On the Variety of Four Dimensional Lie Algebras

Laurent Manivel[*]

Communicated by E. B. Vinberg

Abstract.   Lie algebras of dimension n are defined by their structure constants, which can be seen as sets of $N = n^2(n-1)/2$ scalars (if we take into account the skew-symmetry condition) to which the Jacobi identity imposes certain quadratic conditions. Up to rescaling, we can consider such a set as a point in the projective space $\mathbf{P}^{N-1}$. Suppose $n = 4$, hence $N = 24$. Take a random subspace of dimension 12 in $\mathbf{P}^{23}$, over the complex numbers. We prove that this subspace will contain exactly 1033 points giving the structure constants of some four-dimensional Lie algebras. Among those, 660 will be isomorphic to $\mathbf{bfgl}_2$, 195 will be the sum of two copies of the Lie algebra of one-dimensional affine transformations, 121 will have an abelian three-dimensional derived algebra, and 57 will have for derived algebra the three dimensional Heisenberg algebra. This answers a question of Kirillov and Neretin.
Mathematics Subject Classification 2010: 14C17, 14M99, 17B05.
Key Words and Phrases: Classification of Lie algebras, irreducible component, degree, resolution of singularities.

1.   The variety of Lie algebras

A Lie algebra structure on a k-vector space V_n of finite dimension n is given by a Lie bracket, which can be considered as a linear map from $\wedge^2 V_n$ to V_n, that we will denote by ω:

$$\omega(X \wedge Y) = [X, Y] \qquad \forall X, Y \in V_n.$$

A Lie bracket needs to verify the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \qquad \forall X, Y, Z \in V_n.$$

The left hand side of this identity is skew-symmetric in the three arguments, and what is required in terms of ω is the vanishing of the map $Jacobi(\omega)$ from $\wedge^3 V_n$ obtained as the composition

$$\wedge^3 V_n \hookrightarrow \wedge^2 V_n \otimes V_n \xrightarrow{\omega \otimes id} V_n \otimes V_n \longrightarrow \wedge^2 V_n \xrightarrow{\omega} V_n.$$

The variety of n-dimensional Lie algebras is

$$\mathrm{Lie}_n = \{\omega \in Hom(\wedge^2 V_n, V_n), \quad Jacobi(\omega) = 0\}.$$

Of course this is a cone, with vertex the point defining the abelian Lie algebra $\mathbf{ab}_n$. So we do not loose much information by considering instead the projective variety $\mathbf{P}\mathrm{Lie}_n$. This is a projective variety defined by a collection of quadratic equations, invariant under the natural action of $GL(V_n)$. So it is natural to ask:

1. What are the irreducible components of $\mathbf{P}\mathrm{Lie}_n$?

2. What is the dimension, the degree, more generally the geometry of each component?

3. What is the Lie algebra parametrized by the generic point of each component?

These questions have been addressed in [5], which contains a number of interesting results. For example, the authors prove that the number of irreducible components in $\mathbf{P}\mathrm{Lie}_n$ grows at least exponentially in the square root of n. Moreover they give the list of these components for $n \le 6$, with their dimensions and the Lie algebras corresponding to their generic points. They do not compute their degrees, and in fact they conclude their paper by asking what are the degrees of the four components of $\mathbf{P}\mathrm{Lie}_4$. This is the question we want to answer in this note.

2.   Classification of low dimensional Lie algebras

From now on we work over the complex numbers. The classification of Lie algebras is well known in dimension four [2, 6] (for more information in higher dimension, see for example [7]). We will need some details on this classification in order to construct smooth models of the components of $\mathbf{P}\mathrm{Lie}_4$ in the next section.

In dimension $n = 2$, a Lie bracket is defined as $[X, Y] = \theta(X, Y)V$, where θ is a fixed non zero skewsymmetric form on V_2 and the vector V can be chosen arbitrarily. This means that $\mathrm{Lie}_2 \simeq V_2$, where the origin corresponds to the abelian Lie algebra structure. For any other vector, the resulting algebra is isomorphic to the Lie algebra of the group of affine tranformations of the affine line, that we denote by $\mathbf{aff}_1$. We conclude that $\mathbf{P}\mathrm{Lie}_2 \simeq \mathbf{P}^1$ with its usual transitive action of PGL_2.

In dimension $n = 3$, a Lie bracket is defined by $\omega \in Hom(\wedge^2 V_3, V_3)$. If we fix a non degenerate trilinear non zero skewsymmetric form θ on V_3, it identifies $\wedge^2 V_3$ with V_3^*, and we get an induced identification of $Hom(\wedge^2 V_3, V_3)$ with

$$V_3 \otimes V_3 = Sym^2 V_3 \oplus V_3^*.$$

Therefore ω can be interpreted as $q + u$, with q a symmetric tensor and u a linear form. Moreover, a computation shows that the Jacobi identity translates into the condition that the contraction of q with u is zero. One can then distinguish two cases. Either $u = 0$ and q can be arbitrary, or u is non zero and q must be

a degenerate tensor, with u contained in its kernel. This gives two irreducible components in $\mathbf{P}\mathrm{Lie}_3$. The first one is the linear space $C_1 = \mathbf{P}(Sym^2 V_3) \simeq \mathbf{P}^5$. The second one is

$$C_2 = \mathbf{P}\{q + Ker(q), \quad q \in Sym^2 V_3, \quad \det(q) = 0\}.$$

It is birationally a one-dimensional bundle over the determinantal cubic fourfold parametrizing singular tensors in $\mathbf{P}(Sym^2 V_3)$, which is also the intersection of the two components.

In order to be more intrinsic, we let $Hom_s(\wedge^2 V_3, V_3) \subset Hom(\wedge^2 V_3, V_3)$ denote the space corresponding to symmetric tensors and defining C_1. Generically, such a tensor has rank three and the associated Lie algebra is then isomorphic to $\mathbf{sl}_2$. When the rank is one the associated Lie algebra is the Heisenberg algebra $\mathbf{he}_3$ (denoted Γ_3 in [5]).

Now we turn to the classification in dimension four. The first observation is that if $\mathbf{g}$ is a four dimensional Lie algebra, the derived algebra $\mathbf{h} = [\mathbf{g}, \mathbf{g}]$ is a proper ideal of $\mathbf{g}$. This allows to proceed by induction. In fact $\mathbf{h}$ cannot be any Lie algebra of dimension at most three. A routine examination shows that the only possibilities are $\mathbf{h} = \mathbf{sl}_2, \mathbf{he}_3, \mathbf{ab}_3, \mathbf{ab}_2, \mathbf{ab}_1$ or 0. Moreover the first four possibilities define the four irreducible components of Lie_4 or $\mathbf{P}\mathrm{Lie}_4$.

Proposition 2.1. *The variety $\mathbf{P}\mathrm{Lie}_4$ has four irreducible components, all of dimension 11. They are characterized by the following properties of the Lie algebras defined by their generic points:*

1. *C_1: be isomorphic to $\mathbf{bfgl}_2$,*

2. *C_2: have derived algebra isomorphic to $\mathbf{he}_3$,*

3. *C_3: have derived algebra isomorphic to $\mathbf{ab}_3$,*

4. *C_4: be isomorphic to $2\mathbf{aff}_1$.*

Note that the irreducible components of Lie_n have been classified up to $n = 7$ [3]. Starting from $n = 5$ there exist components of different dimensions.

In the next sections we will compute the degrees of the four irreducible components of $\mathbf{P}\mathrm{Lie}_4$. Desingularizations of the components have been described in [1], and used to find their singular loci. We will give more intrinsic descriptions of smooth models of the components, and use them to compute their degrees.

3. The first component

Suppose that V_4 is endowed with a Lie algebra structure isomorphic to $\mathbf{bfgl}_2$. Let $U \subset V_4$ be the derived algebra, isomorphic to $\mathbf{sl}_2$. For any $T \in V_4$, $ad(T)$ restricts to a derivation of U, so there is a unique vector $p(T) \in U$ such that $ad(T)_{|U} = ad(p(T))$. Or course the restriction of p to U must be the identity. As we have seen, the $\mathbf{sl}_2$-structure on U is defined by a non-degenerate symmetric tensor $\sigma \in Hom_s(\wedge^2 U, U)$.

Conversely, let G_U be the subspace of $Hom(V_4, U)$ consisting of homomorphisms whose restriction to U is a multiple of the identity. Let $H_U = Hom_s(\wedge^2 U, U)$. A pair (p, σ) with $p \in G_U$ and $\sigma \in H_U$ defines a Lie algebra structure on V_4, the corresponding bracket $\omega \in Hom(\wedge^2 V_4, V_4)$ being given by

$$\omega(T, X) = \sigma(p(T), X) \qquad \forall T \in V_4, \ X \in U.$$

Note that G_U and H_U are the fibers of two vector bundles G and H over $\mathbf{P}(V_4^*)$, of respective ranks 4 and 6. The previous discussion implies:

Proposition 3.1. *There exists a birational map*

$$\pi_1 : \mathbf{P}(G) \times_{\mathbf{P}(V_4^*)} \mathbf{P}(H) \longrightarrow \mathbf{P}C_1.$$

Beware that π_1 is not defined on the locus Σ where p vanishes on U, and the line $p(V_4)$ is contained in the kernel of σ. This is a fibered version of the usual desingularisation of the symmetric determinantal cubic in $\mathbf{P}^5$. In particular Σ is smooth, and we would need to blow it up in order to regularize π_1.

The degree of C_1, as a subvariety of $\mathbf{P}^{23}$, can be computed by integrating the first Chern class of the hyperplane line bundle $O(1)$, taken to the power $\dim C_1 = 11$. This integration can be performed on any birational model of C_1. In particular, since $\pi_1^* O(1) = O_{G,H}(1, 1)$, even if π_1 is not a morphism we can compute the degree of $\mathbf{P}C_1$ as

$$deg \, \mathbf{P}C_1 = \int_{\mathbf{P}C_1} c_1(O(1))^{11} = \int_{\mathbf{P}(G) \times_{\mathbf{P}(V_4^*)} \mathbf{P}(H)} c_1(O_{G,H}(1,1))^{11}.$$

Recall that if E is a vector bundle of rank e on a variety X, the projective bundle $\mathbf{P}(E)$ is endowed with a tautological line bundle $O_E(-1)$. If $p : \mathbf{P}(E) \to X$ denotes the projection, $O_E(-1)$ is a sub-bundle of $p^* E$. Let $O_E(1)$ be its dual. Then the Segre classes of E can be obtained by push-forward:

$$p_* c_1(O_E(1))^k = s_{k-e+1}(E),$$

with the convention that the right hand side is zero when $k \leq e-2$. Alternatively, the total Segre class $s(E) = \sum_{j \geq 0} s_j(E)$ is the formal inverse of the total Chern class [4, Chapter 3].

Considering $\mathbf{P}(G) \times_{\mathbf{P}(V_4^*)} \mathbf{P}(H)$ as an iterated projective bundle, and pushing-forward twice, we obtain

$$deg \, \mathbf{P}C_1 = \sum_{i+j=11} \frac{11!}{i!j!} \int_{\mathbf{P}(V_4^*)} s_{i-3}(G) s_{j-5}(H).$$

From the tautological exact sequence on $\mathbf{P}(V_4^*)$ we get that

$$s(G) = (1-h)^{-4}, \qquad s(H) = (1+2h)^4 (1+h)^{-10},$$

in terms of the hyperplane class h. A computation yields:

Proposition 3.2. *The degree of* $\mathbf{P}C_1$ *is* 660.

4. The second component

Suppose that V_4 is endowed with a Lie algebra structure such that the derived algebra is isomorphic to the three dimensional Heisenberg algebra $\mathbf{he}_3$. Denote by U this hyperplane of V_4. The Heisenberg algebra structure on U is determined, up to scalars, by the one-dimensional center $L \subset U$, which is also the derived algebra of U.

Now let T be an arbitrary element of V_4. Then $ad(T)$ restricts to an endomorphism $\theta(T)$ of U which preserves L. We thus get two induced endomorphisms $\theta_{U/L}(T)$ of U/L and $\theta_L(T)$ of L. Of course the latter is just a scalar.

Lemma 4.1. *The Jacobi identity is equivalent to the condition that*

$$\theta_L(T) = \text{trace } \theta_{U/L}(T) \qquad \forall T \in V_4.$$

Conversely, let Ω belong to $Hom_L(\wedge^2 U, L)$, the one dimensional space of two-forms whose kernel contains L. Denote by $End_L(U)$ the vector space of endomorphisms of U preserving L, and by $End_L^0(U)$ the hyperplane defined by the condition of the Lemma. Let $\theta \in Hom(V_4, End_L^0(U))$. Choose some $T \notin U$. Then for any $\alpha \in \wedge^2 V_4$, we can find vectors A, B, C in U such that $\alpha = T \wedge A + B \wedge C$. We would therefore like to define our Lie algebra structure on V_4 by letting

$$\omega(\alpha) = \theta(T)(A) + \Omega(B, C).$$

For this to make sense, it must be independent of T. This is clearly equivalent to the condition that $\theta(D)(A) = \Omega(D, A)$ when D, A belong to U.

Consider therefore the vector space $F_{L,U}$ of pairs (Ω, θ), where Ω belongs to $Hom_L(\wedge^2 U, L)$, θ to $End_L^0(U)$, and the previous compatibility condition is satisfied. This defines a rank seven vector bundle on the flag variety $F(1, 3, V_4)$, which has dimension 5.

Proposition 4.2. *There exists a proper birational map $\pi_2 : \mathbf{P}(F) \longrightarrow \mathbf{P}C_2$.*

Proof. The map π_2 has just been described. It is clearly birational since generically, U can only be the derived algebra, L its center, and the pair (Ω, θ) is completely determined by the Lie bracket. $\square$

Since obviously $\pi_2^* O(1) = O_F(1)$, we can obtain the degree of $\mathbf{P}C_2$ as

$$deg\ \mathbf{P}C_2 = \int_{\mathbf{P}C_2} c_1(O(1))^{11} = \int_{\mathbf{P}(F)} c_1(O_F(1))^{11} = \int_{F(1,3,V_4)} s_5(F).$$

In order to compute this degree, first observe that

$$End_L^0(U) \simeq Hom(U/L, U) \qquad \text{and} \qquad Hom_L(\wedge^2 U, L) \simeq Hom(\wedge^2(U/L), L).$$

Denote the latter line bundle by M. Mapping the pair (Ω, θ) to Ω defines a vector bundle morphism from F to M with kernel $K \simeq Hom(V_4/U, End_L^0(U))$. From the exact sequence

$$0 \longrightarrow End_L^0(U) \longrightarrow End(U) \longrightarrow Hom(L, U) \longrightarrow 0$$

we deduce that the total Segre class of F is

$$s(F) = s(M)s(Hom(V_4/U, End(U)))c(Hom(V_4/U, Hom(L, U))).$$

Let us denote by p the projection from $F(1, 3, V_4)$ to $\mathbf{P}(V_4^*)$, which identifies the flag variety with the projective bundle $\mathbf{P}(U)$. The tautological bundle $O_U(-1)$ is our L. The hyperplane line bundle on $\mathbf{P}(V_4^*)$ is $O(-1) = V_4/U$. Using the projection formula, we get that

$$p_*s(F) = s(End(U)(-1))p_*S,$$

where $S = s(M)c(U(-1)) \otimes L^*)$.

Lemma 4.3. *The push-forward of S is*

$$p_*S = 13 - 90h + 318h^2 - 738h^3.$$

Proof. In order to compute this push-forward, we need to decompose S as $S = \sum_k \ell^k p^* S_k$, where $\ell = c_1(L^*)$, and use the fact that $p_*\ell^k = s_{k-2}(U)$. Note that from the tautological sequence on $\mathbf{P}(V_4^*)$, the total Segre class of U is $s(U) = 1 + h$. Hence $p_*S = S_2 + hS_3$. Let v_1, v_2, v_3 be the Chern roots of $U(-1)$. Then

$$c(U(-1)) \otimes L^*) = \prod_{i=1}^{3}(1 + v_i + \ell) = \sum_{j=0}^{3} e_j \ell^{3-j}.$$

Here we denoted by e_j the degree j elementary symmetric function of $1 + v_1, 1 + v_2, 1 + v_3$. Since $c(M) = 1 - 2\ell + h$, we get that

$$S_k = \sum_{p,j} \frac{(p + k + j - 3)!}{p!(k + j - 3)!}(-1)^p 2^{k+j-3} e_j h^p.$$

The sum over p gives $(1 + h)^{-(k+j-2)}$, and we readily deduce that

$$p_*S = \sum_{j \geq 1} \frac{2^{j-1} e_j}{(1 + h)^j} + h \sum_{j \geq 0} \frac{2^j e_j}{(1 + h)^{j+1}}.$$

In order to evaluate that sum, it is convenient to introduce the stretched variant of the total Chern class

$$c_z(U(-1)) := \sum_{j \geq 0} z^j c_j(U(-1)) = (1 - zh)^4.$$

where the last equality follows from the twisted tautological exact sequence. A formal substitution yields

$$e_z(U(-1)) := \sum_{j \geq 0} z^j e_j = (1 + z)^3 c_{\frac{z}{1+z}}(U(-1)) = (1 + z - zh)^4/(1 + z).$$

We can then get p_*S as

$$p_*S = \frac{1}{2}\left(e_{\frac{2}{1+h}}(U(-1)) - 1\right) - \frac{h}{1 + h} e_{\frac{2}{1+h}}(U(-1)),$$

whence the result. $\blacksquare$

In order to deduce the push-forward of $s(F)$, there remains to observe that, again from the tautological exact sequence, we can deduce that

$$c(End(U)(-1)) = \frac{1-h}{(1+h)^4}.$$

The final result is the following:

Proposition 4.4.   *The degree of* $\mathbf{P}C_2$ *is* 57.

5.   The third component

Suppose V_4 is endowed with a Lie algebra structure such that the derived algebra is abelian of dimension three. More generally we could ask that there exists a hyperplane U of V_4 over which the Lie bracket vanishes identically. In order to define the bracket completely, what remains to do is to prescribe the bracket of an element of U with a given element not in U. More intrinsically, we need to specify a map

$$\theta : V_4/U \longrightarrow End(U),$$

which can be arbitrary: the Jacobi identity will always hold true. As before, denote by U the tautological rank three vector bundle on $\mathbf{P}(V_4^*)$. Let E be the rank 9 vector bundle $Hom(V_4/U \, End(U))$ on $\mathbf{P}(V_4^*)$.

Proposition 5.1.   *There exists a proper birational map* $\pi_3 : \mathbf{P}(E) \longrightarrow \mathbf{P}C_3$.

Proof.   What remains to be proved is that the natural map $\pi_3 : \mathbf{P}(E) \longrightarrow \mathbf{P}C_3$ we have just described is birational. This follows from the obvious fact that the general point of C_3 defines a Lie algebra whose unique abelian codimension one subalgebra is the derived subalgebra. ∎

In particular we can obtain the degree of $\mathbf{P}C_3$ as

$$deg\ \mathbf{P}C_3 = \int_{\mathbf{P}C_3} c_1(O(1))^{11} = \int_{\mathbf{P}(E)} c_1(O_E(1))^{11} = \int_{\mathbf{P}(V_4^*)} s_3(E).$$

In order to compute this number it is convenient to use the Chern character. Indeed we have $E = U^* \otimes \wedge^2 U^*$ and the tautological exact sequence yields, if we denote by h the hyperplane class on $\mathbf{P}(V_4^*)$,

$$ch(U^*) = 4 - e^{-h}, \qquad ch(\wedge^2 U^*) = 6 - 4e^{-h} + e^{-2h}.$$

The Chern character being multiplicative, $ch(E) = ch(U^*)ch(\wedge^2 U^*)$, from which we compute that $p_1(E) = 9h$, $p_2(E) = h^2$, $p_3(E) = -15h^3$, and finally

$$s_3(E) = \frac{1}{6}(p_1(E)^3 + 3p_1(E)p_2(E) + 2p_3(E)) = 121h^3.$$

We have proved that:

Proposition 5.2.   *The degree of* $\mathbf{P}C_3$ *is* 121.

6. The fourth component

Suppose V_4 is endowed with a Lie algebra structure such that the derived algebra is abelian and two-dimensional. Denote the derived algebra by U. The Lie bracket $\omega \in Hom(\wedge^2 V_4, V_4)$ factorizes through U and vanishes on $\wedge^2 U$. Since $V_4 \wedge U / \wedge^2 U \simeq (V_4/U) \otimes U$, the restriction of ω to $V_4 \wedge U$ induces a morphism

$$\tau : V_4/U \longrightarrow End(U).$$

This morphism does not define completely the Lie algebra structure, but observe that $\wedge^2 V_4 / V_4 \wedge U \simeq \wedge^2(V_4/U)$ is one dimensional, so the data that is missing is just a skew-symmetric U-valued two-form on V_4/U. Moreover, the Jacobi identity will not involve this part of the Lie bracket, but only τ:

Lemma 6.1. *The Jacobi identity is equivalent to the vanishing of the composition*

$$\wedge^2(V_4/U) \xrightarrow{\wedge^2 \tau} \wedge^2 End(U) \xrightarrow{[,]} \mathbf{sl}(U).$$

Obviously this only depend on the traceless part τ_0 of τ, and the Jacobi identity imposes that τ_0 belongs to the Segre product

$$\mathbf{P}(V_4/U)^* \times \mathbf{P}(\mathbf{sl}(U)) \subset \mathbf{P}(Hom(V_4/U, \mathbf{sl}(U))).$$

So let S denote this relative Segre product over the Grassmannian $G(2, V_4)$: this is a smooth fiber bundle with fiber $\mathbf{P}^1 \times \mathbf{P}^2$. Let s denote the projection to $G(2, V_4)$. We define a vector bundle P over S as follows: the fiber of P above the line $[\tau_0]$ is the vector space of Lie brackets $\omega \in Hom(\wedge^2 V_4, U) \subset Hom(\wedge^2 V_4, V_4)$ that vanish on $\wedge^2 U$, and such that the traceless part of the associated morphism τ is a multiple of τ_0. By mapping ω to τ we obtain the exact sequence of vector bundles

$$0 \longrightarrow s^* Hom(\wedge^2(V_4/U), U) \longrightarrow P \longrightarrow O_S(-1) \oplus s^*(V_4/U)^* \longrightarrow 0.$$

Proposition 6.2. *There exists a proper birational map $\pi_4 : \mathbf{P}(P) \longrightarrow \mathbf{P}C_4$.*

Proof. The map π_4 has just been described. It is clearly birational since generically, U can only be the derived algebra, and the Lie bracket is exactly a point in the fiber of P over U. ∎

Here again $\pi_4^* O(1) = O_P(1)$, and we can obtain the degree of $\mathbf{P}C_4$ as

$$deg\, \mathbf{P}C_4 = \int_{\mathbf{P}C_4} c_1(O(1))^{11} = \int_{\mathbf{P}(P)} c_1(O_P(1))^{11} = \int_S s_7(P).$$

Observe that in order to compute this number, we may suppose that the exact sequence above is split, hence that $P \simeq O_S(-1) \oplus s^* R$ with $R = Q^* \oplus U(-1)$. Recall that, like the total Chern class, the total Segre class is multiplicative with respect to direct sums, so that, using the projection formula, we get

$$s_*(s(P)) = s_*(s(O_S(-1)))s(R).$$

Since S is a bundle of products of projective spaces over the Grassmannian, the usual formula for projective bundles gives

$$s_*(c_1(O_S(1))^k) = \sum_{i+j=k} \frac{k!}{i!j!} s_{i-1}(Q)s_{j-2}(\mathbf{sl}(U)),$$

and therefore we get the degree of $\mathbf{P}C_4$ as

$$deg\ \mathbf{P}C_4 = \sum_{i,j} \frac{(i+j)!}{i!j!} \int_{G(2,V_4)} s_{i-1}(Q)s_{j-2}(\mathbf{sl}(U))s_{7-i-j}(R).$$

This computation is a straightforward excercise in elementary Schubert calculus. Note in particular that $\mathbf{sl}(U)$ being self-dual has zero Chern and Segre classes in odd degrees. More precisely, in terms of the Chern classes c_1 and c_2 of U^* we get

$$c(\mathbf{sl}(U)) = 1 + 4c_2 - c_1^2, \qquad s(\mathbf{sl}(U)) = 1 - 4c_2 + c_1^2 + 10c_2^2.$$

The Segre classes of the two other bundles are easily computed:

$$s(Q) = 1 - c_1 + c_1^2 - c_2, \qquad s(R) = 1 + 4c_1 + 10c_1^2 + 40c_1c_2 + 70c_2^2.$$

Keeping in mind that $c_2^2 = c_2c_1^2 = 1$ is the punctual class, while $c_1^4 = 2$, we finally obtain that:

Proposition 6.3.   *The degree of* $\mathbf{P}C_4$ *is equal to* 195.

References

[1]   Basili, R., *Resolutions of singularities of varieties of Lie algebras of dimensions 3 and 4*, J. Lie Theory **12** (2002), 397–407.

[2]   Burde, D., and C. Steinhoff, *Classification of orbit closures of 4-dimensional complex Lie algebras*, J. Algebra **214** (1999), 729–739.

[3]   Carles, R., and Y. Diakité, *Sur les variétés d'algèbres de Lie de dimension* ≤ 7, J. Algebra **91** (1984), 53–63.

[4]   Fulton, W., "Intersection theory," Second edition, Springer 1998.

[5]   Kirillov, A. A., and Yu. A. Neretin, *The variety* A_n *of structures of n-dimensional Lie algebras*, in: American Mathematical Society Translations **137**, 1987, 21–30.

[6]   Lauret, J., *On the moment map for the variety of Lie algebras*, J. Funct. Anal. **202** (2003), 392–423.

[7]   Yamaguchi, S., *On some classes of nilpotent Lie algebras and their automorphism groups*, Mem. Fac. Sci. Kyushu Univ. **35** (1981), 341–351.

Laurent Manivel
Institut de Mathématiques
de Marseille
UMR 7373 CNRS/Aix-Marseille
Université Technopôle
Château-Gombert
39 rue Frédéric Joliot-Curie
13453 Marseille Cedex 13, France
laurent.manivel@math.cnrs.fr