

## Divergence and $q$ -Divergence in Depth 2

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**Abstract.** The Kashiwara-Vergne Lie algebra  $\mathfrak{kv}$  encodes symmetries of the Kashiwara-Vergne problem on the properties of the Campbell-Hausdorff series. It is conjectured that  $\mathfrak{kv} \cong \mathbb{K}t \oplus \mathfrak{grt}_1$ , where  $t$  is a generator of degree 1 and  $\mathfrak{grt}_1$  is the Grothendieck-Teichmüller Lie algebra. In the paper, we prove this conjecture in depth 2. The main tools in the proof are the divergence cocycle and the representation theory of the dihedral group  $D_{12}$ . Our calculation is similar to the calculation by Zagier of the graded dimensions of the double shuffle Lie algebra in depth 2.

In analogy to the divergence cocycle, we define the super-divergence and  $q$ -divergence cocycles (here  $q^l = 1$ ) on Lie subalgebras of  $\mathfrak{grt}_1$  which consist of elements with weight divisible by  $l$ . We show that in depth 2 these cocycles have no kernel. This result is in sharp contrast with the fact that the divergence cocycle vanishes on  $[\mathfrak{grt}_1, \mathfrak{grt}_1]$ .

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### 1. Introduction

The Kashiwara-Vergne problem in Lie theory [9] states the existence of an automorphism  $F$  of the free Lie algebra with two generators  $x$  and  $y$  which satisfies the following properties:

- The action of  $F$  preserves the conjugacy classes of generators  $x$  and  $y$ . That is, there exist  $g(x, y)$  and  $h(x, y)$  so as  $F(x) = e^g x e^{-g}$  and  $F(y) = e^h y e^{-h}$ .
- The action of  $F$  maps  $x + y$  to the Campbell-Hausdorff series,  $F(x + y) = \log(e^x e^y)$ .
- The Jacobian cocycle of  $F$  (see [2] for the precise definition) is of the form  $j(F) = f(\log(e^x e^y)) - f(x) - f(y)$ .

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The positive solution of the Kashiwara-Vergne problem (see [1] and [2]) implies the Duflo Theorem on the isomorphism of the center of the universal enveloping algebra and the ring of invariant polynomials, and various generalizations of algebraic and analytic aspects of the Duflo theorem.

The Kashiwara-Vergne Lie algebra  $\mathfrak{kv}$  encodes the symmetry of the Kashiwara-Vergne problem. The corresponding pro-unipotent group acts freely and transitively on the space of solutions of the Kashiwara-Vergne problem. It has been shown in [2] that the Grothendieck-Teichmüller Lie algebra  $\mathfrak{grt}_1$  injects into  $\mathfrak{kv}$ . It is conjectured that

$$\mathfrak{kv} \cong \mathbb{K}t \oplus \mathfrak{grt}_1,$$

where  $t$  is a generator of degree 1. Both Lie algebras  $\mathfrak{kv}$  and  $\mathfrak{grt}_1$  are graded by weight (the total number of  $x$ 's and  $y$ 's) and filtered by depth (the total number of  $y$ 's).

We prove this conjecture in depth 2. That is, we show that  $\mathfrak{kv}$  coincides with  $\mathbb{K}t \oplus \mathfrak{grt}_1$  up to elements of depth 3 for arbitrary weight. The main tools in the proof are the divergence cocycle  $\text{div}$  (the infinitesimal version of the Jacobian cocycle  $j$ ) and the representation theory of the dihedral group  $D_{12}$  acting on polynomials in two variables. It is interesting to note that polynomials satisfying the same symmetry appeared previously in the work of Zagier ([13], see also [8]), in calculation of the graded dimensions of the double shuffle Lie algebra. L. Schneps explained to us that her work [12] implies that the double shuffle Lie algebra injects into  $\mathfrak{kv}$ . This in combination with Zagier's result [13] could provide an alternative proof of the surjection of  $\mathfrak{kv}$  onto  $\mathfrak{grt}_1$  in depth 2.

In analogy to the divergence cocycle, we introduce the super-divergence and  $q$ -divergence cocycles for  $q$  a primitive root of unity of order  $l$ ,  $q^l = 1$ . They are defined on the Lie subalgebras of  $\mathfrak{grt}_1$  which consist of elements with weight divisible by  $l$ . The super-divergence cocycle plays an important role in Rouvière's theory of symmetric spaces (see [11]).

We prove that in depth 2 the super-divergence and  $q$ -divergence cocycles have no kernel. This result is in sharp contrast with the fact that the divergence cocycle vanishes on  $[\mathfrak{grt}_1, \mathfrak{grt}_1]$ . We conjecture that super-divergence and  $q$ -divergence cocycles have no kernel in arbitrary depth.

## 2. Cocycles on Lie algebras of tangential derivations.

**The Lie algebra of tangential derivations and the divergence cocycle.**

Let  $\mathfrak{lie}_n = \mathfrak{lie}(x_1, \dots, x_n)$  denote the degree completion of the free Lie algebra over a field  $\mathbb{K}$  on  $n$  variables, and let  $\text{Ass}_n = U(\mathfrak{lie}_n)$  be its universal enveloping algebra. Then  $\text{Ass}_n$  is a degree completion of the free associative algebra on  $n$  generators. We have

$$\mathfrak{lie}_n = \prod_{k=1}^{\infty} \mathfrak{lie}^k(x_1, \dots, x_n),$$

where  $\mathfrak{lie}^k(x_1, \dots, x_n)$  is spanned by the Lie words of length  $k$ . We will call  $\mathfrak{lie}^k(x_1, \dots, x_n)$  the graded component of  $\mathfrak{lie}_n$  of *weight*  $k$ .

When  $n = 2$ , we will use the variables  $(x, y)$  instead of  $(x_1, x_2)$ .

In [2], the following graded vector space was defined:

$$\mathfrak{tr}_n = \text{Ass}_n^+ / \langle (ab - ba), \quad a, b \in \text{Ass}_n \rangle,$$

where  $\text{Ass}_n^+$  is the augmentation ideal of  $\text{Ass}_n$ .

Recall the Lie algebra of tangential derivations on  $\mathfrak{lie}_n$ : a derivation  $u$  on  $\mathfrak{lie}_n$  is tangential if there exist  $a_1, \dots, a_n \in \mathfrak{lie}_n$  such that  $u(x_i) = [x_i, a_i]$  for all  $i = 1, \dots, n$ . The action of  $u$  on the generators completely determines the derivation.

The tangential derivations  $\mathfrak{tder}_n$  on  $\mathfrak{lie}_n$  form a Lie algebra. Namely, if  $u = (a_1, \dots, a_n)$ ,  $v = (b_1, \dots, b_n) \in \mathfrak{tder}_n$ , then  $[u, v] = (c_1, \dots, c_n)$ , where  $c_k = u(b_k) - v(a_k) + [a_k, b_k]$  for all  $k = 1, \dots, n$ .

Every element  $a \in \text{Ass}_n$  has a unique decomposition

$$a = a_0 + \sum_{k=1}^n (\partial_k a) x_k,$$

where  $a_0 \in \mathbb{K}$ , and  $(\partial_k a) \in \text{Ass}_n$ .

The divergence map  $\text{div} : \mathfrak{tder}_n \rightarrow \mathfrak{tr}_n$  was defined in [2] as follows. Let  $u = (a_1, \dots, a_n) \in \mathfrak{tder}_n$ . Then

$$\text{div}(u) = \sum_{k=1}^n \text{tr}(x_k (\partial_k a_k)) \in \mathfrak{tr}_n.$$

The divergence is a cocycle on  $\mathfrak{tder}_n$ , in the sense that  $u \cdot \text{div}(v) - v \cdot \text{div}(u) = \text{div}([u, v])$  for all  $u, v \in \mathfrak{tder}_n$ . (see [2], Proposition 3.6).

**Even tangential derivations and the superdivergence.** Similarly, we can define an even version of the above mentioned objects. Consider the subspace

$$\mathfrak{lie}_n^{\text{even}} = \prod_{k \in 2\mathbb{N}} \mathfrak{lie}^k(x_1, \dots, x_n).$$

This is a Lie subalgebra of  $\mathfrak{lie}_n$ . Its universal enveloping algebra  $\text{Ass}_n^{\text{even}} = U(\mathfrak{lie}_n^{\text{even}})$  is spanned by the words of even length.

For  $\text{Ass}_n^{\text{even}}$ , the *supertrace* graded vector space can be defined as follows.

**Definition 2.1.** The graded vector space  $\mathfrak{str}_n$  (the *supertrace*) is the quotient

$$\mathfrak{str}_n = \text{Ass}_n^{\text{even}} / \langle (x_i a + a x_i), \quad x_i \in \{x_1, \dots, x_n\}, \quad a \in \text{Ass}_n^+ \rangle.$$

Let  $\text{str}_n : \text{Ass}_n^{\text{even}} \rightarrow \mathfrak{str}_n$  denote the natural projection.

**Example 2.2.** The graded component  $\mathfrak{str}_2^4 \subset \mathfrak{str}_2$  is spanned by  $\{\text{str}(x^4), \text{str}(x^3y) = -\text{str}(yx^3) = -\text{str}(x^2yx) = \text{str}(xyx^2), \text{str}(x^2y^2) = -\text{str}(xy^2x) = \text{str}(y^2x^2) = -\text{str}(yx^2y), \text{str}(xyxy) = -\text{str}(yxyx), \text{str}(xy^3) = -\text{str}(y^3x) = -\text{str}(yx^2y) = \text{str}(y^2xy), \text{str}(y^4)\}$ .

Then we have a natural notion of even tangential derivations.

**Definition 2.3.** The subspace  $\mathfrak{tder}_n^{\text{even}} \subset \mathfrak{tder}_n$  is spanned by the tangential derivations  $u = (a_1, a_2, \dots, a_n)$  such that  $a_i \in \mathfrak{lie}_n^{\text{even}}$  for all  $i = 1, \dots, n$ .

**Proposition 2.4.** *The even tangential derivations  $\mathfrak{tder}_n^{\text{even}}$  form a Lie subalgebra of  $\mathfrak{tder}_n$ .*

**Proof.** If  $a_k, b_k \in \mathfrak{tder}_n^{\text{even}}$  for all  $k = 1, \dots, n$ , then  $c_k = u(b_k) - v(a_k) + [a_k, b_k] \in \mathfrak{tder}_n^{\text{even}}$  for all  $k = 1, \dots, n$ . ■

The even tangential derivations preserve  $\mathfrak{lie}_n^{\text{even}} \subset \mathfrak{lie}_n$  and  $\text{Ass}_n^{\text{even}} \subset \text{Ass}_n$ . The action descends to  $\mathfrak{str}_n$  along the natural projection. If  $u \in \mathfrak{tder}_n^{\text{even}}$ , and  $a \in \mathfrak{str}_n$ , we will denote the result of this action by  $u \cdot a \in \mathfrak{str}_n$ .

Now we can define the superdivergence map for  $\mathfrak{tder}_n^{\text{even}}$ .

**Definition 2.5.** The *superdivergence* is the map  $\text{sdiv} : \mathfrak{tder}_n^{\text{even}} \longrightarrow \mathfrak{str}_n$  given by the formula

$$\text{sdiv}(u = (a_1, \dots, a_n)) = \sum_{k=1}^n \text{str}(x_k(\partial_k a_k)) \in \mathfrak{str}_n.$$

**Proposition 2.6.** *The superdivergence is a cocycle on  $\mathfrak{tder}_n^{\text{even}}$ , in the sense that  $u \cdot \text{sdiv}(v) - v \cdot \text{sdiv}(u) = \text{sdiv}([u, v])$  for all  $u, v \in \mathfrak{tder}_n^{\text{even}}$ .*

**Proof.** See Proposition 2.12 below. ■

**The  $q$ -tangential derivations and the  $q$ -divergence.** More generally, let  $q$  be a primitive  $l$ -th root of unity,  $l \geq 1$ , and assume that the field  $\mathbb{K}$  contains  $q$ . Following the same logic as before, a notion of  $q$ -divergence can be defined.

Let  $\mathfrak{lie}_n^{\text{IN}}$  denote the Lie subalgebra of the free Lie algebra  $\mathfrak{lie}_n$  spanned by the Lie words of length divisible by  $l$ :

$$\mathfrak{lie}_n^{\text{IN}} = \prod_{k \in l\mathbb{N}} \mathfrak{lie}^k(x_1, \dots, x_n),$$

and let  $\text{Ass}_n^{\text{IN}}$  denote its universal enveloping algebra.

**Definition 2.7.** The graded vector space  $\mathfrak{qtr}_n$  (the  $q$ -trace) is the quotient

$$\mathfrak{qtr}_n = \text{Ass}_n^{\text{IN}} / \langle (x_i a - q a x_i), \quad x_i \in \{x_1, \dots, x_n\}, a \in \text{Ass}_n^+ \rangle.$$

Let  $\text{qtr}_n : \text{Ass}_n^{\text{IN}} \longrightarrow \mathfrak{qtr}_n$  denote the natural projection.

**Example 2.8.** Let  $l = 3$ . The graded component  $\mathfrak{qtr}_2^3 \subset \mathfrak{qtr}_2$  is spanned by  $\{\text{qtr}(x^3), \text{qtr}(x^2 y) = q \text{qtr}(x y x) = q^2 \text{qtr}(y x^2), \text{qtr}(x y^2) = q \text{qtr}(y^2 x) = q^2 \text{qtr}(y x y), \text{qtr}(y^3)\}$ .

**Definition 2.9.** The subspace  $\mathfrak{tder}_n^{\mathbb{N}} \subset \mathfrak{tder}_n$  is spanned by the tangential derivations  $u = (a_1, a_2, \dots, a_n)$  such that  $a_i \in \mathfrak{lie}_n^{\mathbb{N}}$  for all  $i = 1, \dots, n$ .

The  $q$ -tangential derivations  $\mathfrak{tder}_n^{\mathbb{N}}$  form a Lie subalgebra of  $\mathfrak{tder}_n$ . The action of  $\mathfrak{tder}_n^{\mathbb{N}}$  preserves  $\mathfrak{lie}_n^{\mathbb{N}} \subset \mathfrak{lie}_n$  and  $\text{Ass}_n^{\mathbb{N}} \subset \text{Ass}_n$ , and descends to  $\mathfrak{qtr}_n$  along the natural projection. If  $u \in \mathfrak{tder}_n^{\mathbb{N}}$ , and  $a \in \mathfrak{qtr}_n$ , we will denote the result of this action by  $u \cdot a \in \mathfrak{qtr}_n$ .

**Definition 2.10.** The  $q$ -divergence is the map  $\text{qdiv} : \mathfrak{tder}_n^{\mathbb{N}} \longrightarrow \mathfrak{qtr}_n$  given by the formula

$$\text{qdiv}(u = (a_1, \dots, a_n)) = \sum_{k=1}^n \text{qtr}(x_k(\partial_k a_k)) \in \mathfrak{qtr}_n.$$

**Remark 2.11.** If  $l = 2$  and  $q = -1$ , we recover the even tangential derivations, the supertrace and the superdivergence. Further on, we will consider the superdivergence as a particular case of  $q$ -divergence with  $l = 2$ .

**Proposition 2.12.** The  $q$ -divergence is a cocycle on  $\mathfrak{qtr}_n$ , in the sense that  $u \cdot \text{qdiv}(v) - v \cdot \text{qdiv}(u) = \text{qdiv}([u, v])$  for all  $u, v \in \mathfrak{tder}_n^{\mathbb{N}}$ .

**Proof.** Let  $u = (a_1, \dots, a_n)$ ,  $v = (b_1, \dots, b_n)$  such that the lengths of  $a_i$  and  $b_i$  are divisible by  $l$  for all  $i = 1, \dots, n$ . Using the computation in the proof of Proposition 3.6 in [2], we obtain:

$$\begin{aligned} \text{qdiv}([u, v]) &= \sum_{k=1}^n \text{qtr}\{x_k u(\partial_k b_k) - x_k v(\partial_k a_k)\} + \\ &+ \sum_{k=1}^n \text{qtr}\{-x_k(\partial_k b_k)a_k + x_k a_k(\partial_k b_k) + x_k(\partial_k a_k)b_k - x_k b_k(\partial_k a_k)\} + \\ &+ \sum_{k,i=1}^n \text{qtr}\{x_k(\partial_i b_k)x_i(\partial_k a_i) - x_k(\partial_i a_k)x_i(\partial_k b_i)\}. \end{aligned}$$

Consider the second line. Since the lengths of  $a_k$  and  $b_k$  are divisible by  $l$ , we have  $\text{qtr}(-x_k(\partial_k b_k)a_k) = \text{qtr}(-a_k x_k(\partial_k b_k))$  and  $\text{qtr}(x_k(\partial_k a_k)b_k) = \text{qtr}(b_k x_k(\partial_k a_k))$ . The third line is zero, since the length of the expression  $x_i(\partial_k a_i)$  is divisible by  $l$ . Therefore, we have

$$\begin{aligned} \text{qdiv}([u, v]) &= \sum_{k=1}^n \text{qtr}\{x_k u(\partial_k b_k) - x_k v(\partial_k a_k) + [x_k, a_k](\partial_k b_k) - [x_k, b_k](\partial_k a_k)\} = \\ &= \sum_{k=1}^n \text{qtr}\{u(x_k(\partial_k b_k)) - v(x_k(\partial_k a_k))\} = \\ &= u \cdot \text{qdiv}(v) - v \cdot \text{qdiv}(u). \quad \blacksquare \end{aligned}$$

**3. The Grothendieck-Teichmüller Lie algebra  $\mathfrak{grt}_1$  and the Kashiwara-Vergne Lie algebra  $\mathfrak{kv}$ .**

**The Grothendieck-Teichmüller Lie algebra  $\mathfrak{grt}_1$ .** Recall the Grothendieck-Teichmüller Lie algebra  $\mathfrak{grt}_1$  first introduced in [3]. First we need to define the  $n$ -strand braid Lie algebra  $\mathfrak{t}_n$ .

**Definition 3.1.** The Lie algebra  $\mathfrak{t}_n$  is generated by  $n(n - 1)/2$  elements  $t^{i,j} = t^{j,i}$ , where  $1 \leq i, j \leq n$  and relations

$$[t^{i,j}, t^{k,l}] = 0 \quad \text{if} \quad \{i, j\} \neq \{k, l\}, \quad \text{and}$$

$$[t^{i,j} + t^{i,k}, t^{j,k}] = 0,$$

for all triples of distinct indices  $i, j, k$ .

**Definition 3.2.** The Lie algebra  $\mathfrak{grt}_1$  is spanned by the elements  $(0, \psi) \in \mathfrak{tdet}_2$ , that satisfy the following relations:

$$\psi(x, y) = -\psi(y, x), \tag{3.1}$$

$$\psi(x, y) + \psi(y, z) + \psi(z, x) = 0 \quad \text{for} \quad x + y + z = 0, \tag{3.2}$$

$$\psi(t^{1,2}, t^{2,34}) + \psi(t^{12,3}, t^{3,4}) = \psi(t^{2,3}, t^{3,4}) + \psi(t^{1,23}, t^{23,4}) + \psi(t^{1,2}, t^{2,3}), \tag{3.3}$$

where the last equation takes values in the Lie algebra  $\mathfrak{t}$ , and  $t^{1,23} = t^{1,2} + t^{1,3}$ , etc.

**Example 3.3.** Let  $\psi = [x, [x, y]] - [y, [y, x]]$ . Then  $(0, \psi) \in \mathfrak{grt}_1$ .

The Lie algebra structure on  $\mathfrak{grt}_1$  is given by the Ihara bracket:

$$\{\psi_1, \psi_2\} = (0, \psi_1)(\psi_2) - (0, \psi_2)(\psi_1) + [\psi_1, \psi_2].$$

In [2], the following statement was shown (Theorem 4.1):

**Theorem 3.4.** *The map  $\nu : \psi \rightarrow (\psi(-x - y, x), \psi(-x - y, y))$  is an injective Lie algebra homomorphism from  $\mathfrak{grt}_1$  to  $\mathfrak{tdet}_2$ .*

**Soulé elements** Consider the free Lie algebra

$$\mathfrak{lie}_2 = \prod_{m=1}^{\infty} \mathfrak{lie}^m(x, y).$$

Let  $\deg(x) = \deg(y) = 1$ . Then  $\mathfrak{lie}^m(x, y)$  is the graded component of weight  $m$ . This graded component can be decomposed further into a direct sum of  $\mathbb{K}$ -vector spaces

$$\mathfrak{lie}^m(x, y) = \bigoplus_{i=1}^{m-1} \mathfrak{lie}^{(i, m-i)}(x, y),$$

the subspace  $\mathfrak{lie}^{(i, m-i)}(x, y)$  is spanned by the elements with  $x$ -degree  $i$  and  $y$ -degree  $(m - i)$ .

Similar decompositions exist for the universal enveloping algebra  $\text{Ass}_2$  with  $\deg(x) = \deg(y) = 1$ . Then we have

$$\text{Ass}_2 = \prod_{m=0}^{\infty} \text{Ass}^m(x, y), \quad \text{Ass}^m(x, y) = \bigoplus_{i=0}^m \text{Ass}^{(i, m-i)}(x, y),$$

where  $\text{Ass}^{(i, m-i)}$  is spanned by the monomials with  $x$ -degree  $i$  and  $y$ -degree  $(m - i)$ . The bigrading factors through to the vector spaces  $\mathfrak{tr}_2$ ,  $\mathfrak{str}_2$  and  $\mathfrak{qtr}_2$ .

**Notation:** We will denote by  $f^{(p,t)}$  the  $(p, t)$ -component of the element  $f$  lying in any of the spaces  $\mathfrak{lie}_2$ ,  $\text{Ass}_2$ ,  $\mathfrak{tr}_2$ ,  $\mathfrak{str}_2$  and  $\mathfrak{qtr}_2$ . For example, if  $f \in \mathfrak{tr}_2$ , then  $f^{(p,t)} = f \cap \mathfrak{tr}_2^{(p,t)}$ .

The following facts are known about  $\mathfrak{grt}_1$  (see, for example, [5]) :

**Theorem 3.5.** *a. The Lie algebra  $\mathfrak{grt}_1 \ni (0, \psi)$  is graded by weight of the elements  $\psi \in \mathfrak{lie}_2$  with  $\deg(x) = \deg(y) = 1$ . The first nontrivial component of  $\mathfrak{grt}_1$  lies in weight  $m = 3$ .*

$$\mathfrak{grt}_1 = \bigoplus_{m=3}^{\infty} \mathfrak{grt}_1^m.$$

*It also admits a decreasing filtration by depth, the smallest  $y$ -degree in any monomial in  $\psi$ . We will denote the associated graded component of depth  $n$  in  $\mathfrak{grt}_1$  by  $\mathfrak{grt}_1^{(n)}$ .*

*b. For each odd integer  $m \geq 3$  there exists a nonzero element  $\sigma_m \in \mathfrak{lie}^m(x, y)$ , called the Soulé element, such that  $(0, \sigma_m) \in \mathfrak{grt}_1^m$ . Hence the dimension  $\dim(\mathfrak{grt}_1^m) \geq 1$  for any odd  $m \geq 3$ .*

*c. For each odd integer  $m \geq 3$ , the element  $\sigma_m$  has depth 1. The  $\mathfrak{lie}^{(m-1,1)}(x, y)$ -component of  $\sigma_m$  is proportional to  $\text{ad}_x^{m-1} y$  with a nonzero coefficient.*

**Example 3.6.** [6] The Soulé elements below are normalized to have integer coefficients.

$$\begin{aligned} \sigma_3 &= [x, [x, y]] - [y, [y, x]]. \\ \sigma_5 &= 2[x, [x, [x, [x, y]]]] - 2[y, [y, [y, [y, x]]]] + 4[x, [x, [y, [x, y]]]] - 4[y, [y, [x, [y, x]]]] \\ &\quad - 3[[x, y], [x, [x, y]]] + 3[[y, x], [y, [y, x]]]. \end{aligned}$$

**Kashiwara-Vergne Lie algebra  $\mathfrak{kv}$ .** The Lie algebra  $\mathfrak{kv}$  was introduced in [2] as the Lie algebra of the group that acts freely and transitively on the space of solutions of the *Kashiwara-Vergne problem* [9]. (In [2], the algebra  $\mathfrak{kv}$  was denoted  $\widehat{\mathfrak{kv}}_2$ ).

**Definition 3.7.**  $\mathfrak{kv} := \{u = (a, b) \in \mathfrak{tdtr}_2$

$$[x, a] + [y, b] = 0, \tag{3.4}$$

$$\text{div}(u) = \text{tr}(-f(x + y) + f(x) + f(y)) \tag{3.5}$$

for some element  $f \in \mathfrak{tr}_1$ .

**Example 3.8.** The element  $t = (y, x)$  belongs to  $\mathfrak{k}\mathfrak{r}\mathfrak{v}$  with  $f = 0$ .

Then  $\mathfrak{k}\mathfrak{r}\mathfrak{v}$  is a Lie subalgebra of the algebra of tangential derivations  $\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_2$  (see [2], p. 14). Moreover, this Lie algebra contains the image of  $\mathfrak{g}\mathfrak{r}\mathfrak{t}_1$  under the map  $\nu$ .

**Theorem 3.9.** *The map  $\nu : \psi \rightarrow (\psi(-x - y, x), \psi(-x - y, y))$  is an injective Lie algebra homomorphism from  $\mathfrak{g}\mathfrak{r}\mathfrak{t}_1$  to  $\mathfrak{k}\mathfrak{r}\mathfrak{v}$ .*

The Lie algebra  $\mathfrak{k}\mathfrak{r}\mathfrak{v}$  is known to contain a one-dimensional central subalgebra generated by the element  $t = (y, x)$ , and a Lie subalgebra isomorphic to  $\mathfrak{g}\mathfrak{r}\mathfrak{t}_1$ . It was conjectured in [2] that in fact  $\mathfrak{k}\mathfrak{r}\mathfrak{v} \cong \mathbb{K}t \oplus \mathfrak{g}\mathfrak{r}\mathfrak{t}_1$ . One of the results of this paper, given below in Theorem 5.5 confirms this conjecture up to the elements of depth  $\geq 3$  in  $\mathfrak{g}\mathfrak{r}\mathfrak{t}_1$ .

To prove this theorem, we will need another presentation of  $\mathfrak{k}\mathfrak{r}\mathfrak{v}$ .

**Proposition 3.10.** *There is a Lie algebra isomorphism between  $\mathfrak{k}\mathfrak{r}\mathfrak{v}$  and the Lie subalgebra  $\mathfrak{k}\mathfrak{r}\mathfrak{v}'$  of  $\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_2$  defined as follows:*

$$\begin{aligned} & \{(0, \psi) \in \mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_2(z, y) : [y, \psi(z, y)] \in \text{im}(\text{ad}_{y+z}), \\ & \text{tr}(y\partial_y\psi_{\geq 2}) = \text{tr}(-f(-z) + f(-y - z) + f(y))\}, \end{aligned}$$

where  $\psi = \psi_1 + \psi_{\geq 2}$  with  $\psi_1 = c(z + y)$  of degree one and  $\psi_{\geq 2}$  of degree at least two.

**Proof.** The change of variables  $y = y, z = -x - y$  in  $\mathfrak{lie}_2$  induces a map taking the derivation

$$x \mapsto [x, a(x, y)], \quad y \mapsto [y, b(x, y)]$$

to the derivation

$$z \mapsto 0, \quad y \mapsto [y, b(-y - z, y)].$$

Let  $\psi(z, y) = b(-y - z, y)$ . Then the first condition (3.4) defining the algebra  $\mathfrak{k}\mathfrak{r}\mathfrak{v}$  is satisfied if and only if  $[y, \psi(z, y)] \in \text{im}(\text{ad}_{y+z})$ .

Now let  $u = (a(x, y), b(x, y))$  be a solution to the equation (3.4). It implies

$$\partial_x([x, a(x, y)] + [y, b(x, y)]) = 0,$$

which yields

$$x\partial_x a(x, y) = a(x, y) - y\partial_x b(x, y).$$

To check condition (3.5), we have to compute the divergence:

$$\text{div}(u)(x, y) = \text{tr}(x\partial_x a(x, y) + y\partial_y b(x, y)) = \text{tr}(a(x, y)) + \text{tr}(y(\partial_y b(x, y) - \partial_x b(x, y)))$$

Changing variables we obtain

$$\begin{aligned} \text{div}(u)(-z - y, y) &= \text{tr}(a(-z - y, y)) + \text{tr}(y(\partial_y b(-z - y, y) - \partial_x b(-z - y, y))) \\ &= \text{tr}(a(-z - y, y)) + \text{tr}(y\partial_y \psi(z, y)). \end{aligned}$$

The trace  $\text{tr}(a(-z - y, y))$  is zero on all elements  $a \in \mathfrak{lie}(z, y)$  of degree at least two, but not on elements of degree one. Writing  $\psi$  as a sum  $\psi = \psi_1 + \psi_{\geq 2}$ , one can easily show that  $\psi_1$  is proportional to  $(z + y)$ , and  $\psi_{\geq 2}$  has to satisfy the condition  $\text{tr}(y\partial_y\psi_{\geq 2}) = \text{tr}(-f(-z) + f(-y - z) + f(y))$ . ■

Morally, the algebra  $\mathfrak{kv}'$  is the same algebra  $\mathfrak{kv}$  viewed in a different coordinate system, obtained by a nonsingular linear change of variables.

**Corollary 3.11.** *The map  $\psi(x, y) \longrightarrow \psi(z, y)$  is an injective homomorphism of Lie algebras  $\mathfrak{grt}_1 \longrightarrow \mathfrak{kv}'$ .*

**Proof.** The map is the composition of the homomorphism

$$\nu : \psi(x, y) \longrightarrow (\psi(-x - y, x), \psi(-x - y, y)) = (a(x, y), b(x, y)) \in \mathfrak{kv}$$

and the isomorphism  $\mathfrak{kv} \cong \mathfrak{kv}'$  of Proposition 3.10 that maps

$$(a(x, y), b(x, y)) \longrightarrow (0, b(-y - z, y)) = (0, \psi(z, y)). \quad \blacksquare$$

**The target space of the injection of  $\mathfrak{grt}_1$  into  $\mathfrak{kv}'$  in depth 2.** Similarly to the algebra  $\mathfrak{grt}_1$ ,  $\mathfrak{kv}'$  admits a grading by weight with  $\deg(y) = \deg(z) = 1$  and a filtration by depth. The depth filtration in  $\mathfrak{kv}'$  is defined as follows:  $\psi$  is of depth  $n$  (denoted  $\psi \in \mathfrak{kv}'^{(n)}$ ) if all Lie words in  $\psi(z, y)$  contain at least  $n$   $y$ 's. The  $n$ -th graded component of the depth-associated graded Lie algebra is defined as

$$\mathfrak{gr}(\mathfrak{kv}')^{(n)} = \mathfrak{kv}'^{(n)} / \mathfrak{kv}'^{(n+1)}.$$

It is clear from the construction that the injective homomorphism  $\mathfrak{grt}_1 \longrightarrow \mathfrak{kv}'$  of Corollary 3.11 is compatible with the filtration by depth.

To see if the injective homomorphism  $\mathfrak{grt}_1 \longrightarrow \mathfrak{kv}'$  is also surjective in depth 2, our strategy will be to compute an upper bound for the dimension of the graded component  $\mathfrak{gr}(\mathfrak{kv}')^{(2)}$ , and compare it with the known lower bound for the dimension of  $\mathfrak{gr}(\mathfrak{grt}_1)^{(2)}$ .

To get upper bounds of the dimensions of the associated graded components  $\mathfrak{gr}(\mathfrak{kv}')^{(n)}$ , it is enough to consider the graded vector space  $\overline{\mathfrak{kv}}$  that is obtained by dropping the terms of depth greater than 1 in the relations for the  $\mathfrak{kv}'$ . Then we have

$$\overline{\mathfrak{kv}} = \{(0, \psi) \in \mathfrak{tder}_2(z, y) : \text{tr}(y\partial_y\psi) = \text{tr}(yg(z))\},$$

where  $g(z)$  is a formal power series in  $z$ . Indeed, for the elements  $\psi$  of depth greater than 1, the relation  $\text{tr}(y\partial_y\psi) = \text{tr}(-f(-z) + f(-y - z) + f(y))$  implies  $\text{tr}(y\partial_y\psi) = 0$ , and for the elements of depth 1 we obtain  $\text{tr}(y\partial_y\psi) = \text{tr}(yg(z))$ . The relation  $[y, \psi(z, y)] \in \text{im}(\text{ad}_{y+z})$  is trivial for  $\psi$  of any given depth  $n$ , because in this case it reads

$$[y, \psi^{(n)}(z, y)] = \text{ad}_y\phi^{(n)}(z, y) + \text{ad}_z\phi^{(n+1)}(z, y),$$

where  $\phi^{(n)}$  and  $\phi^{(n+1)}$  are some elements of depth  $n$  and  $n + 1$  respectively. Such elements always exist (for instance, one can take  $\phi^{(n)} = \psi^{(n)}$  and  $\phi^{(n+1)} = 0$ .)

The graded vector space  $\overline{\mathfrak{krv}}$  admits a filtration by depth, the smallest  $y$ -degree in any monomial of a given expression. There is a natural injection of  $\mathbf{gr}(\mathfrak{krv}') \rightarrow \overline{\mathfrak{krv}}$ : for  $\psi$  of depth  $n$  we simply take its part containing exactly  $n$   $y$ 's. Hence, the dimensions of the graded components of  $\overline{\mathfrak{krv}}$  in each depth give upper bounds for the dimensions of the graded components of  $\mathfrak{krv}'$  in the same depth. In particular, the dimensions of the weight-graded components of  $\overline{\mathfrak{krv}}$  of depth 2 give upper bounds for the dimensions of the corresponding components of  $\mathbf{gr}(\mathfrak{krv}')^{(2)}$ . Note that for  $\psi \in \overline{\mathfrak{krv}}$  of depth 2, the relation  $\mathrm{tr}(y\partial_y\psi) = \mathrm{tr}(yg(x))$  implies  $g(x) = 0$ . Therefore, to estimate the dimensions of the graded components of  $\mathbf{gr}(\mathfrak{krv}')^{(2)}$ , we need to find the dimensions of the graded components of the space

$$\mathbf{gr}(\overline{\mathfrak{krv}})^{(2)} = \{\psi \in \mathfrak{lie}(x, y)^{(2)} / \mathfrak{lie}(x, y)^{(3)} : \mathrm{tr}(y\partial_y\psi) = 0\}.$$

This computation will be carried out in Section 5, as a particular case of a more general computation.

#### 4. The action of the $q$ -divergence cocycle on $\nu(\mathbf{grt}_1)$ .

Recall that for  $q^l = 1$ , with  $l \geq 2$ , the  $q$ -divergence cocycle is defined on the elements of  $\mathfrak{tdet}_n$  with length of words divisible by  $l$ , by the formula  $\mathrm{qdiv}(u = (a_1, \dots, a_n)) = \sum_{k=1}^n \mathrm{qtr}(x_k(\partial_k a_k))$ . In particular, the case  $l = 1$  corresponds to divergence, and  $l = 2$  to the superdivergence. We would like to consider the action of this cocycle on  $\nu(\mathbf{grt}_1) \subset \mathfrak{tdet}_2$ .

#### The $q$ -divergence cocycle on the elements of depth 2.

**Example 4.1.** Let  $\psi = \sigma_3 = [x, [x, y]] - [y, [y, x]] \in \mathbf{grt}_1$ . It is easy to find that  $\nu(\psi) = ([y, [y, x]], [x, [x, y]])$ . Then we have,  $\mathrm{div}(\nu(\psi)) = \mathrm{tr}(xy^2 + yx^2) \neq 0$ . The superdivergence cannot be applied since the length of  $\nu(\psi)$  is odd. If  $q^3 = 1$ , then the  $q$ -divergence can be calculated:  $\mathrm{qdiv}(\psi) = \mathrm{qtr}(xy^2 + yx^2) \neq 0$ .

However, we know that the divergence cocycle has a large kernel on  $\nu(\mathbf{grt}_1)$ :

**Proposition 4.2.** (*[2], page 14 and Theorem 4.1*).  
*The commutant  $\{\mathbf{grt}_1, \mathbf{grt}_1\}$  lies in the kernel of the map*

$$\mathrm{div} \circ \nu : \mathbf{grt}_1 \longrightarrow \mathfrak{tt}_2.$$

We would like to find if, similarly, the  $\mathrm{qdiv}$  cocycle for  $q$  a root of unity of degree  $l \geq 2$  can have nontrivial kernel on  $\nu(\mathbf{grt}_1)$ . It is not hard to see that, in contrast with  $\mathrm{div}$ , the cocycle  $\mathrm{qdiv}$  for  $l \geq 2$  is nonzero on all commutators of the Soulé elements:

$$\mathrm{qdiv}(\nu(\{\sigma_m, \sigma_n\})) \neq 0.$$

Here  $m < n$  are odd numbers,  $m, n \geq 3$  and in case of  $q$ -divergence,  $m + n$  is divisible by  $l$ . For example, Lemma 2.5 in [10] contains an explicit formula for  $\{\sigma_m, \sigma_n\}$  modulo the elements of depth 3, that allows to prove the claim by a

direct computation. Below we will aim to obtain a more general result: if  $q \neq 1$ , the cocycle  $\text{qdiv}$  has a trivial kernel on  $\mathbf{gr}(\nu(\mathbf{grt}_1))^{(2)}$ .

First we will derive a simple formula for the action of  $\text{qdiv}$ , whenever applicable, on the elements of depth 2 in  $\nu(\mathbf{grt}_1)$ . This formula is also applicable in case of  $\text{div}$ .

**Lemma 4.3.** *Suppose  $(0, \psi) \in \mathbf{grt}_1$  is of weight  $2n$  and of depth  $2k$  for some natural  $k < n$ . Then for  $q$  a primitive  $l$ th root of unity with  $l \geq 1$ , whenever the  $q$ -derivation can be applied, we have*

$$\text{qdiv}(\nu(\psi))^{(2n-2k, 2k)} = \text{qtr} \left( y \partial_y \left( \psi(x, y)^{(2n-2k, 2k)} \right) \right).$$

**Proof.** Assume that the weight  $2n$  of  $\psi$  is divisible by the order of  $q$ , so that the  $q$ -derivation can be applied. Then we have

$$\text{qdiv}(\nu(\psi(x, y))) = \text{qtr} \left( x \partial_x \psi(-x - y, x) + y \partial_y \psi(-x - y, y) \right).$$

Using the equations (3.2) and (3.1), we can write

$$\psi(-x - y, y) = \psi(-x - y, x) + \psi(x, y).$$

Then

$$\text{qdiv}(\nu(\psi(x, y))) = \text{qtr} \left( x \partial_x \psi(-x - y, x) + y \partial_y \psi(-x - y, x) + y \partial_y \psi(x, y) \right).$$

The expression  $x \partial_x \psi(-x - y, x) + y \partial_y \psi(-x - y, x)$  differs from  $\psi(-x - y, x)$  by removing the last letter of each monomial and inserting it in front. Under the respective trace maps, it gives

$$\text{qtr} \left( x \partial_x \psi(-x - y, x) + y \partial_y \psi(-x - y, x) \right) = q \cdot \text{qtr}(\psi(-x - y, x)).$$

Now we will use the assumptions on the weight and depth of  $\psi$  to analyze the expression  $\text{qtr}(\psi(-x - y, x))$ . Using the equation (3.2) again, we can write

$$\text{qtr}(\psi(-x - y, x)) = \text{qtr}(\psi(-x - y, y) - \psi(x, y)).$$

We only need to consider the lowest possible  $y$ -degree component of

$$(\psi(-x - y, y) - \psi(x, y)).$$

Since  $\psi(x, y)$  has no component with  $y$ -degree less than  $2k$ , the minimal possible  $y$ -degree in  $\psi(-x - y, y)$  is also  $2k$ , and the component of  $\psi(-x - y, y)$  with the minimal  $y$ -degree is obtained by choosing the summand with  $-x$  each time  $(-x - y)$  occurs in the expression. We have  $\psi(-x - y, y)^{(2n-2k, 2k)}$

$$= \psi(-x, y)^{(2n-2k, 2k)} = (-1)^{2n-2k} \psi(x, y)^{(2n-2k, 2k)} = \psi(x, y)^{(2n-2k, 2k)}.$$

Then we obtain

$$\text{qtr} \left( \psi(-x - y, x)^{(2n-2k, 2k)} \right) = \text{qtr} \left( (\psi(x, y) - \psi(x, y))^{(2n-2k, 2k)} \right) = 0.$$

Finally,

$$\text{qdiv}(\nu(\psi(x, y)))^{(2n-2k, 2k)} = \text{qtr} \left( y \partial_y (\psi(x, y)^{(2n-2k, 2k)}) \right). \quad \blacksquare$$

It is well known that the weight-associated graded component  $\mathbf{gr}(\mathbf{grt}_1)^{(2)}$  is spanned by the corresponding parts of the commutators of the Soulé elements. All these elements have even total weight, and  $y$ -degree equal to 2. Then by Lemma 4.3, studying the action of  $\mathbf{qdiv}$  on this space is equivalent to studying the action of  $\mathbf{qtr}(y\partial_y \cdot)$  on the elements of  $\mathbf{lie}(x, y)$  of even weight and  $y$ -degree 2.

Therefore, we are interested in the dimensions of the weight graded components of the space

$$\ker(\mathbf{qdiv})(\mathbf{gr}(\nu(\mathbf{grt}_1))^{(2)}) = \{\psi \in \mathbf{lie}(x, y)^{(2)} / \mathbf{lie}(x, y)^{(3)} : \mathbf{qtr}(y\partial_y \psi) = 0\}.$$

This computation will be given in the next section.

**5. The upper bound for  $\mathbf{gr}(\mathbf{krv}')^{(2)}$  and the kernel of  $q$ -divergence on  $\mathbf{gr}(\nu(\mathbf{grt}_1))^{(2)}$ .**

The set of relations that determines the upper bound for the graded components of  $\mathbf{gr}(\mathbf{krv}')^{(2)}$  (Section 3), is a particular case (when  $q = 1$ ) of the relations that define the kernel of the  $q$ -divergence on the image of  $\mathbf{grt}_1$  in depth 2 (Section 4). We will use the same combinatorial method to establish the dimensions of the graded components in both cases.

**Combinatorial reformulation of the question.** Let  $q$  be a primitive  $l$ th root of unity with  $l \geq 1$ . We are interested in the dimensions of the graded components of the space

$$\{\psi \in \mathbf{lie}(x, y)^{(2)} / \mathbf{lie}(x, y)^{(3)} : \mathbf{qtr}(y\partial_y \psi) = 0\}.$$

We are interested in the action of  $\mathbf{qtr}(y\partial_y \cdot)$  on the elements of the free Lie algebra in two variables  $(x, y)$  with exactly 2  $y$ 's in any of its Lie words. In other words, we are interested in the kernel of the operator  $\mathbf{qtr}(y\partial_y \cdot)$  acting on the elements of the form

$$\sum_k c_{l,k} [\mathbf{ad}_x^k y, \mathbf{ad}_x^l y],$$

where  $c_{l,k} \in \mathbb{K}$  are some coefficients. Because the operator  $\mathbf{qtr}(y\partial_y \cdot)$  preserves the weight, it is enough to consider its action on the graded components of a given weight  $n$ ,

$$\sum_{k=0}^n c_k [\mathbf{ad}_x^k y, \mathbf{ad}_x^{n-k} y],$$

with some coefficients  $c_k, k = 0, \dots, n$ , for every natural  $n$  divisible by  $l$ .

**Proposition 5.1.** *For each  $n$  divisible by  $l$ , the dimension of the vector space*

$$\mathbf{qker}_n = \{\psi \in \mathbf{lie}(x, y), \psi = \sum_{k=0}^n c_k [\mathbf{ad}_x^k y, \mathbf{ad}_x^{n-k} y], \mathbf{qtr}(y\partial_y \psi) = 0\}$$

*is equal to the dimension of the vector space of homogeneous polynomials of degree  $n$  in two variables  $p(v, w)$  satisfying*

$$\begin{aligned} p(v, w) &= -q \cdot p(v + w, (q - 1)v - w) \\ p(v, w) &= -p(w, v) \end{aligned}$$

In particular, for  $q = 1$ , we obtain the following relations:

$$\begin{aligned} p(v, w) &= -p(v + w, -w) \\ p(v, w) &= -p(w, v) \end{aligned}$$

**Proof.** The generating series for the expressions of the form  $\sum_{k=0}^n c_k [\text{ad}_x^k y, \text{ad}_x^{n-k} y]$  for all natural  $n$  can be written as

$$\phi(\alpha, \beta) = [e^{\alpha \text{ad}_x} y, e^{\beta \text{ad}_x} y].$$

Consider the action of  $\text{qtr}(y\partial_y \cdot)$  on this generating series :

$$\begin{aligned} \text{qtr}(y\partial_y \phi(\alpha, \beta)) &= \text{qtr}(y\partial_y (e^{\alpha x} y e^{-\alpha x} e^{\beta x} y e^{-\beta x} - e^{\beta x} y e^{-\beta x} e^{\alpha x} y e^{-\alpha x})) \\ &= \text{qtr}(y (e^{\alpha x} y e^{-\alpha x} e^{\beta x} - e^{\beta x} y e^{-\beta x} e^{\alpha x})) \\ &= q \cdot \text{qtr}(e^{\alpha x} y e^{-\alpha x} e^{\beta x} y) - q \cdot \text{qtr}(e^{\beta x} y e^{-\beta x} e^{\alpha x} y). \end{aligned}$$

Here we have used  $e^{-\beta x} = (\sum_{m=1}^{\infty} \frac{(-\beta)^m x^m}{m!})$  which implies that  $e^{\alpha x} y e^{-\alpha x} e^{\beta x} y e^{-\beta x}$  only has one term ending with  $y$ . Thus  $\text{qtr}(y\partial_y \phi(\alpha, \beta))$  is an infinite series with terms of the form  $\text{qtr}(x^l y x^m y)$ , that we have to consider up to the  $\text{qtr}$ -symmetry:

$$\text{qtr}(x^m y x^k y) = q^{(m+1)} \text{qtr}(x^k y x^m y) = q \cdot \text{qtr}(x^k y (qx)^m y).$$

These expressions are in one-to-one correspondence with the sums of monomials in two commuting variables,

$$u^m v^k + q \cdot (qv)^m u^k.$$

Applying this correspondence, we transform  $\text{qtr}(y\partial_y \phi(x, y))$  into

$$T(\alpha, \beta; u, v) \equiv q \cdot (e^{\alpha u} e^{(\beta-\alpha)v}) - q \cdot (e^{\beta u} e^{(\alpha-\beta)v}) + q^2 \cdot (e^{\alpha q v} e^{(\beta-\alpha)u}) - q^2 \cdot (e^{\beta q v} e^{(\alpha-\beta)u}).$$

Differentiating  $T(\alpha, \beta; u, v)$  several times with respect to  $\alpha$  and  $\beta$  and setting these parameters equal to zero allows to recover the part of the expression of a given degree in  $u$  and  $v$ . In particular, if we want to recover the action of  $\text{qtr}(y\partial_y \cdot)$  on  $\psi = \sum_{k=0}^n c_k [\text{ad}_x^k y, \text{ad}_x^{n-k} y]$ , we have to apply the differential operator  $p\left(\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}\right) \Big|_{\alpha=0, \beta=0}$ , where  $p(t, s) = \sum_{k=0}^n c_k s^k t^{n-k}$ . Applying  $p\left(\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}\right) \Big|_{\alpha=0, \beta=0}$  to  $T(\alpha, \beta; u, v)$  gives

$$q \cdot p(u - v, v) - q \cdot p(v, u - v) + q^2 \cdot p(qv - u, u) - q^2 \cdot p(u, qv - u).$$

Because of the Lie antisymmetry of  $\psi$ , the polynomial  $p(t, s)$  satisfies the condition  $p(t, s) = -p(s, t)$  with respect to the swap of the variables. Therefore, we have

$$p\left(\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}\right) \Big|_{\alpha=0, \beta=0} (T(\alpha, \beta; u, v) = -2q \cdot p(v, u - v) - 2q^2 \cdot p(u, qv - u).$$

Setting this expression equal to zero, we have the equation

$$p(v, u - v) = -q \cdot p(u, qv - u).$$

With the change of variables  $v \rightarrow v$ ,  $u - v \rightarrow w$ , we obtain the condition

$$p(v, w) = -q \cdot p(v + w, (q - 1)v - w),$$

which together with the antisymmetry yields the system announced in the Proposition. ■

**Upper bounds for the dimensions of the graded components of  $\mathbf{gr}(\mathfrak{kv}')$ <sup>(2)</sup>.**

For each  $n \geq 1$ , denote by  $\ker_n$  the following vector space

$$\ker_n = \{ \psi \in \mathfrak{lie}(x, y), \psi = \sum_{k=0}^n c_k [\text{ad}_x^k y, \text{ad}_x^{n-k} y], \text{tr}(y \partial_y \psi) = 0 \}$$

Then by Proposition 5.1, the dimension of the space  $\ker_n$  is equal to the dimension of the space of homogeneous polynomials of degree  $n$  in two variables, satisfying the relations

$$\begin{aligned} p(v, w) &= -p(v + w, -w) \\ p(v, w) &= -p(w, v) \end{aligned}$$

According to the discussion at the end of Section 3, the dimensions of  $\ker_n$  are equal to the dimensions of the weight associated graded components of  $\mathbf{gr}(\overline{\mathfrak{kv}})^{(2)} = \{ \psi \in \mathfrak{lie}(x, y)^{(2)} / \mathfrak{lie}(x, y)^{(3)} : \text{tr}(y \partial_y \psi) = 0 \}$  and give the upper bounds for the dimensions of the weight associated components of  $\mathbf{gr}(\mathfrak{kv}')$ <sup>(2)</sup>. We will compute these dimensions.

Note that the given transformations of homogeneous polynomials are implemented by the action of the group generated by the following changes of variables:

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} w \\ v \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} w + v \\ -w \end{pmatrix}$$

This is the dihedral group  $D_{12} = \{s^2 = 1, r^6 = 1, srs^{-1} = r^{-1}\}$ , where  $s = a$  and  $r = ba$ .

This group acts on polynomials by  $g \cdot p(v, w) = p(g^{-1}(v, w))$ . Since the action is homogeneous, it induces a representation of  $D_{12}$  on the  $(n + 1)$ -dimensional space of coefficients of degree  $n$  homogeneous polynomials in two variables. Then the dimension of  $\ker_n$  for each natural  $n$  is equal to the dimension of the subrepresentation where both  $a = s$  and  $b = rs$  act as  $-1$ .

**Example 5.2.** Let us consider the action of  $s$  and  $rs \in D_{12}$  on the homogeneous polynomials of degree  $n = 1$ ,  $p(v, w) = av + bw$ . We have  $s \cdot p(v, w) = p(s(v, w)) = p(w, v) = bv + aw$ , and  $rs \cdot p(v, w) = p((rs)^{-1}(v, w)) = p(v + w, -w) = av + (a - b)w$ . For both  $s$  and  $rs$  to act by  $-1$ , the conditions on the coefficients  $(a, b)$  are  $b = -a, a = -b, a = -a, (a - b) = -b$  which yields only the trivial solution. Hence the dimension of  $\ker_1$  is zero.

In general, our strategy will be to decompose the action of  $D_{12}$  on the degree  $n$  homogeneous polynomials in two variables for each natural  $n$  into a direct sum of irreducible representations, and find the dimension of the subspace where both elements  $a = s$  and  $b = rs$  act as  $-1$ .

**Theorem 5.3.** *The dimension of  $\ker_n$  is as follows:*

$$\dim(\ker_n) = \begin{cases} 0, & n \text{ odd;} \\ \lfloor \frac{n}{6} \rfloor & n \text{ even} \end{cases}$$

**Proof.** Here is the character table of  $D_{12}$ , where we denoted by  $\chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}$  the characters of the irreducible representations of degree 1, and by  $\chi_{21}$  and  $\chi_{22}$  those of degree 2:

	$e$	$s$	$r$	$r^2$	$r^3$	$rs$
$\chi_{11}$	1	1	1	1	1	1
$\chi_{12}$	1	1	-1	1	-1	-1
$\chi_{13}$	1	-1	1	1	1	-1
$\chi_{14}$	1	-1	-1	1	-1	1
$\chi_{21}$	2	0	1	-1	-2	0
$\chi_{22}$	2	0	-1	-1	2	0

The only representation that satisfies our requirements is  $\chi_{13}$ , that has the property  $\chi_{13}(s) = \chi_{13}(rs) = -1$ . We will compute the character  $\chi_n$  of the representation of  $D_{12}$  on the  $(n + 1)$ -dimensional space of the coefficients of the homogeneous polynomials in two variables, as defined above. Then for each  $n$  we will look for the multiplicity of  $\chi_{13}$  in  $\chi_n$ .

Computation given in the Appendix leads to the following character table for  $\chi_n$ :

	$e$	$s$	$r$	$r^2$	$r^3$	$rs$
$\chi$	$n + 1$	1	1	1	$n + 1$	1 if $n \equiv 0 \pmod 6$
$\chi$	$n + 1$	0	1	-1	$-(n + 1)$	0 if $n \equiv 1 \pmod 6$
$\chi$	$n + 1$	1	0	0	$n + 1$	1 if $n \equiv 2 \pmod 6$
$\chi$	$n + 1$	0	-1	1	$-(n + 1)$	0 if $n \equiv 3 \pmod 6$
$\chi$	$n + 1$	1	-1	-1	$n + 1$	1 if $n \equiv 4 \pmod 6$
$\chi$	$n + 1$	0	0	0	$-(n + 1)$	0 if $n \equiv 5 \pmod 6$

For each  $n$  we use the orthogonality of characters to compute the multiplicity of  $\psi_{13}$  in the decomposition. We get

$$\langle \chi, \chi_{13} \rangle = \frac{1}{12} \sum_{g \in D_{12}} \chi(g) \cdot \chi_{13}(g^{-1}) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{n}{6} & \text{if } n \equiv 0 \pmod 6 \\ \frac{n-2}{6} & \text{if } n \equiv 2 \pmod 6 \\ \frac{n-4}{6} & \text{if } n \equiv 4 \pmod 6 \end{cases}$$

Therefore, the multiplicity of the character  $\chi_{13}$  in the character  $\chi_n$  for even  $n$  equals  $\lfloor \frac{n}{6} \rfloor$ . This is the dimension of the space of the degree  $n$  homogeneous polynomials in two variables that satisfy the conditions  $p(w + v, -w) = -p(v, w)$  and  $p(w, v) = -p(v, w)$ . Then applying Proposition 5.1, we deduce that the dimension of  $\ker_n$  is  $\lfloor \frac{n}{6} \rfloor$  for even  $n$ , and zero for odd  $n$ . ■

**Remark 5.4.** In [13], Zagier computed graded dimensions of the double shuffle Lie algebra in depth 2. According to [8], his calculation gives rise to a pair of

functional equations

$$p(u, w) = -p(w, u) \quad , \quad p(u + w, u) + p(u + w, w) = 0.$$

These equations do not coincide with ours, but they also lead to a representation of the group  $D_{12}$ , isomorphic to the one we obtained above.

**Surjection  $\mathfrak{grt}_1 \rightarrow \mathfrak{kv}'$  in depth 2.**

**Theorem 5.5.** *The injective Lie algebra homomorphism  $\mathfrak{grt}_1 \rightarrow \mathfrak{kv}'$  is surjective on the elements of depth 2 modulo the elements of higher depth.*

**Proof.** Theorem 5.3 provides an upper bound on the dimensions of the associated graded components of depth 2 in each degree of the algebra  $\mathfrak{kv}'$ . The weight of an element of  $\mathfrak{kv}'$  (total number of  $x$ 's and  $y$ 's in a Lie word) is related to the degree  $n$  as follows:  $k = n + 2$ . Then for the elements of weight  $k$ , this upper bound is equal to  $\lfloor \frac{k-2}{6} \rfloor$  for even  $k$  and 0 for odd  $k$ .

On the other hand, we can compute a lower bound for the dimension of the associated graded components of depth 2 in  $\mathfrak{grt}_1$  for each given weight. We know that all commutators of the Soulé elements  $\{\sigma_i, \sigma_{k-i}\}$ ,  $3 \leq i \leq k-3$ ,  $i$  odd, taken modulo the elements of depth 3, are among the depth 2 elements of weight  $k$ . Taking into account the antisymmetry of the Ihara bracket, there are  $\lfloor \frac{k-4}{4} \rfloor$  different commutators of Soulé elements in the depth-associated graded component of a weight  $k$  and depth 2. In addition, it is shown by Goncharov [4], and independently by Ihara and Takao in [6], that the dimension of the space of all linear relations of the form

$$\sum_{i=3, i \text{ odd}}^{k-3} a_i \{\sigma_i, \sigma_{k-i}\} \equiv 0 \pmod{\mathfrak{grt}_1^{(3)}}$$

is given by :

$$\left\lfloor \frac{k-4}{4} \right\rfloor - \left\lfloor \frac{k-2}{6} \right\rfloor.$$

Therefore, the dimension spanned by the commutators of the Soulé elements of weight  $k$  and depth 2, modulo elements of higher depth, is exactly  $\lfloor \frac{k-2}{6} \rfloor$ . This is the lower bound for the dimension of the depth-associated graded component of  $\mathfrak{grt}_1$  of depth 2 and weight  $k$ . Since  $\mathfrak{grt}_1 \mapsto \mathfrak{kv}'$  is a depth-preserving injection, and the lower bound for  $\mathfrak{grt}_1$  coincides with the upper bound for  $\mathfrak{kv}'$  in the depth-associated graded component of depth 2 in each weight, we have a surjection on the elements of depth 2, modulo the elements of higher depth. ■

**Example 5.6.** The first degree where the dimension of  $\ker_n$  is positive is  $n = 6$ , which corresponds to the Lie words of degree 8. Since we know that  $\mathfrak{grt}_1$  coincides with  $\mathfrak{kv}'$  in depth 2, the depth 2 graded component of  $\mathfrak{kv}'$  of degree 8 is 1-dimensional and is spanned by the element

$$\{\sigma_3, \sigma_5\} = 5[\text{ad}_x^4 y, \text{ad}_x^2 y] + 2[\text{ad}_x^5 y, \text{ad}_x y].$$

**Remark 5.7.** L. Schneps explained to us that her results in [12] imply that the equation  $\text{tr}(y\partial_y\psi) = 0$  in depth 2 is equivalent to the stuffle equation in the definition of the double shuffle Lie algebra. This argument shows that the double shuffle Lie algebra and  $\mathfrak{krv}'$  are isomorphic in depth 2. Hence, the calculation by Zagier [13] of graded dimensions of the double shuffle Lie algebra can be used to give an alternative proof of our result above.

**The kernel of the  $q$ -divergence cocycle on  $\nu(\mathfrak{grt}_1)$  in depth 2.**

**Theorem 5.8.** *Let  $l$  be a positive integer,  $l \geq 2$ , and  $q$  a primitive  $l$ th root of unity. The kernel of the  $q$ -divergence cocycle on the elements of  $\mathfrak{gr}(\nu(\mathfrak{grt}_1))^{(2)}$  with weight divisible by  $l$ , is trivial.*

In view of the discussion in Section 4, and Proposition 5.1, Theorem 5.8 will follow from Proposition 5.9 below.

**Proposition 5.9.** *Let  $q$  be a primitive  $l$ th root of unity with  $l \geq 2$ . No homogeneous polynomial in two variables satisfies the conditions*

$$\begin{aligned} p(v, w) &= -q \cdot p(v + w, (q - 1)v - w), \\ p(v, w) &= -p(w, v). \end{aligned}$$

**Proof.** To find the homogeneous polynomials in each degree divisible by  $l$  that satisfy the obtained conditions, consider the transformation of the variables

$$b : \begin{pmatrix} v \\ w \end{pmatrix} \rightarrow \begin{pmatrix} v + w \\ (q - 1)v - w \end{pmatrix}; \quad a : \begin{pmatrix} v \\ w \end{pmatrix} \rightarrow \begin{pmatrix} w \\ v \end{pmatrix}.$$

Then applying  $(ab)^{-1}$  to the polynomial  $p(v, w)$ , we obtain the following equation for an antisymmetric homogeneous polynomial:

$$p(v, w) = q \cdot (ab)^{-1} \cdot p(v, w) = q \cdot p((ab) \cdot (v, w)) = q \cdot p((q - 1)v - w, v + w).$$

The transformation of variables  $ab$  is represented by the following matrix:

$$ab = \begin{pmatrix} q - 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix is diagonalizable with eigenvalues  $\lambda_{1,2} = \frac{q \pm \sqrt{q^2 - 4q}}{2}$ . The product of these eigenvalues is  $\lambda_1 \cdot \lambda_2 = q$ . Therefore, with a suitable linear change of variables in the polynomial  $p(v, w)$ , we have an equation for another homogeneous polynomial  $P(V, W)$  in variables  $V, W$  of the same total degree as  $p(v, w)$ :

$$P(V, W) = q \cdot P(\lambda_1 V, \lambda_2 W).$$

To solve this equation, we have to satisfy the relations of the form

$$V^t W^s = q \cdot (\lambda_1)^t (\lambda_2)^s V^t W^s$$

for each monomial in  $P(V, W)$ . Suppose that  $t \geq s$  (the case  $t \leq s$  is solved by the same argument), then we have

$$1 = q \cdot (\lambda_1 \lambda_2)^s \lambda_1^{t-s} = q \cdot q^s \lambda_1^{t-s}.$$

If  $t > s$ , then the absolute value of  $\lambda_1$  (and therefore also of  $\lambda_2$ ) has to be 1. Consider the equation

$$\lambda = \frac{q \pm \sqrt{q^2 - 4q}}{2} = e^{i\phi}$$

where  $\phi$  is a real number. The only solution of this equation is  $\phi = \pm \frac{\pi}{3}$  and  $q = 1$ . This corresponds to the case of divergence, that we considered before, and where obtained many nontrivial solutions.

For the  $q$ -divergence with  $l \geq 2$ , we have to have  $t = s$ , and moreover  $1 = q^{s+1}$ . By the assumption on the degree of the polynomial  $p(v, w)$ , we have  $s + t + 2 = 2s + 2 = lk$  for some positive integer  $k$ . Then  $q^{s+1} = q^{\frac{lk}{2}} = 1$  is satisfied if and only if  $k$  is even. We observe that for  $l \geq 2$  the equation

$$P(V, W) = q \cdot P(\lambda_1 V, \lambda_2 W)$$

can have nontrivial solutions only if the total degree of the polynomial is equal to  $lk - 2$  with even  $k$ . These solutions are of the form  $(VW)^s$ , where  $s = \frac{lk}{2} - 1$ . The expression  $VW$  is a quadratic solution of the same equation.

To find  $VW$  in terms of the variables  $(v, w)$ , all we need to do is to find all homogeneous quadratic solutions of the equation  $p(v, w) = q \cdot p((q-1)v-w, v+w)$ . Write  $p(v, w) = c_1 v^2 + c_2 vw + c_3 w^2$ . Then we have the following system of equations for the coefficients  $c_1, c_2, c_3$ :

$$\begin{cases} (q(q-1)^2 - 1)c_1 & +q(q-1)c_2 & +qc_3 & = 0 \\ -2q(q-1)c_1 & +(q^2 - 2q - 1)c_2 & +2qc_3 & = 0 \\ qc_1 & -qc_2 & +(q-1)c_3 & = 0 \end{cases}$$

In particular, the system implies

$$\begin{cases} c_1(q^3 - q^2 - 1) & = -c_3(q^2 - q + 1) \\ 2c_1(q^3 - q^2 - 1) & = -c_2(q^2 + 1) \end{cases}$$

If  $(q^2 - q + 1) = 0$ , then  $q$  is a primitive 6th root of unity, and  $c_1 = 0$ , which immediately implies  $c_2 = c_3 = 0$ . If  $(q^2 + 1) = 0$ , then  $q$  is a primitive 4th root of unity, and again  $c_1 = 0$ , and  $c_2 = c_3 = 0$ . The expression  $q^3 - q^2 - 1 \neq 0$  for any root of unity  $q$ . Further, assuming  $l \neq 4, 6$ , we can express  $c_3$  and  $c_2$  in terms of  $c_1$ . Then the third equation leads to the relation for  $q$ :

$$(q^2 - 1)(q^4 - q^3 + 2q^2 + 1) = 0.$$

We will consider three cases.

$$\underline{q = 1.}$$

This is the case of the divergence that we considered before. The system of equations leads to the solution  $c_1 = c_2 = c_3$ , which results in a symmetric polynomial  $p(v, w) = v^2 + vw + w^2$ . The powers of  $(v^2 + vw + w^2)$  are never

antisymmetric and do not provide solutions for both conditions listed in the Proposition. Instead, the reason for the existence of nontrivial solutions in case of the divergence is that for  $q = 1$ , we have  $|\lambda_1| = |\lambda_2| = 1$ , and so the potential solutions of the polynomial equations are not restricted to the powers of  $(VW)$ .

$$\underline{q = -1.}$$

This is the case of the superdivergence. In this case the system of equations for the coefficients yields the solution  $c_1 = c_3$ ,  $c_2 = 3c_1$ . Therefore, for each even degree  $n$ , the set of homogeneous polynomials of degree  $n$  satisfying the condition  $p(v, w) = -p(v + w, -v - 2w)$  is empty if  $n \equiv 0 \pmod{4}$ , and consists of the polynomials proportional to  $(v^2 + 3vw + w^2)^{\frac{n}{2}}$  if  $n \equiv 2 \pmod{4}$ . All such polynomials are symmetric, and therefore adding the condition  $p(v, w) = -p(w, v)$  implies that there are no homogeneous polynomial satisfies both conditions announced in the Proposition.

$$\underline{q \neq \pm 1.}$$

This is the case of the  $q$ -divergence with  $l \geq 3$ . It is easy to see that solutions of the equation  $q^4 - q^3 + 2q^2 + 1 = 0$  are not roots of unity. Therefore, for  $l \geq 3$ , no quadratic homogeneous polynomial satisfies the equation  $p(v, w) = q \cdot p((q - 1)v - w, v + w)$ , and consequently no homogeneous polynomial of any degree can satisfy this equation.

Finally, we conclude that for  $l \geq 2$ , no homogeneous polynomial in two variables satisfies both conditions of the Proposition. ■

Propositions 5.1 and 5.9 together provide a proof of Theorem 5.8.

## 6. Appendix.

This section contains the computation of the matrix elements and characters of the representation  $\rho_n$  of  $D_{12} = \{s^2 = 1, r^6 = 1, srs^{-1} = r^{-1}\}$  on the space of degree  $n$  homogeneous polynomials in two variables, defined by

$$g \cdot p(v, w) = p(g^{-1}(v, w));$$

$$s \cdot \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} w \\ v \end{pmatrix}$$

and

$$r \cdot \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} w + v \\ -v \end{pmatrix}$$

**Matrix elements of the action** Let us compute the matrix elements  $(\rho_n)_{ij}$  of this representation, where the indices  $i, j$  change from 1 to  $n + 1$ . For the conjugacy classes of  $e, s, r^3$  we have

$$\begin{aligned} \rho_n(e) &= I_{n+1} & \chi_n(e) &= n + 1 \\ \rho_n(s) &= \begin{pmatrix} 0 & & 1 \\ & \cdots & \\ 1 & & 0 \end{pmatrix} & \chi_n(s) &= \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases} \\ \rho_n(r^3) &= \begin{cases} I_{n+1} & \text{if } n \text{ even} \\ -I_{n+1} & \text{if } n \text{ odd} \end{cases} & \chi_n(r^3) &= \begin{cases} n + 1 & \text{if } n \text{ even} \\ -(n + 1) & \text{if } n \text{ odd} \end{cases} \end{aligned}$$

Now let us compute the matrix coefficients of the action of  $rs$ . We will use the following notation for the binomial coefficients:

$$C_n^k = \frac{n!}{k! \cdot (n - k)!},$$

where  $n \geq 0$  and  $0 \leq k \leq n$ . If  $n$  and  $k$  do not satisfy these restrictions, we will assume that  $C_n^k = 0$ .

Let  $\{a_0, a_1, \dots, a_n\}$  denote the coefficients of a homogeneous polynomial of degree  $n$ ,

$$p(x, y) = a_0x^n + a_1x^{n-1}y + \dots + a_{n-1}xy^{n-1} + a_ny^n.$$

The action of  $rs$  is given by

$$\rho_n(rs)(p(x, y)) = p(x+y, -y) = \sum_{j=0}^n a_j(x+y)^{n-j}(-y)^j = \sum_{j=0}^n a_j(-1)^j \sum_{i=0}^{n-j} C_{n-j}^i x^i y^{n-i}.$$

Changing the order of summation, we have

$$\rho_n(rs)(p(x, y)) = \sum_{i=0}^n \left( \sum_{j=0}^{n-i} a_j(-1)^j C_{n-j}^i \right) x^i y^{n-i}.$$

The coefficients

$$(-1)^j C_{n-j}^i$$

span one row of the matrix, the row that corresponds to  $x^i y^{n-i}$ . Since the first row of the matrix corresponds to  $x^n$ , and the  $(n+1)$ st row to  $y^n$ , the row corresponding to  $x^i y^{n-i}$  has number  $n - i + 1$ . Also, the index  $j$  of the element in the row should change from 1 to  $n + 1$  rather than from 0 to  $n$ . Then we get:

$$(\rho_n(rs))_{i,j} = (-1)^{j-1} C_{n-j+1}^{n-i+1} = (-1)^{j-1} C_{n-j+1}^{i-j}.$$

The last equality follows from the symmetry of the binomial coefficients.

For the matrix elements of the action of  $r$ , we have

$$\rho_n(rs) \cdot \rho_n(s) = \rho_n(r).$$

Since  $\rho_n(s)$  is the anti-diagonal matrix with 1s on the anti-diagonal and zeros elsewhere, multiplication of  $\rho_n(rs)$  by this matrix is equivalent to reading the rows of  $\rho_n(rs)$  backwards. All we need to do to obtain  $\rho_n(r)$  is to change variables,  $j \rightarrow n + 2 - j$  in the expressions for the matrix elements of  $\rho_n(rs)$ :

$$(\rho_n(r))_{ij} = (\rho_n(rs))_{i,n+2-j} = (-1)^{n+1-j} C_{j-1}^{n+1-i} = (-1)^{n+1-j} C_{j-1}^{i+j-n-2}.$$

For the action of  $r^2$ , we have:

$$\rho_n(r^2)(p(x, y)) = p(-x - y, x) = (-1)^n p(x + y, -x).$$

Therefore, the matrix of  $r^2$  differs from the matrix of  $rs$  by multiplication by  $(-1)^n$  and reversing the order of elements in each column, which corresponds to the swap of variables  $x \leftrightarrow y$ . The reversing of the order of elements in each column is implemented by the change of variable:  $i \rightarrow n + 2 - i$ . Then the matrix of  $r^2$  is given by

$$(\rho_n(r^2))_{ij} = (-1)^n (\rho_n(rs))_{n+2-i,j} = (-1)^{n+1-j} C_{n-j+1}^{i-1} = (-1)^{n+1-j} C_{n-j+1}^{m+2-i-j}.$$

**Characters of the representation  $\rho_n$ .** For  $\chi_n(rs)$ , we have

$$\chi_n(rs) = \sum_{i=1}^{n+1} (-1)^{i-1} C_{n-i-1}^{m-i-1} = \sum_{i=0}^n (-1)^n = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

For  $\chi_n(r)$ , we compute using suitable changes of variables:

$$\chi_n(r) = (-1)^n \sum_{i=1}^{n+1} (-1)^{i-1} C_{i-1}^{m-(i-1)} \stackrel{k=i-1}{=} (-1)^n \sum_{k=0}^n (-1)^k C_k^{n-k} \stackrel{k \rightarrow n-k}{=} (-1)^n \sum_{k=0}^n (-1)^k C_{n-k}^{n-k}$$

$$(-1)^n \sum_{k=0}^n (-1)^{n-k} C_{n-k}^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_{n-k}^k.$$

The last equality holds because when  $k > \lfloor \frac{n}{2} \rfloor$  we have  $k > n - k$  and  $C_{n-k}^k = 0$ .

For  $\chi_n(r^2)$ , we compute  $\chi_n(r^2) =$

$$(-1)^n \sum_{i=1}^{n+1} (-1)^{i-1} C_{n-(i-1)}^{i-1} \stackrel{k=i-1}{=} (-1)^n \sum_{k=0}^n (-1)^k C_{n-k}^k = (-1)^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_{n-k}^k.$$

Therefore, we have

$$(\rho_n(rs))_{ij} = (-1)^{j-1} C_{n-j+1}^{m-i+1} \quad \chi_n(rs) = \sum_{i=1}^{n+1} (-1)^{i-1} = \begin{cases} 1 \text{ if } n \text{ even} \\ 0 \text{ if } n \text{ odd} \end{cases}$$

$$(\rho_n(r^2))_{ij} = (-1)^{n+1-j} C_{n-j+1}^{i-1} \quad \chi_n(r^2) = (-1)^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_{n-k}^k$$

$$(\rho_n(r))_{ij} = (-1)^{n+1-j} C_{j-1}^{m+1-i} \quad \chi_n(r) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_{n-k}^k$$

To compute the characters  $\chi_n(r^2)$ ,  $\chi_n(r)$  explicitly, we will use the following statement.

**Lemma 6.1.** *Let  $n$  be a nonnegative integer, and denote*

$$f_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_{n-k}^k,$$

where  $C_{n-k}^k = \frac{(n-k)!}{k!(n-2k)!}$  is the binomial coefficient, and  $\lfloor \frac{n}{2} \rfloor$  denotes the integer part of  $\frac{n}{2}$ . Then the following recursive formula holds for all  $n \geq 2$ :

$$f_n = f_{n-1} - f_{n-2}.$$

In particular, the sequence  $\{f_n\}_{n \geq 0}$  is periodic with period 6:

$$1, 1, 0, -1 - 1, 0, (1, 1, 0, -1 - 1, 0), \dots$$

**Proof.** First suppose that  $n \geq 2$  is even. Then we can write

$$f_n = \sum_{k=0}^{\frac{n}{2}} (-1)^k C_{n-k}^k = 1 + \sum_{k=1}^{\frac{n}{2}-1} (-1)^k C_{n-k}^k + (-1)^{\frac{n}{2}}.$$

Using the binomial coefficient identity

$$C_m^l = C_{m-1}^{l-1} + C_{m-1}^l, \quad 1 \leq l \leq m - 1,$$

we have

$$\begin{aligned} f_n &= 1 + (-1)^{\frac{n}{2}} + \sum_{k=1}^{\frac{n}{2}-1} (-1)^k C_{n-1-k}^k + \sum_{k=1}^{\frac{n}{2}-1} (-1)^k C_{n-1-k}^{k-1} = \\ &= \left( 1 + \sum_{k=1}^{\frac{n}{2}-1} (-1)^k C_{(n-1)-k}^k \right) + \left( (-1)^{\frac{n}{2}} + (-1) \sum_{i=0}^{\frac{n}{2}-2} (-1)^i C_{(n-2)-i}^i \right). \end{aligned}$$

The expression in the first parenthesis equals  $\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k C_{n-1-k}^k = f_{n-1}$ . In the second parenthesis, we have

$$(-1)^{\frac{n}{2}} = (-1) \cdot (-1)^{\frac{n-2}{2}} = (-1) \cdot (-1)^{\frac{n-2}{2}} C_{\frac{n-2}{2}}^{\frac{n-2}{2}},$$

and this term can be included in the sum over  $i$  for  $i = \frac{n-2}{2}$ :

$$f_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} C_{(n-1)-k}^k - \left( \sum_{i=0}^{\frac{n-2}{2}} (-1)^i C_{n-2-i}^i \right) = f_{n-1} - f_{n-2}.$$

The proof for an odd  $n$  is similar. Since  $f_0 = f_1 = 1$ , we obtain the required sequence. ■

Then the values of  $\chi_n$  on all conjugacy classes of  $D_{12}$  are given in the table in the proof of Theorem 5.3.

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## References

- [1] Alekseev, A., and E. Meinrenken, *On Kashiwara-Vergne conjecture*, *Inventiones Mathematicae* **164** (2006), 615–634.
- [2] Alekseev, A., and Ch. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, *Annals of Math.* **175** (2012), 415–463.
- [3] Drinfeld V., *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$* , *Leningrad Math. J.* **2** (1991), 829–860.
- [4] Goncharov A., *The dihedral Lie algebras and Galois symmetries of  $\pi_1^{(l)}(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N))$* , *Duke Math. J.* **110** (2001), 397–487.
- [5] Ihara, Y., *The Galois representation arising from  $P^1 - \{0, 1, \infty\}$  and Tate twists of even degree*, in: *Galois groups over  $\mathbb{Q}$* , *Publ. MSRI* **16** (1989), 299–313.
- [6] —, *Some arithmetic aspects of Galois actions on the pro- $p$  fundamental group of  $P^1 - \{0, 1, \infty\}$* , *Proceedings of Symposia in Pure Mathematics* **70** (2002), 247–273.
- [7] Ihara, K., M. Kaneko, and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, *Compositio Math.* **142** (2006), 307–338.
- [8] Ihara, K., and H. Ochiai, *Symmetry on linear relations for multiple zeta values*, *Nagoya Math. J.* **189** (2008), 49–62.
- [9] Kashiwara, M., and M. Vergne, *The Campbell-Hausdorff formula and invariant hyperfunctions*, *Invent. Math.* **47** (1978), 249–272.
- [10] Makoto, M., *On the Galois image in the derivation algebra of  $\pi_1$  of the projective line minus three points*, in: *Contemporary Mathematics* **186** (1995), 201–213.
- [11] Rouvière, F., *Espaces symétriques et méthodes de Kashiwara-Vergne*, *Ann. Sci. de l'Ecole Norm. Sup.* **19** (1986), 553–581.
- [12] Schneps, L., *Double shuffle and Kashiwara-Vergne Lie algebras*, *J. Algebra* **367** (2012), 54–74.

- [13] Zagier, D., *Multiple Zeta Values*, (1995), unpublished.
- [14] —, *Periods of modular forms, trace of Hecke operators, and multiple zeta values*, RIMS Kokyuroku **843** (1993), 162–170.

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