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A Beurling Theorem for Exponential Solvable Lie Groups

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Abstract. We prove in this paper an L^2-version of Beurling's theorem for an arbitrary exponential solvable Lie group G with a non-trivial center, which encompasses the setting of nilpotent connected and simply connected Lie groups. When G has a trivial center, the uncertainty principle may fail to hold and an example is produced. The representation theory and a localized Plancherel formula are fundamental tools in the proof.
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1. Introduction

One of the sharp formalizations of the uncertainty principle in the classical Fourier analysis on $\mathbb{R}$, is the study of how a function f and its Fourier transform $\widehat{f}$ given by:

$$\widehat{f}(x) = \int_{\mathbb{R}} f(y) e^{-2\pi i x y} dy,\ x \in \mathbb{R},$$

cannot both have rapid decay unless f is identically zero. This problem has been intensively studied in many situations. For the real line, Hardy [9], Cowling-Price [5], Morgan [16] and Miyachi [13] are some remarkable examples. In the same context, an important theorem of Beurling on Fourier transform pairs published by Hörmander in the case of the real line (see [10]), says that for any non trivial function f in $L^1(\mathbb{R})$, the function $f(x)\hat{f}(y)$ is never integrable on $\mathbb{R}^2$ with respect to the measure $e^{|xy|}dx\ dy$. A generalization of this result has been proved by Bonami. A and al. (cf. [4], Theorem 1.1). We have:

Theorem 1.1. *(Bonami-Demange-Jaming) Let $f \in L^2(\mathbb{R}^n)$ and suppose that for some $N \geq 0$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+\|x\|)^N}\ \frac{|\widehat{f}(y)|}{(1+\|y\|)^N} e^{|\langle x,y\rangle|}\ dx\ dy < +\infty.$$

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Then $f = 0$ whenever $N \leq n$. If $N > n$, then the above holds if and only if f can be written as

$$f(x) = P(x)e^{\frac{\langle Ax,x\rangle}{2}},$$

where $\|\cdot\|$ and $\langle .,.\rangle$ denote respectively the Euclidean norm and its associated scalar product, A is a real positive definite matrix and P is a polynomial of degree $< \frac{N-n}{2}$.

Concerning Beurling's uncertainty principle and beyond the context of the Abelian groups $\mathbb{R}^N$, $N \geq 1$ in [2], some works extending the context of this theorem have been elaborated by Sarkar and Thangavelu (see [17] and [18]). In [1], the authors defined the following:

Beurling's analogue for nilpotent Lie groups: Let G be a nilpotent connected and simply connected Lie group. Let f be a measurable function on G fulfilling:

$$\int_G \int_{\mathscr{W}} |f(g)| \, \|\pi_\xi(f)\|_{HS} \, e^{\|g\| \, \|\xi\|} d\xi \, dg < +\infty.$$

Is it true that f vanishes almost everywhere on G?

The section $\mathscr{W}$ and the norms $\|g\|$ and $\|\xi\|$ will be defined later. In [1], the authors prove this theorem for a restricted class of nilpotent Lie groups. It concerns the so-called $SNPC$ Lie groups G, which are among Lie groups admitting a common ideal which polarizes all generic elements of $\mathfrak{g}^*$, where $\mathfrak{g}$ designates the Lie algebra of G. This class encompasses the context of Heisenberg and threadlike Lie groups. Later, K. Smaoui published in [14] a proof of such principal for general simply connected nilpotent Lie groups. Though this proof enrolls a pretty technical gap, it could cover some restrictive subclasses.

In this paper, we prove an L^2-version of Beurling's theorem on an arbitrary exponential solvable Lie group provided that it is with a non-trivial center, which is the case for connected nilpotent Lie groups. Otherwise, such a principal may fail as many examples reveal. In the case of the real line, this means that for a non-trivial function f in $L^1(\mathbb{R})$, the function $|f(x)\hat{f}(y)|^2$ is never integrable on $\mathbb{R}^2$ with respect to the measure $e^{2|xy|}dx\, dy$.

Finally, we mention that at present we do not know how to prove the Beurling analogue above, in the context of exponential solvable Lie groups with non-trivial center.

2. Preliminaries and notation

We begin this section by reviewing some useful facts and notations for exponential solvable Lie groups and their representations. For more details, the readers could consult the references [3, 11, 12].

2.1. Exponential Lie groups and norm functions.

Throughout, G will be a connected real exponential solvable Lie group of dimension n, $\mathfrak{g}$ will be the associated Lie algebra. This means that the exponential mapping

$$\exp : \mathfrak{g} \to G$$

is a $C^\infty-$diffeomorphism and then G turns out to be simply connected. Let log designate the inverse mapping of exp and let

$$\mathscr{S} : \mathfrak{s}_0 = \{0\} \subset \mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{g}$$

be a good sequence of subalgebras of $\mathfrak{g}$. This means that $\dim \mathfrak{s}_j/\mathfrak{s}_{j-1} = 1$ $(1 \le j \le n)$, and if $\mathfrak{s}_j$ is not an ideal of $\mathfrak{g}$, then $\mathfrak{s}_{j-1}$ and $\mathfrak{s}_{j+1}$ are both ideals of $\mathfrak{g}$ and the quotient space $\mathfrak{s}_{j+1}/\mathfrak{s}_{j-1}$ is an irreducible $\mathfrak{g}-$module. From $\mathscr{S}$, we single out a Jordan-Hölder basis $\{X_1, ..., X_n\}$ of $\mathfrak{g}$ just by taking $X_j \in \mathfrak{s}_j \setminus \mathfrak{s}_{j-1}$ for $1 \le j \le n$. The map

$$\mathbb{R}^n \to G,\ (t_1, ..., t_n) \mapsto \exp(t_n X_n) \cdots \exp(t_1 X_1) \tag{1}$$

is a $C^\infty-$diffeomorphism and maps the Lebesgue measure $dt_n \cdots dt_1$ to a Haar measure on G (cf. [12]). As such, the group G is therefore identified as a set with $\mathbb{R}^n$ through the map (1). We then introduce *the norm function* $\|\cdot\|$ on G by setting

$$\|x\| = \left(\sum_{i=1}^n t_i^2\right)^{\frac{1}{2}},\ t_j \in \mathbb{R},$$

for any

$$x = \exp(t_n X_n) \cdots \exp(t_1 X_1) \in G.$$

Let $\mathfrak{g}^*$ denote the dual vector space of $\mathfrak{g}$ and $\{X_1^*, ..., X_n^*\}$ the basis of $\mathfrak{g}^*$ dual to the Jordan-Hölder basis $\{X_1, ..., X_n\}$. We hence identify $\mathfrak{g}^*$ with $\mathbb{R}^n$ as well, and any $\xi \in \mathfrak{g}^*$ is written through the dual basis as

$$\xi = \sum_{i=1}^n \xi_i X_i^*.$$

The Euclidean norm with respect to this basis reads:

$$\|\xi\| = \left(\sum_{i=1}^n \xi_i^2\right)^{\frac{1}{2}}.$$

2.2. On induced representations.

Let dg be a left Haar measure on G and Δ_G the modular function of G, which is defined, for all $x \in G$ and f in $\mathscr{C}_c(G)$, the set of continuous functions on G with compact support, by the relation:

$$\int_G f(gx^{-1})\, dg = \Delta_G(x) \int_G f(g)\, dg.$$

It is well known that, for $x \in G$:

$$\Delta_G(x) = |\det \mathrm{Ad}(x)|^{-1} = \exp(-\mathrm{tr}\ \mathrm{ad}(\log x)).$$

Let H be a closed subgroup with corresponding Lie algebra $\mathfrak{h}$. We denote by $\Delta_{H,G}$ the positive character of H defined by:

$$\Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)}.$$

When H is a normal subgroup of G, then $\Delta_{H,G}(h) = 1$ for any $h \in H$. Let $\mathfrak{h}$ be the Lie algebra of H, l a linear form on $\mathfrak{g}$ satisfying $< l, [\mathfrak{h}, \mathfrak{h}] >= \{0\}$ and χ_l the character of H defined as

$$\chi_l(\exp \mathrm{X}) = \mathrm{e}^{-2\pi i < l, X >}, \ X \in \mathfrak{h}. \tag{2}$$

We define the induced representation $\pi_l := \mathrm{Ind}_H^G \chi_l$ of G as the left regular representation of G acting on the Hilbert space $L^2(G/H, \chi_l)$, the completion of the vector space $\mathscr{C}_c(G/H, \chi_l)$ of the complex-valued continuous functions φ of G, which are compactly supported modulo H and which satisfy the covariance condition:

$$\varphi(gh) = \chi_l^{-1}(h)\Delta_{H,G}^{\frac{1}{2}}(h)\varphi(g), \quad g \in G, \ h \in H.$$

2.3. Pukanszky polarizations. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$, i.e.,

$$\mathrm{ad}_{\mathfrak{g}}(X)Y = \mathrm{ad}(X)Y = [X, Y], \quad X, \ Y \in \mathfrak{g}.$$

The group G acts on $\mathfrak{g}$ by the adjoint representation Ad_G, i.e.,

$$\mathrm{Ad}_G(g)Y = \mathrm{Ad}(g)Y = \mathrm{e}^{\mathrm{ad}(X)}Y, \ g = \exp X \in G, \ Y \in \mathfrak{g},$$

and on $\mathfrak{g}^*$ by the coadjoint representation Ad_G^*, i.e.,

$$< \mathrm{Ad}_G^*(g)l, X >=< g \cdot l, X >=< l, \mathrm{Ad}(g^{-1})X >, \ g \in G, \quad l \in \mathfrak{g}^*, \ X \in \mathfrak{g}.$$

The set $G \cdot l = \{g \cdot l, \ g \in G\} =: \Omega_l$ is called the G-orbit of l, we denote by $\mathfrak{g}^*/G$ the space of coadjoint orbits. Let $\mathfrak{g}(l) = \{X \in \mathfrak{g}, \ < l, [X, \mathfrak{g}] >= \{0\} \ \}$ be the stabilizer of $l \in \mathfrak{g}^*$ in $\mathfrak{g}$, it is also the Lie algebra of $G_l = \{g \in G, \ g \cdot l = l \ \}$. For $\mathfrak{p} \subset \mathfrak{g}$, define $\mathfrak{p}^\perp = \{f \in \mathfrak{g}^*, \ f_{|\mathfrak{p}} = 0\}$, the annihilator of $\mathfrak{p}$ in $\mathfrak{g}^*$. A subspace $\mathfrak{b}(l) \subset \mathfrak{g}$ is called a polarization for $l \in \mathfrak{g}^*$, if $\mathfrak{b}(l)$ is a maximal dimensional isotropic subalgebra related to the skew-symmetric bilinear form B_l defined by:

$$B_l(X, Y) =< l, [X, Y] >, \ X, \ Y \in \mathfrak{g}.$$

Moreover, a polarization $\mathfrak{b}(l)$ for l satisfies *Pukanszky's condition* or is a *Pukanszky polarization*, if

$$l + \mathfrak{b}(l)^\perp = \mathrm{Ad}^*(B(l))l = B(l) \cdot l$$

where $B(l) = \exp \mathfrak{b}(l)$ and as above, $l + \mathfrak{b}(l)^\perp$ denotes the set of linear forms on $\mathfrak{g}$ which coincide with l on $\mathfrak{b}(l)$.

2.4. The orbit method theory. The Kirillov-Bernat-Vergne orbit method makes it possible to parameterize the unitary dual $\widehat{G}$ of G by the space of coadjoint orbits of G in $\mathfrak{g}^*$. Let l be in $\mathfrak{g}^*$. We take a polarization $\mathfrak{b} = \mathfrak{b}(l)$ for l satisfying

Pukanszky's condition. So we can consider the unitary character $\chi_l = \chi$ of $B(l)$ associated to l as defined in equation (2). For such a polarization, we define $\pi_{l,\mathfrak{b}}$ by:

$$\pi_{l,\mathfrak{b}} = \mathrm{Ind}_B^G \chi_l, \quad (B = \exp \mathfrak{b}).$$

Then $\pi_{l,\mathfrak{b}}$ is an irreducible representation of G and its equivalence class $[\pi_{l,\mathfrak{b}}]$ depends only upon the coadjoint orbit of l. Every irreducible representation π is equivalent to an induced representation $\pi_{l,\mathfrak{b}}$ from a character χ_l of a Pukanszky polarization for l. Moreover, the following mapping, called the Kirillov-Bernat-Pukanszky-Vergne mapping

$$\begin{array}{rccc} \mathscr{K}: & \mathfrak{g}^*/G & \longrightarrow & \widehat{G} \\ & \Omega_l & \longmapsto & [\pi_{l,\mathfrak{b}}] =: \pi_{G\cdot l} \end{array} \tag{3}$$

is a homeomorphism. For more details, see [12].

2.5. A localized Plancherel formula. The Plancherel formula for exponential Lie groups was first obtained by Duflo and Rais in [8]. Let $Z = \exp \mathfrak{z}$ be the center of G. Let $A = \exp \mathfrak{a}$ be a closed connected subgroup of Z and χ_η the unitary character of A associated to a fixed $\eta \in \mathfrak{a}^*$. Let

$$\mathfrak{g}_\eta^* = \{l \in \mathfrak{g}^* \ : \ l_{|\mathfrak{a}} = \eta\} \text{ and}$$

$$\widehat{G}_{\chi_\eta} = \{\pi \in \widehat{G} \ : \ \pi_{|A} = \chi_\eta \cdot \mathrm{Id}\}.$$

It follows from [12] that the orbit space $\mathfrak{g}_\eta^*/G$ is homeomorphic to $\widehat{G}_{\chi_\eta}$ via the Kirillov mapping (3). For $1 \leq p < +\infty$, the space $L^p(G/A, \chi_\eta)$ is defined as the set of all measurable functions $\varphi : G \to \mathbb{C}$ such that $\varphi(ga) = \overline{\chi_\eta(a)}\varphi(g)$ for all $g \in G$ and all $a \in A$ and

$$\|\varphi\|^p_{L^p(G/A,\chi_\eta)} = \int_{G/A} |\varphi(g)|^p dg < \infty.$$

For $p = 1$ we obtain a $*$-Banach algebra. The convolution here is defined for φ and φ' in $L^1(G/A, \chi_\eta)$ by:

$$\varphi * \varphi'(g) = \int_{G/A} \varphi(u)\varphi'(u^{-1}g)du, \ g \in G/A$$

and the involution * by:

$$f^*(x) = \Delta_G(x^{-1})\overline{f(x^{-1})}, \quad x \in G, f \in L^1(G/A, \chi_\eta).$$

In addition, the space $\widehat{G}_{\chi_\eta}$ is also the dual space of the algebra $L^1(G/A, \chi_\eta)$.

We denote by $\Omega_\eta \in \mathfrak{g}_\eta^*/G$ a coadjoint orbit and by $\pi_{\Omega_\eta} \in \widehat{G}_{\chi_\eta}$ the corresponding representation of G. By [11], there exists a non zero rational function ψ on $\mathfrak{g}^*$ such that:

$$\psi(x \cdot l) = \Delta_G(x^{-1})\psi(l), \text{ for all } x \in G \text{ and } l \in \mathfrak{g}^*. \tag{4}$$

Fix one such function ψ. There is a unique measure $\mu_{\psi,\eta}$ on $\mathfrak{g}_\eta^*/G$ such that, for all Borel function ϕ on $\mathfrak{g}^*$, we have by [8]:

$$\int_{\mathfrak{g}_\eta^*} \phi(l)|\psi(l)|dl = \int_{\mathfrak{g}_\eta^*/G} \int_{\Omega_\eta} \phi(l) d\beta_{\Omega_\eta}(l) d\mu_{\psi,\eta}(\Omega_\eta), \tag{5}$$

where $d\beta_{\Omega_\eta}$ is the canonical measure on Ω_η. Then $d\mu_{\psi,\eta}$ is called the localized Plancherel measure on $\mathfrak{g}_\eta^*/G \simeq \widehat{G}_{\chi_\eta}$.

We recall the fact that, if π is an irreducible representation of G in a Hilbert space $\mathscr{H}_\pi$, then by [7], there exists a unique self-adjoint and positive operator D_π in $\mathscr{H}_\pi$ which is semi-invariant with weight Δ_G^{-1}, which means that

$$\pi(g) D_\pi \pi(g)^{-1} = \Delta_G^{-1}(g) D_\pi,\ g \in G.$$

In the case where $\pi = \pi_{l,\mathfrak{b}}$ and $\Delta_{G|B} \equiv 1$, D_π is nothing else but the operator of multiplication by the function $\widetilde{\psi}$, where $\widetilde{\psi}$ is defined by: $\widetilde{\psi}(x) = \psi(x \cdot l),\ x \in G$ (see [8]). Then, for every $\phi \in C_c^\infty(G)$, for almost all $\Omega_\eta \in \mathfrak{g}_\eta^*/G$, the operator $D_{\pi_{\Omega_\eta}}^{\frac{-1}{2}} \pi_{\Omega_\eta}(\phi) D_{\pi_{\Omega_\eta}}^{\frac{-1}{2}}$ is trace class and

$$\operatorname{tr}(D_{\pi_{\Omega_\eta}}^{\frac{-1}{2}} \pi_{\Omega_\eta}(\phi) D_{\pi_{\Omega_\eta}}^{\frac{-1}{2}}) = \int_{\Omega_\eta} (\Gamma \cdot (\phi \circ \exp))^\wedge(l) |\psi(l)|^{-1} d\beta_{\Omega_\eta}(l),$$

where Γ is a positive $\mathrm{Ad}(G)-$invariant function on $\mathfrak{g}$, which does not depend on ψ.

By [7] and [8] the localized Plancherel formula reads:

$$\|\phi\|_2^2 = \int_{\mathfrak{g}_\eta^*/G} \operatorname{tr}(D_{\pi_{\Omega_\eta}}^{\frac{-1}{2}} \pi_{\Omega_\eta}(\phi^* * \phi) D_{\pi_{\Omega_\eta}}^{\frac{-1}{2}}) d\mu_{\psi,\eta}(\Omega_\eta). \tag{6}$$

On the other hand, we obtain a decomposition of the Plancherel measure: for a measurable function F on $\widehat{G}$, we have:

$$\int_{\mathfrak{g}^*/G} F(\pi_\Omega) d\mu(\Omega) = \int_{\mathfrak{a}^*} \int_{\mathfrak{g}_\eta^*/G} F(\pi_{\Omega_\eta}) d\mu_{\psi,\eta}(\Omega_\eta) d\eta.$$

2.6. Measurable cross-sections. In [6], the author makes use of the above constructions to build an explicit cross-section for generic coadjoint orbits involved in the Plancherel measure spectrum. More precisely, it is shown in ([6], corollary 2.2.6), that there exists an algorithm for constructing in a unique and natural way:

1. an explicit measurable cross-section $\mathscr{W}$ for almost all coadjoint orbits,
2. a Lebesgue measure $d\xi$ on $\mathscr{W}$,
3. a measurable field of irreducible unitary representations $\{\pi_\xi, \mathscr{H}_\xi, \xi \in \mathscr{W}\}$, associated with the parameter ξ via the Kirillov-Bernat correspondence, and for any $\xi \in \mathscr{W}$, a positive, self-adjoint, semi-invariant operator K_ξ acting on $\mathscr{H}_\xi$ such that

$$\phi(e) = \int_{\mathscr{W}} \operatorname{Tr}(K_\xi^{\frac{1}{2}} \pi_\xi(\phi) K_\xi^{\frac{1}{2}}) d\xi$$

for any smooth function ϕ on G having a compact support.

More precisely, for any $\xi \in \mathscr{W}$, one has that $K_\xi = c|Pf(\xi)|D_\xi$ where D_ξ is the multiplication operator determined by the modular function Δ_G on $\mathscr{H}_\xi$, $Pf(\xi)$ is the Pfaffian of an associated skew-symmetric matrix of ξ and c stands for a positive constant. Hence the Plancherel formula for the group G reads:

$$\|\phi\|^2_{L^2(G)} = \int_{\mathscr{W}} \|K_\xi^{\frac{1}{2}} \pi_\xi(\phi)\|^2_{HS} d\xi, \tag{7}$$

for $\phi \in L^1(G) \cap L^2(G)$. Here, for an operator T in a Hilbert space such that $T^* \circ T$ is trace class, the notation $\|T\|_{HS}$ denotes the Hilbert-Schmidt norm of T.

Let now $\mathscr{W}_\eta = \mathscr{W} \cap \mathfrak{g}^*_\eta$. The substitute of formula (6) becomes:

$$\|\phi\|^2_2 = \int_{\mathscr{W}_\eta} \|K_{\xi_\eta}^{\frac{1}{2}} \pi_{\xi_\eta}(\phi)\|^2_{HS} d(\xi_\eta), \ \phi \in L^1(G/A, \chi_\eta) \cap L^2(G/A, \chi_\eta) \tag{8}$$

where the measure $d(\xi_\eta)$ is induced on $\mathscr{W}_\eta$ by the Lebesque measure on $\mathscr{W}$. For the sake of simplicity, the measure $d(\xi_\eta)$ on $\mathscr{W}_\eta$ will merely be noted as $d\xi'$. On the other hand, we obtain a decomposition of the Plancherel measure: for a function $F \in \mathscr{C}_c(\mathscr{W})$, we have:

$$\int_{\mathscr{W}} F(\xi) d\xi = \int_{\mathfrak{a}^*} \int_{\mathscr{W}_\eta} F(\xi_\eta) d\xi_\eta d\eta = \int_{\mathfrak{a}^*} \int_{\mathscr{W}_\eta} F(\xi) d\xi' d\eta. \tag{9}$$

3. The main result

Our main result of this section is the following.

Theorem 3.1. *Let G be an exponential solvable Lie group with a non-trivial center. Let f be a measurable function on G fulfilling:*

$$\int_G \int_{\mathscr{W}} |f(g)|^2 \ \|K_\xi^{\frac{1}{2}} \pi_\xi(f)\|^2_{HS} \ e^{2\|g\| \ \|\xi\|} d\xi \ dg < +\infty. \tag{10}$$

Then f vanishes almost everywhere on G.

Proof. Any measurable function f meeting the condition (10) obviously belongs to $L^1(G) \cap L^2(G)$. As in the second section, we fix a Jordan-Hölder basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ such that $\mathfrak{s}_1 = \mathbb{R}X_1$ is a central ideal of $\mathfrak{g}$. For brevity, we write any function ϕ on G as a function depending on the variables $t_1, ..., t_n$ via the correspondence

$$\phi(t_1, ..., t_n) = \phi(\exp(t_n X_n) \cdots \exp(t_1 X_1)).$$

As X_1 is central in $\mathfrak{g}$, the function f will be a function of two variables (t, y) where $y = (y_2, ..., y_n) \in \mathbb{R}^{n-1}$ and

$$(t, y) := \exp(tX_1) \exp(y_n X_n) \cdots \exp(y_2 X_2). \tag{11}$$

Let $f_y(t) = f(t, y)$ and $f^*_y(t) = \overline{f_y(-t)}$. We define the function g by:

$$g(t) = \int_{\mathbb{R}^{n-1}} f_y * f^*_y(t) \ dy.$$

Then the corresponding Fourier transforms verify:

$$\widehat{g}(\xi) = \int_{\mathbb{R}^{n-1}} |\widehat{f_y}(\xi)|^2 \, dy$$

and

$$\int_{\mathbb{R}} \widehat{g}(\xi) \, d\xi = \int_{\mathbb{R}^n} |f_y(\xi)|^2 \, dy \, d\xi = \|f\|_2^2.$$

So, that will be sufficient to prove that the function g vanishes on $\mathbb{R}$ almost everywhere.

Let $A = \exp(\mathbb{R}X_1) = \exp(\mathfrak{s}_1)$ be the central subgroup of G, and identify A with $\mathbb{R}$. For $t \in \mathbb{R}$, we opt for the notation $\mathscr{W}_t$ instead of $\mathscr{W}_{tX_1^*}$. For a positive integer j, let also $V_j(t) =]t - \frac{1}{2j}, t + \frac{1}{2j}[$.

The following lemma is proved in [15] in the case where G is nilpotent. We here generate a proof for the exponential setting.

Lemma 3.2. *For $t \in \mathbb{R}$, we have:*

$$\widehat{g}(t) = \lim_{j \to +\infty} j \int_{V_j(t)} \int_{\mathscr{W}_\eta} \|K_\xi^{\frac{1}{2}} \pi_\xi(f)\|_{HS}^2 d\xi' d\eta.$$

Proof. For $u \in L^1(A)$, define $u * f$ on G by $u * f(x) = \int_{\mathbb{R}} u(t) f(t^{-1}x) dt$ and then g_u on $\mathbb{R}$ by

$$g_u(t) = \int_{\mathbb{R}^{n-1}} (u * f)_y * (u * f)_y^\star(t) dy$$

Thus g_u is obtained by replacing f with $u * f$ in the definition of the function g. Then it is easily verified that,

$$g_u(t) = \int_{\mathbb{R}^{n-1}} (u * f_y) * (u * f_y)^\star(t) dy.$$

Thus, for every $\eta \in \mathbb{R}$,

$$\begin{aligned} \widehat{g_u}(\eta) = \int_{\mathbb{R}^{n-1}} |\widehat{(u * f)_y}(\eta)|^2 dy &= \int_{\mathbb{R}^{n-1}} |\widehat{u * f_y}(\eta)|^2 dy \\ &= \int_{\mathbb{R}^{n-1}} |\widehat{u}(\eta)|^2 |\widehat{f_y}(\eta)|^2 dy \\ &= |\widehat{u}(\eta)|^2 \int_{\mathbb{R}^{n-1}} |\widehat{f_y}(\eta)|^2 dy \\ &= |\widehat{u}(\eta)|^2 \widehat{g}(\eta). \end{aligned}$$

Now, by the inversion formula for $\mathbb{R}$,

$$\int_{\mathbb{R}} \widehat{g_u}(\eta) d\eta = g_u(0) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |u * f_y(t)|^2 dt dy = \|u * f\|_2^2. \tag{12}$$

Besides, by the Plancherel formula for G,

$$\|u * f\|_2^2 = \int_{\mathscr{W}} \|K_\xi^{\frac{1}{2}} \pi_\xi(u * f)\|_{HS}^2 d\xi. \tag{13}$$

Since A is a central subgroup of G, we have for $\xi \in \mathscr{W}_\eta$,

$$\begin{aligned}\pi_\xi(u * f) &= \pi_\xi(u) \circ \pi_\xi(f) \\ &= \widehat{u}(\eta)\pi_\xi(f).\end{aligned}$$

and hence,

$$\|K_\xi^{\frac{1}{2}}\pi_\xi(u * f)\|_{HS}^2 = |\widehat{u}(\eta)|^2\|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2. \tag{14}$$

This implies using (9) and (14):

$$\begin{aligned}\int_{\mathscr{W}} \|K_\xi^{\frac{1}{2}}\pi_\xi(u * f)\|_{HS}^2 d\xi &= \int_{\mathbb{R}}\int_{\mathscr{W}_\eta} |\widehat{u}(\eta)|^2\|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 d\xi' d\eta \\ &= \int_{\mathbb{R}} |\widehat{u}(\eta)|^2 (\int_{\mathscr{W}_\eta} \|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 d\xi')d\eta.\end{aligned}$$

For $j, m \in \mathbb{N}$, let $u_{m,j} \in L^1(A)$ be such that $0 \leq \widehat{u_{m,j}}(t) \leq 1$ for $t \in \mathbb{R}$, and $(\widehat{u_{m,j}})_{m\in\mathbb{N}}$ converges point-wise to the characteristic function of $V_j(t)$. Since $\widehat{g}$ is continuous and $V_j(t)$ has length $\frac{1}{j}$, it follows that

$$\widehat{g}(t) = \lim_{j\to+\infty} j\int_{V_j(t)} \widehat{g}(\xi_1)d\xi_1 = \lim_{j\to+\infty}\lim_{m\to+\infty} j\int_{\mathbb{R}} \widehat{g_{u_{m,j}}}(\xi_1)d\xi_1.$$

But, by equations (12) and (13):

$$\begin{aligned}\|u_{m,j} * f\|_2^2 &= \int_{\mathbb{R}} \widehat{g_{u_{m,j}}}(\xi_1)d\xi_1 \\ &= \int_{\mathbb{R}} \widehat{g}(\xi_1)|\widehat{u_{m,j}}(\xi_1)|^2 d\xi_1 \\ &= \int_{\mathscr{W}} \|K_\xi^{\frac{1}{2}}\pi_\xi(u_{m,j} * f)\|_{HS}^2 d\xi \\ &= \int_{\mathbb{R}} |\widehat{u_{m,j}}(\eta)|^2 (\int_{\mathscr{W}_\eta} \|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 d\xi')d\eta.\end{aligned}$$

Then for $t \in \mathbb{R}$,

$$\begin{aligned}\widehat{g}(t) &= \lim_{j\to+\infty}\lim_{m\to+\infty} j\int_{\mathbb{R}} |\widehat{u_{m,j}}(\eta)|^2 \int_{\mathscr{W}_\eta} \|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 d\xi' d\eta \\ &= \lim_{j\to+\infty} j\int_{V_j(t)}\int_{\mathscr{W}_\eta} \|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 d\xi' d\eta.\end{aligned}$$

■

Lemma 3.3. *Let f meet the condition (10) of theorem 3.1. Then for $t \in \mathbb{R}$,*

$$\widehat{g}(t) = \int_{\mathscr{W}_t} \|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 \ d\xi'.$$

Proof. We have from lemma 3.2, that

$$\widehat{g}(t) = \lim_{j\to+\infty} j\int_{V_j(t)}\int_{\mathscr{W}_\eta} \|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 d\xi' \ d\eta.$$

Then using the localized Plancherel formula for G/A as developed in subsection (1), we get

$$\widehat{g}(t)=\lim_{j\to+\infty} j\int_{V_j(t)}\int_{G/A}|f^{\eta_1}(g_1)|^2 d\eta_1\, dg_1,$$

where $f^{\eta_1}(g_1)=\widehat{f}^1(\exp(\eta_1 X_1)g_1)$ is the Fourier transform with respect to the first central variable $\exp(\eta_1 X_1)$. Now,

$$j\int_{V_j(t)}|f^{\eta_1}(g_1)|^2 d\eta_1=|f^{c_j}(g_1)|^2$$

for some $c_j\in V_j(t)$. On the other hand, due to the identification (11),

$$|f^{c_j}(g_1)|^2=|\int_{\mathbb{R}} f(s,y)e^{-2\pi i s c_j}ds|^2\le\ (\int_{\mathbb{R}}|f(s,y)|ds)^2$$

for $g_1=\exp(y_n X_n)\cdots\exp(y_2X_2)$. Making use of the dominated convergence theorem, it is therefore enough to prove that the integral

$$E=\int_{\mathbb{R}^{n-1}}(\int_{\mathbb{R}}|f(s,y)|ds)^2 dy=\int_{G/A}(\int_{\mathbb{R}}|f(g_1e^{sX}|)ds)^2 dg_1 \tag{15}$$

converges. Let $\xi_0\in\mathscr{W}$ such that

$$\int_G|f(g)|^2e^{2\|g\|\ \|\xi_0\|}dg=\int_{\mathbb{R}^n}|f(s,y)|^2e^{2\|(s,y)\|\ \|\xi_0\|}dsdy<+\infty.$$

It follows that:

$$E=\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}e^{-|s|\ \|\xi_0\|}e^{|s|\ \|\xi_0\|}|f(s,y)|ds\right)^2 dy\le M\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}e^{2|s|\ \|\xi_0\|}|f(s,y)|^2dsdy<+\infty,$$

for some positive constant M, and this is using Hölder inequality. ■

We now come back to the proof of theorem 3.1. Let φ be a Schwartz function on $\mathbb{R}^{n-1}$ and F the function defined as:

$$F(t)=\int_{\mathbb{R}^{n-1}}f(t,y)\ \varphi(y)dy.$$

Then

$$\widehat{F}(t)=\int_{\mathbb{R}^{n-1}}\widehat{f}_y(t)\ \varphi(y)\ dy.$$

We get therefore

$$|\widehat{F}(t)|^2\le M\int_{\mathbb{R}^{n-1}}|\widehat{f}_y(t)|^2\ dy=M\ \widehat{g}(t)=M\ \int_{\mathscr{W}_t}\|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2\ d\xi',$$

for some positive constant M. On the other hand, let

$$B=\int_{\mathbb{R}^2}\frac{|F(t)|}{(1+|t|)}\ \frac{|\widehat{F}(\lambda)|}{1+|\lambda|}e^{|t\lambda|}dt\ d\lambda.$$

Then:

$$
\begin{aligned}
B &\leq \sqrt{M}\int_{\mathbb{R}}\frac{|F(t)|}{(1+|t|)}\int_{\mathbb{R}}\Big(\int_{\mathscr{W}_\lambda}\|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 e^{|2t\lambda|}d\xi'\Big)^{\frac{1}{2}}\frac{d\lambda}{1+|\lambda|}dt \\
&\quad \text{(we use Hölder inequality)} \\
&\leq c\int_{\mathbb{R}}\frac{|F(t)|}{1+|t|}\Big(\int_{\mathbb{R}}\int_{\mathscr{W}_\lambda}\|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 e^{2|t\lambda|}d\xi'\,d\lambda\Big)^{\frac{1}{2}}dt \\
&\quad \text{(we use again Hölder inequality)} \\
&\leq c'\Big(\int_{\mathbb{R}}|F(t)|^2\int_{\mathscr{W}}\|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 e^{2|t\lambda|}d\xi\,dt\Big)^{\frac{1}{2}} \\
&\leq c'\Big(\int_{\mathbb{R}}(\int_{\mathbb{R}^{n-1}}|\varphi(y)||f(t,y)|dy)^2dt\int_{\mathscr{W}}\|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 e^{2|t\lambda|}d\xi\Big)^{\frac{1}{2}} \\
&\quad \text{(we use once again Hölder inequality)} \\
&\leq c''\Big(\int_G\int_{\mathscr{W}}|f(g)|^2\|K_\xi^{\frac{1}{2}}\pi_\xi(f)\|_{HS}^2 e^{2\|\xi\|\,\|g\|}dg\,d\xi\Big)^{\frac{1}{2}} < +\infty,
\end{aligned}
$$

for some constants c, c' and c''. By theorem 1.1, we get $F = 0$ almost everywhere on $\mathbb{R}$. As φ is taken arbitrarily as a Schwartz function on $\mathbb{R}^{n-1}$, this is enough to conclude. ■

Remark 3.4. We now consider the exponential solvable two dimensional Lie group $G^2 := ax+b$ which is of a trivial center, whose Lie algebra $\mathfrak{g}$ admits a basis $\{X, Y\}$ such that $[X, Y] = Y$. This group only admits two infinite dimensional representations π_+ and π_- associated respectively to the linear forms $+Y^*$ and $-Y^*$ whose co-adjoint orbits are open sets in $\mathfrak{g}^*$. In fact, the representation $\pi_\pm = \mathrm{Ind}_N^G(\chi_{\pm Y^*})$ is induced from the Pukanszky polarization $N = \exp(\mathbb{R}Y)$ with the unitary character $\chi_{\pm Y^*}$ and is realized on the space $L^2(\mathbb{R}, \chi_{\pm Y^*})$ of square integrable functions fulfilling the covariance relation with respect to the character $\chi_{\pm Y^*}$. As such, the unitary dual of G^2 is then described as

$$\widehat{G^2} = \{\pi_\pm\} \amalg \{\chi_{\lambda X^*}, \lambda \in \mathbb{R}\}.$$

On the other hand, the map $\mathbb{R}^2 \to G^2$, $(t, s) \mapsto \exp(tX)\exp(sY)$ is a C^∞ diffeomorphism and maps the Lebesgue measure $dtds$ to a Haar measure on G^2. As above any function ϕ on G is then written as a function depending on the variables t and s via the correspondence $\phi(t, s) = \phi(\exp(tX)\exp(sY))$. The following proposition shows the failure of theorem 3.1 in this case.

Proposition 3.5. *For a positive real number α, the function defined on G^2 by $\phi(t, s) = e^{-\alpha\|(t,s)\|^2}\sin(s)$ satisfies condition (10) of theorem 3.1.*

Proof. For any $\pi\in\widehat{G^2}$, it is easy to check that $\pi(\phi)\neq 0$ if and only if $\pi\simeq\pi_\pm$. ■

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