accordingly, we will require the following analogue of Theorem 2.2 of [10].

Lemma 4.5. *Let $S \subseteq T$ be a subtorus. The image of $(G/B)^S$ under the fibration $G/B \xrightarrow{\varphi} G/P$ is $(G/P)^S$.*

Proof. Consider the fibration $\psi : K/T \to K/(K \cap P)$. By Theorem 2.2 of [10], $\psi((K/T)^S) = (K/(K \cap P))^S$. Since each of the maps in the commutative diagram

$$\begin{array}{ccc} K/T & \xrightarrow{\psi} & K/(K \cap P) \\ \downarrow \cong & & \downarrow \cong \\ G/B & \xrightarrow{\varphi} & G/P \end{array}$$

is T-equivariant, the desired result follows. ■

We immediately obtain a description of $(G/P)^T$. Indeed, note that $(G/B)^T = \{[k] : k \in N_K(T)\}$. Hence, $(G/P)^T$ is identified with $N_K(T)/(N_K(T) \cap P) \cong W/W_P$, where $W_P = W_\Lambda$ (see [3]). Let us now determine the special two-spheres in G/P.

Lemma 4.6. *A submanifold $X \subseteq G/P$ is a special two-sphere if and only if it is related by the action of $N_K(T)$ to a special two-sphere containing the identity* *coset* $[e]$.

Proof. Suppose that X is a two-sphere arising as a component of $(G/P)^S$ for some codimension-one subtorus $S \subseteq T$. Note that $X^T = \{[k_1],[k_2]\}$ for some $k_1, k_2 \in N_K(T)$. Furthermore, $R := k_1^{-1}Sk_1$ is a codimension-one subtorus of T and $k_1^{-1}X \cong S^2$ is a component of $(G/P)^R$ containing $[e]$. The proof of the converse is then a simple reversal of this argument. ∎

Accordingly, we will temporarily restrict our attention to the special two-spheres in G/P containing $[e]$. Let $X \subseteq G/P$ denote one such two-sphere. Note that

$$T_{[e]}(G/P) \cong \mathfrak{g}/\mathfrak{p} \cong \bigoplus_{\beta\in\Delta\setminus\Delta_P} \mathfrak{g}_\beta$$

as complex T-modules, where Δ_P is the set of roots whose root spaces belong to $\mathfrak{p}$. Since $T_{[e]}X$ is a complex one-dimensional T-invariant subspace of $T_{[e]}(G/P)$, $T_{[e]}X \cong \mathfrak{g}_\beta$ for some $\beta \in \Delta \setminus \Delta_P$. In [10], it is then concluded that $X^T = \{[e],[s_\beta]\} \subseteq W/W_P$. Furthermore, by associating to X the weight of $T_{[e]}X$, we obtain a bijection between $\Delta \setminus \Delta_P$ and the special two-spheres containing $[e]$.

Given $[w] \in W/W_P$, choose a representative $k \in N_K(T)$ of w. Note that the special two-spheres containing $[w]$ are then the left-translates by k of the special two-spheres containing $[e]$.

Lemma 4.7. *Let $X \subseteq G/P$ be a special two-sphere containing $[e]$, so that $Y := kX$ is a special two-sphere containing $[w]$. If $\beta \in \Delta$ is the weight with which T acts on $T_{[e]}X$, then $w \cdot \beta$ is the weight with which T acts on $T_{[w]}Y$.*

Proof. Consider the automorphism $\phi : G/P \to G/P$, $[g] \mapsto [kg]$, noting that $(d_{[e]}\phi)|_{T_{[e]}X} : T_{[e]}X \to T_{[w]}Y$ is a complex vector space isomorphism. Furthermore, $\phi(t[g]) = (ktk^{-1})\phi([g])$ for all $t \in T$ and $g \in G$, so that $d_{[e]}\phi((k^{-1}tk)v) = td_{[e]}\phi(v)$ for all $v \in T_{[e]}(G/P)$. Hence, if $u \in T_{[w]}Y$, then $u = d_{[e]}\phi(v)$ for some $v \in T_{[e]}X$ and

$$tu = td_{[e]}\phi(v) = d_{[e]}\phi((k^{-1}tk)v) = d_{[e]}\phi(\beta(k^{-1}tk)v) = \beta(k^{-1}tk)d_{[e]}\phi(v) = (w\cdot\beta)(t)u$$

for all $t \in T$. ∎

We have the following summary of the material considered in this section.

Theorem 4.8. *(i) There is a natural bijection $W/W_P \cong (G/P)^T$.*

(ii) Fix $[w] \in W/W_P \cong (G/P)^T$. Given $\beta \in \Delta \setminus \Delta_P$, there exists a unique special two-sphere $X \subseteq G/P$ with $X^T = \{[w],[ws_\beta]\}$, and with the property that $w \cdot \beta$ is the weight of $T_{[w]}X$. Every special two-sphere containing $[w]$ arises in this way

(iii) We have a graded algebra isomorphism

$$H_T^*(G/P) \cong \{f \in \mathrm{Map}(W/W_P \to H_T^*(pt)) : (w\cdot\beta)|(f([w]) - f([ws_\beta])) \\ \forall w \in W, \beta \in \Delta \setminus \Delta_P\}$$

.

4.5. A Description of $\mathcal{O}_{\mathbf{min}}$ and $\mathbb{P}(\mathcal{O}_{\mathbf{min}})$. We devote this section to giving explicit descriptions of $\mathcal{O}_{\min}$ and $\mathbb{P}(\mathcal{O}_{\min})$ as homogeneous G-varieties. As noted earlier, the latter space is G-equivariantly isomorphic to a partial flag variety G/P. Accordingly, we shall begin by finding a parabolic subgroup $P \subseteq G$ with this property. In order to proceed, however, we will require the following result.

Theorem 4.9. *Let Φ be an irreducible root system with collection of simple roots $\Sigma \subseteq \Phi$.*

(i) There exists a unique maximal root $\beta \in \Phi$ (called the highest root).

(ii) This root is long.

(iii) We have $\langle \beta, \gamma \rangle \geq 0$ for all $\gamma \in \Sigma$.

For a proof, the reader might refer to Propositions 19 and 23 in [12].

Denote by $\alpha \in \Delta_+$ the highest root, and choose a root vector $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$. Note that $[e_\alpha] = \mathfrak{g}_\alpha \in \mathbb{P}(\mathcal{O}_{\min})^T$. Let $C_{\mathfrak{g}}(e_\alpha)$ denote the centralizer of e_α with respect to the adjoint representation of $\mathfrak{g}$. Note that $C_{\mathfrak{g}}(e_\alpha)$ is $\mathfrak{t}$-invariant, meaning that it is a sum of $\mathfrak{t}$-submodules of the $\mathfrak{t}$-weight spaces occurring in the adjoint representation of $\mathfrak{t}$ on $\mathfrak{g}$. The summand coming from the trivial weight space $\mathfrak{h}$ is just $\ker(d_e\alpha)$, where we regard $d_e\alpha$ as belonging to $\mathfrak{h}^*$ instead of $\mathfrak{t}^*$. Furthermore, if $\beta \in \Delta$, then $\mathfrak{g}_\beta \subseteq C_{\mathfrak{g}}(e_\alpha)$ if and only if $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$. Hence, we have established that

$$C_{\mathfrak{g}}(e_\alpha) = \ker(d_e\alpha) \oplus \bigoplus_{\{\beta \in \Delta : [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}\}} \mathfrak{g}_\beta.$$

Now, let $C_G(e_\alpha)$ and $Q := C_G([e_\alpha])$ be the G-centralizers of $e_\alpha \in \mathcal{O}_{\min}$ and $[e_\alpha] \in \mathbb{P}(\mathcal{O}_{\min})$, respectively. The inclusion $C_G(e_\alpha) \subseteq Q$ yields an inclusion of Lie algebras $C_{\mathfrak{g}}(e_\alpha) \subseteq \mathfrak{q} := \mathrm{Lie}(Q)$. Since $\dim_{\mathbb{C}} \mathfrak{q} = \dim_{\mathbb{C}} C_{\mathfrak{g}}(e_\alpha) + 1$ (a consequence of comparing the dimensions of $\mathcal{O}_{\min}$ and $\mathbb{P}(\mathcal{O}_{\min})$), and since $\mathfrak{h} \subseteq \mathfrak{q}$ (as H stabilizes $[e_\alpha]$), we must have

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\{\beta \in \Delta : [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}\}} \mathfrak{g}_\beta. \tag{4}$$

In light of our having chosen α to be the highest root, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$ for all $\beta \in \Delta_+$. It thus remains to determine those negative roots whose root spaces appear as summands of $\mathfrak{q}$. To do this, we will require the following standard fact.

Lemma 4.10. *If $\beta \in \Delta_-$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$ if and only if $\langle \alpha, \beta \rangle = 0$.*

Now, suppose that

$$\beta = \sum_{\gamma \in \Pi} a_{\gamma\beta} \gamma,$$

$a_{\gamma\beta} \in \mathbb{Z}_{\leq 0}$, is the expression of $\beta \in \Delta_-$ as a linear combination of simple roots. By Lemma 4.10, $\mathfrak{g}_\beta$ is a summand appearing in (4) if and only if $\langle \alpha, \beta \rangle = 0$. Since

$\langle \alpha, \gamma \rangle \geq 0$ for all $\gamma \in \Pi$, we see that $\langle \alpha, \beta \rangle = 0$ if and only if $\langle \alpha, \gamma \rangle = 0$ whenever $a_{\gamma\beta} \neq 0$. In other words, $\langle \alpha, \beta \rangle = 0$ if and only if β is a linear combination of those simple roots orthogonal to α.

Accordingly, let us set

$$\Xi := \{\beta \in \Pi : \langle \alpha, \beta \rangle = 0\}.$$

We have shown that $Q = P_\Xi$. Hence, we have the following description of $\mathbb{P}(\mathcal{O}_{\min})$ as a G-variety.

Theorem 4.11. *There is a G-variety isomorphism $\mathbb{P}(\mathcal{O}_{min}) \cong G/P_\Xi$.*

Let us now address the G-variety structure of $\mathcal{O}_{\min}$. To this end, we denote by $\mathcal{L} \xrightarrow{\pi} \mathbb{P}(\mathfrak{g})$ the tautological line bundle over $\mathbb{P}(\mathfrak{g})$. Recall that for $\xi \in \mathfrak{g} \setminus \{0\}$, we have $\pi^{-1}([\xi]) = \operatorname{span}_\mathbb{C}\{\xi\}$. Furthermore, the tautological bundle is G-equivariant, with the G-action on the total space $\mathcal{L}$ given by

$$g : ([\xi], v) \mapsto ([\operatorname{Ad}_g(\xi)], \operatorname{Ad}_g(v)),$$

$g \in G$, $\xi \in \mathfrak{g} \setminus \{0\}$, $v \in \operatorname{span}_\mathbb{C}\{\xi\}$.

Let $\mathcal{E} \xrightarrow{\varphi} \mathbb{P}(\mathcal{O}_{\min})$ denote the pullback of $\mathcal{L}$ along the inclusion $\mathbb{P}(\mathcal{O}_{\min}) \hookrightarrow \mathbb{P}(\mathfrak{g})$. Note that $\mathcal{E}$ inherits from $\mathcal{L}$ the structure of a G-equivariant line bundle over $\mathbb{P}(\mathcal{O}_{\min})$. Furthermore, $\mathcal{O}_{\min}$ G-equivariantly (and also $\mathbb{C}^*$-equivariantly) includes into $\mathcal{E}$ as a smooth open subvariety, namely the complement $\mathcal{E}^*$ of the zero-section. Accordingly, we will describe $\mathcal{O}_{\min}$ by more closely examining $\mathcal{E}$.

Since $\mathbb{P}(\mathcal{O}_{\min})$ is the homogeneous G-variety G/P_Ξ, we may exhibit $\mathcal{E}$ as an associated bundle for the one-dimensional P_Ξ-representation $\varphi^{-1}([e_\alpha]) = \mathfrak{g}_\alpha$. More precisely, let $G \times_{P_\Xi} \mathfrak{g}_\alpha$ denote the quotient of $G \times \mathfrak{g}_\alpha$ by the equivalence relation

$$(gp, v) \sim (g, \operatorname{Ad}_p(v)),$$

$p \in P_\Xi$, $g \in G$, $v \in \mathfrak{g}_\alpha$. Consider the map $G \times_{P_\Xi} \mathfrak{g}_\alpha \to G/P_\Xi$ given by projection from the first component, whose fibres are then naturally complex vector spaces. The bundle $G \times_{P_\Xi} \mathfrak{g}_\alpha \to G/P_\Xi$ is G-equivariant by virtue of the left-multiplicative G-action on the first component of $G \times_{P_\Xi} \mathfrak{g}_\alpha$.

We have an isomorphism $\mathcal{E} \cong G \times_{P_\Xi} \mathfrak{g}_\alpha$ of G-equivariant holomorphic line bundles over G/P_Ξ, where we are regarding $\mathcal{E}$ as a line bundle over G/P_Ξ. We therefore have the following description of $\mathcal{O}_{\min}$.

Theorem 4.12. *There is an isomorphism of G-equivariant holomorphic principal $\mathbb{C}^*$-bundles over G/P_Ξ between $\mathcal{O}_{min}$ and $(G \times_{P_\Xi} \mathfrak{g}_\alpha)^*$.*

4.6. The T-Equivariant Cohomology of $\mathcal{O}_{\min}$.

Let us use the description of $\mathcal{O}_{\min}$ provided in Theorem 4.12 to compute $H_T^*(\mathcal{O}_{\min})$. To this end, we have the equivariant Thom-Gysin sequence

$$\cdots \to H_T^{i-2}(G/P_\Xi) \to H_T^i(G \times_{P_\Xi} \mathfrak{g}_\alpha) \to H_T^i((G \times_{P_\Xi} \mathfrak{g}_\alpha)^*) \to \cdots \qquad (5)$$

associated with the zero-section G/P_Ξ in $G\times_{P_\Xi}\mathfrak{g}_\alpha$ and its complement $(G\times_{P_\Xi}\mathfrak{g}_\alpha)^*$. We can say considerably more about this sequence in our context, but it will require a brief computation of the T-equivariant first Chern class $c_1^T(N)\in H_T^2(G/P_\Xi)$ of the normal bundle $N\cong G\times_{P_\Xi}\mathfrak{g}_\alpha$ of the zero-section in $G\times_{P_\Xi}\mathfrak{g}_\alpha$. Given a fixed point $[w]\in W/W_\Xi\cong(G/P_\Xi)^T$, consider the inclusion $i_{[w]}:\{[w]\}\hookrightarrow G/P_\Xi$ and the induced restriction map

$$i_{[w]}^*:H_T^*(G/P_\Xi)\to H_T^*(\mathrm{pt})$$

on T-equivariant cohomology. The following lemma computes the restriction of $c_1^T(G\times_{P_\Xi}\mathfrak{g}_\alpha)$ to each T-fixed point in G/P_Ξ.

Lemma 4.13. *If $w\in W$, then $i_{[w]}^*(c_1^T(G\times_{P_\Xi}\mathfrak{g}_\alpha))=w\cdot\alpha$.*

Proof. Note that

$$\begin{aligned} i_{[w]}^*(c_1^T(G\times_{P_\Xi}\mathfrak{g}_\alpha)) &= c_1^T((i_{[w]})^*(G\times_{P_\Xi}\mathfrak{g}_\alpha)) \\ &= c_1^T((G\times_{P_\Xi}\mathfrak{g}_\alpha)_{[w]}), \end{aligned}$$

where $(G\times_{P_\Xi}\mathfrak{g}_\alpha)_{[w]}$ is the fibre over $[w]$, viewed as a T-equivariant vector bundle over a point. Recalling (1), the T-equivariant first Chern class of this bundle is precisely its weight as a representation of T. To compute this weight, choose a representative $k\in N_K(T)$ of w. Note that any element of $(G\times_{P_\Xi}\mathfrak{g}_\alpha)_{[w]}$ is of the form $[(k,\xi)]$, $\xi\in\mathfrak{g}_\alpha$. For $t\in T$, we have

$$\begin{aligned} t\cdot[(k,\xi)] = [(tk,\xi)] = [(k(k^{-1}tk),\xi)] = [(k,(k^{-1}tk)\cdot\xi)] &= [(k,\alpha(k^{-1}tk)\xi)] \\ &= (w\cdot\alpha)(t)[(k,\xi)]. \end{aligned}$$

Hence, $w\cdot\alpha=c_1^T((G\times_{P_\Xi}\times\mathfrak{g}_\alpha)_{[w]})=i_{[w]}^*(c_1^T(G\times_{P_\Xi}\mathfrak{g}_\alpha))$. ∎

Now, note that

$$H_T^*((G/P_\Xi)^T)=\bigoplus_{[w]\in W/W_\Xi}H_T^*(\mathrm{pt}) \tag{6}$$

as $\mathbb{Q}$-algebras. Lemma 4.13 is then seen to imply that the image of $c_1^T(G\times_{P_\Xi}\mathfrak{g}_\alpha)$ under the restriction map

$$H_T^*(G/P_\Xi)\to H_T^*((G/P_\Xi)^T)$$

has a non-zero projection to each direct summand appearing in (6). Since restriction gives an inclusion of $H_T^*(G/P_\Xi)$ into $H_T^*((G/P_\Xi)^T)$ as a subalgebra, and since $H_T^*(\mathrm{pt})$ has no zero-divisors, we conclude that $c_1^T(G\times_{P_\Xi}\mathfrak{g}_\alpha)$ is not a zero-divisor in $H_T^*(G/P_\Xi)$. It follows that our Thom-Gysin sequence splits into the short-exact sequences

$$0\to H_T^{i-2}(G/P_\Xi)\to H_T^i(G\times_{P_\Xi}\mathfrak{g}_\alpha)\to H_T^i((G\times_{P_\Xi}\mathfrak{g}_\alpha)^*)\to 0.$$

(For a proof, see [1].)

For a second useful refinement of our Thom-Gysin sequence, we note that restriction to the zero-section gives a T-equivariant homotopy equivalence between $G \times_{P_\Xi} \mathfrak{g}_\alpha$ and G/P_Ξ. It follows that the associated restriction map $H_T^*(G \times_{P_\Xi} \mathfrak{g}_\alpha) \to H_T^*(G/P_\Xi)$ is an isomorphism. Using this isomorphism, we shall replace $H_T^*(G \times_{P_\Xi} \mathfrak{g}_\alpha)$ in our short-exact sequences to obtain

$$0 \to H_T^{i-2}(G/P_\Xi) \to H_T^i(G/P_\Xi) \to H_T^i((G \times_{P_\Xi} \mathfrak{g}_\alpha)^*) \to 0.$$

The map $H_T^{i-2}(G/P_\Xi) \to H_T^i(G/P_\Xi)$ is multiplication by $c_1^T(G \times_{P_\Xi} \mathfrak{g}_\alpha)$ (see [4], for instance). Furthermore, the map $H_T^i(G/P_\Xi) \to H_T^i((G \times_{P_\Xi} \mathfrak{g}_\alpha)^*)$ is the map ψ^* on equivariant cohomology induced by the projection $\psi : (G \times_{P_\Xi} \mathfrak{g}_\alpha)^* \to G/P_\Xi$. (This follows from the fact that the bundle projection $G \times_{P_\Xi} \mathfrak{g}_\alpha \to G/P_\Xi$ and zero-section $G/P_\Xi \to G \times_{P_\Xi} \mathfrak{g}_\alpha$ give inverse maps on equivariant cohomology.)

The above analysis yields two immediate corollaries. Firstly, the T-equivariant Betti numbers $b_T^i(\mathcal{O}_{\min})$ of $\mathcal{O}_{\min}$ are given by

$$b_T^i(\mathcal{O}_{\min}) = b_T^i(G/P_\Xi) - b_T^{i-2}(G/P_\Xi).$$

Secondly, $\psi^* : H_T^*(G/P_\Xi) \to H_T^*(\mathcal{O}_{\min})$ is a surjective graded algebra morphism. Its kernel is $\langle c_1^T(G \times_{P_\Xi} \mathfrak{g}_\alpha)\rangle$, the ideal of $H_T^*(G/P_\Xi)$ generated by the equivariant first Chern class $c_1^T(G \times_{P_\Xi} \mathfrak{g}_\alpha) \in H_T^2(G/P_\Xi)$. In particular, there is a graded algebra isomorphism

$$H_T^*(\mathcal{O}_{\min}) \cong H_T^*(G/P_\Xi)/\langle c_1^T(G \times_{P_\Xi} \mathfrak{g}_\alpha)\rangle.$$

Using Lemma 4.13 and Theorem 4.8, we obtain the following more explicit description of $H_T^*(\mathcal{O}_{\min})$.

Theorem 4.14. *$H_T^*(\mathcal{O}_{min})$ is isomorphic to the quotient of*

$$\{f \in \mathrm{Map}(W/W_\Xi, H_T^*(pt)) : (w \cdot \beta) | (f([w]) - f([ws_\beta])) \\ \forall w \in W, \beta \in \Delta_-, \ \langle \alpha, \beta \rangle \neq 0\}$$

by the ideal generated by the map $W/W_\Xi \to H_T^*(pt)$, $[w] \mapsto w \cdot \alpha$.

Remark 4.15. In [11], D. Juteau used a non-equivariant version of the Thom-Gysin sequence (5) to help compute the ordinary integral cohomology groups of $\mathcal{O}_{\min}$. However, there is an interesting difference between the equivariant and non-equivariant cases. Indeed, Juteau found that multiplication by the ordinary first Chern class $c_1(G \times_{P_\Xi} \mathfrak{g}_\alpha) \in H^2(G/P_\Xi; \mathbb{Z})$ gave rise to a non-injective map $H^{i-2}(G/P_\Xi; \mathbb{Z}) \to H^i(G/P_\Xi; \mathbb{Z})$ for some values of i. This is in contrast to the equivariant setup, as $c_1^T(G \times_{P_\Xi} \mathfrak{g}_\alpha)$ is not a zero-divisor in $H_T^*(G/P_\Xi)$.

4.7. An Example. Let us compute the equivariant cohomology of the minimal nilpotent orbit of $G = \mathrm{SL}_2(\mathbb{C})$. To this end, let $T \subseteq G$ be the compact real form of the standard maximal torus of G. Note that $\Delta = \{-2, 2\} \subseteq \mathbb{Z} \cong \mathrm{Hom}(T, \mathrm{U}(1))$ is the resulting collection of roots. Letting $B \subseteq G$ be the Borel subgroup of upper-triangular matrices, we find that $\alpha = 2$ is the highest root. It is not orthogonal to any of the simple roots, so that $\Xi = \emptyset$. Hence, $P_\Xi = B$ and $\Delta_{P_\Xi} = \{2\}$. The Weyl group W is $\mathbb{Z}/2\mathbb{Z}$, and the generator acts by negation on the weight lattice. The subgroup W_Ξ is trivial. In particular, G/P_Ξ has two T-fixed points.

Since α is identified with $2x \in \mathbb{Q}[x] \cong H_T^*(\mathrm{pt})$, Theorem 4.8 implies that $H_T^*(G/P_\Xi)$ includes into $H_T^*(\mathrm{pt})^{\oplus 2} \cong \mathbb{Q}[x]^{\oplus 2}$ as the subalgebra

$$\begin{aligned} H_T^*(G/P_\Xi) &\cong \{(f_1(x), f_2(x)) \in \mathbb{Q}[x]^{\oplus 2} : x | (f_1(x) - f_2(x))\} \\ &= \{(f_1(x), f_2(x)) \in \mathbb{Q}[x]^{\oplus 2} : f_1(0) = f_2(0)\}. \end{aligned}$$

Indeed, we have recovered the U(1)-equivariant cohomology of the two-sphere with the rotation action of U(1).

Lemma 4.13 tells us that $c_1^T(N) = (2x, -2x)$ when included into $\mathbb{Q}[x]^{\oplus 2}$. Hence,

$$H_T^*(\mathcal{O}_{\min}) \cong \frac{\{(f_1(x), f_2(x)) \in \mathbb{Q}[x]^{\oplus 2} : f_1(0) = f_2(0)\}}{\langle (x, -x) \rangle}.$$

Note that this is generated as a $\mathbb{Q}$-algebra by $y := [(x, 0)]$. The relation is $y^2 = 0$, so that $H_T^*(\mathcal{O}_{\min}) \cong \mathbb{Q}[y]/\langle y^2 \rangle$ with y an element of grading degree two.

We remark that this is consistent with our findings in Section 3. Indeed, if $G = \mathrm{SL}_2(\mathbb{C})$, then $\mathcal{O}_{\min} = \mathcal{O}_{\mathrm{reg}}$. Hence, $H_T^*(\mathcal{O}_{\min}) = H_T^*(\mathcal{O}_{\mathrm{reg}})$, and the latter is isomorphic to the ordinary cohomology of $G/B \cong \mathbb{P}^1$.

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