

## Invariant Connections with Skew-Torsion and $\nabla$ -Einstein Manifolds

Ioannis Chrysikos\*

Communicated by B. Ørsted

**Abstract.** For a compact connected Lie group  $G$  we study the class of bi-invariant affine connections whose geodesics through  $e \in G$  are the 1-parameter subgroups. We show that the bi-invariant affine connections which induce derivations on the corresponding Lie algebra  $\mathfrak{g}$  coincide with the bi-invariant metric connections. Next we describe the geometry of a naturally reductive space  $(M = G/K, g)$  endowed with a family of  $G$ -invariant connections  $\nabla^\alpha$  whose torsion is a multiple of the torsion of the canonical connection  $\nabla^c$ . For the spheres  $S^6$  and  $S^7$  we prove that the space of  $G_2$  (respectively,  $\text{Spin}(7)$ )-invariant affine or metric connections consists of the family  $\nabla^\alpha$ . Then we examine the “constancy” of the induced Ricci tensor  $\text{Ric}^\alpha$  and prove that any compact isotropy irreducible standard homogeneous Riemannian manifold, which is not a symmetric space of Type I, is a  $\nabla^\alpha$ -Einstein manifold for any  $\alpha \in \mathbb{R}$ . We also provide examples of  $\nabla^{\pm 1}$ -Einstein structures for a class of compact homogeneous spaces  $M = G/K$  with two isotropy summands.

*Mathematics Subject Classification 2010:* 53C025, 53C30, 22E46.

*Key Words and Phrases:* Invariant connection with skew-symmetric torsion, naturally reductive space, Killing metric,  $\nabla$ -Einstein structure.

### Introduction

Given a Riemannian manifold  $(M^n, g)$ , metric connections whose torsion is a 3-form are geometrically the connections which have the same geodesics as the Levi-Civita connection. These connections play a crucial role in the theory of non-integrable geometries and admit physical applications in type II string theory, see for example [1, 15, 2]. A very remarkable example is the so-called canonical connection  $\nabla^c$  on a naturally reductive space. This article is a contribution to the geometry of naturally reductive manifolds and Lie groups with respect to an invariant metric connection with skew-torsion. Our approach is fundamental and it mainly relies on the homogeneous structure that such a manifold carries. We begin by describing bi-invariant affine connections on a compact connected Lie group  $G$ . Among the different bi-invariant connections that one can consider on

---

\* This work has been fully supported by GAČR (Czech Science Foundation) under the grant no. GP14-24642P.

$G$ , we are concerned with those for which the Nomizu map  $\Lambda : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  satisfies the property  $\Lambda(X)X = 0$ , for any  $X \in \mathfrak{g} \cong T_e G$ . Although any bi-invariant *metric* connection has this property, the converse is not necessarily true; counterexamples are known for  $G = U(n)$  [23, 5]. It is therefore natural to ask what conditions we have to impose on the Nomizu map in order to establish a possible correspondence. For this, we propose a formula which relates an equivariant derivation of  $\mathfrak{g}$  with the torsion and the curvature of a bi-invariant connection on  $G$ , satisfying  $\Lambda(X)X = 0$  ( $X \in \mathfrak{g}$ ). This observation enables us to classify the  $\text{Ad}(G)$ -equivariant derivations  $D : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$  carrying the property  $D(X)X = 0$ . Then, we show that the class of bi-invariant affine connections which induce derivations  $\Lambda : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$  on the corresponding Lie algebra  $\mathfrak{g}$  coincides with the class of bi-invariant metric connections on  $G$  (see Theorem 2.9).

After this description, a natural step is the investigation of naturally reductive Riemannian manifolds  $(M = G/K, g)$  endowed with  $G$ -invariant connections whose torsion is a multiple of the torsion  $T^c$  of the canonical connection  $\nabla^c$ , say  $T^\alpha = \alpha \cdot T^c$  for some parameter  $\alpha$ . For an irreducible symmetric space  $(M = G/K, g)$  of Type I one can show that the space of  $G$ -invariant *metric* connections consists only of the canonical connection  $\nabla^c \equiv \nabla^g$  (see [24, Thm. 2.1] and [5, Rem. 3.2]). Here, we primarily focus on symmetric spaces which can be (re)presented as cosets of distinct Lie groups, e.g. the spheres  $S^6$  and  $S^7$ . In terms of representation theory we show that the space of  $\text{Spin}(7)$ -invariant affine (or metric) connections on the 7-sphere  $S^7 = \text{Spin}(7)/G_2$  is 1-dimensional; it consists of the family  $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$  described above. The same is true for the space of  $G_2$ -invariant affine (or metric) connections on the sphere  $S^6 = G_2/\text{SU}(3)$ , with the difference that  $\alpha \in \mathbb{C}$ , i.e. there is a 2-dimensional family of  $G_2$ -invariant affine connections on  $S^6 = G_2/\text{SU}(3)$ , which has the same geodesics with the Levi-Civita connection (see Theorem 3.4). These invariant connections occur since the cosets  $\text{Spin}(7)/G_2$  and  $G_2/\text{SU}(3)$ , although diffeomorphic to a symmetric space, do not provide us with symmetric pairs (see [10, 7.107 Table 6]).

The rest of the article is a detailed study of  $\nabla$ -Einstein structures with skew-torsion on compact naturally reductive spaces. Given a Riemannian manifold  $(M^n, g)$  ( $n \geq 3$ ) equipped with a metric connection  $\nabla$  with non-trivial skew-torsion  $T \in \Lambda^3(M^n)$ , a  *$\nabla$ -Einstein structure with skew-torsion*, or in short a  *$\nabla$ -Einstein structure*, is a generalization of the Riemannian Einstein condition, given by a tuple  $(M^n, g, \nabla, T)$  satisfying the equation  $\text{Ric}_S = (\text{Scal} \cdot g)/n$ . Here,  $\text{Ric}_S$  denotes the symmetric part of the Ricci tensor associated to  $\nabla$  and  $\text{Scal}$  is the corresponding scalar curvature. Solutions of this equation naturally appear in the context of non-integrable geometries, where a metric connection different than the Levi-Civita connection is adapted to the geometry under consideration, the so-called characteristic connection [15, 2]. A variational principle has been recently deduced in [3], proving that  $\nabla$ -Einstein structures are optimal between the different metric connections with skew-torsion that one can define by choosing pairs  $(g, T)$  of metrics and compatible totally skew-symmetric torsion tensors. In this paper we describe the  $\nabla^\alpha$ -Einstein equation on *compact* naturally reductive Riemannian manifolds in terms of Casimir elements. We prove that any compact isotropy irreducible standard homogeneous Riemannian manifold  $(M^n = G/K, g)$

of a compact connected semi-simple Lie group  $G$ , which is not a symmetric space of Type I, is a  $\nabla^\alpha$ -Einstein manifold for *any*  $\alpha \in \mathbb{R}$  (see Theorem 4.7). In particular, symmetric spaces of Type I are never  $\nabla^\alpha$ -Einstein for some  $\alpha \neq 0, 1$ , since  $\nabla^\alpha \equiv \nabla^c \equiv \nabla^g$ , for any  $\alpha$ . In contrast, symmetric spaces of Type II, i.e. compact connected simple Lie groups are  $\nabla^\alpha$ -Einstein with parallel torsion for any  $\alpha \in \mathbb{R}$ , with the well-known flat  $\pm 1$ -connections of Cartan-Schouten being the trivial members (see [3, Lemma 1.8] and Example 4.6).

In the final section we extend our study on compact homogeneous Riemannian manifolds  $M = G/K$  of a compact connected semi-simple Lie group  $G$ , whose isotropy representation decomposes into two non-trivial irreducible and inequivalent  $K$ -submodules, such that

$$[\mathfrak{k}, \mathfrak{m}_i] \subset \mathfrak{m}_i, \quad [\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{k} \oplus \mathfrak{m}_2, \quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1, \quad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}. \quad (1)$$

As a first step, we characterize the invariant metric connections on  $T(G/K)$  which have skew-torsion with respect to a 1-parameter family of  $G$ -invariant metrics  $\{g_t : t \in \mathbb{R}_+\}$ . They exist only for the Killing metric  $t = 1/2$ , under the further assumption that the associated Nomizu map  $\Lambda_{\mathfrak{m}} : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$  satisfies  $\Lambda_{\mathfrak{m}}(X)X = 0$ , for any  $X \in \mathfrak{m}$  (see Theorem 5.2). Based on this characterization, we introduce a new 2-parameter family of  $G$ -invariant metric connections, say  $\nabla^{s,t}$  with  $s \in \mathbb{R}$  and  $t > 0$ , which joins the connections  $\nabla^t$  and  $\nabla^c$ , and for the Killing metric gives rise to a family of invariant connections with skew-torsion, namely  $\{\nabla^{s, \frac{1}{2}} : s \in \mathbb{R}\}$ . For completeness, we examine the full algebraic type of the torsion  $T^{s,t}$ ; we show (even by a theory based on Dirac operators) that it does not contain any component of vectorial type. Then we describe the  $\nabla^{s, \frac{1}{2}}$ -Einstein condition in terms of the Casimir eigenvalues  $\text{Cas}_1$  and  $\text{Cas}_2$ . We prove that  $M = G/K$  is a  $\nabla^s$ -Einstein manifold with skew-torsion for the values  $s = 0$  or  $s = 2$  if and only if the Killing metric  $g_B \equiv g_{1/2}$  is Einstein, i.e.  $\text{Cas}_1 = \text{Cas}_2$  (see Theorem 5.14). Finally, based on a previous work [8] related to author's PhD thesis, we provide a series of examples of homogeneous spaces carrying  $\nabla^s$ -Einstein structures with skew-torsion, for the values  $s = 0, 2$ . These are flag manifolds and they are the first known examples of *infinite families of non-isotropy irreducible* homogeneous Riemannian manifolds, admitting  $\nabla$ -Einstein structures with skew-torsion.

## 1. Homogeneous Riemannian manifolds and invariant connections

**1.1.  $G$ -invariant connections.** Consider a connected homogeneous Riemannian manifold  $(M = G/K, g)$ , where  $G \subset I(M)$  is a closed subgroup of the isometry group and  $K$  is the isotropy subgroup of some point of  $M$ . Assume for simplicity that the transitive  $G$ -action is effective and  $K$  is connected. We shall denote by  $\mathfrak{g}, \mathfrak{k}$  the corresponding Lie algebras. Because  $K$  is compact, one can always fix an  $\text{Ad}(K)$ -invariant splitting  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ , i.e.  $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ . Then,  $\mathfrak{m}$  is identified with the tangent space  $T_oM$  of  $M$  ( $o = eK \in M$ ) and the isotropy representation  $\chi : K \rightarrow \text{SO}(\mathfrak{m}) \subset \text{Aut}(\mathfrak{m})$  coincides with the restriction of the adjoint representation  $\text{Ad}|_K$  on  $\mathfrak{m}$ , see [10]. Let us denote by  $\text{aff}_G(P)$  the space of  $G$ -invariant affine connections on a homogeneous principal bundle  $P \rightarrow G/K$  over  $M = G/K$ . Let also  $\text{Hom}_K^{\mathbb{R}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$  be the space of  $K$ -intertwining maps

$\mathfrak{m} \otimes \mathfrak{m} \rightarrow \mathfrak{m}$ . By a theorem of H. C. Wang [29] it is well-known that a linear  $G$ -invariant connection  $\nabla : \Gamma(TM) \rightarrow \Gamma(T^*M \otimes TM)$  is described by a  $\mathbb{R}$ -linear map  $\Lambda_{\mathfrak{m}} : \mathfrak{m} \rightarrow \mathfrak{gl}(\mathfrak{m})$  which is equivariant under the isotropy representation, i.e.  $\Lambda_{\mathfrak{m}}(\text{Ad}(k)X) = \text{Ad}(k)\Lambda_{\mathfrak{m}}(X)\text{Ad}(k)^{-1}$  for any  $X \in \mathfrak{m}$  and  $k \in K$ . Writing  $\Lambda_{\mathfrak{m}}(X)Y = \eta(X, Y)$  for some  $\text{Ad}(K)$ -equivariant bilinear map  $\eta : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ , i.e.  $\eta(\text{Ad}(k)X, \text{Ad}(k)Y) = \text{Ad}(k)\eta(X, Y)$  for any  $X, Y \in \mathfrak{m}$  and  $k \in K$ , one can finally establish the identification (see [11, 24])

$$\text{aff}_G(F(G/K)) \cong \text{Hom}_K^{\mathbb{R}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}).$$

The linear map  $\Lambda_{\mathfrak{m}}$  is commonly referred to us as the *Nomizu map* (for details see [6, 22]) and it nicely describes the properties of  $\nabla$ . For example, pull back the Riemannian metric  $g := \langle \cdot, \cdot \rangle_o$  on  $T_oM$  to an  $\text{Ad}(K)$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{m}$ . Then,  $\nabla$  is metric, i.e.  $\Lambda_{\mathfrak{m}}(X)$  lies in  $\mathfrak{so}(\mathfrak{m})$  for any  $X \in \mathfrak{m}$  if and only if  $\langle \Lambda_{\mathfrak{m}}(X)Y, Z \rangle + \langle Y, \Lambda_{\mathfrak{m}}(X)Z \rangle = 0$  for any  $X, Y, Z \in \mathfrak{m}$ . Furthermore, the torsion and curvature are given by

$$\left. \begin{aligned} T(X, Y)_o &= \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X - [X, Y]_{\mathfrak{m}} \\ R(X, Y)_o &= [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X, Y]_{\mathfrak{m}}) - \text{ad}([X, Y]_{\mathfrak{t}}) \end{aligned} \right\}. \quad (2)$$

Viewing the torsion as a  $(3, 0)$ -tensor  $T(X, Y, Z) := \langle T(X, Y), Z \rangle$  we will call  $T$  the *torsion form* if and only if it is skew-symmetric in  $Y$  and  $Z$  (and hence totally skew-symmetric).

Recall that  $M = G/K$  carries a distinguished invariant connection, the so-called *canonical connection*  $\nabla^c$ . This is the unique  $G$ -invariant connection whose Nomizu map  $\Lambda_{\mathfrak{m}} : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$  is the zero map, i.e.  $\Lambda_{\mathfrak{m}}(X) = 0$ , for any  $X \in \mathfrak{m}$  [6, 22]. The canonical connection depends on the choice of  $\mathfrak{m} \cong T_oM$ , for example its torsion is given by  $T^c(X, Y) = -[X, Y]_{\mathfrak{m}}$ . Moreover, any  $G$ -invariant tensor field is  $\nabla^c$ -parallel, in particular  $\nabla^c T^c = 0 = \nabla^c R^c$ . A homogeneous Riemannian manifold  $(M = G/K, g)$  is called *naturally reductive* with respect to  $G$  if and only if the torsion of  $\nabla^c$  is a 3-form on  $\mathfrak{m}$ , i.e.  $T^c(X, Y, Z) := \langle T^c(X, Y), Z \rangle \in \Lambda^3(\mathfrak{m})$  for any  $X, Y, Z \in \mathfrak{m}$ . Geometrically, the notion of natural reductivity is equivalent to say that for each vector  $X \in \mathfrak{m}$  the orbit  $\gamma(t) := \exp(tX)o$  is a geodesic on  $M$ , which means the Riemannian geodesics coincide with the  $\nabla^c$ -geodesics.

**1.2. Bi-invariant connections.** A compact connected Lie group  $M = G$  with a bi-invariant metric  $\rho$  can be viewed as a symmetric space of the form  $(G \times G)/\Delta G$ . The Cartan decomposition is given by  $\mathfrak{g} \oplus \mathfrak{g} = \Delta \mathfrak{g} \oplus \mathfrak{p}$ , where both

$$\Delta \mathfrak{g} := \{(X, X) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g}\} \text{ and } \mathfrak{p} := \{(X, -X) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g}\}$$

are isomorphic to  $\mathfrak{g}$ , as  $G$ -modules. The isotropy representation is the adjoint representation of  $G$ , i.e.  $\chi(g, g)(X, -X) := (\text{Ad}(g)X, -\text{Ad}(g)X)$ . Hence, as a symmetric space,  $G$  is isotropy irreducible if and only if  $G$  is simple. In this note we are interested in *bi-invariant* connections on  $G$ , i.e.  $(G \times G)$ -invariant affine connections. Such a connection, say  $\nabla^\eta$ , is completely described by a bilinear map  $\eta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  such that  $\eta(\text{Ad}(g)X, \text{Ad}(g)Y) = \text{Ad}(g)\eta(X, Y)$ , for any  $g \in G$  and  $X, Y \in \mathfrak{g}$  [23]. The associated Nomizu map  $\Lambda^\eta : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$  is given by  $\Lambda^\eta(X)Y := \eta(X, Y)$  and the equivariant condition is expressed by  $\Lambda^\eta(\text{Ad}(g)X) = \text{Ad}(g)\Lambda^\eta(X)\text{Ad}(g)^{-1}$ . For a (compact) *simple* Lie group  $G$  there

exists a 1-dimensional family of *canonical connections* which joins the Levi-Civita connection with the flat  $\pm 1$ -connections of Cartan-Schouten (see [26, Rem. 6.1] or [5, p. 18]). To be more precise, it is induced from the reductive decomposition  $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathfrak{p}_\alpha$ , with  $\alpha \in \mathbb{R}$  and

$$\mathfrak{p}_\alpha := \left\{ X_\alpha := \left( \frac{\alpha+1}{2}X, \frac{\alpha-1}{2}X \right) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g} \right\} \cong \mathfrak{g},$$

for example. Then, one computes  $[X_\alpha, Y_\alpha]_{\mathfrak{p}_\alpha} = \alpha \left( \frac{\alpha+1}{2}[X, Y], \frac{\alpha-1}{2}[X, Y] \right)$  and hence the torsion of the induced connection is given by  $T^\alpha(X, Y) := -[X_\alpha, Y_\alpha]_{\mathfrak{p}_\alpha} = -\alpha[X, Y]$ , for any  $X, Y \in \mathfrak{g}$ . To summarise:

**Theorem 1.1.** *On a compact simple Lie group  $G \cong (G \times G)/\Delta G$  with a bi-invariant metric  $\rho$  there exists a 1-dimensional family of bi-invariant canonical connections, namely  $\nabla_X^\alpha Y = \eta^\alpha(X, Y) = \frac{1-\alpha}{2}[X, Y]$  ( $\alpha \in \mathbb{R}$ ) (up to scale). The curvature has the form  $R^\alpha(X, Y)Z = (1 - \alpha^2)[Z, [X, Y]]/4$  for any  $X, Y, Z \in \mathfrak{g}$ . Thus,  $(G, \rho)$  endowed with one of the connections  $\nabla^{\pm 1}$  becomes flat, i.e.  $R^{\pm 1} \equiv 0$ . Moreover, the torsion  $T^\alpha$  is  $\nabla^\alpha$ -parallel for any  $\alpha \in \mathbb{R}$  (by the Jacobi identity).*

## 2. Metric bi-invariant connections and derivations

We recall the classification of *metric* bi-invariant connections on a *compact* Lie group  $G$  by [5]. For the sake of completeness, and since we will use this result, we explain the main idea of the proof (adapted in our notation). This is essentially based on the classification of bi-invariant affine connections given in [23].

**Theorem 2.1.** ([5, Thm. 3.1]) *Consider a compact connected Lie group  $G$  with a bi-invariant metric  $\rho$  and let  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$  be the decomposition of the Lie algebra  $\mathfrak{g} = T_e G$  into its centre  $\mathfrak{g}_0$  and simple ideals  $\mathfrak{g}_i$  ( $1 \leq i \leq r$ ). Then, a bi-invariant metric connection on  $G$  is given by (up to scale)*

$$\nabla_X^\alpha Y := \eta^\alpha(X, Y) = \sum_{1 \leq i \leq r} ((1 - \alpha_i)/2) \cdot [X, Y]_{\mathfrak{g}_i}, \quad (3)$$

for any  $X, Y \in \mathfrak{g}$ , where  $\alpha := (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$ . The torsion and the curvature of this  $r$ -parameter family are  $T^\alpha(X, Y) = -\sum_{1 \leq i \leq r} \alpha_i \cdot [X, Y]_{\mathfrak{g}_i}$  and  $R^\alpha(X, Y)Z = \sum_{1 \leq i \leq r} ((1 - \alpha_i^2)/4) \cdot [Z, [X, Y]_{\mathfrak{g}_i}]_{\mathfrak{g}_i}$ , respectively.

**Proof.** Consider first a bilinear  $\text{Ad}(G)$ -equivariant map  $\eta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  corresponding to a bi-invariant metric connection  $\nabla$  on  $G$ . Since  $\nabla$  is metric with respect to  $\rho$ ,  $\eta$  is skew-symmetric with respect to the induced  $\text{Ad}(G)$ -invariant inner product  $\langle \cdot, \cdot \rangle$ , i.e.  $\eta_X := \Lambda(X) \in \mathfrak{so}(\mathfrak{g})$ , for any  $X \in \mathfrak{g}$ . Thus, a bi-invariant metric connection  $\nabla$  has skew-torsion  $T \in \Lambda^3(\mathfrak{g})$  if and only if  $\eta(X, X) = \Lambda(X)X = 0$  for any  $X \in \mathfrak{g}$  and this corrects a small error in [5, Lem. 3.1], see also [1, Lem. 2.1]. Obviously, a connection induced by the adjoint representation of  $\mathfrak{g}$  verifies this condition. Hence, the most interesting part of the proof is that of uniqueness. We break the argument up into two steps.

*1st Step:* We begin with the additional assumption that  $G$  is simple. By Theorem 1.1 we know that the bilinear map  $\lambda(X, Y) = (1 - \alpha)[X, Y]/2$  defines a 1-dimensional family of bi-invariant metric connections on  $G$  with torsion  $T^\alpha(X, Y) = -\alpha[X, Y]$ , for any  $X, Y \in \mathfrak{g}$ . We need now to show that this is the unique family (up to scale). The space of bi-invariant affine connections on  $G$  is isomorphic to the space  $\text{Hom}_G^{\mathbb{R}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ . Since  $\mathfrak{g}$  is irreducible (and of real type), it is sufficient to compute the multiplicity of  $\mathfrak{g}$  inside  $\mathfrak{g} \otimes \mathfrak{g} = S^2(\mathfrak{g}) \oplus \Lambda^2(\mathfrak{g})$ . In [23] H. T. Laquer confirms that for any compact simple Lie group the multiplicity of  $\mathfrak{g}$  in  $\Lambda^2(\mathfrak{g})$  is one and only for  $SU(n)$  ( $n \geq 3$ ) there is a *new* copy of  $\mathfrak{g}$  inside  $S^2(\mathfrak{g})$  with the same multiplicity. Thus, for  $G$  simple with  $G \not\cong SU(n)$  ( $n \geq 3$ ), the unique family of bi-invariant affine connections is determined by the bilinear map  $\lambda$  (up to scale). For  $SU(n)$  the “exceptional” family corresponds to the symmetric bilinear map  $\eta^{\text{exc}}(X, Y) = i(XY + YX - (2/n)\text{tr}(XY) \cdot I)$ , where  $I$  is the  $n \times n$  identity matrix. However, the induced affine connection is not metric with respect to a bi-invariant metric, e.g. the negative of the Killing form [5]. This proves the claim.

*2nd Step:* Let us drop now the latter condition and explain the more general case of a compact Lie group  $G$ . Consider the decomposition of the corresponding Lie algebra  $\mathfrak{g} = T_e G$  into its centre and simple ideals  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$  and write  $X = X_0 + X_1 + \cdots + X_r$ . For any simple ideal  $\mathfrak{g}_i$  one can apply the method described in the first step, by using the bi-invariant connection  $\nabla^{\alpha_i}$  induced by the bilinear map  $\eta^{\alpha_i}(X, Y) = ((1 - \alpha_i)/2) \cdot [X, Y]_{\mathfrak{g}_i}$  for some  $\alpha_i \in \mathbb{R}$ , where  $[X, Y]_{\mathfrak{g}_i} := [X_i, Y_i]$ . Obviously,  $\nabla^{\alpha_i}$  is metric with respect to the restriction  $\langle \cdot, \cdot \rangle_{\mathfrak{g}_i} = x_i \rho|_{\mathfrak{g}_i}$ , where  $x_i$  are real positive numbers for any  $i = 1, \dots, r$ . Consider now some scalar product  $b$  on the centre  $\mathfrak{g}_0$  and notice that  $\eta^{\alpha_0} \equiv 0$ . The  $\text{Ad}(G)$ -invariant scalar product  $\langle \cdot, \cdot \rangle$  can be expressed by  $\langle \cdot, \cdot \rangle = b|_{\mathfrak{g}_0} + x_1 \rho|_{\mathfrak{g}_1} + \cdots + x_r \rho|_{\mathfrak{g}_r}$  for some  $x_i \in \mathbb{R}_+$ . Hence, it is not difficult to see that the map defined by  $\eta^\alpha(X, Y) := \sum_{i=1}^r \eta^{\alpha_i}(X, Y)$  with  $\alpha := (\alpha_1, \dots, \alpha_r)$ , induces a family of bi-invariant connections on  $G$  which are metric with respect to  $\langle \cdot, \cdot \rangle$ . The associated torsion is given by  $T^\alpha(X, Y) = -\sum_{i=1}^r \alpha_i \cdot [X, Y]_{\mathfrak{g}_i}$  and due to the  $\text{ad}(\mathfrak{g})$ -invariance of  $\langle \cdot, \cdot \rangle$ , the induced 3-tensor is a 3-form on  $\mathfrak{g}$ . On the other hand, by [23] it is known that besides  $SU(n)$  ( $n \geq 3$ ), only for  $U(n)$  ( $n \geq 2$ ) one can construct affine bi-invariant connections corresponding to  $\text{Ad}(U(n))$ -equivariant bilinear maps different from the Lie bracket (for details see [23, Thm. 9.1] and Proposition 2.8). However, as for  $SU(n)$ , in [5, Thm 3.1] it was explained that the induced connections fail to carry the metric property. Using now [23, Thm. 9.1], we conclude that for an arbitrary compact Lie group  $G$  a bi-invariant metric connection necessarily corresponds to a copy of  $\mathfrak{g}$  inside  $\Lambda^2(\mathfrak{g})$ , and this is given by (3) (up to scale).  $\blacksquare$

**Corollary 2.2.** *Any bi-invariant metric connection  $\nabla$  on a compact connected Lie group  $G$  endowed with a bi-invariant metric, has (totally) skew-symmetric torsion  $T \in \Lambda^3(\mathfrak{g})$ .*

**Question 1.** Let  $G$  be a compact connected Lie group with Lie algebra  $\mathfrak{g}$ . Given an arbitrary bi-invariant *affine* connection  $\nabla$  whose Nomizu map  $\Lambda : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$  satisfies the equation

$$\Lambda(X)X = 0, \quad \forall X \in \mathfrak{g}, \quad (4)$$

is it true that  $\nabla$  is metric with respect to a bi-invariant metric? In other words, are the conditions  $\Lambda(X)X = 0$  and  $\Lambda(X) \in \mathfrak{so}(\mathfrak{g})$  equivalent for *any* bi-invariant *affine* connection on  $G$ ?

A bi-invariant connection satisfying (4) has as geodesics orbits of the one-parameter subgroups of  $G$  (in the simple case the same geodesics with the 1-parameter family of canonical connections on  $G$ , see [22, Prop. 2.9, Ch. X]). Hence, as we explained before, if  $\nabla$  is metric with respect to a bi-invariant metric on  $G$  then its torsion must be a 3-form on  $\mathfrak{g}$ , i.e.  $\Lambda(X)X = 0$ . However, the “converse” is not true, i.e. the previous question admits a *negative* answer with counterexamples appearing for  $U(n)$  (see [23, 5] and for details the proof of Proposition 2.8). Hence, we ask:

**Question 2.** Which further conditions do we have to impose on the Nomizu map  $\Lambda : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$  of a bi-invariant *affine* connection on  $G$  satisfying (4) in order to be metric with respect to a bi-invariant metric? In other words, which subclass of bi-invariant affine connections satisfying (4) can be identified with the class of bi-invariant *metric* connections on  $G$ ?

Our answer relates the *flat connections* of this type, which coincide with the  $\pm 1$ -connections discussed in Theorem 1.1. We should emphasize once more that here we drop the condition that  $G$  is simple.

**Lemma 2.3.** *Let  $G$  be a compact connected Lie group and let  $\nabla$  be a bi-invariant affine connection with  $\Lambda(X)X = 0$ , for any  $X \in \mathfrak{g}$ . Then the following are equivalent:*

- (a)  $\nabla$  is flat  $R \equiv 0$ , i.e.  $\Lambda(X) : \mathfrak{g} \rightarrow \mathfrak{g}$  is a representation for any  $X \in \mathfrak{g}$ .
- (b)  $\Lambda(X) = \text{ad}(X)$ , or  $\Lambda(X) = 0$  for any  $X \in \mathfrak{g}$ , and these are the unique bi-invariant linear connections which satisfy (a).

**Proof.** By definition,  $R(X, Y) = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y])$  and thus  $\nabla$  is flat  $R \equiv 0$  if and only if  $\Lambda : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  is a representation (for example, the Riemannian connection does not induce a representation). Assume that  $R \equiv 0$ , i.e.  $\Lambda(X)\Lambda(Y)Z - \Lambda(Y)\Lambda(X)Z - \Lambda([X, Y])Z = 0$ , for any  $X, Y, Z \in \mathfrak{g}$ . By polarization, equation (4) is equivalent to  $\Lambda(X)Y + \Lambda(Y)X = 0$  for any  $X, Y \in \mathfrak{g}$ . Thus  $\Lambda(Y)[X, Y] = -\Lambda([X, Y])Y$  and after setting  $Y = Z$  inside the equation  $R \equiv 0$ , it follows that  $0 = -\Lambda(Y)\Lambda(X)Y - \Lambda([X, Y])Y = -\Lambda(Y)\Lambda(X)Y + \Lambda(Y)[X, Y] = -\Lambda(Y)(\Lambda(X)Y - [X, Y])$ . Therefore  $\Lambda(X) = \text{ad}(X)$  or  $\Lambda(X) = 0$ , for any  $X \in \mathfrak{g}$ . The converse is trivial.  $\blacksquare$

From now on we shall denote the special connections presented in Lemma 2.3, (b) by  $\nabla^+$  and  $\nabla^-$ , respectively. The torsion is given by  $T^\pm(X, Y) = \pm[X, Y]$ . Both  $\nabla^\pm$  can be viewed as special members of these bi-invariant linear connections on  $G$ , whose Nomizu map induces *derivations* on the corresponding Lie algebra  $\mathfrak{g}$  (for  $\nabla^-$  trivially). In the sequel we show that this is the desired condition that answers Question 2. First we propose a formula which allows us to characterize the

$\text{Ad}(G)$ -equivariant derivations on  $\mathfrak{g}$  in terms of the curvature and the covariant derivative of the torsion of a bi-invariant connection satisfying (4).

**Proposition 2.4.** *Let  $G$  be a compact connected Lie group endowed with a bi-invariant affine connection  $\nabla$  whose Nomizu map  $\Lambda : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$  satisfies (4). Then,  $\Lambda$  is a derivation of  $\mathfrak{g}$ , i.e.  $\Lambda : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$ , if and only if the curvature  $R$  and the covariant derivative of the torsion  $T$  of  $\nabla$  satisfy the following relation:*

$$(\nabla_Z T)(X, Y) = 2\{R(Z, X)Y - \Lambda(Y)([Z, X] - \Lambda(Z)X)\}, \quad \forall X, Y, Z \in \mathfrak{g}. \quad (5)$$

**Proof.** The proof is direct. Crucial is our assumption  $\Lambda(X)X = 0$  and hence we mention that for bi-invariant connections without this property our claim fails. For simplicity set  $D(Z, X, Y) := \Lambda(Z)[X, Y] - [\Lambda(Z)X, Y] - [X, \Lambda(Z)Y]$  and notice that the endomorphism  $\Lambda(Z) : \mathfrak{g} \rightarrow \mathfrak{g}$  is a derivation if and only if  $D(Z, X, Y) = 0$ , for any  $X, Y, Z \in \mathfrak{g}$ . Now, for any  $Z \in \mathfrak{g}$  we view the covariant derivative of the torsion  $T(X, Y) = 2\Lambda(X)Y - [X, Y]$  as a bilinear map  $\nabla_Z T : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ . Then, because  $\Lambda(\Lambda(Z)X)Y = -\Lambda(Y)\Lambda(Z)X$  and  $\Lambda([Z, X])Y = -\Lambda(Y)[Z, X]$  for any  $X, Y, Z \in \mathfrak{g}$ , we obtain that

$$\begin{aligned} (\nabla_Z T)(X, Y) &= \Lambda(Z)T(X, Y) - T(\Lambda(Z)X, Y) - T(X, \Lambda(Z)Y) \\ &= 2\Lambda(Z)\Lambda(X)Y - \Lambda(Z)[X, Y] - 2\Lambda(\Lambda(Z)X)Y \\ &\quad + [\Lambda(Z)X, Y] - 2\Lambda(X)\Lambda(Z)Y + [X, \Lambda(Z)Y] \\ &= 2\Lambda(Z)\Lambda(X)Y + 2\Lambda(Y)\Lambda(Z)X - 2\Lambda(X)\Lambda(Z)Y - D(Z, X, Y) \\ &\stackrel{(\dagger)}{=} 2R(Z, X)Y + 2\Lambda([Z, X])Y + 2\Lambda(Y)\Lambda(Z)X - D(Z, X, Y) \\ &= 2R(Z, X)Y - 2\Lambda(Y)([Z, X] - \Lambda(Z)X) - D(Z, X, Y), \end{aligned}$$

where in  $(\dagger)$  we used  $R(Z, X)Y = \Lambda(Z)\Lambda(X)Y - \Lambda(X)\Lambda(Z)Y - \Lambda([Z, X])Y$ .  $\blacksquare$

**Corollary 2.5.** *Let  $G$  be a compact connected Lie group endowed with a bi-invariant affine connection  $\nabla$  whose Nomizu map  $\Lambda : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g}) \subset \text{End}(\mathfrak{g})$  is a derivation and satisfies (4). If  $\nabla$  is flat, i.e.  $R \equiv 0$ , or  $R(Z, X)Y = \Lambda(Y)([Z, X] - \Lambda(Z)X)$ , then the corresponding torsion  $T$  is  $\nabla$ -parallel.*

**Proof.** If  $R \equiv 0$ , then equation (5) reduces to

$$(\nabla_Z T)(X, Y) = -2\Lambda(Y)([Z, X] - \Lambda(Z)X).$$

Simultaneously, Lemma 2.3 ensures that  $\Lambda = \Lambda^\pm$  and then for the left hand side one gets  $-2\Lambda(Y)([Z, X] - \Lambda(Z)X) = 0$ . Hence  $\nabla T = 0$ , i.e.  $\nabla^\pm T^\pm = 0$ . For example, this is the case if  $G$  is semi-simple, since then any derivation is inner (however notice that in the compact case this argument fails, see Proposition 2.7). Now, if  $R(Z, X)Y = \Lambda(Y)([Z, X] - \Lambda(Z)X)$ , then it is immediate from (5) that  $\nabla T \equiv 0$ . An alternative way that avoids (5) but includes a few more computations occurs due to the following observation. For a bi-invariant affine connection on  $G$  satisfying our assumptions, it is not difficult to prove that the equation  $R(Z, X)Y = \Lambda(Y)([Z, X] - \Lambda(Z)X)$  is equivalent to  $[\Lambda(Z), \Lambda(Y)] = \Lambda(\Lambda(Z)Y)$  (as an endomorphism of  $\mathfrak{g}$ ), or in other words  $[\Lambda(Z), \Lambda(Y)]X = \Lambda(\Lambda(Z)Y)X = -\Lambda(X)\Lambda(Z)Y$  for any  $X, Y, Z \in \mathfrak{g}$ . By using this relation and the properties of

$\Lambda$ , a straightforward computation shows that  $(\nabla_Z T)(X, Y) = 0$  for any  $X, Y, Z \in \mathfrak{g}$ .  $\blacksquare$

By combining this with Lemma 2.3 we conclude that

**Corollary 2.6.** *On a compact connected Lie group  $G$  there exist exactly two bi-invariant affine connections satisfying (4), which are flat and have parallel torsion. These are the connections  $\nabla^\pm$  described in Lemma 2.3 and they coincide with the  $\pm 1$ -connections of Cartan-Schouten.*

**Proposition 2.7.** *Let  $\mathfrak{g}$  be a reductive Lie algebra. Then, any derivation  $D : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$  is given by  $D(X) = \phi(Z) \oplus \text{ad}(X_s)$  for some linear map  $\phi : \mathfrak{g}_0 \rightarrow \text{End}(\mathfrak{g}_0)$  in the centre  $\mathfrak{g}_0$ . In particular,  $H^1(\mathfrak{g}, \mathfrak{g}) \cong \text{End}(\mathfrak{g}_0)$ .*

**Proof.** Consider the decomposition of  $\mathfrak{g}$  into its centre and semi-simple part, i.e.  $\mathfrak{g} = \mathfrak{g}_0 \oplus [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_0 \oplus \mathfrak{g}_{ss}$  and express any  $X \in \mathfrak{g}$  in a unique way by  $X = Z + X_s$ , where  $Z \in \mathfrak{g}_0$  and  $X_s \in \mathfrak{g}_{ss}$ . Then, define  $D : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$  by  $D(X) := \phi(Z) + \text{ad}(X_s) (\equiv \phi(Z) + \text{ad}(X))$  for some linear map  $\phi : \mathfrak{g}_0 \rightarrow \text{End}(\mathfrak{g}_0)$ . Obviously, this is a derivation of  $\mathfrak{g}$  and in order to prove our claim it is sufficient to show that for any  $Z \in \mathfrak{g}_0$  and  $X_s \in \mathfrak{g}_{ss}$  derivations of the form  $D_1(Z) : \mathfrak{g}_0 \rightarrow \mathfrak{g}_{ss}$  and  $D_2(X_s) : \mathfrak{g}_{ss} \rightarrow \mathfrak{g}_0$  are necessarily trivial. This follows easily for  $D_1$ , because the centre of any Lie algebra is a characteristic ideal, i.e. remains invariant under derivations. Consider now some  $\alpha, \beta \in \mathfrak{g}$  with  $[\alpha, \beta] \in \mathfrak{g}_{ss}$  and assume that for any  $X_s \in \mathfrak{g}_{ss}$  the linear map  $D_2(X_s) : \mathfrak{g}_{ss} \rightarrow \mathfrak{g}_0$  is a non-trivial derivation. Then,  $D_2(X_s)$  acts on  $[\alpha, \beta]$  as an inner derivation, i.e.  $D_2(X_s)[\alpha, \beta] = \text{ad}_{X_s}[\alpha, \beta] = [X_s, [\alpha, \beta]]$ . On the other hand we have that  $D_2(X_s)[\alpha, \beta] = [D_2(X_s)\alpha, \beta] + [\alpha, D_2(X_s)\beta] = 0$ , since  $D_2(X_s)\alpha, D_2(X_s)\beta \in \mathfrak{g}_0$ . Because  $X_s, [\alpha, \beta] \in \mathfrak{g}_{ss}$ , this gives a contradiction. In this way we conclude that the spaces  $\text{Der}(\mathfrak{g}_0, \mathfrak{g}_{ss})$  and  $\text{Der}(\mathfrak{g}_{ss}, \mathfrak{g}_0)$  must be trivial and for the Lie algebra  $\text{Der}(\mathfrak{g})$  we get the direct sum decomposition

$$\text{Der}(\mathfrak{g}) = \text{Der}(\mathfrak{g}_0) \oplus \text{Der}(\mathfrak{g}_{ss}) = \text{End}(\mathfrak{g}_0) \oplus \text{ad}(\mathfrak{g}) = \text{Out}(\mathfrak{g}) \oplus \text{Inn}(\mathfrak{g}),$$

where  $\text{Inn}(\mathfrak{g}) \cong \mathfrak{g} \backslash \mathfrak{g}_0 := \{\text{ad}(X) : X \in \mathfrak{g}\} = \text{ad}(\mathfrak{g})$  denotes the space of all inner derivations (the adjoint algebra) and  $\text{Out}(\mathfrak{g})$  is the quotient algebra of outer derivations, i.e.  $\text{Out}(\mathfrak{g}) \cong \text{Der}(\mathfrak{g}) \backslash \text{Inn}(\mathfrak{g})$ . The algebra  $\text{Out}(\mathfrak{g})$  coincides with the first cohomology  $H^1(\mathfrak{g}, \mathfrak{g})$  of  $\mathfrak{g}$  acting on itself by the adjoint representation, see [18, p. 57]. Hence  $H^1(\mathfrak{g}, \mathfrak{g}) \cong \text{End}(\mathfrak{g}_0)$ .  $\blacksquare$

Therefore, given a compact connected Lie group  $G$  and an arbitrary derivation  $D : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$ , the relation  $D(X)X = 0$  is *not* necessarily true for any  $X \in \mathfrak{g}$ . Next we will show that if  $D : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$  is an  $\text{Ad}(G)$ -equivariant derivation with  $D(X)X = 0$  for any  $X \in \mathfrak{g}$ , then  $\phi$  must be trivial  $\phi \equiv 0$ , i.e.  $D$  is an inner derivation. Although for non-central elements  $g \notin Z(G)$  one can prove this result easily, for central elements the equivariance condition does not provide any further information and a proof of the claim seems difficult. Proposition 2.4 allows us to overpass this problem. In fact, we provide two different proofs with the first one being independent of Laquer's classification results [23].

**Proposition 2.8.** *Let  $G$  be a compact connected Lie group and let  $D : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$  be a derivation of  $\mathfrak{g} = T_e G$ . Assume that  $D(\text{Ad}(g)X) = \text{Ad}(g)D(X)\text{Ad}(g)^{-1}$  for any  $g \in G$ ,  $X \in \mathfrak{g}$  and that  $D(X)X = 0$  for any  $X \in \mathfrak{g}$ . Then  $D$  is an inner derivation.*

**Proof.** **1st way:** By Proposition 2.7, write  $D = \phi \oplus \text{ad}$  for some linear map  $\phi : \mathfrak{g}_0 \rightarrow \text{End}(\mathfrak{g}_0)$ . Because  $\text{Ad}(G)\mathfrak{g}_0 = \mathfrak{g}_0$  and the adjoint representation of  $\mathfrak{g}$  is  $\text{Ad}(G)$ -equivariant, it turns out that  $D$  has the same property if and only if  $\phi(\text{Ad}(g)Z) = \phi(Z)$  ( $\equiv \text{Ad}(g)\phi(Z)\text{Ad}(g)^{-1}$ ), for any  $g \in G$  and  $Z \in \mathfrak{g}_0$  (where we view  $\text{Ad}(g)\phi(Z)\text{Ad}(g)^{-1}$  as an endomorphism  $\mathfrak{g}_0 \rightarrow \mathfrak{g}_0$ ). In addition, the condition  $D(X)X = 0$  for any  $X \in \mathfrak{g}$  is equivalent to  $\phi(Z)Z = 0$  for any  $Z \in \mathfrak{g}_0$ . Now, it is sufficient to prove that  $\phi \equiv 0$ . Assume in contrast that  $\phi(Y) \neq 0$  for some  $Y \in \mathfrak{g}_0$ . We view the centre  $Z(G)$  of  $G$  as a compact Lie group itself and we identify  $T_e(Z(G)) = \mathfrak{g}_0$  (the centre  $Z(G)$  is closed subgroup of  $G$ ). Because for any  $Z \in \mathfrak{g}_0$  the endomorphism  $\phi(Z) : \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$  is (trivially) a derivation which satisfies the properties of Proposition 2.4, the associated bi-invariant affine connection on  $Z(G)$  satisfies (5) for any  $X, Y, Z \in \mathfrak{g}_0$ . Let us denote this connection by  $\nabla^\phi$ . Obviously  $R^\phi \equiv 0$  (since  $\mathfrak{g}_0$  is abelian) and  $T^\phi(Z, Z') = \phi(Z)Z' - \phi(Z')Z$  for any  $Z, Z' \in \mathfrak{g}_0$ . An easy computation also shows that

$$\begin{aligned} (\nabla_Z^\phi T^\phi)(X, Y) &= \phi(Z)(T^\phi(X, Y)) - T^\phi(\phi(Z)X, Y) - T^\phi(X, \phi(Z)Y) \\ &= \phi(\phi(Z)Y)X - \phi(\phi(Z)X)Y, \end{aligned}$$

since for example  $\phi(Z)\phi(X) = \phi(X)\phi(Z)$  for any  $Z, X \in \mathfrak{g}_0$ . Finally, relation (5) becomes

$$\phi(\phi(Z)Y)X - \phi(\phi(Z)X)Y = 2\phi(Y)\phi(Z)X \quad \forall X, Y, Z \in \mathfrak{g}_0.$$

Now, for  $X = Z$  it reduces to  $\phi(\phi(X)Y)X = 0$ , for any  $X, Y \in \mathfrak{g}_0$ . Because the identity  $\phi(Z)Z = 0$  is equivalent to  $\phi(X)Y + \phi(Y)X = 0$  for any  $X, Y \in \mathfrak{g}_0$ , we can write  $\phi(\phi(Y)X)X = 0$ . Hence, it must be  $\phi(Y)X = X$ , i.e.  $\phi(Y) : \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$  is the identity map for any  $Y \in \mathfrak{g}_0$ . But then  $\phi(Y)Y = Y$ , i.e.  $\phi(Y)Y \neq 0$  for any  $Y \in \mathfrak{g}_0$ , which gives rise to a contradiction. Thus  $\phi \equiv 0$  and  $D \equiv \text{ad}$ , as claimed.

**2nd way:** Proposition 2.4 characterizes the  $\text{Ad}(G)$ -equivariant derivations  $D : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$  on the Lie algebra  $\mathfrak{g}$  of a compact connected Lie group  $G$  satisfying the condition  $D(X)X = 0$  for any  $X \in \mathfrak{g}$ . Such derivations correspond to bi-invariant affine connections of  $G$ , whose Nomizu map  $\Lambda : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  satisfies the relations (4) and (5). For a compact simple Lie group, except  $G = \text{SU}(n)$  ( $n \geq 3$ ), the bi-invariant affine connections are described (up to scale) by the 1-parameter family  $\Lambda^\alpha : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$  with  $\Lambda^\alpha := ((1 - \alpha)/2) \cdot \text{ad}$ , for some  $\alpha \in \mathbb{R}$ , which is obviously an  $\text{Ad}(G)$ -equivariant derivation. For the general compact case, by adopting the notation of Theorem 2.1 it is easy to see that for some  $\alpha_i \in \mathbb{R}$  the expression  $D(X) := \sum_{i=1}^r \Lambda^{\alpha_i}(X) = \sum_{i=1}^r ((1 - \alpha_i)/2) \cdot \text{ad}(X)|_{\mathfrak{g}_i}$  is a derivation on  $\mathfrak{g}$ , which turns out to be inner (by linearity of  $\text{ad}$ ). In order to prove our claim, there remains to exclude the exotic connections of  $\text{SU}(n)$  and  $\text{U}(n)$ . Indeed, a routine computation shows that these are not derivations, in particular: *the unique bi-invariant linear connections of a compact connected Lie group which*

induce derivations on the corresponding Lie algebra are induced by the Lie bracket. For example, for  $SU(n)$  the bilinear map  $\eta^{\text{exc}}$  described in Theorem 2.1 does not induce a derivation nor does it satisfy (4). The special families for  $U(n)$  are more complicated. For  $n = 2$ , aside the skew-symmetric map induced by the Lie bracket, the new families of symmetric bilinear  $\text{Ad}(U(n))$ -equivariant maps span a 3-dimensional space [23]. However, neither these are derivations and the condition  $\eta(X, X) = 0$  also fails. The same is true for  $n \geq 3$ ; there is a 3-dimensional space generated by symmetric bilinear maps  $\eta_i : \mathfrak{u}(n) \times \mathfrak{u}(n) \rightarrow \mathfrak{u}(n)$  which do not induce derivations, namely  $\eta_1(X, Y) := i(X \cdot Y + Y \cdot X)$ ,  $\eta_2(X, Y) := \text{tr}(X \cdot Y) \cdot iI$ , and  $\eta_3(X, Y) := \text{tr}(X)\text{tr}(Y) \cdot iI$ , but also the skew-symmetric map  $\mu(X, Y) = i(\text{tr}(Y)X - \text{tr}(X)Y)$  (see [23, Thm. 10.1] or [5, Thm. 3.1]). Because  $\mu(X, X) = 0$  for any  $X \in \mathfrak{u}(n)$ ,  $\mu$  is at least a candidate of Proposition 2.4. However a quick check implies that neither this is a derivation. Now, although a linear combination  $\eta_c(X, Y) := c_1\eta_1(X, Y) + c_2\eta_2(X, Y) + c_3\eta_3(X, Y) + c\mu(X, Y)$  gives rise to an  $\text{Ad}(U(n))$ -equivariant bilinear map on  $\mathfrak{g}$ , the condition  $\eta_c(X, X) = 0$  for any  $X \in \mathfrak{u}(n)$  is true, if and only if,  $c_1 = c_2 = c_3 = 0$ . Because on an arbitrary compact Lie group  $G$  these connections exhaust all possible bi-invariant affine connections [23, Thm. 9.1], the proof is complete. ■

Based on Proposition 2.8, we are now able to present the main theorem of this section.

**Theorem 2.9.** *Let  $G$  be a compact connected Lie group with a bi-invariant metric  $\rho$  and let  $\nabla$  be a bi-invariant affine connection corresponding to a linear  $\text{Ad}(G)$ -equivariant map  $\Lambda : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  satisfying (4). Then  $\Lambda$  is a derivation if and only if  $\nabla$  is metric with respect to  $\rho$ . In particular, the class of bi-invariant affine connections which induce derivations  $\Lambda : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$  on the corresponding Lie algebra  $\mathfrak{g}$  coincides with the class of bi-invariant metric connections on  $G$ .*

**Proof.** For the first part we need only to prove the one direction, since the converse is obvious due to Theorem 2.1. Recall that the Killing form of a Lie algebra  $\mathfrak{g}$  (which here we denote by  $B \equiv B_{\mathfrak{g}}$ ) satisfies the relation  $B(AX, AY) = B(X, Y)$ , for any automorphism  $A : \mathfrak{g} \rightarrow \mathfrak{g}$ , see [18, p. 13]. If  $\Lambda(X) \in \text{Der}(\mathfrak{g})$ , then  $\exp(t\Lambda(X)) \in \text{Aut}(\mathfrak{g})$ . Thus, the derivative of the relation  $B(AX, AY) = B(X, Y)$  at  $t = 0$  for  $A = \exp(t\Lambda(X))$  implies that  $B(\Lambda(X)Y, Z) + B(Y, \Lambda(X)Z) = 0$  for any  $X, Y, Z \in \mathfrak{g}$ . If the Lie group  $G$  is simple, then any  $\text{Ad}(G)$ -invariant inner product is a multiple of  $-B$ , so  $\Lambda(X) \in \mathfrak{so}(\mathfrak{g})$ . If  $G$  is just compact, then we express the  $\text{Ad}(G)$ -invariant inner product associated to  $\rho$  by  $\langle \cdot, \cdot \rangle = b|_{\mathfrak{g}_0} - \sum_{i=1}^r c_i \cdot B|_{\mathfrak{g}_i}$  for some  $c_i > 0$  (see Theorem 2.1). This is possible, because  $\rho|_{\mathfrak{g}_i} = \text{multiple of } -B_i$  where  $B_i \equiv B_{\mathfrak{g}_i} = B|_{\mathfrak{g}_i}$ , for any  $i = 1, \dots, r$ . As we explained above, for any simple ideal  $\mathfrak{g}_i$  ( $1 \leq i \leq r$ ) (inner) derivations become metric with respect to  $B|_{\mathfrak{g}_i}$ . For the centre  $\mathfrak{g}_0$  not all the derivations are necessarily metric with respect to the scalar product  $b$ . However,  $\Lambda : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$  is an  $\text{Ad}(G)$ -equivariant derivation with  $\Lambda(X)X = 0$  and Proposition 2.8 guarantees that this is inner. Hence the centre has no contribution and we finally obtain  $\Lambda(X) \in \mathfrak{so}(\mathfrak{g})$  for any  $X \in \mathfrak{g}$ . Now, the final result is valid if one can drop the condition  $\Lambda(X)X = 0$ . Indeed, this is the case because a routine computation shows that

a linear combination of the exotic connections on  $U(n)$  ( $n \geq 2$ ) fails to induce a non-trivial derivation (nor satisfies (4) as we explained in the proof of Proposition 2.8). The same time, the endomorphism  $\Lambda(X) := \sum_{i=1}^r ((1 - \alpha_i)/2) \cdot \text{ad}(X)|_{\mathfrak{g}_i}$  is an equivariant derivation which trivially verifies the condition  $\Lambda(X)X = 0$ . ■

### 3. Invariant metric connections with skew-torsion on naturally reductive spaces

Let  $(M = G/K, g)$  be a connected naturally reductive Riemannian manifold. We shall study  $G$ -invariant metric connections whose torsion is proportional to the torsion of the canonical connection  $\nabla^c$ . Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$  be a reductive decomposition and let  $\Lambda^g : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$  be the Nomizu map of the Levi-Civita connection.

**Proposition 3.1.** (a) *For any  $\alpha \in \mathbb{R}$  there is a bijective correspondence between linear  $\text{Ad}(K)$ -equivariant maps  $\Lambda^\alpha : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$ , defined by*

$$\Lambda^\alpha(X)Y = \frac{1 - \alpha}{2}[X, Y]_{\mathfrak{m}} = (1 - \alpha)\Lambda^g(X)Y, \quad \forall X, Y \in \mathfrak{m}, \quad (6)$$

and  $G$ -invariant metric connections  $\nabla^\alpha$  on  $T(G/K)$  with skew-symmetric torsion  $T^\alpha \in \Lambda^3(\mathfrak{m})$  such that  $T^\alpha = \alpha \cdot T^c$ .

(b) *If the Lie group  $G$  is compact and simple, then the family  $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$  is naturally induced by a bi-invariant metric connection of  $G$ .*

**Proof.** (a) The direct statement is well-known [1]. The converse is also very easy. Because  $(M = G/K, g)$  is naturally reductive with respect to  $G$ ,  $\nabla^\alpha$  is a  $G$ -invariant metric connection with skew-torsion  $T^\alpha \in \Lambda^3(\mathfrak{m})$  if and only if the corresponding Nomizu map, say  $\Lambda_{\mathfrak{m}} : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$ , is such that  $\Lambda_{\mathfrak{m}}(X)Y + \Lambda_{\mathfrak{m}}(Y)X = 0$  for any  $X, Y \in \mathfrak{m}$ , see [1, Lem. 2.1]. Because  $T^\alpha = \alpha \cdot T^c$ , a simple application of (2) shows that  $\Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X = (1 - \alpha)[X, Y]_{\mathfrak{m}}$ , for any  $X, Y \in \mathfrak{m}$  and the claim follows.

(b) In [24, Thm. 6.1] it is proved that there is a natural mapping

$$\text{aff}_{G \times G}(F(G)) \rightarrow \text{aff}_G(F(G/K)), \quad \eta \mapsto \pi_*\eta, \quad \text{with } (\pi_*\eta)(X, Y) := \eta(X, Y)_{\mathfrak{m}},$$

for any  $X, Y \in \mathfrak{m}$ . Here,  $\eta$  is a bi-invariant linear connection on  $G$  and  $\pi_* \equiv d\pi_e : \mathfrak{g} \rightarrow \mathfrak{m}$  is the differential of  $\pi$  at  $e$ .

If  $G$  is compact and  $\eta$  is a bi-invariant *metric* connection on  $G$ , then the induced  $G$ -invariant connection on  $M = G/K$  will be also metric, since the inner product on  $\mathfrak{m}$  is the restriction of an  $\text{Ad}(G)$ -invariant inner product of  $\mathfrak{g}$ . In our case, and since  $g$  has been assumed to be naturally reductive,  $\eta$  oughts to induce a  $G$ -invariant metric connection with skew-torsion. For  $G$  compact and simple, any bi-invariant metric connection is given by the map  $\eta^\alpha(X, Y) = ((1 - \alpha)/2)[X, Y]$  for some  $\alpha \in \mathbb{R}$  (up to scale). Consider the composition  $\pi_*\eta^\alpha : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{m}$  and write  $X = X_{\mathfrak{m}} + X_{\mathfrak{k}}$ . Then, by restricting  $\pi_*\eta^\alpha$  on  $\mathfrak{m} \times \mathfrak{m}$  we obtain a well-defined bilinear map  $\lambda^\alpha : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$  with  $\lambda^\alpha(X_{\mathfrak{m}}, Y_{\mathfrak{m}}) = (\pi_*\eta^\alpha)(X_{\mathfrak{m}}, Y_{\mathfrak{m}}) := \eta^\alpha(X_{\mathfrak{m}}, Y_{\mathfrak{m}})_{\mathfrak{m}} = ((1 - \alpha)/2)[X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}}$ . This is an  $\text{Ad}(K)$ -equivariant map satisfying  $\langle \lambda^\alpha(X_{\mathfrak{m}}, Y_{\mathfrak{m}}), Z_{\mathfrak{m}} \rangle + \langle Y_{\mathfrak{m}}, \lambda^\alpha(X_{\mathfrak{m}}, Z_{\mathfrak{m}}) \rangle = 0$ . The associated Nomizu map  $\Lambda^\alpha : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$  is the family discussed in (a). ■

We recall now the case of a symmetric space of Type I.

**Theorem 3.2.** ([24, 5]) *Let  $(M = G/K, g)$  be an (irreducible) Riemannian symmetric space of Type I. Then*

(a) *If  $\nabla$  is a  $G$ -invariant metric connection with torsion a multiple of the torsion of the canonical connection, then necessarily  $\nabla \equiv \nabla^c (\equiv \nabla^g)$ .*

(b) *The space of  $G$ -invariant metric connections consists of just the canonical connection  $\nabla^c \equiv \nabla^g$ .*

**Remark 3.3.** According to [26, Thm. 1.2], given a compact naturally reductive Riemannian manifold  $(M = G/K, g)$  (locally irreducible) the canonical connection is *unique* under the assumption that  $M$  is not isometric to a sphere, a real projective space, or a compact simple Lie group with a bi-invariant metric. Viewing the sphere  $S^n$  as a compact quotient  $M = G/K$ , this anomaly appears since  $G$  is not necessarily equal to the full isometry group  $\text{Iso}(M)$  or its connected component  $\text{Iso}_0(M)$ , in contrast to a symmetric space of Type I. Actually, let  $(M = G/K, g_B)$  be an (effective) simply connected normal homogeneous manifold with  $G$  being a compact, connected, simple Lie group and assume that the isotropy representation is (strongly) irreducible. Then  $G = \text{Iso}_0(M)$ , unless  $M = G_2/\text{SU}(3) = S^6$  or  $M = \text{Spin}(7)/G_2 = S^7$  where  $\text{Iso}_0(M, g_B) = \text{SO}(7), \text{SO}(8)$ , respectively (see [32, Thm. 17.1] or [30, p. 623]). It is well-known that there are more spheres that can be represented as quotients of distinct Lie groups [10]. In particular, the theory of enlargements of transitive actions (developed by A. L. Oniřik) describes all simple compact Lie algebras  $\mathfrak{g}$  which can be written as a direct sum  $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$  of two Lie subalgebras  $\mathfrak{k}_1, \mathfrak{k}_2$  (see [19, 21]). If  $G$  is the compact simply connected Lie group corresponding to  $\mathfrak{g}$  and  $K_1, K_2 \subset G$  are the Lie subgroups associated to  $\mathfrak{k}_1, \mathfrak{k}_2$ , then it holds that  $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$  if and only if  $K_1$  acts transitively on  $G/K_2$ . Hence, in the Lie group level we have the identifications  $G/K_1 = K_2/(K_1 \cap K_2)$  (and  $G/K_2 = K_1/(K_1 \cap K_2)$ ). Oniřik's list (for symmetric cosets) contains several spheres. Let us present them.

	$G/K_1$	$\mathfrak{p}$	$K_2/K_1 \cap K_2$	$\mathfrak{m}$
$S^{4n-1}$	$\text{SO}(4n)/\text{SO}(4n-1)$	irred.	$\text{Sp}(n)/\text{Sp}(n-1)$	$\mathfrak{m}_1 \oplus \mathfrak{m}_2$
$S^{4n-1}$	$\text{SO}(4n)/\text{SO}(4n-1)$	irred.	$\text{Sp}(n) \text{U}(1)/\text{Sp}(n-1) \text{U}(1)$	$\mathfrak{m}_1 \oplus \mathfrak{m}_2$
$S^{4n-1}$	$\text{SO}(4n)/\text{SO}(4n-1)$	irred.	$\text{Sp}(n) \text{Sp}(1)/\text{Sp}(n-1) \text{Sp}(1)$	$\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$
$S^{2n-1}$	$\text{SO}(2n)/\text{SO}(2n-1)$	irred.	$\text{U}(n)/\text{U}(n-1)$	$\mathfrak{m}_1 \oplus \mathfrak{m}_2$
$S^{2n-1}$	$\text{SO}(2n)/\text{SO}(2n-1)$	irred.	$\text{SU}(n)/\text{SU}(n-1)$	$\mathfrak{m}_1 \oplus \mathfrak{m}_2$
$S^6$	$\text{SO}(7)/\text{SO}(6)$	irred.	$G_2/\text{SU}(3)$	irred.
$S^7$	$\text{SO}(8)/\text{SO}(7)$	irred.	$\text{Spin}(7)/G_2$	irred.
$S^{15}$	$\text{SO}(16)/\text{SO}(15)$	irred.	$\text{Spin}(9)/\text{Spin}(7)$	$\mathfrak{m}_1 \oplus \mathfrak{m}_2$

In this table, any symmetric space  $M = G/K_1$  is isotropy irreducible, but for the presentations  $K_2/(K_1 \cap K_2)$  only them of  $S^6$  and  $S^7$  are (strongly) isotropy irreducible. Another fact that deserves our attention is that although the cosets  $K_2/(K_1 \cap K_2)$  are diffeomorphic to a Riemannian symmetric space, namely a sphere, the pairs  $(K_2, K_1 \cap K_2)$  are not necessarily symmetric. For example

$$S^7 = \text{SO}(8)/\text{SO}(7) \cong \text{U}(4)/\text{U}(3) \cong \text{Sp}(2) \text{U}(1)/\text{Sp}(1) \text{U}(1) \cong \text{Spin}(7)/G_2,$$

but taking a reductive decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  the relation  $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$  holds only for the first presentation. Due to this observation and since  $K_2 \subset G$  and  $M = G/K_1 = K_2/(K_1 \cap K_2)$ , one may expect more  $K_2$ -invariant affine connections on  $M$  than  $G$ -invariant connections. Let us examine this interesting problem for the irreducible cosets  $K_2/(K_1 \cap K_2)$  appearing above.

**Theorem 3.4.** *The space of Spin(7)-invariant affine (or metric) connections on the 7-sphere  $S^7 = \text{Spin}(7)/G_2$  is 1-dimensional; it consists of the family  $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$  described in Proposition 3.1, i.e.  $\dim_{\mathbb{R}}(\mathbf{aff}_{\text{Spin}(7)}(F(S^7))) = 1$ . Similarly, the space of  $G_2$ -invariant affine (or metric) connections on the sphere  $S^6 = G_2/\text{SU}(3)$  consists of the same family  $\nabla^\alpha$ , but in this case the parameter  $\alpha$  is a complex number  $\alpha \in \mathbb{C}$ . Hence  $\dim_{\mathbb{R}}(\mathbf{aff}_{G_2}(F(S^6))) = 2$ .*

**Proof.** The non-symmetric presentations of  $S^6$  and  $S^7$  are still (strongly) isotropy irreducible. Therefore, in order to compute the dimensions of the spaces  $\mathbf{aff}_{G_2}(F(G_2/\text{SU}(3)))$  and  $\mathbf{aff}_{\text{Spin}(7)}(F(\text{Spin}(7)/G_2))$ , it is sufficient to find the multiplicity of the corresponding isotropy representation  $\mathfrak{m}$  inside  $\mathfrak{m} \otimes \mathfrak{m} = \Lambda^2(\mathfrak{m}) \oplus \text{Sym}^2(\mathfrak{m})$ . However, we need now to view  $\mathfrak{m}$  as a  $\text{SU}(3)$ - (resp.  $G_2$ -) module. Consider first the 7-sphere  $S^7 \subset \mathbb{R}^8$  and identify  $\mathbb{R}^8 \cong \mathbb{O}$ , where  $\mathbb{O}$  are the Cayley numbers. We view  $G_2 \cong \text{Aut}(\mathbb{O})$  as a subgroup of  $\text{Spin}(7) \subset \text{Cl}(\mathbb{R}^7)$  preserving the spinor  $\psi_0 = (1, 0, \dots, 0)^t \in \Delta_7$ , where  $\Delta_7 := \mathbb{R}^8$  is the 8-dimensional spin representation of  $\text{Spin}(7)$ . Because  $\text{Spin}(7)$  acts transitively on  $S^7$  we get the diffeomorphism  $S^7 \cong \text{Spin}(7)/G_2$ , see [16]. As usual, we write  $V^{a,b}$  for the irreducible representation of  $G_2$  corresponding to highest weight  $(a, b)$ , where both  $a, b$  are non-negative integers; for example  $V^{0,0} \cong \mathbb{R}$  is the trivial representation,  $\phi_7 := V^{1,0} \cong \mathbb{R}^7$  is the standard representation of  $G_2$  and  $V^{0,1} \cong \mathfrak{g}_2$  is its adjoint representation. Let now  $\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}$  be a reductive decomposition. The isotropy representation  $\mathfrak{m}$  coincides with the standard representation  $\mathfrak{m} \cong \phi_7 = \{X \lrcorner \omega : X \in \mathbb{R}^7\} \cong \mathbb{R}^7$ , where  $\omega$  states for the (generic) 3-form on  $\mathbb{R}^7$  preserved by  $G_2$ , see [16, 15]. For the (real)  $G_2$ -modules  $\Lambda^2(\mathfrak{m})$  and  $\text{Sym}^2(\mathfrak{m})$  we get the decompositions (we use the Lie software package, for  $\Lambda^2(\mathfrak{m})$  see also [16, 15]):

$$\Lambda^2(\mathfrak{m}) \cong \mathfrak{so}(7) = V^{0,1} \oplus V^{1,0} = \mathfrak{g}_2 \oplus \phi_7 = \mathfrak{g}_2 \oplus \mathfrak{m}, \quad \text{Sym}^2(\mathfrak{m}) = V^{2,0} \oplus \mathbb{R},$$

where  $V^{2,0} \cong S_0^2 \mathbb{R}^7$  with  $\dim V^{2,0} = 27$ . Thus, there is only one copy of  $\mathfrak{m}$  inside the  $G_2$ -module  $\mathfrak{m} \otimes \mathfrak{m}$ , lying in  $\Lambda^2(\mathfrak{m})$ . In other words, there is skew-symmetric bilinear  $\text{Ad}(G_2)$ -equivariant map  $\eta : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$  which induces a 1-dimensional family of  $\text{Spin}(7)$ -invariant affine connections on  $S^7$ . Because  $\mathfrak{m}$  is irreducible, Schur's lemma tells us that  $\eta$  must be a multiple of the Lie bracket, say  $\eta(X, Y) = \frac{(1-\alpha)}{2} \cdot [X, Y]_{\mathfrak{m}}$  for some  $\alpha \in \mathbb{R}$ , with  $X, Y \in \mathfrak{m}$ . This defines the family  $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$  discussed in Proposition 3.1.

Let us treat now the 6-sphere. Recall that  $G_2$  preserves the imaginary octonions  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$  and acts transitively on  $S^6 \subset \text{Im}(\mathbb{O})$  with stabilizer diffeomorphic to  $\text{SU}(3)$ , i.e.  $S^6 \cong G_2/\text{SU}(3)$ , see [19, Lemma. 5.1]. The weights of  $\text{SU}(3)$  are also given by pairs of non-negative integers  $(a, b)$  and irreducible  $\text{SU}(3)$ -representations will be labeled again by  $V^{a,b}$ . In particular, it

is  $\dim_{\mathbb{C}} V^{a,b} = \frac{1}{2}(a+1)(b+1)(a+b+2)$ . Obviously,  $V^{1,0} \cong \mathbb{C}^3 := \mu_3$  is the standard (complex) representation of  $SU(3)$ ,  $V^{0,1} \cong \overline{\mathbb{C}^3} = \overline{\mu_3}$  is its conjugate and  $V^{1,1} \cong \mathfrak{su}(3)^{\mathbb{C}}$  is the complexified adjoint representation. Let  $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{m}$  be a reductive decomposition. It follows that  $\mathfrak{m} = [\mu_3]_{\mathbb{R}}$ , where for a complex representation  $V$  we denote by  $[V]_{\mathbb{R}}$  the underlying real representation (whose real dimension is twice the complex dimension of  $V$ ). Thus, it is more convenient to use the complexified isotropy representation, which splits now into two conjugate (inequivalent) submodules:  $\mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C} = \mu_3 \oplus \overline{\mu_3} = \mathbb{C}^3 \oplus \overline{\mathbb{C}^3}$ . Then, for the  $SU(3)$ -module  $\Lambda^2(\mathfrak{m}) \otimes_{\mathbb{R}} \mathbb{C}$  we get

$$\begin{aligned} \Lambda^2(\mathfrak{m}) \otimes_{\mathbb{R}} \mathbb{C} &= \Lambda^2(\mathfrak{m}^{\mathbb{C}}) = \Lambda^2(\mathbb{C}^3 \oplus \overline{\mathbb{C}^3}) = \Lambda^2(\mathbb{C}^3) \oplus \Lambda^2(\overline{\mathbb{C}^3}) \oplus (\mathbb{C}^3 \otimes \overline{\mathbb{C}^3}) \\ &= (V^{1,0} \oplus V^{0,1}) \oplus V^{1,1} \oplus \mathbb{C}, \end{aligned}$$

since  $\mathbb{C} \cong V^{0,0}$ ,  $\Lambda^2(\mathbb{C}^3) \cong \Lambda^2(\mu_3) \cong V^{1,0} = \mu_3$  (see also [20, p. 125]) and  $\Lambda^2(\overline{\mathbb{C}^3}) \cong \overline{\Lambda^2(\mathbb{C}^3)} \cong \overline{V^{1,0}} \cong V^{0,1} = \overline{\mu_3}$ . Hence we finally conclude that

$$\Lambda^2(\mathfrak{m}) = [V^{1,0}]_{\mathbb{R}} \oplus \mathfrak{su}(3) \oplus \mathbb{R} = [\mu_3]_{\mathbb{R}} \oplus \mathfrak{su}(3) \oplus \mathbb{R} = \mathfrak{m} \oplus \mathfrak{su}(3) \oplus \mathbb{R}.$$

Under the action of  $SU(3)$ , we also get

$$\begin{aligned} \text{Sym}^2(\mathfrak{m}) \otimes_{\mathbb{R}} \mathbb{C} &= \text{Sym}^2(\mathfrak{m}^{\mathbb{C}}) = \text{Sym}^2(\mathbb{C}^3 \oplus \overline{\mathbb{C}^3}) \\ &= \text{Sym}^2(\mathbb{C}^3) \oplus \text{Sym}^2(\overline{\mathbb{C}^3}) \oplus (\mathbb{C}^3 \otimes \overline{\mathbb{C}^3}) \\ &= (V^{2,0} \oplus V^{0,2}) \oplus V^{1,1} \oplus \mathbb{C}, \end{aligned}$$

with  $V^{2,0} \cong \text{Sym}^2(\mathbb{C}^3)$ . Consequently, the decomposition of  $\text{Sym}^2(\mathfrak{m})$  into irreducible  $SU(3)$ -submodules is given by  $\text{Sym}^2(\mathfrak{m}) = [V^{2,0}]_{\mathbb{R}} \oplus \mathfrak{su}(3) \oplus \mathbb{R}$ . Similarly with  $S^7$ , the copy of  $\mathfrak{m}$  inside  $\Lambda^2(\mathfrak{m})$  defines a skew-symmetric bilinear  $\text{Ad}(SU(3))$ -equivariant map  $\eta : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ , which by Schur's lemma, must be proportional to the Lie bracket restricted on  $\mathfrak{m}$ . In fact, in this case the parameter  $\alpha$  is a complex number, i.e.  $\eta(X, Y) = \frac{(1-\alpha)}{2} \cdot [X, Y]_{\mathfrak{m}}$  for some  $\alpha \in \mathbb{C}$ . This proves the claim for the affine case. Now, the assertion about the metric property is simple. Any  $\text{Spin}(7)$ -invariant metric on  $S^7 = \text{Spin}(7)/G_2$  must be a multiple of the negative of the Killing form of  $\text{Spin}(7)$ , restricted on  $\mathfrak{m}$ . Hence, the family  $\{\nabla^{\alpha} : \alpha \in \mathbb{R}\}$  is necessarily metric. Similarly for  $S^6 = G_2/SU(3)$ . ■

**Remark 3.5.** The embedding of  $S^7$  inside the spin representation  $\Delta_7 \cong \mathbb{R}^8$  induces on  $S^7$  an affine connection  $\nabla^{\text{flat}}$  which is metric and has (non-parallel) skew-torsion  $T^{\text{flat}} \neq 0$  [4]. The 7-sphere endowed with this connection and a Riemannian metric of constant sectional curvature becomes flat, and together with the compact (simple) Lie groups endowed with a bi-invariant metric and one of the  $\pm 1$ -connections, exhaust all Riemannian manifolds carrying a flat metric connection with non-trivial skew-torsion (Cartan-Schouten theorem). Viewing the sphere  $S^7 = \text{Spin}(7)/G_2$  as a  $G_2$ -manifold, I. Agricola and Th. Friedrich [4, pp. 7–9] described this connection as a connection whose torsion  $T^{\text{flat}}$  does not have *constant* coefficients. Hence,  $\nabla^{\text{flat}}$  is *not* an invariant connection and this is the reason that it does not appear in Theorem 3.4 (for example, the difference  $\nabla^{\text{flat}} - \nabla^c$  is not an  $\text{Ad}(G_2)$ -invariant tensor and hence given a reductive decomposition  $\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}$ , the relation  $T^{\text{flat}} = \alpha \cdot T^c$  fails for any  $\alpha$ ).<sup>1</sup>

<sup>1</sup>The author thanks I. Agricola for this remark.

#### 4. $\nabla^\alpha$ -Einstein naturally reductive manifolds with skew-torsion

In this section we describe the geometry of a naturally reductive manifold endowed with a family of invariant metric connections whose torsion is such that  $T^\alpha = \alpha \cdot T^c$ , for some parameter  $\alpha$ . We are mainly interested to answer the following question: *For which values of  $\alpha$  ( $\alpha \neq 0$ ) the induced Ricci tensor  $\text{Ric}^\alpha$  is “proportional” to the naturally reductive metric  $g$ ?*

**4.1.  $\nabla$ -Einstein manifolds.** With the aim to give a precise definition of a  $\nabla$ -Einstein structure, it is useful to recall identities of the Ricci tensor and the scalar curvature of a metric connection  $\nabla$  with skew-symmetric torsion  $0 \neq T \in \Lambda^3(T^*M)$ . We limit ourselves only in a few details and for a general picture we refer to [15, 3, 2]. As usual, we write  $g(\nabla_X Y, Z) = g(\nabla_X^g Y, Z) + \frac{1}{2}T(X, Y, Z)$ , where  $\nabla^g$  denotes the Levi-Civita connection of the fixed Riemannian manifold  $(M^n, g)$ . Let  $\{e_1, \dots, e_n\}$  be a (local) orthonormal frame of  $M$ . In terms of the co-differential  $\delta T$  and the normalized length  $\|T\|^2 := (1/6) \sum_{i,j} g(T(e_i, e_j), T(e_i, e_j))$  of  $T$ , one has the formulas:  $\text{Scal} = \text{Scal}^g - \frac{3}{2}\|T\|^2$  and

$$\text{Ric}(X, Y) = \text{Ric}^g(X, Y) - \frac{1}{4} \sum_{i=1}^n g(T(e_i, X), T(e_i, Y)) - \frac{1}{2}(\delta^g T)(X, Y).$$

The co-differential (with respect to  $\nabla$ ) of a  $n$ -form  $\omega$  on  $M$  is given by  $\delta^\nabla \omega := -\sum_i e_i \lrcorner \nabla_{e_i} \omega$ . For the torsion 3-form it holds that  $\delta^\nabla T = \delta^g T$ . We emphasize that the Ricci tensor of  $\nabla$  is not necessarily symmetric; it decomposes into a symmetric and antisymmetric part  $\text{Ric} = \text{Ric}_S + \text{Ric}_A$ , given by

$$\text{Ric}_S(X, Y) := \text{Ric}^g(X, Y) - \frac{1}{4} \sum_{i=1}^n g(T(e_i, X), T(e_i, Y)),$$

and  $\text{Ric}_A(X, Y) := -\frac{1}{2}(\delta^g T)(X, Y)$ , respectively. In analogy to compact Einstein manifolds,  $\nabla$ -Einstein manifolds with skew-torsion admit a variational approach based on the functional

$$(g, T) \mapsto \int_M (\text{Scal} - 2\Lambda) d \text{vol}_g,$$

where  $\Lambda$  is a constant. In particular, by [3, Thm. 2.1] it is known that critical points of this functional are pairs  $(g, T)$  as above, satisfying the equation

$$-\text{Ric}_S + \frac{1}{2} \text{Scal} \cdot g - \Lambda \cdot g = 0.$$

For this reason, one has the following formal definition:

**Definition 4.1.** We call a 4-tuple  $(M^n, g, \nabla, T)$  a  $\nabla$ -Einstein manifold with skew-torsion  $T$ , or in short, a  $\nabla$ -Einstein manifold, if the symmetric part of the Ricci tensor of  $\nabla$  satisfies the equation  $\text{Ric}_S = \frac{\text{Scal}}{n} g$ .

In contrast to the Riemannian case, for a  $\nabla$ -Einstein manifold the scalar curvature is not necessarily constant, see [3]. For parallel torsion  $T$  one has

$\delta^\nabla T = 0$  and the Ricci tensor becomes symmetric  $\text{Ric} = \text{Ric}_S$ . If in addition  $\delta \text{Ric}^g = 0$ , then the scalar curvature is constant, similarly with an Einstein manifold. This is the case for any  $\nabla$ -Einstein manifold  $(M, g, \nabla, T)$  with parallel skew-torsion [3, Prop.2.7].

**4.2. The  $\nabla^\alpha$ -Einstein condition on naturally reductive spaces.** From now on we assume that  $(M^n = G/K, g)$  ( $n \geq 3$ ) is a naturally reductive manifold, endowed with an effective transitive action of a connected Lie group  $G$  and a reductive decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$  such that  $\mathfrak{g} = \tilde{\mathfrak{g}} := \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ , see [30, p. 569] or [10, p. 196].

**Remark 4.2.** Given a homogeneous space  $M = G/K$  with a reductive decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ , the ideal  $\tilde{\mathfrak{g}} := \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$  is identified with the Lie algebra  $\mathfrak{tr}(\nabla^c)$  of the transvection group  $\text{Tr}(\nabla^c)$  of the canonical connection associated to  $\mathfrak{m}$  [28, Rem. 4.1]. The group  $\text{Tr}(\nabla^c)$  is a connected and normal subgroup of  $\text{Aff}_0(\nabla^c)$  (the connected component of the affine group of  $\nabla^c$ ), which consists of all  $\nabla^c$ -affine transformations that preserve any  $\nabla^c$ -holonomy subbundle of the orthonormal frame bundle. Hence, our assumption equivalently says that  $M = G/K$  is a naturally reductive manifold with respect to the decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ , where  $G = \text{Tr}(\nabla^c)$  is the group of transvections of the canonical connection  $\nabla^c$  associated to  $\mathfrak{m}$ . In general  $\text{Tr}(\nabla^c) \subset G$ . However, if  $M = G/K$  is a compact normal homogeneous space, then  $G = \text{Tr}(\nabla^c)$  [28, Prop. 4.2] and hence any such space gives rise to a (compact) homogeneous manifold satisfying our assumption.

Next we shall use the  $\text{Ad}(G)$ -invariant extension  $Q$  of the (naturally reductive) inner product  $\langle \cdot, \cdot \rangle$  on whole  $\mathfrak{g}$ . Let  $Q_{\mathfrak{k}}$  be the restriction of  $Q$  on  $\mathfrak{k}$ , i.e.  $Q_{\mathfrak{k}}(X, Y) = Q(X_{\mathfrak{k}}, Y_{\mathfrak{k}})$  where  $X_{\mathfrak{k}}$  is the  $\mathfrak{k}$ -component of  $X \in \mathfrak{g}$ . A customary trick is to associate with  $Q_{\mathfrak{k}}$  and the isotropy representation  $\chi_* : \mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{m})$ , the Casimir element; this is the linear operator  $C_\chi \equiv C_{\chi, Q_{\mathfrak{k}}} : \mathfrak{m} \rightarrow \mathfrak{m}$  defined by  $C_\chi := -\sum_q^{\dim \mathfrak{k}} \chi_*(k_q) \circ \chi_*(k'_q)$ , where  $\{k_q, k'_q\}$  are dual bases of  $\mathfrak{k}$  with respect to  $Q_{\mathfrak{k}}$ . If  $\chi$  is an irreducible representation, then  $C_\chi$  is a scalar operator, i.e.  $C_\chi = \text{Cas} \cdot \text{Id}_{\mathfrak{m}}$  for some constant  $\text{Cas} \in \mathbb{R}$  (Casimir constant). We introduce also the symmetric bilinear map  $A$  on  $\mathfrak{m}$ , given by  $A(X, Y) = \langle C_\chi X, Y \rangle$  for any  $X, Y \in \mathfrak{m}$  and we denote by  $B$  the negative of the restriction of the Killing form of  $\mathfrak{g}$  on  $\mathfrak{m}$ . Then, the following relations are standard (see for example [30, 10, 1])

$$\begin{aligned} A(X, Y) &= \sum_j Q_{\mathfrak{k}}([X, Z_j], [Y, Z_j]), \\ B(X, Y) &= \sum_i \langle [X, Z_i]_{\mathfrak{m}}, [Y, Z_i]_{\mathfrak{m}} \rangle + 2A(X, Y). \end{aligned} \quad (7)$$

Consider now the family of  $G$ -invariant metric connections  $\nabla^\alpha$  described in Proposition 3.1. We present formulas for the Ricci tensor and the scalar curvature associated to  $\nabla^\alpha$  (see also [1, Lem. 2.2, Thm. 4.4] for similar expressions).

**Theorem 4.3.** *The Ricci curvature of the naturally reductive Riemannian man-*

ifold  $(M = G/K, g)$  endowed with the family  $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$  is given by

$$\begin{aligned} \text{Ric}^\alpha(X, Y) &= \frac{1 - \alpha^2}{4} \sum_{1 \leq i \leq n} \langle [X, Z_i]_{\mathfrak{m}}, [Y, Z_i]_{\mathfrak{m}} \rangle + A(X, Y) \\ &= \frac{1 - \alpha^2}{4} B(X, Y) + \frac{1 + \alpha^2}{2} A(X, Y). \end{aligned}$$

The corresponding scalar curvature  $\text{Scal}^\alpha : M \rightarrow \mathbb{R}$  has the form

$$\begin{aligned} \text{Scal}^\alpha &= \frac{1 - \alpha^2}{4} \sum_{1 \leq i, j \leq n} \|[Z_i, Z_j]_{\mathfrak{m}}\|^2 + \sum_{1 \leq i, j \leq n} Q_{\mathfrak{k}}([Z_i, Z_j], [Z_i, Z_j]) \\ &= \frac{1 - \alpha^2}{4} \sum_{1 \leq i, j \leq n} \|[Z_i, Z_j]_{\mathfrak{m}}\|^2 + \sum_{1 \leq i \leq n} A(Z_i, Z_i), \end{aligned}$$

where  $\|Z\| := \sqrt{\langle Z, Z \rangle}$  is the norm of a vector  $Z \in \mathfrak{m}$  with respect to the  $\text{Ad}(K)$ -invariant inner product  $\langle \cdot, \cdot \rangle$ .

For a moment, notice that

$$\text{Ric}^0(X, Y) = \frac{1}{4} \sum_{i=1}^n \langle [X, Z_i]_{\mathfrak{m}}, [Y, Z_i]_{\mathfrak{m}} \rangle + A(X, Y) = \frac{1}{4} B(X, Y) + \frac{1}{2} A(X, Y),$$

which is the classical formula of the Riemannian Ricci tensor of a naturally reductive space  $(M = G/K, g)$  with  $\mathfrak{g} = \tilde{\mathfrak{g}}$ , see [30, Prop. 1.9, pp. 569–570] or [10, (7.89b)]. For  $G$  compact and semi-simple, one can replace  $Q_{\mathfrak{k}}$  with the restriction  $B_{\mathfrak{k}} := B|_{\mathfrak{k} \times \mathfrak{k}}$ ; then  $A(X, Y) := B(C_{X, B_{\mathfrak{k}}} X, Y)$  and in this case it is clear that the Killing metric  $g_B$  is Einstein if and only if  $C_{X, B_{\mathfrak{k}}} = \mu \cdot \text{Id}$  for some constant  $\mu$ . In contrast to the compact case and  $C_{X, B_{\mathfrak{k}}}$ , the restriction  $Q_{\mathfrak{k}}$  is not necessarily positive definite. Thus, in the general case the Casimir operator  $C_{X, Q_{\mathfrak{k}}}$  can have eigenvalues of either sign; the same is true for the Ricci tensor and therefore the Einstein condition for the Killing metric is not anymore equivalent to the relation  $C_{X, Q_{\mathfrak{k}}} = \mu \cdot \text{Id}$ .

Now, it is well-known that  $\delta^\alpha T^\alpha = 0$  for any  $\alpha \in \mathbb{R}$ , see [1]. Therefore, a simple combination with Theorem 4.3 shows that

**Corollary 4.4.** *The Ricci tensor  $\text{Ric}^\alpha$  associated to the family  $\nabla^\alpha$  is symmetric for any  $\alpha \in \mathbb{R}$ , i.e.  $\text{Ric}^\alpha \equiv \text{Ric}_S^\alpha = \text{Ric}^g - \frac{1}{4} S^\alpha$ , where  $S^\alpha$  is the symmetric tensor defined by  $S^\alpha(X, Y) = \sum_i \langle T^\alpha(Z_i, X), T^\alpha(Z_i, Y) \rangle$ . In full details*

$$S^\alpha(X, Y) = \alpha^2 \sum_i \langle [X, Z_i]_{\mathfrak{m}}, [Y, Z_i]_{\mathfrak{m}} \rangle = \alpha^2 \left\{ B(X, Y) - 2A(X, Y) \right\}.$$

We proceed with a few remarks for the well-known flat case  $R^\alpha = 0$ , see [12, 25, 4]. Let  $\text{Jac}_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$  be the trilinear map defined by  $\text{Jac}_{\mathfrak{m}}(X, Y, Z) := \mathfrak{S}^{X, Y, Z} [X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}}$ , where  $\mathfrak{S}$  denotes the cyclic sum over the vectors  $X, Y, Z \in \mathfrak{m}$ . For a symmetric space  $(M = G/K, g)$  of Type I, it is  $\text{Jac}_{\mathfrak{m}} \equiv 0$  identically, since  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ . By using Theorem 3.2 we can prove that

**Lemma 4.5.** *Let  $(M^n = G/K, g)$  be a compact connected naturally reductive Riemannian manifold endowed with a  $G$ -invariant metric connection  $\nabla^\alpha$  whose torsion is such that  $T^\alpha = \alpha \cdot T^c$ , for some  $\alpha \in \mathbb{R} \setminus \{0, 1\}$ . Then the following are equivalent:*

- (a)  $\text{Jac}_m \equiv 0$  identically,
- (b)  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$ ,
- (c)  $M^n \cong G$  is isometric to a compact Lie group with a bi-invariant metric.

**Proof.** The less trivial part is the correspondence between (a) and (b). By Proposition 3.1 we write  $\nabla_X^\alpha Y = \nabla_X^c Y + \Lambda^\alpha(X)Y$ , where  $\Lambda^\alpha(X)Y = \frac{1-\alpha}{2}[X, Y]_m$  with  $\alpha \neq 0, 1$ . Since  $\alpha \neq 1$ ,  $\nabla^\alpha$  cannot be the canonical connection associated to  $\mathfrak{m}$  and by Theorem 3.2,  $M = G/K$  cannot be a symmetric space of Type I. First we prove that the condition  $\text{Jac}_m \equiv 0$  implies the relation  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$ . In contrast, assume that there exist some  $X, Y \in \mathfrak{m}$  such that  $[X, Y] \notin \mathfrak{m}$ , i.e.  $[X, Y] = [X, Y]_{\mathfrak{k}}$ . Computing the curvature  $R^\alpha$  of  $\nabla^\alpha$ , we see that

$$R^\alpha(X, Y)Z = \frac{(1-\alpha)^2}{4} \text{Jac}_m(X, Y, Z) + \frac{1-\alpha^2}{4} [Z, [X, Y]_m]_m - [[X, Y]_{\mathfrak{k}}, Z],$$

which finally reduces to  $R^\alpha(X, Y)Z = -[[X, Y]_{\mathfrak{k}}, Z] = -[[X, Y], Z]$ , i.e.  $R^\alpha$  is identical with the curvature associated to the canonical connection. Then, it is easy to prove that  $\nabla^\alpha R^\alpha = 0 = \nabla^\alpha T^\alpha$ , but only the canonical connection has this property (Ambrose-Singer theorem). Since  $\alpha \neq 1$  we obtain a contradiction. Conversely, notice that the relation  $\mathfrak{g} = \tilde{\mathfrak{g}}$  implies that the  $\mathfrak{k}$ -part of the commutator  $[\mathfrak{m}, \mathfrak{m}]$  spans all of  $\mathfrak{k}$ . Assuming that  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$ , it means that the isotropy algebra  $\mathfrak{k}$  is trivial, i.e.  $K = \{e\}$  and  $M^n \cong G$ . Hence  $\mathfrak{g} = \mathfrak{m}$  and by the Jacobi identity we see that  $\text{Jac}_m \equiv 0$ . ■

In the naturally reductive case, the flatness condition  $R^\alpha \equiv 0$  for some  $\alpha \neq 0$  has as consequence the parallelism of the associated torsion form, i.e.  $\nabla^\alpha T^\alpha = 0$  for any  $\alpha \neq 0$  (see for example [12, Prop. 3.7, (d)] or [4, p. 4]). Notice that  $(\nabla_Z^\alpha T^\alpha)(X, Y) = (\alpha(\alpha - 1)/2) \text{Jac}_m(X, Y, Z)$  [1]. Thus, in the compact case Lemma 4.5 can be used to recover in a Lie theoretic way a part of the classical Cartan-Schouten theorem, namely: *If  $(M = G/K, g)$  is a de Rham irreducible, compact, connected naturally reductive manifold as in our assumption, which is flat with respect to the family  $\nabla^\alpha$  for some  $\alpha \in \mathbb{R} \setminus \{0\}$ , then  $M^n \cong G$  is isometric to a compact simple Lie group endowed with a bi-invariant metric and one of the  $\pm 1$ -connections.* The reason that  $\text{Spin}(7)/G_2$  does not appear here, is due to our invariant-torsion scenario which does not allow non-invariant connections, see Remark 3.5.

Let us now have a closer look at the  $\nabla^\alpha$ -Einstein condition on  $(M^n = G/K, g)$ , namely the polynomial equation

$$\frac{1-\alpha^2}{4} B(X, Y) + \frac{1+\alpha^2}{2} A(X, Y) = \frac{\text{Scal}^\alpha}{n} \langle X, Y \rangle. \quad (8)$$

**Example 4.6.** Consider a compact connected Lie group  $M^n \cong G$  with a bi-invariant metric  $g$ . Then, the  $\nabla^\alpha$ -Einstein condition is equivalent to the original

Einstein condition [3, Lemma 2.18]. Indeed, in this case  $A$  is identically equal to zero, thus the Ricci tensor associated to  $\nabla^\alpha$  is proportional to Riemannian Ricci curvature. For simplicity, let us use the family  $\nabla_X^\alpha Y = \eta^\alpha(X, Y) := ((1 - \alpha)/2)[X, Y]$ . Then

$$S^\alpha(X, Y) = \alpha^2 \sum_i \langle [X, Z_i], [Y, Z_i] \rangle, \quad \text{Ric}^\alpha(X, Y) = \frac{1 - \alpha^2}{4} \sum_i \langle [X, Z_i], [Y, Z_i] \rangle,$$

or in other words  $\text{Ric}^\alpha = (1 - \alpha^2)\text{Ric}^0$ , where  $\text{Ric}^0 \equiv \text{Ric}^g$  is the Riemannian Ricci curvature. A little computation also shows that  $\text{Scal}^\alpha = \frac{1 - \alpha^2}{4} \sum_{i,j} \|[Z_i, Z_j]\|^2$  and  $\|T^\alpha\|^2 = \frac{\alpha^2}{6} \sum_{i,j} \|[Z_i, Z_j]\|^2$ . Thus, if  $g$  is a bi-invariant Einstein metric with Einstein constant  $c$ , then  $G$  is  $\nabla^\alpha$ -Einstein with (constant) scalar curvature  $\text{Scal}^\alpha = n(1 - \alpha^2)c$  and torsion  $T^\alpha$  such that  $6\|T^\alpha\|^2 = 4cn\alpha^2$ . Conversely, if  $M^n = G$  is  $\nabla^\alpha$ -Einstein for  $\alpha \neq \pm 1$ , then its scalar curvature  $\text{Scal}^\alpha$  is constant (since  $\nabla^\alpha T^\alpha = 0$  for any  $\alpha \in \mathbb{R}$ ) and the bi-invariant metric  $g$  is Einstein with Einstein constant  $c = \frac{\text{Scal}^\alpha}{n(1 - \alpha^2)}$ . By Corollary 2.6 we know that  $G$  becomes  $\nabla^{\pm 1}$ -flat for  $\alpha = \pm 1$ , in particular it is  $\text{Ric}^{\pm 1}$ -flat and thus it is trivially a  $\nabla$ -Einstein manifold, see also [3]. When  $G$  is simple, the Killing metric  $g_B = -B$  is Einstein with  $c = 1/4$  [30]; hence any compact simple Lie group  $G$  is a  $\nabla^\alpha$ -Einstein manifold with constant scalar curvature  $\text{Scal}^\alpha = n(1 - \alpha^2)/4$  and skew-torsion  $T^\alpha$  such that  $\|T^\alpha\|^2 = n\alpha^2/6$ . Because  $\text{SU}(2) \cong \text{S}^3$ , we also conclude that the 3-sphere  $\text{S}^3$  is a  $\nabla^\alpha$ -Einstein manifold with respect to 1-parameter family  $\nabla^\alpha$  of bi-invariant metric connections. To summarise: *Any compact connected simple Lie group  $G$  is a  $\nabla$ -Einstein manifold with parallel torsion with respect to a 1-parameter family of bi-invariant metric connections, namely the family  $\nabla^\alpha$  described in Theorem 1.1. In particular, the  $\nabla^{\pm 1}$ -Einstein structures are flat.*

Fix now a compact simply-connected homogeneous Riemannian manifold  $(M = G/K, g)$  endowed with an effective  $G$ -action and assume that the isotropy representation of  $K$  on  $T_oG/K \cong \mathfrak{g}/\mathfrak{k}$  is irreducible over  $\mathbb{R}$ . Because  $G$  is compact and its semi-simple part acts transitively on  $G/K$ , in addition one can assume that  $G$  is a compact, connected, semi-simple Lie group. If  $M$  is an isotropy irreducible homogeneous Riemannian manifold, both its universal covering and the product  $M \times \cdots \times M$  are isotropy irreducible (the latter with the product metric). Hence we shall focus in the case that  $M = G/K$  de Rham irreducible, with the aim to describe specific solutions of the  $\nabla^\alpha$ -Einstein condition. Since we are treating invariant connections whose torsion is a multiple of the canonical torsion, by Proposition 3.2 we have to exclude symmetric spaces of Type I. On the other hand, notice that  $(M = G/K, g)$  admits a unique (up to scale)  $G$ -invariant (Einstein) metric, the Killing metric  $g_B$ , see for example [10, Prop. 7.91, p. 198]. Hence,  $(M = G/K, g_B)$  is naturally reductive, in particular *standard* (in terms of [10, Def. 7.90]); this means that  $\mathfrak{m}$  is chosen so that  $B(\mathfrak{k}, \mathfrak{m}) = 0$ . As we said in Remark 3.3, classical examples of isotropy irreducible spaces are the strongly isotropy irreducible spaces where for the non-symmetric case,  $G$  is always simple [32, Thm. 1.1, p. 62]. The isotropy irreducible homogeneous Riemannian manifolds which are not strongly isotropy irreducible were classified in [31]. Finally, we remark that a non-compact isotropy irreducible space is necessarily symmetric,

see [10, Prop. 7.46].

**Theorem 4.7.** *Let  $(M^n = G/K, g_B)$  be a compact simply-connected isotropy irreducible standard homogeneous Riemannian manifold, endowed with an effective action of a compact connected simple Lie group  $G$ . Assume that  $M = G/K$  is not a symmetric space of Type I. Then,  $(M^n = G/K, g)$  is a  $\nabla^\alpha$ -Einstein manifold for any  $\alpha \in \mathbb{R}$ . The same holds if  $G$  is semi-simple but not simple, i.e. when  $M$  is isometric to the coset  $(G/\Delta Z)/(\Delta H/\Delta Z)$  with  $G := H \times \dots \times H$  ( $q$ -times) for a compact simply-connected simple Lie group  $H$ .*

**Proof.** Assume first that  $G$  is simple. By Schur's lemma it follows that  $g_B := B|_{\mathfrak{m} \times \mathfrak{m}}$  is the unique  $G$ -invariant (Einstein) metric. Consider the Casimir operator  $C_\chi \equiv C_{\chi, B_\mathfrak{k}} : \mathfrak{m} \rightarrow \mathfrak{m}$  associated to the isotropy representation  $\chi$  and the positive definite restriction  $B_\mathfrak{k}$ . Then,  $C_\chi = \text{Cas} \cdot \text{Id}_\mathfrak{m}$  with  $\text{Cas} \in \mathbb{R}_+$ . In particular, given a  $B$ -orthonormal basis  $\{Z_1, \dots, Z_n\}$  of  $\mathfrak{m}$ , it is  $B(C_\chi Z_i, Z_i) = B(\text{Cas} \cdot Z_i, Z_i) = \text{Cas} \cdot B(Z_i, Z_i) = \text{Cas}$  and since  $A(X, Y) = B(C_\chi X, Y)$  we get  $\text{Cas} = A(Z_i, Z_i) = \sum_j B_\mathfrak{k}([Z_i, Z_j], [Z_i, Z_j])$ . Now, due to Theorem 4.3 the  $\nabla^\alpha$ -Einstein condition (8) can be expressed by

$$\frac{1 - \alpha^2}{4} B(X, Y) + \frac{1 + \alpha^2}{2} A(X, Y) = \frac{f(\alpha)}{4n} B(X, Y), \quad (9)$$

where  $f(\alpha) := (1 - \alpha^2) \sum_{i,j} \|[Z_i, Z_j]_\mathfrak{m}\|^2 + 4 \sum_i A(Z_i, Z_i)$  is such that  $f(1) = f(-1) = 4 \sum_i A(Z_i, Z_i) = 4n \cdot \text{Cas}$ . Looking for  $\nabla^\alpha$ -Einstein structures with skew-torsion for the values  $\alpha = \pm 1$ , this formula reduces to

$$A(X, Y) = \frac{f(\pm 1)}{4n} B(X, Y) = \frac{4n \cdot \text{Cas}}{4n} B(X, Y) = B(\text{Cas} \cdot X, Y) = B(C_\chi X, Y),$$

which is an identity. In fact, we will show that the  $\nabla^\alpha$ -Einstein condition is an identity for *any*  $\alpha \in \mathbb{R}$ . Indeed, (9) is nothing than the equation  $\mu(X, Y) \cdot \alpha^2 = \nu(X, Y)$ , where for any  $X, Y \in \mathfrak{m}$  we set

$$\begin{aligned} \mu(X, Y) &:= 2nA(X, Y) + \left( \sum_{i,j} \|[Z_i, Z_j]_\mathfrak{m}\|^2 - n \right) B(X, Y), \\ \nu(X, Y) &:= -2nA(X, Y) + \left( \sum_{i,j} \|[Z_i, Z_j]_\mathfrak{m}\|^2 - n + 4n \cdot \text{Cas} \right) B(X, Y). \end{aligned}$$

However, it is easy to see that  $\mu(X, Y) \equiv \nu(X, Y)$  and  $\mu(X, Y) \equiv 0$ , identically. For example, since  $\mathfrak{m}$  is isotropy irreducible, for any  $X, Y \in \mathfrak{m}$  we have  $\mu(X, Y) =$

$$\begin{aligned} &A(X, Y) + \left( \sum_{i,j} \|[Z_i, Z_j]_\mathfrak{m}\|^2 - n \right) B(X, Y) \\ &= 2nB(C_\chi X, Y) + \left( \sum_{i,j} \|[Z_i, Z_j]_\mathfrak{m}\|^2 - n \right) B(X, Y) \\ &= 2n \text{Cas} B(X, Y) + \left( \sum_{i,j} \|[Z_i, Z_j]_\mathfrak{m}\|^2 - n \right) B(X, Y) \\ &= (2n \text{Cas} + \sum_{i,j} \|[Z_i, Z_j]_\mathfrak{m}\|^2 - n) B(X, Y). \end{aligned}$$

But after considering the sum of relation (7) with respect to the orthonormal basis  $\{Z_i\}$ , we deduce that  $2n \text{Cas} + \sum_{i,j} \|[Z_i, Z_j]_\mathfrak{m}\|^2 - n = 0$ , since  $\sum_i A(Z_i, Z_i) = n \text{Cas}$ . Thus  $\mu(X, Y) \equiv 0$  and our claim follows.

We proceed now with the non-simple case. In [31, Thm. 2.2] it was shown that the unique example of a de Rham irreducible, compact, simply connected

isotropy irreducible homogeneous Riemannian manifold  $M = G/K$  with  $\mathfrak{g}$  semi-simple but not simple, is the coset  $G/\Delta H$  where  $H$  is a compact simply connected simple Lie group and  $\Delta H$  the diagonal subgroup of  $G := H \times \cdots \times H$  ( $q$ -times). The effective version of  $G/\Delta H$  has the form  $(G/\Delta Z)/(\Delta H/\Delta Z)$ , where  $Z$  is the finite centre of  $H$  and due to our assumptions,  $M$  is isometric to  $(G/\Delta Z)/(\Delta H/\Delta Z)$ . The isotropy representation is given by  $\chi := \bigoplus_{i=1}^{q-1} \text{ad}_{\mathfrak{h}}$  with  $\mathfrak{m} := \{(X_1, \dots, X_q) : \sum X_i = 0, X_i \in \mathfrak{h}\}$ .  $M$  is a symmetric space of Type II for  $q = 2$ ; for  $q \geq 3$  it is isotropy irreducible but not strongly isotropy irreducible. The negative  $B$  of the Killing form of  $H$  is Einstein, i.e.  $C_{\chi, B} = \text{Cas} \cdot \text{Id}$  for some  $\text{Cas} > 0$  and our result above shows that for  $q \geq 2$  the standard homogeneous space  $M = (G/\Delta Z)/(\Delta H/\Delta Z)$  is a  $\nabla^\alpha$ -Einstein manifold for any  $\alpha \in \mathbb{R}$ . For  $q = 2$ , this assertion follows also from the summary in Example 4.6.  $\blacksquare$

Among the  $\nabla^\alpha$ -Einstein structures described above, there are two special members, namely the structures defined by the canonical connection ( $\alpha = 1$ ) and the anti-canonical connection ( $\alpha = -1$ ). These have identical Ricci tensor  $\text{Ric}^{\pm 1}(X, Y) = A(X, Y) = \text{Cas} B(X, Y)$  with  $S^{\pm 1}(X, Y) = B(X, Y) - 2A(X, Y) = (1 - 2\text{Cas})B(X, Y)$  and the scalar curvature is given by  $\text{Scal}^{\pm 1} = n \text{Cas}$ . Notice however that  $\nabla^1 T^1 = 0 \neq \nabla^{-1} T^{-1}$  [1]. Moreover,

$$R^1(X, Y, Z) = -[[X, Y]_{\mathfrak{t}}, Z], \quad R^{-1}(X, Y, Z) = \text{Jac}_{\mathfrak{m}}(X, Y, Z) - [[X, Y]_{\mathfrak{t}}, Z].$$

Hence, geometrically the Jacobian  $\text{Jac}_{\mathfrak{m}}$  represents the difference of the curvatures associated to the canonical and the anti-canonical connection, i.e.  $R^{-1}(X, Y, Z) - R^1(X, Y, Z) = \text{Jac}_{\mathfrak{m}}(X, Y, Z)$ . Of course, the case changes for a compact simple Lie group  $G$ , where the  $\nabla^{\pm 1}$ -Einstein structures are necessarily flat, i.e.  $R^{\pm 1} \equiv 0$  (see Theorem 1.1, Lemma 4.5 and Example 4.6).

**Example 4.8.** We conclude that the spheres  $S^6 = G_2/\text{SU}(3)$  and  $S^7 = \text{Spin}(7)/G_2$  are  $\nabla^\alpha$ -Einstein manifolds for any  $\alpha \in \mathbb{R}$ . Notice that the  $\nabla^1$ -Einstein structure on  $S^6$  coincides with the one induced by the Gray connection (parallel skew-torsion), see [2]. On the other hand, the  $\nabla^\alpha$ -Einstein structures on  $S^7$  differ from the trivial  $\nabla^{\text{flat}}$ -Einstein structure associated to the flat connection, see Remark 3.5. For an alternative study of homogeneous  $\nabla$ -Einstein structures on odd-dimensional spheres we refer to [14].

**Example 4.9.** Let  $G$  be a compact simple Lie group whose simple roots are all of the same length; this means that  $G$  is one of the groups  $\text{SU}(\ell), \text{SO}(2\ell), E_6, E_7, E_8$ . Consider the full flag manifold  $M = G/T$ , where  $T$  is a maximal torus in  $G$ . Let  $\Delta$  be the root system of the complexification  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$  with respect to the Cartan subalgebra  $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ , where  $\mathfrak{t}$  is the Lie algebra of  $T$ . Fix a Weyl basis  $\{E_\alpha : \alpha \in \Delta\}$  of  $\mathfrak{g}^{\mathbb{C}}$  and for any positive root  $\alpha \in \Delta^+$ , set  $A_\alpha = E_\alpha + E_{-\alpha}$ ,  $B_\alpha = i(E_\alpha - E_{-\alpha})$  and  $\mathfrak{m}_\alpha := \mathbb{R}A_\alpha + \mathbb{R}B_\alpha$ . The root space decomposition of  $\mathfrak{g}$  has the form  $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{m}_\alpha = \mathfrak{t} \oplus \mathfrak{m}$ , i.e.  $\mathfrak{m} \cong T_o(G/T) = \bigoplus_{\alpha \in \Delta^+} \mathfrak{m}_\alpha$ . Any  $\mathfrak{m}_\alpha$  is a 2-dimensional real irreducible  $\text{Ad}(T)$ -module, which simultaneously can be viewed as a complex plane on which  $T$  acts by rotations. The Weyl group of  $G$  acts transitively on the factors  $\mathfrak{m}_\alpha$  (since acts transitively on roots of the same length) and by adding these isometries to  $G$ , the full flag manifold  $G/T$  becomes

an isotropy irreducible space [31, p. 235] with the Killing metric  $g_B$  being Einstein [30, Cor. 1.5]. By Theorem 4.7 it follows that these full flag manifolds are also  $\nabla^\alpha$ -Einstein manifolds for any  $\alpha \in \mathbb{R}$ .

**Example 4.10.** Consider the 7-dimensional Berger sphere  $B^7 = \text{SO}(5)/\text{SO}(3)_{\text{ir}}$ . The embedding  $\text{SO}(3)_{\text{ir}} \subset \text{SO}(5)$  is given by the unique 5-dimensional  $\text{SO}(3)$ -irreducible representation which is defined by the action of  $\text{SO}(3)$  on the set of  $3 \times 3$  symmetric traceless matrices  $S_0^2(\mathbb{R}^3) \cong \mathbb{R}^5$ , via conjugation. The isotropy representation coincides with the unique 7-dimensional  $\text{SO}(3)$ -irreducible representation, and this defines an embedding of  $\text{SO}(3)$  inside  $G_2$  and thus a  $G_2$ -structure. The Killing metric  $g_B$  is the unique  $\text{SO}(5)$ -invariant (Einstein) metric (up to scale). By Theorem 4.7 we conclude that  $B^7$  is also a  $\nabla^\alpha$ -Einstein manifold for any  $\alpha \in \mathbb{R}$ . In fact, the  $G_2$ -structure is (proper) nearly parallel, i.e. the coset  $\text{SO}(5)/\text{SO}(3)_{\text{ir}}$  admits a unique real Killing spinor [16]. Hence, the canonical connection  $\nabla^1$  coincides with the characteristic connection  $\nabla^c$  preserving this structure, in particular the  $\nabla^1$ -Einstein structure is well-known, see [15, p. 318].

## 5. Homogeneous Riemannian manifolds with two isotropy summands

In this final section we study compact connected homogeneous Riemannian manifolds  $(M = G/K, g)$  whose isotropy representation  $\chi : K \rightarrow \text{SO}(\mathfrak{m})$  decomposes into two (non-trivial) inequivalent and irreducible  $K$ -submodules satisfying (1). Well-known examples are: connected semi-simple Lie groups, flag manifolds with two isotropy summands, 3- and 4-symmetric spaces (see [7, 8] and the references therein). Without loss of generality we assume that the compact Lie group  $G$  is connected and semi-simple and that  $K$  is connected, see [10, 30]. We consider the 1-parameter family of  $G$ -invariant Riemannian metrics on  $M = G/K$ , given by  $g_t = B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + 2t \cdot B|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$  for some  $t \in \mathbb{R}_+$ , where  $B$  denotes the negative of the Killing form of  $\mathfrak{g}$ . It follows that any  $G$ -invariant Riemannian metric on  $M = G/K$  is a multiple of  $g_t$ . The value  $t = 1/2$  defines the Killing metric  $g_{1/2} = g_B = B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + B|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$ .

**Remark 5.1.** There is a natural construction that gives rise to compact homogeneous spaces with two isotropy summands satisfying (1). Consider a semi-simple Lie algebra  $\mathfrak{g}$  and assume that the pairs  $(\mathfrak{g}, \mathfrak{k} \oplus \mathfrak{m}_2)$  and  $(\mathfrak{k} \oplus \mathfrak{m}_2, \mathfrak{k})$  are orthogonal symmetric pairs such that  $\mathfrak{m}_1$  be an orthogonal complement of  $\mathfrak{k} \oplus \mathfrak{m}_2$  in  $\mathfrak{g}$ , with respect to the Killing form of  $\mathfrak{g}$ . Then, by setting  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$  one can easily verify the inclusions given by (1). In this way we obtain a Riemannian submersion  $U/K \rightarrow G/K \rightarrow G/U$  where  $U$  is the connected Lie group generated by the Lie algebra  $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{m}_2$ , and both (effective) quotients  $U/K$  and  $G/U$  are symmetric spaces. If  $\mathfrak{p}$  denotes an orthogonal complement of  $\mathfrak{u}$  in  $\mathfrak{g}$ , i.e.  $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ , then we may identify  $\mathfrak{p} = T_oG/U = \mathfrak{m}_1$ ,  $\mathfrak{m}_2 = T_oU/K$  and  $\mathfrak{m} = T_oG/K = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ .

Next we shall characterize the  $G$ -invariant metric connections of  $M = G/K$  which have (totally) skew-symmetric torsion. Consider a connected Riemannian manifold  $(M^n, g)$  carrying a metric connection  $\nabla$ . Via  $g$  we shall identify  $TM \cong$

$T^*M$ . Set

$$\mathcal{A}^g := \{A \in \otimes^3 TM : A(X, Y, Z) + A(X, Z, Y) = 0\} \cong TM \otimes \Lambda^2(TM).$$

Since  $\nabla$  is metric we write  $\nabla_X Y - \nabla_Y X = A(X, Y)$  for a  $(2, 1)$ -tensor field  $A \in \mathcal{A}^g$ . Given a (local) orthonormal frame  $\{e_i\}$  of  $M$ , let  $\Phi : \mathcal{A}^g \rightarrow T^*M$  the map defined by  $\Phi(A)(Z) = \sum_i A(e_i, e_i, Z)$ , where  $A(X, Y, Z) := g(A(X, Y), Z)$  is the induced  $(3, 0)$ -tensor field. It is well-known that  $\mathcal{A}^g$  coincides with the space of torsion tensors and under the action of  $O(n)$  it decomposes into three irreducible representations, i.e.  $\mathcal{A}^g = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$  (see for example [2, 27]). These are explicitly given by

$$\mathcal{A}_1 := \{A \in \mathcal{A}^g : A(X, Y, Z) = g(X, Y)g(V, Z) - g(V, Y)g(X, Z), V \in \Gamma(TM)\},$$

$$\mathcal{A}_2 := \{A \in \mathcal{A}^g : A(X, Y, Z) + A(Y, X, Z) = 0\},$$

$$\mathcal{A}_3 := \{A \in \mathcal{A}^g : \mathfrak{S}^{X, Y, Z} A(X, Y, Z) = 0, \Phi(A) = 0\}.$$

We say that the torsion  $T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$  is of vectorial type (and the same for  $\nabla$ ) if  $A \in \mathcal{A}_1 \cong TM$ , (totally) skew-symmetric if  $A \in \mathcal{A}_2 \cong \Lambda^3(T^*M)$  and finally of Cartan type if  $A \in \mathcal{A}_3$ . For  $n = 2$ ,  $\mathcal{A}^g \cong \mathbb{R}^2$  is  $O(2)$ -irreducible.

**Theorem 5.2.** *Let  $\nabla$  be a  $G$ -invariant metric connection of the homogeneous Riemannian manifold  $(M^n = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_t)$  ( $n \geq 3$ ) with non-trivial torsion  $T \neq 0$ . Then,  $T$  is totally skew-symmetric if and only if  $t = 1/2$  and  $\Lambda(X)X = 0$  for any  $X \in \mathfrak{m}$ , where  $\Lambda : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$  denotes the associated Nomizu map.*

**Proof.** The tensor field  $T(X, Y, Z) = g_t(T(X, Y), Z)$  which occurs from  $T$  by contraction with  $g_t$  is already skew-symmetric with respect to  $X, Y$ . Set  $\mathfrak{T}(X, Y, Z) := T(X, Y, Z) + T(X, Z, Y)$ . Then, the condition  $T \in \Lambda^3(\mathfrak{m})$  is equivalent to  $\mathfrak{T}(X, Y, Z) = 0$ , for any  $X, Y, Z \in \mathfrak{m}$ . We compute

$$\begin{aligned} \mathfrak{T}(X, Y, Z) &= \underbrace{g_t(\Lambda(X)Y, Z)}_{(\alpha')} - \underbrace{g_t(\Lambda(Y)X, Z)}_{(\beta')} - \underbrace{g_t([X, Y]_{\mathfrak{m}}, Z)}_{(\gamma')} \\ &\quad + \underbrace{g_t(\Lambda(X)Z, Y)}_{(\delta')} - \underbrace{g_t(\Lambda(Z)X, Y)}_{(\varepsilon')} - \underbrace{g_t([X, Z]_{\mathfrak{m}}, Y)}_{(\zeta')}, \end{aligned}$$

for any  $X, Y, Z \in \mathfrak{m}$ . Since  $\Lambda(X) \in \mathfrak{so}(\mathfrak{m})$  for all  $X \in \mathfrak{m}$ , we see that the terms  $(\alpha')$  and  $(\delta')$  cancel one another:  $g_t(\Lambda(X)Y, Z) + g_t(\Lambda(X)Z, Y) = -g_t(\Lambda(X)Z, Y) + g_t(\Lambda(X)Z, Y) = 0$ . For the same reason it is

$$\begin{aligned} -(\beta') - (\varepsilon') &= -g_t(\Lambda(Y)X, Z) - g_t(\Lambda(Z)X, Y) = g_t(\Lambda(Y)Z, X) + g_t(\Lambda(Z)Y, X) \\ &= g_t(\Lambda(Y)Z + \Lambda(Z)Y, X). \end{aligned}$$

Hence, the equation  $\mathfrak{T}(X, Y, Z) = 0$  becomes equivalent to

$$\underbrace{g_t(\Lambda(Y)Z + \Lambda(Z)Y, X)}_{(\alpha)} - \underbrace{g_t([X, Y]_{\mathfrak{m}}, Z)}_{(\beta)} - \underbrace{g_t([X, Z]_{\mathfrak{m}}, Y)}_{(\gamma)} = 0, \quad \forall X, Y, Z \in \mathfrak{m}.$$

By using this formula one needs to examine each possible case separately. Consider for example some non-zero vectors  $X, Y \in \mathfrak{m}_1$  and  $Z \in \mathfrak{m}_2$ . Then

$$\begin{aligned} -(\beta) - (\gamma) &= g_t([Y, X]_{\mathfrak{m}_2}, Z) + g_t([Z, X], Y) = 2tB([Y, X]_{\mathfrak{m}_2}, Z) + B([Z, X], Y) \\ &= -(2t - 1)B([Y, Z], X) = -g_t((2t - 1)[Y, Z], X). \end{aligned}$$

Hence the skew-symmetry of  $T$  reduces to the equation  $g_t(\Lambda(Y)Z + \Lambda(Z)Y - (2t - 1)[Y, Z], X) = 0$ , for any  $X \in \mathfrak{m}_1$ ,  $Y \in \mathfrak{m}_1$  and  $Z \in \mathfrak{m}_2$ , which means that  $\Lambda(Y)Z + \Lambda(Z)Y = (2t - 1)[Y, Z]$ , for all  $Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2$ . Let now  $X \in \mathfrak{m}_2$ ,  $Y \in \mathfrak{m}_1$  and  $Z \in \mathfrak{m}_2$ . Then we get  $\Lambda(Y)Z + \Lambda(Z)Y = 0$ , for any  $Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2$ , and this gives rise to the following system of equations  $\left\{ \Lambda(Y)Z + \Lambda(Z)Y = (2t - 1)[Y, Z], \quad \Lambda(Y)Z + \Lambda(Z)Y = 0, \quad \forall Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2 \right\}$ . Thus  $t = 1/2$  and  $\Lambda(Y)Z + \Lambda(Z)Y = 0$  for all  $Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2$ . One can obtain the same result by comparing the cases  $X \in \mathfrak{m}_1, Y \in \mathfrak{m}_2, Z \in \mathfrak{m}_1$  and  $X, Y \in \mathfrak{m}_2, Z \in \mathfrak{m}_1$ , respectively, i.e.  $t = 1/2$  and  $\Lambda(Y)Z + \Lambda(Z)Y = 0$ , for any  $Y \in \mathfrak{m}_2, Z \in \mathfrak{m}_1$ . Similar are treated the other cases. The converse direction follows by [1, Lem. 2.1], since for  $t = 1/2$  we obtain the Killing metric  $g_{1/2}$  which is naturally reductive.  $\blacksquare$

**Proposition 5.3.** ([9, Lem. 10, p. 141]) *The Nomizu map  $\Lambda_t : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$  associated to the Levi-Civita connection  $\nabla^t (\equiv \nabla^{g_t})$  ( $t > 0$ ) of the homogeneous Riemannian manifold  $(M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_t)$  is defined by the following relations*

$$\begin{aligned} \Lambda_t(\mathfrak{m}_1)\mathfrak{m}_1 &= (1/2)[\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{m}_2}, & \Lambda_t(\mathfrak{m}_2)\mathfrak{m}_1 &= (1 - t)[\mathfrak{m}_2, \mathfrak{m}_1], \\ \Lambda_t(\mathfrak{m}_1)\mathfrak{m}_2 &= t[\mathfrak{m}_1, \mathfrak{m}_2], & \Lambda_t(\mathfrak{m}_2)\mathfrak{m}_2 &= 0. \end{aligned}$$

We fix finally some notation that will be used throughout this section. We shall denote by  $D_{i,j}$  the  $(n \times n)$ -matrix having 1 in the  $(i, j)$ -entry and zeros elsewhere and we set  $E_{i,j} = -D_{i,j} + D_{j,i}$ , with  $1 \leq i \neq j \leq n$ . We also set  $d_i := \dim \mathfrak{m}_i$  for any  $i = 1, 2$  and fix a  $B$ -orthonormal basis of  $\mathfrak{m}$  adapted to the decomposition  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ , that is

$$\{0 \neq X_i \in \mathfrak{m}_1 : 1 \leq i \leq d_1\} \sqcup \{0 \neq Y_k \in \mathfrak{m}_2 : 1 \leq k \leq d_2\},$$

such that  $\mathfrak{m}_1 = \text{span}_{\mathbb{R}}\{X_i\}_{i=1}^{d_1}$ ,  $\mathfrak{m}_2 = \text{span}_{\mathbb{R}}\{Y_k\}_{k=1}^{d_2}$ , with  $B(X_i, X_j) = \delta_{i,j}$ ,  $B(Y_k, Y_l) = \delta_{k,l}$ ,  $B(X_i, Y_k) = 0$ . The associated  $g_t$ -orthonormal bases are of the form  $\mathfrak{m}_1 = \text{span}_{\mathbb{R}}\{V_i := X_i : 1 \leq i \leq d_1\}$ ,  $\mathfrak{m}_2 = \text{span}_{\mathbb{R}}\{W_k := Y_k/\sqrt{2t} : 1 \leq k \leq d_2\}$ .

**5.1. A new family of  $G$ -invariant metric connections.** For a new parameter  $s \in \mathbb{R}$  consider the map  $\Lambda_{s,t}^{\mathfrak{m}} \equiv \Lambda_{s,t} : \mathfrak{m} \rightarrow \mathfrak{so}(\mathfrak{m})$  with  $\Lambda_{s,t}(X)Y := s \cdot \Lambda_t(X)Y$ , i.e.

$$\begin{aligned} \Lambda_{s,t}(\mathfrak{m}_1)\mathfrak{m}_1 &= (s/2)[\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{m}_2}, & \Lambda_{s,t}(\mathfrak{m}_2)\mathfrak{m}_1 &= s(1 - t)[\mathfrak{m}_2, \mathfrak{m}_1], \\ \Lambda_{s,t}(\mathfrak{m}_1)\mathfrak{m}_2 &= st[\mathfrak{m}_1, \mathfrak{m}_2], & \Lambda_{s,t}(\mathfrak{m}_2)\mathfrak{m}_2 &= 0. \end{aligned} \quad (10)$$

Obviously,  $\Lambda_{s,t}$  is an  $\text{Ad}(K)$ -equivariant linear map such that  $\Lambda_{s,t}(X) \in \mathfrak{so}(\mathfrak{m})$  for any  $X \in \mathfrak{m}$ . Thus, it induces a 2-parameter family of  $G$ -invariant metric connections  $\{\nabla^{s,t} : s \in \mathbb{R}, t \in \mathbb{R}_+\}$ , which after identifying  $\mathfrak{m} = T_oG/K$  can be explicitly described by

$$\nabla_X^{s,t} Y := \nabla_X^0 Y + \Lambda_{s,t}(X)Y = \nabla_X^0 Y + s \cdot \Lambda_t(X)Y, \quad \forall X, Y \in \mathfrak{m}. \quad (11)$$

Notice that  $\nabla^{s,t}$  joins the canonical connection  $\nabla^{0,t} \equiv \nabla^0 \equiv \nabla^c$  ( $s = 0$ ) and the Levi-Civita connection  $\nabla^{1,t} \equiv \nabla^t$  ( $s = 1$ ). Now, using (2) it follows that

**Lemma 5.4.** *The torsion  $T^{s,t}$  is given as follows:*

$$\begin{aligned} T^{s,t}(\mathfrak{m}_1, \mathfrak{m}_1) &= (s - 1)[\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{m}_2}, & T^{s,t}(\mathfrak{m}_2, \mathfrak{m}_1) &= (s - 1)[\mathfrak{m}_2, \mathfrak{m}_1], \\ T^{s,t}(\mathfrak{m}_1, \mathfrak{m}_2) &= (s - 1)[\mathfrak{m}_1, \mathfrak{m}_2], & T^{s,t}(\mathfrak{m}_2, \mathfrak{m}_2) &= 0. \end{aligned}$$

Consequently, the 3-tensor  $T^{s,t}(X, Y, Z) := g_t(T^{s,t}(X, Y), Z)$  is such that

$$\begin{aligned} T^{s,t}(\mathbf{m}_1, \mathbf{m}_1, \mathbf{m}_2) &= 2t(s-1)B([\mathbf{m}_1, \mathbf{m}_1]_{\mathbf{m}_2}, \mathbf{m}_2), \\ T^{s,t}(\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_1) &= (s-1)B([\mathbf{m}_2, \mathbf{m}_1], \mathbf{m}_1), \\ T^{s,t}(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_1) &= (s-1)B([\mathbf{m}_1, \mathbf{m}_2], \mathbf{m}_1), \end{aligned} \quad (12)$$

and all the other combinations are zero.

Although  $T^s(X, Y) = -(s-1)T^0(X, Y)$ , the induced 3-tensor  $T^{s,t}(X, Y, Z)$  is not a 3-form for any  $s, t$ ; Theorem 5.2 states that this is possible only for  $t = 1/2$  and under the further assumption  $\Lambda_{s,t}(X)X = 0$  for any  $X \in \mathfrak{m}$ . Writing  $\mathfrak{m} \ni X = V + Z$  with  $V \in \mathfrak{m}_1, Z \in \mathfrak{m}_2$ , the latter condition reduces to  $s(2t-1)[V, Z] = 0$ . This is always satisfied for  $t = 1/2$ , i.e.  $\Lambda_{s, \frac{1}{2}}(X)X = 0$  for any  $X \in \mathfrak{m}$ , as required. In fact, by (12) and for non-zero vectors  $X \in \mathfrak{m}_1, Y \in \mathfrak{m}_1$  and  $Z \in \mathfrak{m}_2$ , it follows that the equation  $T^{s,t}(X, Y, Z) + T^{s,t}(X, Z, Y) = 0$  is equivalent to  $(s-1)(2t-1)B([X, Y]_{\mathbf{m}_2}, Z) = 0$ . Consequently, there are two possibilities:  $s = 1$  or  $t = 1/2$ . The first fails, since it corresponds to the Riemannian connection. Hence  $t = 1/2$  which corresponds to the Killing metric. Similarly are treated the other cases. We conclude that

**Corollary 5.5.** *For any  $s \neq 1$  it holds that  $0 \neq T^{s,t}(X, Y, Z) \in \Lambda^3(\mathfrak{m})$  for any  $X, Y, Z \in \mathfrak{m} \Leftrightarrow t = 1/2$ .*

Let us describe now the algebraic type of the torsion  $T^{s,t}$  for any  $s \in \mathbb{R}, t > 0$ . It is useful to present the skew-symmetric endomorphism  $\Lambda^{s,t}(X) \in \mathfrak{so}(\mathfrak{m})$  ( $X \in \mathfrak{m}$ ) in terms of the matrices  $E_{i,j}$ . This is given by

$$\begin{aligned} (a) \quad \Lambda^{s,t}(X) &= \frac{st}{\sqrt{2t}} \sum_{i,k} B([X, X_i]_{\mathbf{m}_2}, Y_k) E_{i,k}, \quad \text{for } X \in \mathfrak{m}_1, \\ (b) \quad \Lambda^{s,t}(X) &= \frac{s(1-t)}{2} \sum_{i,j} B([X, X_i], X_j) E_{i,j}, \quad \text{for } X \in \mathfrak{m}_2. \end{aligned}$$

**Theorem 5.6.** *For any  $s \in \mathbb{R}$  and  $t \in \mathbb{R}_+ - \{1/2\}$ , the 2-parameter family of  $G$ -invariant metric connections  $\nabla^{s,t}$  has torsion of mixed type  $\mathcal{A}_2 \oplus \mathcal{A}_3$ . For  $t = 1/2$  it reduces to 1-parameter family of metric connections with skew-torsion.*

**Proof.** **1st way.** It is known that given a connected Riemannian manifold  $(M^n, g)$  endowed with a metric connection  $\nabla$ , then the condition  $A := \nabla - \nabla^g \in \mathcal{A}_2 \oplus \mathcal{A}_3$  is equivalent to say that  $\Phi(A)(X) = 0$  for any vector field  $X$  on  $M$ , or in other words that the  $\nabla$ -divergence of  $X$  coincides with the Riemannian divergence [27, Corol. 4.6]. Therefore, it is sufficient to show that  $\text{div}^{s,t}(X) - \text{div}^t(X) = 0$ , for any  $X \in \mathfrak{m}$ , where  $\text{div}^{s,t}$  and  $\text{div}^t$  are the divergences with respect to  $\nabla^{s,t}$  and  $\nabla^t$ , respectively. Because  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$  is an orthogonal splitting, we write  $X = X^\alpha + X^\beta$  for some  $X^\alpha \in \mathfrak{m}_1$  and  $X^\beta \in \mathfrak{m}_2$ . Set  $\mathfrak{D}(X) := \text{div}^{s,t}(X) - \text{div}^t(X)$ .

Since  $A^{s,t} := \nabla^{s,t} - \nabla^t = (s-1)\Lambda^t$ , an easy computation proves our claim:

$$\begin{aligned}
\mathfrak{D}(X) &= \sum_{1 \leq i \leq d_1} g_t(X_i, \nabla_{X_i}^{s,t} X^\alpha - \nabla_{X_i}^t X^\alpha) + \sum_{1 \leq k \leq d_2} g_t(W_k, \nabla_{W_k}^{s,t} X^\alpha - \nabla_{W_k}^t X^\alpha) \\
&\quad + \sum_{1 \leq i \leq d_1} g_t(X_i, \nabla_{X_i}^{s,t} X^\beta - \nabla_{X_i}^t X^\beta) + \sum_{1 \leq k \leq d_2} g_t(W_k, \nabla_{W_k}^{s,t} X^\beta - \nabla_{W_k}^t X^\beta) \\
&= \sum_i g_t(X_i, (s-1)\Lambda^t(X_i)X^\alpha) + \sum_k g_t(W_k, (s-1)\Lambda^t(W_k)X^\alpha) \\
&\quad + \sum_i g_t(X_i, (s-1)\Lambda^t(X_i)X^\beta) + \sum_k g_t(W_k, (s-1)\Lambda^t(W_k)X^\beta) \\
&= \sum_i g_t(X_i, \frac{(s-1)}{2}[X_i, X^\alpha]_{\mathfrak{m}_2}) + \sum_k g_t(W_k, (s-1)(1-t)[W_k, X^\alpha]) \\
&\quad + \sum_i g_t(X_i, (s-1)t[X_i, X^\beta]) \\
&\stackrel{\mathfrak{m}_1 \perp \mathfrak{m}_2}{=} t(s-1) \sum_i B(X_i, [X_i, X^\beta]) = -t(s-1) \sum_i B([X_i, X_i]_{\mathfrak{m}_2}, X^\beta) = 0.
\end{aligned}$$

**2nd way.** Given a Riemannian spin manifold  $(M^n, g)$  with a metric connection  $\nabla$ , the Dirac operator  $D\varphi := \sum_i e_i \cdot \nabla_{e_i} \varphi$  associated to  $\nabla$  is formally self-adjoint if and only if  $\operatorname{div}^\nabla(X) = \operatorname{div}^g(X)$  for any vector field  $X$  [17, Satz. 2]. Hence, after assuming that  $(M^n = G/K, g_t)$  carries an invariant spin structure, one can show that  $\nabla^{s,t}$  has no vectorial component, i.e.  $A^{s,t} \in \mathcal{A}_2 \oplus \mathcal{A}_3$  by proving the identification  $(D^{s,t})^* = D^{s,t}$ , where  $D^{s,t}$  is the Dirac operator associated to the 2-parameter family  $\nabla^{s,t}$ . A  $G$ -invariant spin structure on  $M = G/K$  corresponds to a lift of the isotropy representation  $\chi$  into the spin group  $\operatorname{Spin}(\mathfrak{m})$ , i.e. a homomorphism  $\tilde{\chi} : K \rightarrow \operatorname{Spin}(\mathfrak{m})$  such that  $\chi = \lambda \circ \tilde{\chi}$ , where  $\lambda : \operatorname{Spin}(\mathfrak{m}) \rightarrow \operatorname{SO}(\mathfrak{m})$  is the double covering. Because the tangent bundle splits  $T(G/K) = (G \times_{\chi^1} \mathfrak{m}_1) \oplus (G \times_{\chi^2} \mathfrak{m}_2)$ , the Clifford algebra  $(\operatorname{Cl}(\mathfrak{m}), B)$  of  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$  with respect to  $B$  is the graded tensor product  $\operatorname{Cl}(\mathfrak{m}) = \operatorname{Cl}(\mathfrak{m}_1 \oplus \mathfrak{m}_2) = \operatorname{Cl}(\mathfrak{m}_1) \hat{\otimes} \operatorname{Cl}(\mathfrak{m}_2)$ , where  $\chi^i : K \rightarrow \operatorname{SO}(\mathfrak{m}_i)$  and  $\operatorname{Cl}(\mathfrak{m}_i)$  denote the Clifford algebras of  $\mathfrak{m}_i$  with respect to the inner products  $B_{\mathfrak{m}_i} = B|_{\mathfrak{m}_i \times \mathfrak{m}_i}$ , for  $i = 1, 2$ . We shall denote by  $\kappa_n : \operatorname{Cl}^{\mathbb{C}}(\mathfrak{m}) \xrightarrow{\sim} \operatorname{End}(\Delta_{\mathfrak{m}})$  the Clifford representation and by  $\mu(X \otimes \phi) := \kappa_n(X)\psi = X \cdot \psi$  the Clifford multiplication between vectors and spinors, see [1, 2, 9] for more details. Set  $\rho := \kappa \circ \tilde{\chi} : K \rightarrow \operatorname{Aut}(\Delta_{\mathfrak{m}})$ , where  $\kappa = \kappa_n|_{\operatorname{Spin}(\mathfrak{m})} : \operatorname{Spin}(\mathfrak{m}) \rightarrow \operatorname{Aut}(\Delta_{\mathfrak{m}})$  is the spin representation. The spinor bundle  $\Sigma \rightarrow G/K$  is the homogeneous vector bundle associated to the  $\operatorname{Spin}(\mathfrak{m})$ -principal bundle  $P := G \times_{\tilde{\chi}} \operatorname{Spin}(\mathfrak{m})$  via the representation  $\rho$ , i.e.  $\Sigma = G \times_{\rho} \Delta_{\mathfrak{m}}$ . Therefore we may identify sections of  $\Sigma$  with smooth functions  $\varphi : G \rightarrow \Delta_{\mathfrak{m}}$  such that

$$\varphi(gk) = \kappa(\tilde{\chi}(k^{-1}))\varphi(g) = \rho(k^{-1})\varphi(g), \quad \forall g \in G, k \in K.$$

It is useful to fix spin endomorphisms induced by the  $B$ -orthonormal vectors  $\{X_i\}_{i=1}^{d_1}$ ,  $\{Y_k\}_{k=1}^{d_2}$  of  $\mathfrak{m}_1, \mathfrak{m}_2$  and interpret the Clifford relations by

$$\begin{aligned}
\kappa_n(X_i)\kappa_n(X_j) + \kappa_n(X_j)\kappa_n(X_i) &= X_i \cdot X_j + X_j \cdot X_i = -2\delta_{i,j}, \\
\kappa_n(Y_k)\kappa_n(Y_l) + \kappa_n(Y_l)\kappa_n(Y_k) &= Y_k \cdot Y_l + Y_l \cdot Y_k = -2\delta_{k,l}, \\
\kappa_n(X_i)\kappa_n(Y_k) + \kappa_n(Y_k)\kappa_n(X_i) &= X_i \cdot Y_k + Y_k \cdot X_i = 0.
\end{aligned} \tag{13}$$

Due to the definition of  $\lambda_*$  it is also  $\lambda_*^{-1}(E_{i,j}) = (X_i \cdot X_j)/2$ ,  $\lambda_*^{-1}(E_{k,l}) = (Y_k \cdot Y_l)/2$  and  $\lambda_*^{-1}(E_{i,k}) = (X_i \cdot Y_k)/2$ . Consider now the lift  $\widetilde{\Lambda}_{s,t} := \lambda_*^{-1} \circ \Lambda_{s,t} : \mathfrak{m} \rightarrow \mathfrak{spin}(\mathfrak{m})$  of the Nomizu map into  $\mathfrak{spin}(\mathfrak{m})$ . Then, by using the expression of  $\Lambda^{s,t}(X)$  described above, we conclude that

$$\begin{aligned} (a) \text{ If } X \in \mathfrak{m}_1, \quad \text{then} \quad \widetilde{\Lambda}_{s,t}(X) &= \frac{st}{2\sqrt{2t}} \sum_{i,k} B([X, X_i]_{\mathfrak{m}_2}, Y_k) X_i \cdot Y_k, \\ (b) \text{ If } X \in \mathfrak{m}_2, \quad \text{then} \quad \widetilde{\Lambda}_{s,t}(X) &= \frac{s(1-t)}{4} \sum_{i,j} B([X, X_i], X_j) X_i \cdot X_j. \end{aligned} \quad (14)$$

The lifted connection on spinors reads by (we use the same notation)  $\nabla_X^{s,t} \varphi = X(\varphi) + \widetilde{\Lambda}_{s,t}(X)\varphi$  and the Dirac operator acts on spinor fields as (see [1, 9]),

$$D^{s,t}(\varphi) = \sum_{i=1}^{d_1} \kappa_n(X_i) \{X_i(\varphi) + \widetilde{\Lambda}_{s,t}(X_i)\varphi\} + \sum_{k=1}^{d_2} \kappa_n(Y_k) \left\{ \frac{Y_k}{\sqrt{2t}}(\varphi) + \widetilde{\Lambda}_{s,t}\left(\frac{Y_k}{\sqrt{2t}}\right)\varphi \right\}.$$

Using the relations (13) and (14), it finally takes the form

$$D^{s,t} = D^0(\varphi) + \frac{s(1+t)}{4\sqrt{2t}} \sum_{i,j,k} B([X_i, X_j]_{\mathfrak{m}_2}, Y_k) X_i \cdot X_j \cdot Y_k \cdot \varphi,$$

where  $D^0(\varphi) := \sum_{1 \leq i \leq d_1} X_i \cdot X_i(\varphi) + \frac{1}{\sqrt{2t}} \sum_{1 \leq k \leq d_2} Y_k \cdot Y_k(\varphi)$  is the Dirac operator associated to the canonical connection. The adjoint operator of the Dirac operator  $D^{s,t}$  is given by (see [32, 17])  $(D^{s,t})^*(\varphi) =$

$$\begin{aligned} &= \sum_i \{ \nabla_{X_i}^{s,t} + \text{div}^{gt}(X_i) \} (\kappa_n(X_i)\varphi) + \sum_k \{ \nabla_{W_k}^{s,t} + \text{div}^{gt}(W_k) \} (\kappa_n(Y_k)\varphi) \\ &= \sum_i \{ \nabla_{X_i}^{s,t} + \text{div}^{gt}(X_i) \} (X_i \cdot \varphi) + \sum_k \{ \nabla_{W_k}^{s,t} + \text{div}^{gt}(W_k) \} (Y_k \cdot \varphi), \end{aligned}$$

where in a line with the definition of  $D^{s,t}$  we interpret the Clifford multiplication only in terms of  $B$ -orthonormal vectors. Since  $\nabla^{s,t}$  is metric, one has (see [9, 2])

$$\nabla_{X_i}^{s,t}(X_i \cdot \varphi) = \nabla_{X_i}^{s,t} X_i \cdot \varphi + X_i \cdot \nabla_{X_i}^{s,t} \varphi, \quad \nabla_{W_k}^{s,t}(Y_k \cdot \varphi) = \nabla_{W_k}^{s,t} Y_k \cdot \varphi + Y_k \cdot \nabla_{W_k}^{s,t} \varphi, \quad \forall \varphi \in \Gamma(\Sigma).$$

But by the definition of  $\nabla^{s,t}$  one has  $\nabla_{X_i}^{s,t} X_i = 0 = \nabla_{W_k}^{s,t} Y_k$ . Hence

$$\{D^{s,t} - (D^{s,t})^*\}(\varphi) = \sum_{1 \leq i \leq d_1} \text{div}^{gt}(X_i)(X_i \cdot \varphi) + \frac{1}{\sqrt{2t}} \sum_{1 \leq k \leq d_2} \text{div}^{gt}(Y_k)(Y_k \cdot \varphi).$$

However, it is very easy to prove that  $\text{div}^{gt}(X_i) = 0 = \text{div}^{gt}(Y_k)$  and thus  $(D^{s,t})^* = D^{s,t}$  for any  $s \in \mathbb{R}$  and  $t > 0$ . We finally mention that an alternative method available to prove the identification  $(D^{s,t})^* = D^{s,t}$ , relies on [1, Prop. 3.1].  $\blacksquare$

We want now to describe the Ricci tensor and the scalar curvature of  $(M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_t)$  with respect to the family  $\{\nabla^{s,t} : s \in \mathbb{R}, t > 0\}$ . Let  $B_{\mathfrak{k}} := B|_{\mathfrak{k} \times \mathfrak{k}}$  be the restriction of  $B$  on the Lie subalgebra  $\mathfrak{k}$ .

**Lemma 5.7.** a) Let  $X \in \mathfrak{m}_1$  and  $Y \in \mathfrak{m}_1$ . Then, for  $Z \in \mathfrak{m}_1$  one has

$$g_t(R^{s,t}(X, Z)Z, Y) = \frac{(s^2t - 2s + 2st)}{2} B([X, Z]_{\mathfrak{m}_2}, Z, Y) - B_{\mathfrak{k}}([X, Z], [Z, Y]),$$

while for  $Z \in \mathfrak{m}_2$  it holds that

$$g_t(R^{s,t}(X, Z)Z, Y) = st(s - st - 1)B([X, Z], Z, Y).$$

b) Let  $X \in \mathfrak{m}_1$  and  $Y \in \mathfrak{m}_2$ , or  $X \in \mathfrak{m}_2$  and  $Y \in \mathfrak{m}_1$ . Then, one has  $g_t(R^{s,t}(X, Z)Z, Y) = 0$ , for any  $Z \in \mathfrak{m}$ .

c) Let  $X \in \mathfrak{m}_2$  and  $Y \in \mathfrak{m}_2$ . Then

$$\begin{aligned} g_t(R^{s,t}(X, Z)Z, Y) &= st(s - st - 1)B([X, Z], Z, Y), \quad \text{if } Z \in \mathfrak{m}_1, \\ g_t(R^{s,t}(X, Z)Z, Y) &= -2tB_{\mathfrak{k}}([X, Z], [Z, Y]), \quad \text{if } Z \in \mathfrak{m}_2. \end{aligned}$$

It is useful to express the splitting  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$  by  $\chi_* = \chi_*^1 \oplus \chi_*^2$ , where the sub-representations  $\chi_*^i : \mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{m}_i)$  are given by  $\chi_*^i(Y) := \text{ad}(Y)|_{\mathfrak{m}_i}$ , for any  $Y \in \mathfrak{k}$ . Then, for the Casimir element  $C_{\chi, B_{\mathfrak{k}}} : \mathfrak{m} \rightarrow \mathfrak{m}$  we write

$$C_{\chi} = C_{\chi_1} \oplus C_{\chi_2}, \quad \text{with } C_{\chi_i} : \mathfrak{m}_i \rightarrow \mathfrak{m}_i \quad (i = 1, 2)$$

such that  $C_{\chi_1} = -\sum_{a=1}^{\dim_{\mathbb{R}} \mathfrak{k}} \chi_*^1(k_a) \circ \chi_*^1(k'_a)$  and  $C_{\chi_2} = -\sum_{a=1}^{\dim_{\mathbb{R}} \mathfrak{k}} \chi_*^2(k_a) \circ \chi_*^2(k'_a)$ , respectively. Here,  $\{k_a, k'_a\}$  are dual bases of  $\mathfrak{k}$  with respect to  $B_{\mathfrak{k}}$ . Because  $B$  is the Killing metric, it is necessarily  $C_{\chi_i} = \text{Cas}_i \cdot \text{Id}_{\mathfrak{m}_i}$  with  $\text{Cas}_i = B(\lambda_i, \lambda_i + 2\delta) > 0$ , where  $\lambda_i$  is the dominant weight of the  $K$ -module  $\mathfrak{m}_i$  and  $\delta$  denotes the half of the sum of positive roots of  $\mathfrak{k} \otimes \mathbb{C}$  [30]. In other words (see [30] or [10, p. 197])

$$\begin{aligned} \text{Cas}_1 &= B(C_{\chi_1} X_j, X_j) = \sum_{i=1}^{d_1} B_{\mathfrak{k}}([X_j, X_i], [X_j, X_i]) = A_1(X_j, X_j), \\ \text{Cas}_2 &= B(C_{\chi_2} Y_l, Y_l) = \sum_{k=1}^{d_2} B_{\mathfrak{k}}([Y_l, Y_k], [Y_l, Y_k]) = A_2(Y_l, Y_l), \end{aligned}$$

where we define symmetric bilinear maps  $A_i$  on  $\mathfrak{m}_i$  ( $i = 1, 2$ ) by

$$\begin{aligned} A_1(X, Y) &:= B(C_{\chi_1} X, Y) = \sum_{i=1}^{d_1} B_{\mathfrak{k}}([X, X_i], [Y, X_i]), \quad X, Y \in \mathfrak{m}_1, \\ A_2(X, Y) &:= B(C_{\chi_2} X, Y) = \sum_{k=1}^{d_2} B_{\mathfrak{k}}([X, Y_k], [Y, Y_k]), \quad X, Y \in \mathfrak{m}_2. \end{aligned}$$

**Theorem 5.8.** The Ricci tensor  $\text{Ric}^{s,t}$  of the homogeneous Riemannian manifold  $(M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_t)$  endowed with the family of  $G$ -invariant metric connections  $\{\nabla^{s,t} : s \in \mathbb{R}, t \in \mathbb{R}_+\}$ , is expressed as follows:

(a) Let  $X, Y \in \mathfrak{m}_1$ . Then

$$\begin{aligned} \text{Ric}^{s,t}(X, Y) &= \sum_{i=1}^{d_1} \frac{(s^2t - 2s + 2st)}{2} B([X, X_i]_{\mathfrak{m}_2}, X_i, Y) \\ &\quad + \sum_{k=1}^{d_2} \frac{(s^2 - s^2t - s)}{2} B([X, Y_k], Y_k, Y) + A_1(X, Y). \end{aligned}$$

- (b) Let  $X \in \mathfrak{m}_1$ ,  $Y \in \mathfrak{m}_2$ , or  $X \in \mathfrak{m}_2$ ,  $Y \in \mathfrak{m}_1$ . Then  $\text{Ric}^{s,t}(X, Y) = 0$ .  
(c) Let  $X, Y \in \mathfrak{m}_2$ . Then

$$\text{Ric}^{s,t}(X, Y) = \sum_{i=1}^{d_1} (s^2t - s^2t^2 - st)B([X, X_i], X_i)_{\mathfrak{m}_2, Y} + A_2(X, Y).$$

**Proof.** Given some  $g_t$ -orthonormal vectors  $V_i \in \mathfrak{m}_1$ ,  $W_k \in \mathfrak{m}_2$ , it is  $R^{s,t}(X, V_i)V_i = R^{s,t}(X, X_i)X_i$  and  $R^{s,t}(X, W_k)W_k = (1/2t)R^{s,t}(X, Y_k)Y_k$ , respectively. Thus, for any  $X, Y \in \mathfrak{m}$  we compute

$$\text{Ric}^{s,t}(X, Y) = \sum_{i=1}^{d_1} g_t(R^{s,t}(X, X_i)X_i, Y) + (1/2t) \sum_{k=1}^{d_2} g_t(R^{s,t}(X, Y_k)Y_k, Y)$$

and the result follows by Lemma 5.7.  $\blacksquare$

**Corollary 5.9.** *The scalar curvature  $\text{Scal}^{s,t}$  of  $(M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_t, \nabla^{s,t})$  is the function on  $M$  given by*

$$\begin{aligned} \text{Scal}^{s,t} &= -\frac{(s^2t - 2s + 2st)}{2} \sum_{i,j} \|[X_i, X_j]_{\mathfrak{m}_2}\|^2 - \sum_{i,k} (s^2 - s^2t - s) \|[X_i, Y_k]\|^2 \\ &\quad + \sum_i A_1(X_i, X_i) + \frac{1}{2t} \sum_k A_2(Y_k, Y_k). \end{aligned}$$

**5.2. The homogeneous Einstein equation.** Let us shortly illustrate the traditional homogeneous Einstein equation  $\text{Ric}^{1,t} = cg_t$  (where  $c \in \mathbb{R}_+$  is the Einstein constant). We need the components  $r_1 = \text{Ric}^{1,t}(V_j, V_j)$  and  $r_2 = \text{Ric}^{1,t}(W_l, W_l)$  of the Riemannian Ricci tensor, for some  $g_t$ -orthonormal vectors  $V_j \in \mathfrak{m}_1$  and  $W_l \in \mathfrak{m}_2$ , respectively. Because  $\mathfrak{m}_1 \not\cong \mathfrak{m}_2$  as  $K$ -representations, it is  $\text{Ric}^{1,t}(\mathfrak{m}_1, \mathfrak{m}_2) = 0$  and all homogeneous Einstein metrics, if existent, appear as real positive solutions of the equation  $r_1 - r_2 = 0$ . As a first step, by Theorem 5.8 we see that

**Corollary 5.10.** *Let  $\{X_i\}_{i=1}^{d_1}$  and  $\{Y_k\}_{k=1}^{d_2}$  be the  $B$ -orthonormal bases of  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$ , respectively. Then we have  $\text{Ric}^{s,t}(X_i, Y_k) = 0 = \text{Ric}(Y_k, X_i)$ , and*

$$\begin{aligned} \text{Ric}^{s,t}(X_j, X_j) &= -\frac{(s^2t - 2s + 2st)}{2} \sum_{i=1}^{d_1} \|[X_j, X_i]_{\mathfrak{m}_2}\|^2 \\ &\quad - \frac{(s^2 - s^2t - s)}{2} \sum_{k=1}^{d_2} \|[X_j, Y_k]\|^2 + \text{Cas}_1, \\ \text{Ric}^{s,t}(Y_l, Y_l) &= -st(s - st - 1) \sum_{i=1}^{d_1} \|[Y_l, X_i]\|^2 + \text{Cas}_2. \end{aligned}$$

The Riemannian Ricci tensor occurs for  $s = 1$ , i.e.  $r_1 = \text{Ric}^{1,t}(V_j, V_j) = \text{Ric}^{1,t}(X_j, X_j)$  and  $r_2 = \text{Ric}^{1,t}(W_l, W_l) = (1/2t) \text{Ric}^{1,t}(Y_l, Y_l)$ , respectively. Since

$t \neq 0$ , we conclude that the homogeneous Einstein equation is the quadratic equation  $\alpha \cdot t^2 + \beta \cdot t + \gamma = 0$ , where  $\alpha, \beta, \gamma$  are given by

$$\begin{aligned} \alpha &:= -3 \sum_i \|[X_j, X_i]_{\mathfrak{m}_2}\|^2 + \sum_k \|[X_j, Y_k]\|^2 - \sum_i \|[Y_l, X_i]\|^2, \\ \beta &:= 2 \left( \sum_i \|[X_j, X_i]_{\mathfrak{m}_2}\|^2 + \text{Cas}_1 \right), \quad \gamma := -\text{Cas}_2. \end{aligned}$$

Since the sign of  $\alpha$  depends on the underlying manifold, all that we can state is a bound of the number of invariant Einstein metrics  $\mathbf{ein}(M)$ , namely  $0 \leq \mathbf{ein}(M) \leq 2$ , see also [13] and [10, p. 263-264].

**Example 5.11.** Consider the complex projective space  $\mathbb{C}P^3 = \text{SO}(5)/\text{U}(2)$ . For the Lie algebra  $\mathfrak{so}(5)$  we fix a reductive decomposition related to the twistor fibration of  $\mathbb{C}P^3$  over the 4-sphere  $S^4$  (for details we refer to [9]). Recall that the matrices  $\{E_{i,j} : i < j\}$  form an orthonormal basis of  $\mathfrak{so}(n)$  with respect to the scalar product  $B' = -(1/2) \text{tr} AB$  (which is such that  $B_{\text{SO}(n)} = 2(n-2)B'$ ). By definition, it is  $\mathfrak{so}(5) = \text{span}_{\mathbb{R}}\{E_{1,2}, \dots, E_{4,5}\}$ . Set

$$\mathfrak{k} := \text{span}_{\mathbb{R}}\{k_1 := E_{1,2}, k_2 := E_{3,4}, k_3 := (E_{1,3} - E_{2,4})/\sqrt{2}, k_4 := (E_{1,4} + E_{2,3})/\sqrt{2}\} \cong \mathfrak{u}(2).$$

Notice that  $B'(k_i, k_j) = \delta_{i,j}$ , for any  $1 \leq i, j \leq 4$ . Let  $\mathfrak{m}$  be the invariant  $B'$ -orthogonal complement of  $\mathfrak{k}$  into  $\mathfrak{so}(5)$ ; an orthonormal basis with respect to  $B'$  is given by the vectors  $e_1 := E_{1,5}, e_2 := E_{2,5}, e_3 := E_{3,5}, e_4 := E_{4,5}, e_5 := (E_{1,3} + E_{2,4})/\sqrt{2}$ , and  $e_6 := (E_{1,4} - E_{2,3})/\sqrt{2}$ . We also set

$$\mathfrak{m}_1 := \text{span}_{\mathbb{R}}\{e_1, e_2, e_3, e_4\}, \quad \mathfrak{m}_2 := \text{span}_{\mathbb{R}}\{e_5, e_6\},$$

such that  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ . Then, the inclusions given by (1) can be easily checked by computing the Lie brackets  $\text{ad}_{i,j} := \text{ad}(e_i)e_j = [e_i, e_j]$  ( $1 \leq i, j \leq 6$ ) of the base vectors:

	$\text{ad}_{1,j}$	$\text{ad}_{2,j}$	$\text{ad}_{3,j}$	$\text{ad}_{4,j}$	$\text{ad}_{5,j}$	$\text{ad}_{6,j}$
$e_1$	0	$-k_1$	$-E_{1,3}$	$-E_{1,4}$	$(\sqrt{2}/2)e_3$	$(\sqrt{2}/2)e_4$
$e_2$	$k_1$	0	$-E_{2,3}$	$-E_{2,4}$	$(\sqrt{2}/2)e_4$	$-(\sqrt{2}/2)e_3$
$e_3$	$E_{1,3}$	$E_{2,3}$	0	$-k_2$	$-(\sqrt{2}/2)e_1$	$(\sqrt{2}/2)e_2$
$e_4$	$E_{1,4}$	$E_{2,4}$	$k_2$	0	$-(\sqrt{2}/2)e_2$	$-(\sqrt{2}/2)e_1$
$e_5$	$-(\sqrt{2}/2)e_3$	$-(\sqrt{2}/2)e_4$	$(\sqrt{2}/2)e_1$	$(\sqrt{2}/2)e_2$	0	$k_1 - k_2$
$e_6$	$-(\sqrt{2}/2)e_4$	$(\sqrt{2}/2)e_3$	$-(\sqrt{2}/2)e_2$	$(\sqrt{2}/2)e_1$	$-k_1 + k_2$	0

Observe that  $E_{1,3}, E_{1,4}, E_{2,3}, E_{2,4} \in \mathfrak{k} \oplus \mathfrak{m}_2$ . The restrictions separately on  $\mathfrak{k}$  and  $\mathfrak{m}_2$ , are given by

$$\begin{aligned} E_{1,3}|_{\mathfrak{m}_2} &= (\sqrt{2}/2)e_5, & E_{1,4}|_{\mathfrak{m}_2} &= (\sqrt{2}/2)e_6, & E_{2,3}|_{\mathfrak{m}_2} &= -(\sqrt{2}/2)e_6, \\ E_{1,3}|_{\mathfrak{k}} &= (\sqrt{2}/2)k_3, & E_{1,4}|_{\mathfrak{k}} &= (\sqrt{2}/2)k_4, & E_{2,3}|_{\mathfrak{k}} &= (\sqrt{2}/2)k_4, \\ E_{2,4}|_{\mathfrak{m}_2} &= (\sqrt{2}/2)e_5, & E_{2,4}|_{\mathfrak{k}} &= -(\sqrt{2}/2)k_3, \end{aligned}$$

Now, up to scaling, any invariant Riemannian metric on  $\mathbb{C}P^3$  has the form  $g_t = B'|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + 2tB'|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$ , for some  $t \in \mathbb{R}_+$ . For  $X_j = e_1$  and  $Y_l = e_5$ ,

the coefficients  $\alpha, \beta, \gamma$  are given respectively by  $\alpha = -3 \sum_{i=1}^4 \|[e_1, e_i]_{\mathfrak{m}_2}\|^2 + \sum_{k=5}^6 \|[e_1, e_k]\|^2 - \sum_{i=1}^4 \|[e_5, e_i]\|^2$ ,  $\beta = 2 \left( \sum_{i=1}^4 \|[e_1, e_i]_{\mathfrak{m}_2}\|^2 + B'(C_{\chi_1} e_1, e_1) \right)$  and  $\gamma = -B'(C_{\chi_2} e_5, e_5)$ . We compute  $\gamma = -B'(C_{\chi_2} e_5, e_5) = -2 (= -B'(C_{\chi_1} e_1, e_1))$ ,  $\alpha = -4$  and  $\beta = 6$ . Therefore, on  $\mathbb{C}P^3$  the Einstein equation  $\alpha \cdot t^2 + \beta \cdot t + \gamma = 0$  has two positive solutions, namely  $t = 1$  and  $t = 1/2$ . The first value defines the Kähler-Einstein metric  $g_1 = B'|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + 2B'|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$  and the second one corresponds to the Killing metric  $g_{1/2} = B'|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + B'|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$ , which is a homogeneous Einstein metric for  $\mathbb{C}P^3$  (in fact nearly-Kähler), see also [9, 8].

**5.3.  $\nabla^{s, \frac{1}{2}}$ -Einstein structures.** Corollary 5.5 ensures that for  $s \neq 1$  and  $t = 1/2$  the family  $\{\nabla^{s, \frac{1}{2}} : s \in \mathbb{R}\}$  has non-trivial skew-symmetric torsion  $T^{s, \frac{1}{2}} \in \Lambda^3(\mathfrak{m})$ . Thus, if  $\nabla^{g_{1/2}} \equiv \nabla^{1, \frac{1}{2}}$  denotes the Levi-Civita connection on the standard homogeneous Riemannian manifold  $(M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_{1/2})$ , one can write  $B(\nabla_X^s Y, Z) = B(\nabla_X^{g_{1/2}} Y, Z) + \frac{1}{2} T^s(X, Y, Z)$ . Hence, for the Killing metric  $g_{1/2} \equiv B|_{\mathfrak{m} \times \mathfrak{m}}$  it makes sense to examine the existence of a  $\nabla^{s, \frac{1}{2}}$ -Einstein structure with skew-torsion. Because the value  $t = 1/2$  will be fixed from now on, for simplicity we will write  $\nabla^{s, \frac{1}{2}} \equiv \nabla^s$ ,  $T^{s, \frac{1}{2}} \equiv T^s$ , e.t.c. Let us present some structural properties of the torsion form  $T^s \in \Lambda^3(\mathfrak{m})$ . It is useful to introduce the following trilinear maps:

$$\begin{aligned} \text{Jac}_{\mathfrak{m}}(X_j, X_r, X_s) &:= \mathfrak{S}_{j,r,s}[X_j, [X_r, X_s]_{\mathfrak{m}_2}], \\ \text{Jac}_{\mathfrak{m}}(X_i, X_j, Y_k) &:= [X_i, [X_j, Y_k]]_{\mathfrak{m}_2} + [X_j, [Y_k, X_i]]_{\mathfrak{m}_2} + [Y_k, [X_i, X_j]_{\mathfrak{t}}], \\ \text{Jac}_{\mathfrak{m}}(X_i, Y_k, Y_l) &:= \mathfrak{S}_{i,k,l}[X_i, [Y_k, Y_l]]. \end{aligned}$$

Here, the vectors  $\{X_i\}_{i=1}^{d_1}$  and  $\{Y_k\}_{k=1}^{d_2}$  stand for the fixed  $B$ -orthonormal bases. Although these maps are different each other, we use the same notation since in any case their definition is obvious by (1). In fact, it is easy to see that the mixed Jacobians  $\text{Jac}_{\mathfrak{m}}(X_i, X_j, Y_k)$  and  $\text{Jac}_{\mathfrak{m}}(X_i, Y_k, Y_l)$  are identically equal to zero. Indeed, by viewing the vectors  $X_i, Y_k, Y_l \in \mathfrak{m}$  as left-invariant vectors fields, it follows that  $\text{Jac}_{\mathfrak{m}}(Y_k, Y_l, X_i) = 0$ . Then, because  $B(\text{Jac}_{\mathfrak{m}}(X_i, X_j, Y_k), Y_l) = B(\text{Jac}_{\mathfrak{m}}(X_i, Y_k, Y_l), X_j)$  it also follows that  $\text{Jac}_{\mathfrak{m}}(X_i, X_j, Y_k) = 0$ . This observation simplifies the calculations. In the following, for some vectors  $A \in \mathfrak{m}_{\alpha}, B \in \mathfrak{m}_{\beta}, C \in \mathfrak{m}_{\gamma}$  with  $1 \leq \alpha, \beta, \gamma \leq 2$ , we shall use the convention  $\nabla_{\alpha\beta\gamma}^s := (\nabla_C^s T^s)(A, B)$ .

**Lemma 5.12.** *The covariant derivative  $\nabla_Z^s T^s : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$  of the torsion form  $T^{s, \frac{1}{2}} \equiv T^s$  is given by*

$$\begin{aligned} \nabla_{111}^s &:= (\nabla_{X_r}^s T^s)(X_i, X_j) = \frac{s(s-1)}{2} \text{Jac}_{\mathfrak{m}}(X_i, X_j, X_r), \\ \nabla_{112}^s &:= (\nabla_{Y_k}^s T^s)(X_i, X_j) = -\frac{s(s-1)}{2} [Y_k, [X_i, X_j]_{\mathfrak{t}}], \\ \nabla_{221}^s &:= (\nabla_{X_i}^s T^s)(Y_k, Y_l) = -\frac{s(s-1)}{2} [X_i, [Y_k, Y_l]], \end{aligned}$$

with  $\nabla_{112}^s = -\nabla_{121}^s = \nabla_{211}^s$ ,  $\nabla_{221}^s = -\nabla_{212}^s = \nabla_{122}^s$  and all the other combinations are zero. On the other hand, the co-differential  $\delta^s T^s$  vanishes for any  $s \in \mathbb{R}$ .

The vanishing of the co-differential  $\delta^s T^s$  ensures that the Ricci tensor  $\text{Ric}^s$  is symmetric. In full details

**Proposition 5.13.** *Let  $\{X_i\}_{i=1}^{d_1}$  and  $\{Y_k\}_{k=1}^{d_2}$  be the  $B$ -orthonormal bases of  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$ , respectively. Then, the Ricci tensor associated to the 1-parameter family  $\{\nabla^s \equiv \nabla^{s, \frac{1}{2}} : s \in \mathbb{R}\}$  satisfies the following relations:  $\text{Ric}^s(X_i, Y_l) = 0$  and*

$$\begin{aligned} \text{Ric}^s(X_j, X_j) &= \text{Ric}^{g_{1/2}}(X_j, X_j) - \frac{1}{4}S^s(X_j, X_j), \\ \text{Ric}^s(Y_l, Y_l) &= \text{Ric}^{g_{1/2}}(Y_l, Y_l) - \frac{1}{4}S^s(Y_l, Y_l), \end{aligned}$$

where the non-zero parts of the symmetric tensor  $S^s$  are of the form

$$\begin{aligned} S^s(X_j, X_j) &= (s-1)^2 \left\{ \sum_i \|[X_j, X_i]_{\mathfrak{m}_2}\|^2 + \sum_k \|[X_j, Y_k]\|^2 \right\}, \\ S^s(Y_l, Y_l) &= (s-1)^2 \sum_i \|[Y_l, X_i]\|^2. \end{aligned}$$

**Proof.** By Corollary 5.10, the Ricci tensor  $\text{Ric}^s \equiv \text{Ric}^{s, \frac{1}{2}}$  with respect to  $\nabla^s$  has the form

$$\begin{aligned} \text{Ric}^s(X_j, X_j) &= -\frac{(s^2-2s)}{4} \left\{ \sum_{i=1}^{d_1} \|[X_j, X_i]_{\mathfrak{m}_2}\|^2 + \sum_{k=1}^{d_2} \|[X_j, Y_k]\|^2 \right\} + \text{Cas}_1, \\ \text{Ric}^s(Y_l, Y_l) &= -\frac{(s^2-2s)}{4} \sum_{i=1}^{d_1} \|[Y_l, X_i]\|^2 + \text{Cas}_2. \end{aligned}$$

Using now the Riemannian Ricci tensor  $\text{Ric}^{1, \frac{1}{2}} \equiv \text{Ric}^1 \equiv \text{Ric}^{g_{1/2}}$  and the definition of the symmetric tensor  $S^s$ , one can obtain the given expressions. For example, let  $0 \neq X, Y \in \mathfrak{m}_1$ . Then

$$\begin{aligned} S^s(X, Y) &:= \sum_{j=1}^{d_1} B(T^s(X_j, X), T^s(X_j, Y)) + \sum_{l=1}^{d_2} B(T^s(Y_l, X), T^s(Y_l, Y)) \\ &= (s-1)^2 \left\{ \sum_j B([X_j, X]_{\mathfrak{m}_2}, [X_j, X]_{\mathfrak{m}_2}) + \sum_l B([Y_l, X], [Y_l, Y]) \right\}. \end{aligned}$$

Similarly, for  $X, Y \in \mathfrak{m}_2$  we get  $S^s(X, Y) = \sum_j B(T^s(X_j, X), T^s(X_j, Y)) = (s-1)^2 \sum_j B([X_j, X], [X_j, Y])$ , and finally for  $X \in \mathfrak{m}_1, Y \in \mathfrak{m}_2$  it is  $S^s(X, Y) = 0 = S^s(Y, X)$ .  $\blacksquare$

**Theorem 5.14.** *Let  $(M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_{1/2})$  be a compact connected homogeneous Riemannian manifold with two isotropy summands satisfying (1). Then,  $M = G/K$  is a  $\nabla^s$ -Einstein manifold with skew-torsion  $0 \neq T^s \in \Lambda^3(\mathfrak{m})$  for the values  $s = 0$  or  $s = 2$ , if and only if, the Killing metric  $g_B \equiv g_{1/2}$  is a  $G$ -invariant Einstein metric, i.e.  $\text{Cas}_1 = \text{Cas}_2$ .*

**Proof.** A  $\nabla^s$ -Einstein structure on  $(M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_{1/2})$  is given as a solution (with respect to  $s$ ) of the following system:

$$\left\{ \text{Ric}^s(X_j, X_j) = \frac{\text{Scal}^s}{n} B(X_j, X_j) = \frac{\text{Scal}^s}{n}, \text{Ric}^s(Y_l, Y_l) = \frac{\text{Scal}^s}{n} B(Y_l, Y_l) = \frac{\text{Scal}^s}{n} \right\}.$$

This is equivalent to the equation  $\text{Ric}^s(X_j, X_j) - \text{Ric}^s(Y_l, Y_l) = 0$ , namely

$$(s^2 - 2s) \left\{ \sum_{i=1}^{d_1} \|[X_j, X_i]_{\mathfrak{m}_2}\|^2 + \sum_{k=1}^{d_2} \|[X_j, Y_k]\|^2 - \sum_{i=1}^{d_1} \|[Y_l, X_i]\|^2 \right\} = 4(\text{Cas}_1 - \text{Cas}_2).$$

Since  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$  is a  $B$ -orthogonal decomposition of  $\mathfrak{m} = T_oG/K$  and both  $\mathfrak{m}_1, \mathfrak{m}_2$  have been assumed to be irreducible and non-equivalent, by [30, Thm. 1.11] it is known that  $M = G/K$  is a standard homogeneous Einstein manifold if and only if the Casimir constants coincide  $\text{Cas}_1 = \text{Cas}_2$ , and our assertion follows. ■

Recall that  $T^s(X, Y) = (s-1)[X, Y]_{\mathfrak{m}}$ , hence the torsion of the connections  $\nabla^0$  and  $\nabla^2$  are such that  $T^0 = -T^2$ . Although  $\nabla^0 T^0 = 0$ , Lemma 5.12 ensures that  $\nabla^2 T^2 \neq 0$  (in analogy to the isotropy irreducible case).

**Remark 5.15.** The  $\nabla^s$ -Einstein condition on  $(M = G/K, g_{1/2})$  is the following quadratic equation:

$$\mathfrak{c}s^2 - 2\mathfrak{c}s - 4(\text{Cas}_1 - \text{Cas}_2) = 0,$$

where  $\mathfrak{c} := (\sum_{i=1}^{d_1} \|[X_j, X_i]_{\mathfrak{m}_2}\|^2 + \sum_{k=1}^{d_2} \|[X_j, Y_k]\|^2 - \sum_{i=1}^{d_1} \|[Y_l, X_i]\|^2)$ . The discriminant is given by  $\Delta = 4\mathfrak{c}(\mathfrak{c} + 4(\text{Cas}_1 - \text{Cas}_2))$ . Obviously, if  $\mathfrak{c} > 0$  and  $\text{Cas}_1 \geq \text{Cas}_2$ , then  $\Delta > 0$  and there are two solutions, i.e. two  $\nabla^s$ -Einstein structures, defined by

$$s = 1 \pm \sqrt{\frac{\mathfrak{c} + 4(\text{Cas}_1 - \text{Cas}_2)}{\mathfrak{c}}}.$$

The same is true if  $\mathfrak{c} < 0$  and  $\text{Cas}_2 \geq \text{Cas}_1$ . Assuming that the Killing metric  $g_B$  is Einstein, then we recover the values  $s = 0, 2$  described in Theorem 5.14. In general, if  $\mathfrak{c} > 0$  with  $\mathfrak{c} > 4(\text{Cas}_2 - \text{Cas}_1)$ , or  $\mathfrak{c} < 0$  with  $\mathfrak{c} < 4(\text{Cas}_1 - \text{Cas}_2)$ , then  $\Delta > 0$  and the  $\nabla^s$ -Einstein structures described above are still available. Hence, one can theoretically describe  $\nabla^s$ -Einstein structures which are different than the  $\nabla^s$ -Einstein structures associated to the canonical ( $s = 0$ ) and the anti-canonical connection ( $s = 2$ ). The same time, the case  $\Delta < 0$  is still possible. In fact, if  $\text{Cas}_1 \neq \text{Cas}_2$ , only such examples we are able to construct, i.e. cosets  $(M = G/K, g_{1/2})$  with two isotropy summands satisfying (1), which do not admit any  $\nabla^s$ -Einstein structure. Explicit examples of cosets  $(M = G/K, g_{1/2})$  carrying  $\nabla^s$ -Einstein structures with  $s \neq 0, 2$  are still missing; this interesting topic will be addressed in a forthcoming work.

**5.4. Examples.** Let us present now a series of manifolds that Theorem 5.14 can be applied. We focus on flag manifolds and we prove that several of them carry  $\nabla^s$ -Einstein structures for  $s = 0, 2$ . Let us fix a compact *simple* Lie group  $G$  and let  $M = G/K$  be a flag manifold with two isotropy summands, say  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ . Such spaces have been classified in terms of painted Dynkin diagrams in [7]. Since both  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$  are irreducible and inequivalent, any  $G$ -invariant Riemannian metric on  $M = G/K$  is a multiple of  $g_t$ . In [8, Thm 1.1] it was shown that  $M$  admits precisely two  $G$ -invariant Einstein metrics; one of them is Kähler and corresponds to the value  $t = 1$ ; the other one is given by  $t = \frac{2d_2}{d_1 + 2d_2}$ , i.e.  $g_{\frac{2d_2}{d_1 + 2d_2}} = B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + \frac{4d_2}{d_1 + 2d_2} B|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$ .

**Theorem 5.16.** ([8]) *Let  $G$  be a compact connected simple Lie group. A generalized flag manifold  $M = G/K$  whose isotropy representation is such that  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ , is a standard homogeneous Einstein manifold if and only if  $d_1 = 2d_2$ , where  $d_i = \dim \mathfrak{m}_i$  for  $i = 1, 2$ .*

**Example 5.17.** Consider the complex projective space  $\mathbb{C}P^3 = \text{SO}(5)/\text{U}(2) = \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$ . It is  $d_1 = 4 = 2d_2$ , hence  $\mathbb{C}P^3$  is standard Einstein, i.e.  $\text{Cas}_1 = \text{Cas}_2$ , see also Example 5.11. Thus,  $(\mathbb{C}P^3, g_{1/2})$  admits exactly two  $\nabla^{s, \frac{1}{2}}$ -Einstein structures with skew-torsion, namely these which occurs for  $s = 0, 2$ . The  $\nabla^{0, \frac{1}{2}}$ -Einstein structure is well-known; it is related with the homogeneous nearly-Kähler structure that  $(\mathbb{C}P^3, g_{1/2})$  admits, in particular the canonical connection  $\nabla^{0, \frac{1}{2}}$  coincides with the characteristic connection  $\nabla^c$  (Gray connection), see [3, 2].

A quick check of the dimensions of the isotropy summands implies that there are no exceptional flag manifolds, with  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ , for which the Killing metric can be a  $G$ -invariant Einstein metric (see [8, p. 245]). However, several examples appear for adjoint orbits corresponding to the classical Lie groups  $B_\ell = \text{SO}(2\ell+1)$ ,  $C_\ell = \text{Sp}(\ell)$ , or  $D_\ell = \text{SO}(2\ell)$ .

**Example 5.18.** For the family  $B(\ell, p) := \text{SO}(2\ell+1)/\text{U}(p) \times \text{SO}(2(\ell-p)+1)$  ( $2 \leq p \leq \ell$ ,  $\ell \geq 2$ ) we compute  $d_1 = 4p(\ell-p) + 2p$  and  $d_2 = p(p-1)$ . According to Theorem 5.16, the Killing metric  $g_B$  is Einstein if and only if  $p = 2(\ell+1)/3 \in \mathbb{Z}_+$ . Hence we conclude that the manifold  $B(\ell, 2(\ell+1)/3) = \text{SO}(2\ell+1)/\text{U}(2(\ell+1)/3) \times \text{SO}(2(\ell-2)/3+1)$ , with  $\ell = 2+3k$  and  $k = 0, 1, 2, 3, \dots$ , is a  $\nabla^s$ -Einstein manifold for the values  $s = 0, 2$ . Let us list the first examples:

$\ell$	$p = 2(\ell+1)/3 : p \in \mathbb{Z}_+$	$(M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_B)$
$\ell = 2$	$p = 2$	$\mathbb{C}P^3 = \text{SO}(5)/\text{U}(2)$
$\ell = 5$	$p = 4$	$\text{SO}(11)/\text{U}(4) \times \text{SO}(3)$
$\ell = 8$	$p = 6$	$\text{SO}(17)/\text{U}(6) \times \text{SO}(5)$
$\vdots$	$\vdots$	$\vdots$

**Example 5.19.** For the space  $C(\ell, p) := \text{Sp}(\ell)/\text{U}(p) \times \text{Sp}(\ell-p)$  ( $1 \leq p \leq \ell-1$ ,  $\ell \geq 2$ ) it is  $d_1 = 4p(\ell-p)$  and  $d_2 = p(p+1)$  and the condition  $d_1 = 2d_2$  takes the form  $p = (2\ell-1)/3 \in \mathbb{Z}_+$ . Thus the family  $C(\ell, (2\ell-1)/3) = \text{Sp}(\ell)/\text{U}((2\ell-1)/3) \times \text{Sp}((\ell+1)/3)$ , with  $\ell = 2+3k$  and  $k = 0, 1, 2, 3, \dots$ , is a standard homogeneous Einstein manifold. Moreover, for  $s = 0, 2$  it becomes a  $\nabla^s$ -Einstein manifold with skew-torsion.

$\ell$	$p = (2\ell-1)/3 : p \in \mathbb{Z}_+$	$(M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_B)$
$\ell = 2$	$p = 1$	$\mathbb{C}P^3 = \text{Sp}(2)/\text{U}(1) \times \text{Sp}(1)$
$\ell = 5$	$p = 3$	$\text{Sp}(5)/\text{U}(3) \times \text{Sp}(2)$
$\ell = 8$	$p = 5$	$\text{Sp}(8)/\text{U}(5) \times \text{Sp}(3)$
$\vdots$	$\vdots$	$\vdots$

**Example 5.20.** For the flag manifold  $D(\ell, p) := \text{SO}(2\ell)/\text{U}(p) \times \text{SO}(2(\ell-p))$  ( $2 \leq p \leq \ell-2$ ,  $\ell \geq 4$ ) it is  $d_1 = 4p(\ell-p)$  and  $d_2 = p(p-1)$ . Hence  $D(\ell, p)$  is

a standard homogeneous Einstein manifold if and only if  $p = (2\ell + 1)/3 \in \mathbb{Z}_+$ . It follows that the family  $D(\ell, (2\ell + 1)/3) = \text{SO}(2\ell)/\text{U}((2\ell + 1)/3) \times \text{SO}(2(\ell - 1)/3)$ , with  $\ell = 4 + 3k$  and  $k = 0, 1, 2, 3, \dots$ , admits  $\nabla^s$ -Einstein structures with skew-torsion for the values  $s = 0, 2$ .

$\ell$	$p = (2\ell + 1)/3 : p \in \mathbb{Z}_+$	$(M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_B)$
$\ell = 4$	$p = 3$	$\text{SO}(8)/\text{U}(3) \times \text{SO}(2)$
$\ell = 7$	$p = 5$	$\text{SO}(14)/\text{U}(5) \times \text{SO}(4)$
$\ell = 10$	$p = 7$	$\text{SO}(20)/\text{U}(7) \times \text{SO}(6)$
$\vdots$	$\vdots$	$\vdots$

**Acknowledgments** The author warmly acknowledges I. Agricola, D. V. Alekseevsky, A. Arvanitoyeorgos, S. Chiossi, C. Draper, Th. Friedrich, and Y. Sakane for useful discussions and comments. Special thanks go to the anonymous referee for the referee’s report, which has contributed to improvements of the paper, as well as for his/her suggestions for further research.

### References

- [1] Agricola, I., *Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory*, Comm. Math. Phys. **232** (2003), 535–563.
- [2] —, *The Srní lectures on non-integrable geometries with torsion. With an appendix by M. Kassuba*, Arch. Math. **42** (2006), 5–84.
- [3] Agricola, I., and A. C. Ferreira, *Einstein manifolds with skew torsion*, Oxford Quart. J. **65** (2014), 717–741.
- [4] Agricola, I., and Th. Friedrich, *A note on flat metric connections with anti-symmetric torsion*, Diff. Geom. Appl. **28** (2010), 480–487.
- [5] Agricola, I., Th. Friedrich, and J. Höll, *Sp(3)-structures on 14-dimensional manifolds*, J. Geom. Phys. **69** (2013), 12–30.
- [6] Alekseevsky, D. V., A. M. Vinogradov and V. V. Lychagin, “Geometry I - Basic Ideas and Concepts of Differential Geometry,” Encyclopaedia of Mathematical Sciences **28**, Springer-Verlag, Berlin, 1991.
- [7] Arvanitoyeorgos, A., and I. Chrysikos, *Motion of charged particles and homogeneous geodesics in Kähler C-spaces with two isotropy summands*, Tokyo J. Math. **32** (2009), 487–500.
- [8] Arvanitoyeorgos, A., and I. Chrysikos, *Invariant Einstein metrics on generalized flag manifolds with two isotropy summands*, J. Austral. Math. Soc. **90** (2011), 237–251.
- [9] Baum, H., Th. Friedrich, R. Grunewald and I. Kath, ‘ “Twistors and Killing spinors on Riemannian manifolds,” B. G. Teubner Verlagsgesellschaft, Stuttgart etc., 1991.

- [10] Besse, A. L., “Einstein Manifolds,” Springer-Verlag, Berlin, 1986.
- [11] Čap, A., and J. Slovák, “Parabolic geometries I: Background and General Theory,” Mathematical Surveys and Monographs **154**, Amer. Math. Soc., Providence, RI, 2009.
- [12] D’Atri, J. E., and H. N. Nickerson, *The existence of special orthonormal frames*, J. Differ. Geom. **2** (1968), 933–946.
- [13] Dickinson, W., and M. Kerr, *The geometry of compact homogeneous spaces with two isotropy summands*, Ann. Glob. Anal. Geom. **34** (2008), 329–350.
- [14] Draper, C., A. Garvin and F. J. Palomo, *Invariant affine connections on odd dimensional spheres*,  
arXiv: 1503.08401v1.
- [15] Friedrich, Th., and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), 303–335.
- [16] Friedrich, Th., I. Kath, A. Moroianu, and U. Semmelmann, *On nearly parallel  $G_2$ -structures*, J. Geom. Phys. **23** (1997), 259–286.
- [17] Friedrich, Th., and S. Sulanke, *Ein Kriterium für die formale Selbstadjungiertheit des Dirac-Operators*, Coll. Math. **bf40** (1979), 239–247.
- [18] Gortzevich, V. V., A. L. Onishchik, and E. B. Vinberg, “Lie Groups and Lie Algebras III,” Encyclopaedia of Mathematical Sciences **20**, Springer-Verlag, Berlin, 1993.
- [19] Kerr, M., *Some new homogeneous Einstein metrics on symmetric spaces*, Trans. Amer. Math. Soc. **348** (1996), 153–171.
- [20] Kerr, M., *New examples of homogeneous Einstein metrics*, Michigan J. Math. **45** (1998), 115–134.
- [21] Kerin, M., and K. Shankar, *Riemannian submersions from simple, compact Lie groups*, Münster J. of Math. **5** (2012), 25–40.
- [22] Kobayashi, S., and K. Nomizu, “Foundations of Differential Geometry Vol II”, Wiley - Interscience, New York, 1969.
- [23] Laquer, H. T., *Invariant affine connections on Lie groups*, Trans. Am. Math. Soc. **331** (1992), 541–551.
- [24] —, *Invariant affine connections on symmetric spaces*, Proc. Amer. Math. Soc. **115** (1992), 447–454.
- [25] Nagy, P-A., *Skew-symmetric prolongations of Lie algebras and applications*, J. of Lie Theory **23** (2013), 1–33.
- [26] Olmos, C., and S. Reggiani, *The skew-torsion holonomy theorem and naturally reductive spaces*, J. Reine Angew. Math. **664** (2012), 29–53.

- [27] Pfäffle, F., and C. Stephan, *On gravity, torsion and the spectral action principle*, J. Funct. Anal. **262** (2012), 1529–1565.
- [28] Reggiani, S., *On the affine group of a normal homogeneous manifold*, Ann. Glob. Anal. Geom. **37** (2010), 351–359.
- [29] Wang, H. C., *On invariant connections over a principal fibre bundle*, Nagoya Math. J. **13** (1958), 1–19.
- [30] Wang, M. Y., and W. Ziller, *On normal homogeneous Einstein manifolds*, Ann. Scient. Éc. Norm. Sup. **18** (1985), 563–633.
- [31] —, *On isotropy irreducible Riemannian manifolds*, Acta. Math. **166** (1991), 223–261.
- [32] Wolf, J. A., *The geometry and the structure of isotropy irreducible homogeneous spaces*, Acta. Math. **120** (1968), 59–148; correction: Acta Math. **152** (1984), 141–142.
- [33] —, *Essential self adjointness for the Dirac operator and its square*, Indiana Univ. Math. J. **22** (1973), 611–640.

Ioannis Chrysi­kos  
Department of Mathematics  
and Statistics  
Masaryk University  
Brno 611 37  
Czech Republic  
chrysi­kosi@math.muni.cz

Received September 18, 2014  
and in final form October 29, 2015