Journal of Lie Theory
Volume **26** (2015) 97–115
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Projective Oscillator Representations of sl$(n+1)$ and sp$(2m+2)$

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Communicated by F. Knop

Abstract. The n-dimensional projective group gives rise to a one-parameter family of inhomogeneous first-order differential operator representations of sl$(n+1)$. By partially swapping differential operators and multiplication operators, we obtain more general differential operator representations of sl$(n+1)$. Letting these differential operators act on the corresponding polynomial algebra and the space of exponential-polynomial functions, we construct new multi-parameter families of explicit infinite-dimensional irreducible representations for sl$(n+1)$ and sp$(2m+2)$ when $n = 2m+1$. Our results can be viewed as extensions of Howe's oscillator construction of infinite-dimensional multiplicity-free irreducible representations for sl(n). They can also be used to study free bosonic field irreducible representations of the corresponding affine Kac-Moody algebras.
Mathematics Subject Classification 2010: Primary 17B10, secondary 17B20.
Key Words and Phrases: Special linear Lie algebra, symplectic Lie algebra, oscillator representation, irreducible module, polynomial algebra, exponential-polynomial function.

1. Introduction

A module of a finite-dimensional simple Lie algebra is called a *weight module* if it is a direct sum of its weight subspaces. A module of a finite-dimensional simple Lie algebra is called *cuspidal* if it is not induced from its proper parabolic subalgebras. Infinite-dimensional irreducible weight modules of finite-dimensional simple Lie algebras with finite-dimensional weight subspaces have been intensively studied by the authors in [1, 2, 3, 4, 5, 6, 8, 12]. In particular, Fernando [6] proved that such modules must be cuspidal or parabolically induced. Moreover, such cuspidal modules exist only for special linear Lie algebras and symplectic Lie algebras. A similar result was independently obtained by Futorny [8]. Mathieu [12] proved that such cuspidal modules are irreducible components in the tensor modules of their multiplicity-free modules with finite-dimensional modules. Although the structures of irreducible weight modules of finite-dimensional simple

[*] This work is supported by Chinese National Science Foundation NSFC Grants 11171324 and 11321101.

ISSN 0949–5932 / \$2.50 © Heldermann Verlag

Lie algebras with finite-dimensional weight subspaces were essentially determined by Fernando's result in [6] and Methieu's result in [12], explicit structures of such modules are not that known. It is important to find explicit natural realizations of them.

Let $\mathbb{F}$ be a field with characteristic 0 (say, $\mathbb{Q},\mathbb{R},\mathbb{C}$) and let $n\geq 2$ be an integer. A projective transformation on $\mathbb{F}^n$ is given by

$$u\mapsto \frac{Au+\vec{b}}{\vec{c}^{\,t}u+d}\qquad \text{for}\ \ u\in\mathbb{F}^n, \tag{1.1}$$

where all the vectors in $\mathbb{F}^n$ are in column form and the $(n+1)\times(n+1)$ matrix

$$\begin{pmatrix} A & \vec{b}\\ \vec{c}^{\,t} & d\end{pmatrix}\in GL(n). \tag{1.2}$$

It is well-known that a transformation of mapping straight lines to lines must be a projective transformation. The above transformations give rise to an inhomogeneous representation of the Lie algebra $sl(n+1,\mathbb{F})$ on the polynomial functions of the projective space. Using Shen's mixed product for Witt algebras in [13] and the above representation, Zhao and the author [14] constructed a new functor from $gl(n,\mathbb{F})$-**Mod** to $sl(n+1)$-**Mod** and found a condition for the functor to map a finite-dimensional irreducible $gl(n,\mathbb{F})$-module to an infinite-dimensional irreducible $sl(n+1,\mathbb{F})$-module. Our general frame also gave a direct polynomial extension from irreducible $gl(n,\mathbb{F})$-modules to irreducible $sl(n+1,\mathbb{F})$-modules.

The work [14] lead to a one-parameter family of inhomogeneous first-order differential operator (oscillator) representations of $sl(n+1,\mathbb{F})$. By partially swapping differential operators and multiplication operators, we obtain more general differential operator (oscillator) representations of $sl(n+1,\mathbb{F})$. In this paper, we construct new multi-parameter families of explicit infinite-dimensional irreducible representations for $s(n+1,\mathbb{F})$ and $sp(2m+2)$ when $n=2m+1$ by letting these differential operators act on the corresponding polynomial algebra and the space of exponential-polynomial functions. Some of the corresponding modules are explicit infinite-dimensional irreducible weight modules with finite-dimensional weight subspaces. Our results can be viewed as extensions of Howe's oscillator construction of infinite-dimensional multiplicity-free irreducible representations for $sl(n,\mathbb{F})$ (cf. [11]), where he used the symmetric tensor over several copies of the natural module and its dual to construct the representations. Indeed, Howe's result plays an important role in proving the irreducibility of the representations for $sl(n+1,\mathbb{F})$. The results on symplectic Lie algebras in this paper can be used to study the irreducible representations of the other simple Lie algebra via Howe's theta correspondence technique. Free field representations of affine Kac-Moody algebras are closely related to oscillator representations of finite-dimensional simple Lie algebras. Our results can also be used to study free bosonic field irreducible representations of the corresponding affine Kac-Moody algebras.

Let $E_{r,s}$ be the $(n+1)\times(n+1)$ matrix with 1 as its (r,s)-entry and 0 as the others. The special linear algebra

$$sl(n+1,\mathbb{F})=\sum_{1\leq i<j\leq n+1}(\mathbb{F}E_{i,j}+\mathbb{F}E_{j,i})+\sum_{r=1}^{n}\mathbb{F}(E_{r,r}-E_{r+1,r+1}). \tag{1.3}$$

For any two integers $p \leq q$, we denote $\overline{p,q} = \{p, p+1, \cdots, q\}$. Set $D = \sum_{s=1}^{n} x_s\partial_{x_s}$. According to Zhao and the author's work [14], we have the following one-parameter generalization π_c of the projective representation of $sl(n+1,\mathbb{F})$:

$$\pi_c(E_{i,j}) = x_i\partial_{x_j},\ \ \pi_c(E_{i,n+1}) = x_i(D+c),\ \ \pi_c(E_{n+1,i}) = -\partial_{x_i}, \tag{1.4}$$

$$\pi_c(E_{i,i}-E_{j,j}) = x_i\partial_{x_i} - x_j\partial_{x_j},\ \ \pi_c(E_{n,n}-E_{n+1,n+1}) = D + c + x_n\partial_{x_n} \tag{1.5}$$

for $i, j \in \overline{1,n}$ with $i \neq j$, where $c \in \mathbb{F}$.

Let S be a subset of $\overline{1,n}$. Note the symmetry:

$$[\partial_{x_r}, x_r] = 1 = [-x_r, \partial_{x_r}]. \tag{1.6}$$

Changing operators $\partial_{x_r} \mapsto -x_r$ and $x_r \mapsto \partial_{x_r}$ for $r \in S$ in (1.4) and (1.5), we get another differential-operator representation $\pi_{c,S}$ of $sl(n+1,\mathbb{F})$. We treat $\pi_{c,\emptyset} = \pi_c$ and call $\pi_{c,S}$ *projective oscillator representations* in terms of physics terminology. For $\vec{a} = (a_1, a_2, ..., a_n)^t \in \mathbb{F}^n$, we denote $\vec{a}\cdot\vec{x} = \sum_{i=1}^{n} a_i x_i$. Let $\mathscr{A} = \mathbb{F}[x_1, x_2, ..., x_n]$ be the algebra of polynomials in $x_1, x_2, ..., x_n$. Moreover, we set

$$\mathscr{A}_{\vec{a}} = \{fe^{\vec{a}\cdot\vec{x}} \mid f \in \mathscr{A}\}. \tag{1.7}$$

Denote by $\pi_{c,S}^{\vec{a}}$ the representation $\pi_{c,S}$ of $sl(n+1,\mathbb{F})$ on $\mathscr{A}_{\vec{a}}$ and by $\mathbb{N}$ the set of nonnegative integers. In [14], Zhao and the author proved that the representation $\pi_{c,\emptyset}^{\vec{0}}$ of $sl(n+1,\mathbb{F})$ is irreducible if and only if $c \notin -\mathbb{N}$. Moreover, $\mathscr{A}$ has a composite series of length 2 when $c \in -\mathbb{N}$. In this paper, we prove:

Theorem 1.1. *Let S be a proper subset of $\overline{1,n}$. The representation $\pi_{c,S}^{\vec{0}}$ is irreducible for any $c \in \mathbb{F}\setminus\mathbb{Z}$, and the underlying module $\mathscr{A}$ is an infinite-dimensional weight $sl(n+1,\mathbb{F})$-module with finite-dimensional weight subspaces. If $a_i \neq 0$ for some $i \in \overline{1,n}\setminus S$, then the representation $\pi_{c,S}^{\vec{a}}$ of $sl(n+1,\mathbb{F})$ is always irreducible for any $c \in \mathbb{F}$. When $|S| > 1$, $\vec{a} \neq 0$ and $a_i = 0$ for any $i \in \overline{1,n}\setminus S$, the representation $\pi_{c,S}^{\vec{a}}$ of $sl(n+1,\mathbb{F})$ is irreducible for $c \in \mathbb{F}\setminus\mathbb{Z}$*

Suppose that $n = 2m+1 > 1$ is an odd integer and the subset S satisfies:

$$m+1 \notin S \ \text{ and for } i \in \overline{1,m}, \text{ at most one of } i \text{ and } i+m+1 \text{ in } S. \tag{1.8}$$

Our second main theorem in this paper is as follows.

Theorem 1.2. *If $c \notin -\mathbb{N}$, the restricted representation $\pi_{c,\emptyset}^{\vec{0}}$ of $sp(2m+2,\mathbb{F})$ is irreducible. When $c \in -\mathbb{N}$, the $sp(2m+2,\mathbb{F})$-module $\mathscr{A}$ has a composite series of length 2 with respect to the restricted representation $\pi_{c,\emptyset}^{\vec{0}}$.*

The restricted representation $\pi_{c,S}^{\vec{0}}$ of $sp(2m+2,\mathbb{F})$ with $S \neq \emptyset$ is irreducible for any $c \in \mathbb{F}\setminus\mathbb{Z}$. Suppose that $\vec{a} \neq \vec{0}$, $a_{m+1} = 0$, $a_{i_0} \neq 0$ for some $m+1+i_0 \in S\cap\overline{m+2,2m+1}$ if $S\cap\overline{m+2,2m+1} \neq \emptyset$, and $a_{m+1+j_0} \neq 0$ for some $j_0 \in S\cap\overline{1,m+1}$ if $S\cap\overline{1,m+1} \neq \emptyset$, then the restricted representation $\pi_{c,S}^{\vec{a}}$ of $sp(2m+2,\mathbb{F})$ is irreducible for any $c \in \mathbb{F}$.

With respect to the restricted representation $\pi_{c,S}^{\vec{0}}$, $\mathscr{A}$ is an infinite-dimensional weight $sp(2m+2,\mathbb{F})$-module with finite-dimensional weight subspaces.

In Section 2, we prove Theorem 1. The proof of Theorem 2 is given in Section 3.

2. Proof of Theorem 1

In this section, we will prove Theorem 1 case by case.

Case 1. The representation $\pi^{\vec{0}}_{c,S}$ with $S \neq \emptyset, \overline{1,n}$.

Without loss of generality, we assume $S = \overline{1,n_1}$ for some $n_1 \in \overline{1,n-1}$. Set

$$\tilde{D} = \sum_{r=n_1+1}^{n} x_r\partial_{x_r} - \sum_{i=1}^{n_1} x_i\partial_{x_i}. \tag{2.1}$$

Then the representation $\pi^{\vec{0}}_{c,S}$ of $sl(n+1,\mathbb{F})$ is the representation $\pi_{c,S}$ on $\mathscr{A}$ with

$$\pi_{c,S}(E_{i,j}) = \begin{cases} -x_j\partial_{x_i} - \delta_{i,j} & \text{if } i,j \in \overline{1,n_1}; \\ \partial_{x_i}\partial_{x_j} & \text{if } i \in \overline{1,n_1},\ j \in \overline{n_1+1,n}; \\ -x_ix_j & \text{if } i \in \overline{n_1+1,n},\ j \in \overline{1,n_1}; \\ x_i\partial_{x_j} & \text{if } i,j \in \overline{n_1+1,n}, \end{cases} \tag{2.2}$$

$$\pi_{c,S}(E_{i,n+1}) = \begin{cases} (\tilde{D}+c-n_1-1)\partial_{x_i} & \text{if } i \leq n_1, \\ x_i(\tilde{D}+c-n_1) & \text{if } i > n_1, \end{cases} \tag{2.3}$$

$$\pi_{c,S}(E_{n+1,i}) = \begin{cases} x_i & \text{if } i \leq n_1, \\ -\partial_{x_i} & \text{if } i > n_1, \end{cases} \tag{2.4}$$

$$\pi_{c,S}(E_{n,n} - E_{n+1,n+1}) = \tilde{D} - n_1 + c + x_n\partial_{x_n} \tag{2.5}$$

For any $k \in \mathbb{Z}$, we denote

$$\mathscr{A}_{\langle k\rangle} = \text{Span}\,\{x^\alpha = \prod_{i=1} x_i^{\alpha_i} \mid \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n; \sum_{i=1}^{n_1} \alpha_i - \sum_{r=n_1+1}^{n} \alpha_r = k\}. \tag{2.6}$$

Then $\mathscr{A} = \bigoplus_{k\in\mathbb{Z}} \mathscr{A}_{\langle k\rangle}$ and

$$\mathscr{A}_{\langle k\rangle} = \{f \in \mathscr{A} \mid \tilde{D}(f) = kf\}. \tag{2.7}$$

Note that

$$\mathscr{G}_0 = \sum_{1\leq i<j\leq n} (\mathbb{F}E_{i,j} + \mathbb{F}E_{j,i}) + \sum_{r=1}^{n-1} \mathbb{F}(E_{r,r} - E_{r+1,r+1}) \tag{2.8}$$

is a Lie subalgebra of $sl(n+1,\mathbb{F})$ isomorphic to $sl(n,\mathbb{F})$. The following result was due to Howe [11], where he used the symmetric tensor over several copies of the natural module and its dual to construct the representation. Since the structure constants are integers, the result works for any field with characteristic 0. Denote by λ_i the ith fundamental weight of $sl(n+1,\mathbb{F})$. For convenience, we treat $\lambda_0 = 0$.

Lemma 2.1. *Let $\ell_1, \ell_2 \in \mathbb{N}$ with $\ell_1 > 0$, $\mathscr{A}_{\langle -\ell_1\rangle}$ is an irreducible highest-weight $\mathscr{G}_0$-submodule with highest weight $\ell_1\lambda_{n_1-1} - (\ell_1+1)\lambda_{n_1}$ and $\mathscr{A}_{\langle \ell_2\rangle}$ is an irreducible highest-weight $\mathscr{G}_0$-submodule with highest weight $-(\ell_2+1)\lambda_{n_1} + \ell_2\lambda_{n_1+1}$.*

Proof. Let $f \in \mathscr{A}$ be a singular vector of $\mathscr{G}_0$. Then

$$E_{i,n_1}(f) = -x_{n_1}\partial_{x_i}(f) = 0 \qquad \text{for } i \in \overline{1, n_1 - 1} \tag{2.9}$$

and

$$E_{n_1+1,j}(f) = x_{n_1+1}\partial_{x_j}(f) = 0 \qquad \text{for } j \in \overline{n_1 + 2, n} \tag{2.10}$$

by (2.2). Thus $f = g(x_{n_1}, x_{n_1+1})$ only depends on x_{n_1} and x_{n_1+1}. Moreover,

$$0 = E_{n_1,n_1+1}(f) = \partial_{x_{n_1}}\partial_{x_{n_1+1}}(g) \tag{2.11}$$

again by (2.2). This shows that g is a one-variable function in x_{n_1} or x_{n_1+1}. Since f is a weight vector, $f = x_{n_1}^{m_1}$ or $x_{n_1+1}^{m_2}$ up to a scalar multiple for $m_1 \in \mathbb{N}$ and $0 \neq m_2 \in \mathbb{N}$. Note $x_{n_1}^{m_1} \in \mathscr{A}_{\langle -m_1\rangle}$ and $x_{n_1+1}^{m_2} \in \mathscr{A}_{\langle m_2\rangle}$. So $\mathscr{A}_{\langle k\rangle}$ has a unique singular vector for any $k \in \mathbb{Z}$. Then the irreducibility of $\mathscr{A}_{\langle k\rangle}$ follows from the fact that the singular vector is not isotropic with respect the following invariant symmetric bilinear form determined by

$$(x^\alpha | x^\beta) = \delta_{\alpha,\beta}(-1)^{\sum_{i=1}^{n_1}\alpha_i}\alpha! \qquad \text{for } \alpha, \beta \in \mathbb{N}^n, \tag{2.12}$$

where

$$\alpha! = \prod_{i=1}^{n} \alpha_i! \tag{2.13}$$

(cf. (2.6)). The highest-weights are calculated by (2.2). ∎

Now we have the first result in this section.

Theorem 2.2. *The representation $\pi_{c,S}^{\vec{0}}$ of $sl(n+1,\mathbb{F})$ is irreducible for any $c \in \mathbb{F} \setminus \mathbb{Z}$.*

Proof. Let k be any integer. For any $0 \neq f \in \mathscr{A}_{\langle k\rangle}$, we have

$$0 \neq E_{n+1,1}(f) = x_1 f \in \mathscr{A}_{\langle k-1\rangle} \tag{2.14}$$

by (2.4), and

$$0 \neq E_{n,n+1}(f) = (k + c - n_1)x_n f \in \mathscr{A}_{\langle k+1\rangle} \tag{2.15}$$

by (2.3). Let $\mathscr{M}$ be a nonzero $sl(n+1,\mathbb{F})$-submodule of $\mathscr{A}$. If $k_1, k_2 \in \mathbb{Z}$ with $k_1 \neq k_2$, then the highest weights of $\mathscr{A}_{\langle k_1\rangle}$ and $\mathscr{A}_{\langle k_2\rangle}$ are different as $\mathscr{G}_0$-modules by Lemma 2.1. So $\mathscr{A}_{\langle k_0\rangle} \subset \mathscr{M}$ for some $k_0 \in \mathbb{Z}$. Moreover, (2.14) and (2.15) imply $\mathscr{A}_{\langle k\rangle} \subset \mathscr{M}$ for any $k \in \mathbb{Z}$. Hence $\mathscr{M} = \mathscr{A}$. ∎

Expressions (2.2)-(2.5) imply the above representation is not of highest-weight type. Moreover, $\mathscr{A}$ is a weight $sl(n+1,\mathbb{F})$-module with finite-dimensional weight subspaces.

Case 2. The representation $\pi_{c,\emptyset}^{\vec{a}}$ with $\vec{0} \neq \vec{a} \in \mathbb{F}^n$.

In this case,

$$E_{n+1,i}(fe^{\vec{a}\cdot\vec{x}}) = -(\partial_{x_i} + a_i)(f)e^{\vec{a}\cdot\vec{x}} \qquad \text{for } i \in \overline{1,n},\ f \in \mathscr{A}. \tag{2.16}$$

Thus

$$(E_{n+1,i}+a_i)(fe^{\vec{a}\cdot\vec{x}})=-\partial_{x_i}(f)e^{\vec{a}\cdot\vec{x}}\qquad\text{for }\; i\in\overline{1,n},\; f\in\mathscr{A}.\tag{2.17}$$

The second result in this section.

Theorem 2.3. *The representation $\pi^{\vec{a}}_{c,\emptyset}$ with $\vec{0}\neq\vec{a}\in\mathbb{F}^n$ is an irreducible representation of $sl(n+1,\mathbb{F})$ for any $c\in\mathbb{F}$.*

Proof. Let $\mathscr{A}_k$ be the subspace of homogeneous polynomials with degree k. Set

$$\mathscr{A}_{\vec{a},k}=\mathscr{A}_k e^{\vec{a}\cdot\vec{x}}\qquad\text{for } k\in\mathbb{N}.\tag{2.18}$$

Without loss of generality, we assume $a_1\neq 0$. Let $\mathscr{M}$ be a nonzero $sl(n+1,\mathbb{F})$-submodule of $\mathscr{A}_{\vec{a}}$. Take any $0\neq fe^{\vec{a}\cdot\vec{x}}\in\mathscr{M}$ with $f\in\mathscr{A}$. By (2.17),

$$\partial_{x_i}(f)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\qquad\text{for }\; i\in\overline{1,n}.\tag{2.19}$$

By induction, we have $e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}$; that is, $\mathscr{A}_{\vec{a},0}\subset\mathscr{M}$.

Suppose $\mathscr{A}_{\vec{a},\ell}\subset\mathscr{M}$ for some $\ell\in\mathbb{N}$. For any $ge^{\vec{a}\cdot\vec{x}}\in\mathscr{A}_{\vec{a},\ell}$,

$$E_{i,1}(ge^{\vec{a}\cdot\vec{x}})=x_i(\partial_{x_1}+a_1)(g)e^{\vec{a}\cdot\vec{x}}=a_1x_ige^{\vec{a}\cdot\vec{x}}+x_i\partial_{x_1}(g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\tag{2.20}$$

for $i\in\overline{2,n}$ by (1.4). Since $x_i\partial_{x_1}(g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{A}_{\vec{a},\ell}\subset\mathscr{M}$, we have

$$x_ige^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\qquad\text{for }\; i\in\overline{2,n}.\tag{2.21}$$

On the other hand,

$$(E_{1,1}-E_{2,2})(ge^{\vec{a}\cdot\vec{x}})=a_1x_1ge^{\vec{a}\cdot\vec{x}}+(x_1\partial_{x_1}-x_2\partial_{x_2}-a_2x_2)(g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\tag{2.22}$$

by (1.5). Our assumption says that $(x_1\partial_{x_1}-x_2\partial_{x_2})(g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{A}_{\vec{a},\ell}\subset\mathscr{M}$. According to (2.21), $-a_2x_2(g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}$. Therefore,

$$x_1ge^{\vec{a}\cdot\vec{x}}\in\mathscr{M}.\tag{2.23}$$

Expressions (2.22) and (2.23) imply $\mathscr{A}_{\vec{a},\ell+1}\subset\mathscr{M}$. By induction, $\mathscr{A}_{\vec{a},\ell}\subset\mathscr{M}$ for any $\ell\in\mathbb{N}$. So $\mathscr{A}_{\vec{a}}=\mathscr{M}$. Hence $\mathscr{A}_{\vec{a}}$ is an irreducible $sl(n+1,\mathbb{F})$-module. ∎

Case 3. The representation $\pi^{\vec{a}}_{c,S}$ with $a_i\neq 0$ for some $i\in\overline{1,n}\setminus S$.

The following is the third result in this section.

Theorem 2.4. *Under the above assumption, the representation $\pi^{\vec{a}}_{c,S}$ with $\vec{0}\neq\vec{a}\in\mathbb{F}^n$ is an irreducible representation of $sl(n+1,\mathbb{F})$ for any $c\in\mathbb{F}$.*

Proof. Without loss of generality, we assume $S=\overline{1,n_1}$ for some $n_1\in\overline{1,n-1}$. Let $\mathscr{M}$ be a nonzero $sl(n+1,\mathbb{F})$-submodule of $\mathscr{A}_{\vec{a}}$. By (2.4) and (2.16)-(2.19), there exists $0\neq fe^{\vec{a}\cdot\vec{x}}\in\mathscr{M}$ with $f\in\mathbb{F}[x_1,...,x_{n_1}]$. By symmetry, we can assume $a_n\neq 0$. According to (2.2),

$$E_{i,n}(fe^{\vec{a}\cdot\vec{x}})=(\partial_{x_i}+a_i)(\partial_{x_n}+a_n)(f)e^{\vec{a}\cdot\vec{x}}=a_ia_nfe^{\vec{a}\cdot\vec{x}}+a_n\partial_{x_i}(f)e^{\vec{a}\cdot\vec{x}}\tag{2.24}$$

for $i\in\overline{1,n_1}$. Thus

$$(a_n^{-1}E_{i,n}-a_i)(fe^{\vec{a}\cdot\vec{x}})=\partial_{x_i}(f)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\;\;\text{for }i\in\overline{1,n_1}. \tag{2.25}$$

By induction on the degree of f, we get $e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}$; that is, $\mathscr{A}_{\vec{a},0}\subset\mathscr{M}$.

The arguments in (2.20)-(2.23) yield

$$\mathbb{F}[x_{n_1+1},...,x_n]e^{\vec{a}\cdot\vec{x}}\subset\mathscr{M}. \tag{2.26}$$

According to (2.4),

$$\begin{aligned}&E_{n+1,1}^{\ell_1}\cdots E_{n+1,n_1}^{\ell_{n_1}}(\mathbb{F}[x_{n_1+1},...,x_n]e^{\vec{a}\cdot\vec{x}})\\ =\;& x_1^{\ell_1}\cdots x_{n_1}^{\ell_{n_1}}(\mathbb{F}[x_{n_1+1},...,x_n]e^{\vec{a}\cdot\vec{x}})\subset\mathscr{M}\end{aligned} \tag{2.27}$$

for $\ell_i\in\mathbb{N}$ with $i\in\overline{1,n_1}$. Thus $\mathscr{A}_{\vec{a}}=\mathscr{M}$. So $\mathscr{A}_{\vec{a}}$ is $\mathscr{A}_{\vec{a}}$ is an irreducible $sl(n+1,\mathbb{F})$-module. ■

Case 4. $|S|>1$, $\vec{a}\neq 0$ *and* $a_i=0$ *for any* $i\in\overline{1,n}\setminus S$.

The following is the fourth result in this section.

Theorem 2.5. *Under the above assumption, the representation* $\pi_{c,S}^{\vec{a}}$ *is an irreducible representation of* $sl(n+1,\mathbb{F})$ *for any* $\kappa\in\mathbb{F}\setminus\mathbb{Z}$.

Proof. Without loss of generality, we assume $S=\overline{1,n_1}$ for some $n_1\in\overline{2,n-1}$. Applying the transformation

$$\vec{x}\mapsto T\vec{x},\;\;A\mapsto TAT^{-1},\;\;E_{n,n}-E_{n+1,n+1}\mapsto E_{n,n}-E_{n+1,n+1}. \tag{2.28}$$

$$(E_{1,n+1},...,E_{n,n+1})\mapsto(E_{1,n+1},...,E_{n,n+1})T^{-1}, \tag{2.29}$$

$$(E_{n+1,1},...,E_{n+1,n})\mapsto(E_{n+1,n},...,E_{n+1,n})T^{-1} \tag{2.30}$$

with $A\in sl(n,\mathbb{F})$ for some $n\times n$ orthogonal matrix T, we can assume $a_1\neq 0$ and $a_i=0$ for $i\in\overline{2,n}$.

Let $\mathscr{M}$ be a nonzero $sl(n+1,\mathbb{F})$-submodule of $\mathscr{A}_{\vec{a}}$. Take any $0\neq fe^{\vec{a}\cdot\vec{x}}\in\mathscr{M}$. Note that

$$E_{1,2}(fe^{\vec{a}\cdot\vec{x}})=-x_2(\partial_{x_1}+a_1)(f)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M} \tag{2.31}$$

by (2.2) and

$$E_{n+1,2}(fe^{\vec{a}\cdot\vec{x}})=x_2fe^{\vec{a}\cdot\vec{x}}\in\mathscr{M} \tag{2.32}$$

by (2.4). Thus

$$x_2\partial_{x_1}(f)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}. \tag{2.33}$$

Repeatedly applying (2.33) if necessary, we can assume $f\in\mathbb{F}[x_2,....,x_n]$. We apply the arguments in the proof of Theorem 2.2 to the Lie subalgebra

$$\mathscr{L}=\sum_{2\leq i<j\leq n+1}(\mathbb{F}E_{i,j}+\mathbb{F}E_{j,i})+\sum_{r=2}^n\mathbb{F}(E_{r,r}-E_{r+1,r+1}) \tag{2.34}$$

and obtain

$$\mathbb{F}[x_2, ..., x_n]e^{\vec{a}\cdot\vec{x}} \subset \mathscr{M} \tag{2.35}$$

if $c \notin \mathbb{Z}$. According to (2.4),

$$E_{n+1,1}^{\ell}(\mathbb{F}[x_2, ..., x_n]e^{\vec{a}\cdot\vec{x}}) = x_1^{\ell}(\mathbb{F}[x_2, ..., x_n]e^{\vec{a}\cdot\vec{x}}) \subset \mathscr{M} \tag{2.36}$$

for any $\ell \in \mathbb{N}$. Therefore $\mathscr{M} = \mathscr{A}_{\vec{a}}$. So $\mathscr{A}_{\vec{a}}$ is an irreducible $sl(n+1, \mathbb{F})$-module. ■

With the representation $\pi_{c,S}^{\vec{0}}$, $\mathscr{A}$ is an infinite-dimensional weight $sl(n+1, \mathbb{F})$-module with finite-dimensional weight subspaces by (2.2) and (2.5). Now Theorem 1 follows from Theorems 2.2-2.5.

3. Proof of Theorem 2

Assume that $n = 2m + 1 > 1$ is an odd integer. In this section, we will give the proof of Theorem 2.

Recall that the symplectic Lie algebras

$$\begin{aligned} sp(2m+2, \mathbb{F}) &= \sum_{1\leq r\leq s\leq m+1} [\mathbb{F}(E_{r,m+1+s} + E_{s,m+1+r}) + \mathbb{F}(E_{m+1+r,s} + E_{m+1+s,r})] \\ &\quad + \sum_{i,j=1}^{m+1} \mathbb{F}(E_{i,j} - E_{m+1+j,m+1+i}). \end{aligned} \tag{3.1}$$

For convenience, we rednote

$$x_0 = x_{m+1}, \quad y_i = x_{m+1+i} \qquad \text{for } i \in \overline{1, m}. \tag{3.2}$$

In particular,

$$D = \sum_{i=0}^{m} x_i \partial_{x_i} + \sum_{r=1}^{m} y_r \partial_{y_r}. \tag{3.3}$$

According to (1.4) and (1.5), we have the representation π_c of $sp(2m+2, \mathbb{F})$:

$$\pi_c(E_{i,j} - E_{m+1+j,m+1+i}) = x_i \partial_{x_j} - y_j \partial_{y_i}, \tag{3.4}$$

$$\pi_c(E_{i,m+1+j} + E_{j,m+1+i}) = x_i \partial_{y_j} + x_j \partial_{y_i}, \tag{3.5}$$

$$\pi_c(E_{2m+2,m+1}) = -\partial_{x_0},\ \pi_c(E_{m+1,i} - E_{m+1+i,2m+2}) = x_0 \partial_{x_i} - y_i(D + c), \tag{3.6}$$

$$\pi_c(E_{i,m+1} - E_{2m+2,m+1+i}) = x_i \partial_{x_0} + \partial_{y_i}, \tag{3.7}$$

$$\pi_c(E_{2m+2,i} + E_{m+1+i,m+1}) = y_i \partial_{x_0} - \partial_{x_i}, \tag{3.8}$$

$$\pi_c(E_{m+1+i,j} + E_{m+1+j,i}) = y_i \partial_{x_j} + y_j \partial_{x_i}, \tag{3.9}$$

$$\pi_c(E_{m+1,m+1} - E_{2m+2,2m+2}) = D + x_0 \partial_{x_0} + c, \tag{3.10}$$

$$\pi_c(E_{m+1,m+1+i} + E_{i,2m+2}) = x_0 \partial_{y_i} + x_i(D + c),\ \ \pi_c(E_{m+1,2m+2}) = x_0(D + c) \tag{3.11}$$

for $i, j \in \overline{1, m}$.

Denote

$$\begin{aligned}\mathscr{K} &= \sum_{1\leq r\leq s\leq m}[\mathbb{F}(E_{r,m+1+s}+E_{s,m+1+r})+\mathbb{F}(E_{m+1+r,s}+E_{m+1+s,r})]\\ &\quad+\sum_{i,j=1}^{m}\mathbb{F}(E_{i,j}-E_{m+1+j,m+1+i}),\end{aligned}\tag{3.12}$$

which is a Lie subalgebra of $sp(2m+2,\mathbb{F})$ isomorphic to $sp(2m,\mathbb{F})$. We will prove Theorem 2 case by case.

Case 1. $\vec{a}=\vec{0}$ *and* $S=\emptyset$.

Let $\mathscr{B}=\mathbb{F}[x_1,...,x_m,y_1,...,y_m]$. Denote by $\mathscr{B}_k$ the subspace of homogeneous polynomials with degree k. First we have the following well-known result (e.g., cf. [7]).

Lemma 3.1. *For any* $k\in\mathbb{N}$, $\mathscr{B}_k$ *forms a finite-dimensional irreducible* $\mathscr{K}$*-module with highest weight* $k\lambda_1$.

Denote by $\mathscr{A}_i$ the subspace of homogeneous polynomials in $\mathscr{A}$ with degree i. Set

$$\mathscr{A}_{(\ell)}=\sum_{i=0}^{\ell}\mathscr{A}_i\qquad\text{for }\ \ell\in\mathbb{N}.\tag{3.13}$$

Take the Cartan subalgebra

$$H=\sum_{i=1}^{m+1}\mathbb{F}(E_{i,i}-E_{m+1+i,m+1+i})\tag{3.14}$$

of $sp(2m+2,\mathbb{F})$. Define $\{\varepsilon_1,...,\varepsilon_{m+1}\}\subset H^*$ by:

$$\varepsilon_j(E_{i,i}-E_{m+1+i,m+1+i})=\delta_{i,j}.\tag{3.15}$$

Recall that the representation $\pi_{c,\emptyset}^{\vec{0}}$ of $sp(2m+2,\mathbb{F})$ is the representation π_c (cf. (3.4)-(3.8)) on the space $\mathscr{A}=\mathbb{F}[x_0,...,x_m,y_1,...,y_m]$. Then we have:

Theorem 3.2. *If* $c\not\in-\mathbb{N}$, *the representation* $\pi_{c,\emptyset}^{\vec{0}}$ *of* $sp(2m+2,\mathbb{F})$ *given in (3.4)-(3.8) is a highest-weight irreducible representation with highest weight* $-c\lambda_1$. *When* $-c=\ell\in\mathbb{N}$, $\mathscr{A}_{(\ell)}$ *is a finite-dimensional irreducible* $sp(2m+2,\mathbb{F})$*-module with highest weight* $\ell\lambda_n$ *and* $\mathscr{A}/\mathscr{A}_{(\ell)}$ *is an irreducible highest weight* $sp(2m+2,\mathbb{F})$*-module with highest weight* $-(\ell+2)\lambda_1+(\ell+1)\lambda_2$, *where* λ_i *is the* i*th fundamental weight of* $sp(2m+2,\mathbb{F})$.

Proof. Observe that

$$\mathscr{A}_k=\sum_{s=0}^{k}x_0^s\mathcal{B}_{k-s}\qquad\text{for }\ k\in\mathbb{N}.\tag{3.16}$$

Let $\mathscr{M}$ be a nonzero $sp(2m+2,\mathbb{F})$-submodule of $\mathscr{A}$. Take any $0 \neq f \in \mathscr{M}$. Repeatedly applying the first equation in (3.6), (3.7) and (3.8) to f, we obtain $1 \in \mathscr{M}$. Note

$$(E_{m+1,2m+2})^k(1) = [\prod_{r=0}^{k-1}(r+c)]x_0^k \in \mathscr{M} \tag{3.17}$$

by (3.11) and

$$(E_{1,m+1} - E_{2m+2,m+1+1})^s(x_0^k) = [\prod_{i=0}^{s-1}(k-i)x_0^{k-s}x_1^s \qquad \text{for } s \in \overline{1,k} \tag{3.18}$$

by (3.7). Suppose $c \notin -\mathbb{N}$. Then (3.17) yields

$$x_0^k \in \mathscr{M} \qquad \text{for } k \in \mathbb{N}. \tag{3.19}$$

Moreover, (3.18) with $k = r+s$ gives

$$x_0^r x_1^s \in V \qquad \text{for } r,s \in \mathbb{N}. \tag{3.20}$$

Furthermore,

$$U(\mathcal{K})(x_0^r x_1^s) = x_0^r \mathscr{B}_s \subset \mathscr{M} \tag{3.21}$$

by Lemma 3.1. Thus

$$\mathscr{A} = \sum_{r,s=0}^{\infty} x_0^r \mathscr{B}_s \subset \mathscr{M}; \tag{3.22}$$

that is, $\mathscr{M} = \mathscr{A}$. So $\mathscr{A}$ is an irreducible $sp(2m+2,\mathbb{F})$-module and 1 is its highest-weight vector with weight $-c\lambda_1$ with respect to the following simple positive roots

$$\{\varepsilon_{n+1} - \varepsilon_n, \varepsilon_n - \varepsilon_{n-1}, ..., \varepsilon_2 - \varepsilon_1, 2\varepsilon_1\}. \tag{3.23}$$

Next we assume $c = -\ell$ with $\ell \in \mathbb{N}$. Since

$$E_{m+1,2m+2}|_{\mathscr{A}_\ell} = 0, \;\; (E_{m+1,i} - E_{m+1+i,2m+2})|_{\mathscr{A}_\ell} = x_0\partial_{x_i}, \tag{3.24}$$

$$(E_{m+1,m+1+i} + E_{i,2m+2})|_{\mathscr{A}_\ell} = x_0\partial_{y_i} \tag{3.25}$$

by (3.6), (3.11) and (3.12), $\mathscr{A}_{(\ell)}$ is a finite-dimensional $sp(2m+2,\mathbb{F})$-module. Let $\mathscr{M}$ be a nonzero $sp(2m+2,\mathbb{F})$-submodule of $\mathscr{A}_{(\ell)}$. By (3.17),

$$x_0^k \in \mathscr{M} \qquad \text{for } k \in \overline{0,\ell}. \tag{3.26}$$

Moreover, (3.19) with $k = r+s$ gives

$$x_0^r x_1^s \in \mathscr{M} \qquad \text{for } r,s \in \overline{0,\ell} \text{ such that } r+s \leq \ell. \tag{3.27}$$

Thus

$$\mathscr{A}_{(\ell)} = \sum_{r=0}^{\ell}\sum_{s=0}^{\ell-r} x_0^r \mathcal{B}_s \subset \mathscr{M} \tag{3.28}$$

by Lemma 3.1; that is, $\mathscr{M} = \mathscr{A}_{(\ell)}$. So $\mathscr{A}_{(\ell)}$ is an irreducible $sp(2m+2,\mathbb{F})$-module and 1 is again its highest-weight vector.

Consider the quotient $sp(2m+2,\mathbb{F})$-module $\mathscr{A}/\mathscr{A}_{(\ell)}$. Let $W \supset \mathscr{A}_{(\ell)}$ be an $sp(2m+2,\mathbb{F})$-submodule of $\mathscr{A}$ such that $W \neq \mathscr{A}_{(\ell)}$. Take any $f \in W \setminus \mathscr{A}_{(\ell)}$. Repeatedly applying (3.7), (3.8) and the first equation in (3.6) to f if necessary, we can assume $f \in \mathcal{B}_{\ell+1}$. Since $\mathcal{B}_{\ell+1}$ is an irreducible $\mathscr{K}$-module, we have

$$\mathcal{B}_{\ell+1} \subset W. \tag{3.29}$$

In particular, $x_1^{\ell+1} \in W$. According to (3.11),

$$(E_{m+1,m+2} + E_{1,2m+2})^r(x_1^{\ell+1}) = r!x_1^{\ell+1+r} \in W \qquad \text{for } 0 < r \in \mathbb{Z}. \tag{3.30}$$

Since $\mathcal{B}_{\ell+1+r} \ni x_1^{\ell+1+r}$ is an irreducible $\mathscr{K}$-module, we have

$$\mathcal{B}_{\ell+1+r} \subset W. \tag{3.31}$$

Suppose that

$$x_0^r\mathcal{B}_s \subset W \qquad \text{for } r \in \overline{0,k} \text{ and } s \in \mathbb{N} \text{ such that } r+s \geq \ell+1. \tag{3.32}$$

Fix such r and s. Observe $x_0^r x_1^{s-1} y_1 \in x_0^r\mathcal{B}_s \subset W$. Using the first equation in (3.11), we get

$$(E_{m+1,m+2} + E_{1,2m+2})(x_0^r x_1^{s-1} y_1) = (r+s-\ell)x_0^r x_1^s y_1 + x_0^{r+1}x_1^{s-1} \in W. \tag{3.33}$$

By the assumption (3.32), $(r+s-\ell)x_0^r x_1^s y_1 \in x_0^r\mathcal{B}_{s+1} \subset W$. So

$$x_0^{r+1}x_1^{s-1} \in W \cap x_0^{r+1}\mathcal{B}_{s-1}. \tag{3.34}$$

Since $x_0^{r+1}\mathcal{B}_{s-1}$ is an irreducible $\mathscr{K}$-module, we get

$$x_0^{r+1}\mathcal{B}_{s-1} \subset W. \tag{3.35}$$

By induction on r, we prove

$$x_0^r\mathcal{B}_s \subset W \qquad \text{for } r, s \in \mathbb{N} \text{ such that } r+s \geq \ell+1. \tag{3.36}$$

According to (3.16),

$$\sum_{k=\ell+1}^{\infty} \mathscr{A}_k \subset W. \tag{3.37}$$

Since $W \supset \mathscr{A}_{(\ell)}$, we have $W = \mathscr{A}$. So $\mathscr{A}/\mathscr{A}_{(\ell)}$ is an irreducible $sp(2m+2,\mathbb{F})$-module. Moreover, $x_n^{\ell+1}$ is a highest weight vector of weight $-(\ell+2)\lambda_1 + (\ell+1)\lambda_2$ with respect to (3.23). ■

Case 2. $\vec{a} \neq \vec{0}$, $a_{m+1} = 0$ *and* $S = \emptyset$.

For simplicity, we redenote

$$b_i = a_{m+1+i} \qquad \text{for } i \in \overline{1,m}. \tag{3.38}$$

Recall that the representation $\pi_{c,\emptyset}^{\vec{a}}$ of $sp(2m+2,\mathbb{F})$ is the representation π_c (cf. (3.4)-(3.12)) on the space $\mathscr{A}_{\vec{a}}$ (cf. (1.7)). Our second result in this section is:

Theorem 3.3. *The representation $\pi_{c,\emptyset}^{\vec{a}}$ with $\vec{0}\neq\vec{a}\in\mathbb{F}^n$ and $a_{m+1}=0$ is an irreducible representation of $sp(2m+2,\mathbb{F})$ for any $c\in\mathbb{F}$.*

Proof. . By symmetry, we may assume $a_1\neq 0$. Let $\mathscr{M}$ be a nonzero $sp(2m+2,\mathbb{F})$-submodule of $\mathscr{A}_{\vec{a}}$. Take any $0\neq fe^{\vec{a}\cdot\vec{x}}\in\mathscr{M}$ with $f\in\mathscr{A}$. By the assumption $a_0=0$ and (3.6)-(3.8),

$$E_{2m+2,m+1}(fe^{\vec{a}\cdot\vec{x}})=-\partial_{x_0}(f)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M},\tag{3.39}$$

$$(E_{i,m+1}-E_{2m+2,m+1+i}-b_i)(fe^{\vec{a}\cdot\vec{x}})=[\partial_{y_i}(f)+x_i\partial_{x_0}(f)]e^{\vec{a}\cdot\vec{x}}\in\mathscr{M},\tag{3.40}$$

$$(E_{2m+2,i}+E_{m+1+i,m+1}+a_i)(fe^{\vec{a}\cdot\vec{x}})=[-\partial_{x_i}(f)+y_i\partial_{x_0}(f)]e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\tag{3.41}$$

for $i\in\overline{1,m}$. Repeatedly applying (3.39)-(3.41), we obtain $e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}$. Equivalently, $\mathscr{A}_{\vec{a},0}\subset\mathscr{M}$ (cf. (2.18)).

Suppose $\mathscr{A}_{\vec{a},\ell}\subset\mathscr{M}$ for some $\ell\in\mathbb{N}$. For any $ge^{\vec{a}\cdot\vec{x}}\in\mathscr{A}_{\vec{a},\ell}$,

$$(E_{i,1}-E_{m+2,m+1+i})(ge^{\vec{a}\cdot\vec{x}})=[a_1x_i-b_iy_1+x_i\partial_{x_1}-y_1\partial_{x_i}](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\tag{3.42}$$

by (3.4) and

$$(E_{m+1+i,1}+E_{m+2,i})(ge^{\vec{a}\cdot\vec{x}})=[a_1y_i+a_iy_1+y_i\partial_{x_1}+y_1\partial_{x_i}](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\tag{3.43}$$

by (3.9), where $i\in\overline{1,m}$. Since

$$(x_i\partial_{x_1}-y_1\partial_{x_i})(g)e^{\vec{a}\cdot\vec{x}},\ (y_i\partial_{x_1}+y_1\partial_{x_i})(g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{A}_{\vec{a},\ell}\subset\mathscr{M},\tag{3.44}$$

we have

$$(a_1x_i-b_iy_1)ge^{\vec{a}\cdot\vec{x}},\ (a_1y_i+a_iy_1)ge^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\tag{3.45}$$

for $i\in\overline{1,m}$. The above second equation with $i=1$ gives

$$2a_1y_1ge^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\Rightarrow y_1ge^{\vec{a}\cdot\vec{x}}\in\mathscr{M}.\tag{3.46}$$

Thus (3.45) yields

$$x_ige^{\vec{a}\cdot\vec{x}},y_ige^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\qquad\text{for } i\in\overline{1,m}.\tag{3.47}$$

According to the second equation in (3.6),

$$\begin{aligned}&(E_{m+1,1}-E_{m+2,2m+2})(ge^{\vec{a}\cdot\vec{x}})\\=\;&[a_1x_0-\sum_{i=1}^m(a_ix_i+b_iy_i)y_1+x_0\partial_{x_1}-y_1(D+c)](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}.\end{aligned}\tag{3.48}$$

Replacing $ge^{\vec{a}\cdot\vec{x}}\in\mathscr{A}_{\vec{a},\ell}$ by $ge^{\vec{a}\cdot\vec{x}}\in\sum_{i=1}^m(x_i\mathscr{A}_{\vec{a},\ell}+y_i\mathscr{A}_{\vec{a},\ell})$ in (3.42)-(3.47), we obtain

$$x_iy_1ge^{\vec{a}\cdot\vec{x}},y_iy_1ge^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\qquad\text{for } i\in\overline{1,m}.\tag{3.49}$$

Since $D(g)=\ell g$ and $x_0\partial_{x_1}(g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{A}_{\vec{a},\ell}\subset\mathscr{M}$, we have

$$[-\sum_{i=1}^m(a_ix_i+b_iy_i)y_1+x_0\partial_{x_1}-y_1(D+c)](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}.\tag{3.50}$$

Hence (3.48) yields $x_0ge^{\vec{a}\cdot\vec{x}}\in\mathscr{M}$. Therefore, $\mathscr{A}_{\vec{a},\ell+1}\subset\mathscr{M}$. By induction, $\mathscr{A}_{\vec{a},\ell}\subset\mathscr{M}$ for any $\ell\in\mathbb{N}$. So $\mathscr{A}_{\vec{a}}=\mathscr{M}$. Hence $\mathscr{A}_{\vec{a}}$ is an irreducible $sp(2m+2,\mathbb{F})$-module. ∎

Case 3. $\vec{a}=\vec{0}$ *and* $S\neq\emptyset$.

By symmetry and the assumption (1.8), we can assume

$$S=\overline{1,m_1}\cup\overline{m_2+1,m},\qquad m_1,m_2\in\overline{1,m}\text{ and }m_1\leq m_2, \tag{3.51}$$

where we treat $\overline{m+1,m}=\emptyset$ when $m_2=m$. Set

$$\tilde{D}=x_0\partial_{x_0}+\sum_{r=m_1+1}^{m}x_r\partial_{x_r}-\sum_{i=1}^{m_1}x_i\partial_{x_i}+\sum_{i=1}^{m_2}y_i\partial_{y_i}-\sum_{r=m_2+1}^{m}y_r\partial_{y_r} \tag{3.52}$$

and

$$\tilde{c}=c+m_2-m_1-m. \tag{3.53}$$

Then we have the following representation $\pi_{c,S}$ of the Lie algebra $sp(2m+2,\mathbb{F})$ determined by

$$\pi_{c,S}(E_{i,j}-E_{m+1+j,m+1+i})=E^x_{i,j}-E^y_{j,i} \tag{3.54}$$

with

$$E^x_{i,j}=\begin{cases}-x_j\partial_{x_i}-\delta_{i,j} & \text{if } i,j\in\overline{1,m_1};\\ \partial_{x_i}\partial_{x_j} & \text{if } i\in\overline{1,m_1},\ j\in\overline{m_1+1,m},\\ -x_ix_j & \text{if } i\in\overline{m_1+1,m},\ j\in\overline{1,m_1},\\ x_i\partial_{x_j} & \text{if } i,j\in\overline{m_1+1,m}\end{cases} \tag{3.55}$$

and

$$E^y_{i,j}=\begin{cases}y_i\partial_{y_j} & \text{if } i,j\in\overline{1,m_2};\\ -y_iy_j & \text{if } i\in\overline{1,m_2},\ j\in\overline{m_2+1,m},\\ \partial_{y_i}\partial_{y_j} & \text{if } i\in\overline{m_2+1,m},\ j\in\overline{1,m_2},\\ -y_j\partial_{y_i}-\delta_{i,j} & \text{if } i,j\in\overline{m_2+1,m},\end{cases} \tag{3.56}$$

and

$$\pi_{c,S}(E_{i,m+1+j})=\begin{cases}\partial_{x_i}\partial_{y_j} & \text{if } i\in\overline{1,m_1},\ j\in\overline{1,m_2},\\ -y_j\partial_{x_i} & \text{if } i\in\overline{1,m_1},\ j\in\overline{m_2+1,m},\\ x_i\partial_{y_j} & \text{if } i\in\overline{m_1+1,m},\ j\in\overline{1,m_2},\\ -x_iy_j & \text{if } i\in\overline{m_1+1,m},\ j\in\overline{m_2+1,m},\end{cases} \tag{3.57}$$

$$\pi_{c,S}(E_{m+1+i,j})=\begin{cases}-x_jy_i & \text{if } j\in\overline{1,m_1},\ i\in\overline{1,m_2},\\ -x_j\partial_{y_i} & \text{if } j\in\overline{1,m_1},\ i\in\overline{m_2+1,m},\\ y_i\partial_{x_j} & \text{if } j\in\overline{m_1+1,m},\ i\in\overline{1,m_2},\\ \partial_{x_j}\partial_{y_i} & \text{if } j\in\overline{m_1+1,m},\ i\in\overline{m_2+1,m},\end{cases} \tag{3.58}$$

$$\pi_{c,S}(E_{2m+2,m+1})=-\partial_{x_0},\quad \pi_{c,S}(E_{m+1,2m+2})=x_0(\tilde{D}+\tilde{c}), \tag{3.59}$$

$$\pi_{c,S}(E_{i,m+1}-E_{2m+2,m+1+i})=\begin{cases}\partial_{x_0}\partial_{x_i}+\partial_{y_i} & \text{if } i\in\overline{1,m_1},\\ x_i\partial_{x_0}+\partial_{y_i} & \text{if } i\in\overline{m_1+1,m_2},\\ x_i\partial_{x_0}-y_i & \text{if } i\in\overline{m_2+1,m},\end{cases} \tag{3.60}$$

$$\pi_{c,S}(E_{2m+2,i}+E_{m+1+i,m+1})=\begin{cases}y_i\partial_{x_0}+x_i & \text{if } i\in\overline{1,m_1},\\ y_i\partial_{x_0}-\partial_{x_i} & \text{if } i\in\overline{m_1+1,m_2},\\ \partial_{x_0}\partial_{y_i}-\partial_{x_i} & \text{if } i\in\overline{m_2+1,m},\end{cases} \tag{3.61}$$

$$\pi_{c,S}(E_{m+1,i} - E_{m+1+i,2m+2})$$
$$= \begin{cases} -x_0x_i - y_i(\tilde{D} + \tilde{c}) & \text{if } i \in \overline{1, m_1}, \\ x_0\partial_{x_i} - y_i(\tilde{D} + \tilde{c}) & \text{if } i \in \overline{m_1 + 1, m_2}, \\ x_0\partial_{x_i} - (\tilde{D} + \tilde{c} - 1)\partial_{y_i} & \text{if } i \in \overline{m_2 + 1, m}, \end{cases} \tag{3.62}$$

$$\pi_{c,S}(E_{m+1,m+1+i} + E_{i,2m+2})$$
$$= \begin{cases} x_0\partial_{y_i} + (\tilde{D} + \tilde{c} - 1)\partial_{x_i} & \text{if } i \in \overline{1, m_1}, \\ x_0\partial_{y_i} + x_i(\tilde{D} + \tilde{c}) & \text{if } i \in \overline{m_1 + 1, m_2}, \\ -x_0y_i + x_i(\tilde{D} + \tilde{c}) & \text{if } i \in \overline{m_2 + 1, m}, \end{cases} \tag{3.63}$$

$$\pi_{c,S}(E_{m+1,m+1} - E_{2m+2,2m+2}) = \tilde{D} + x_0\partial_{x_0} + \tilde{c}, \tag{3.64}$$

for $i, j \in \overline{1, m}$.

Recall $\mathscr{B} = \mathbb{F}[x_1, ..., x_m, y_1, ..., y_m]$. Set

$$\mathscr{B}_{\langle k\rangle} = \mathscr{A}_{\langle k\rangle} \cap \mathscr{B} \qquad \text{for } k \in \mathbb{Z} \tag{3.65}$$

(cf. (2.6)). Then $\mathscr{B} = \bigoplus_{k\in\mathbb{Z}} \mathscr{B}_{\langle k\rangle}$ is a $\mathbb{Z}$-graded space. The following result is due to [10]:

Lemma 3.4. *Assume $m \geq 2$. Let $k \in \mathbb{Z}$. If $m_1 < m_2$ or $k \neq 0$, the subspace $\mathscr{B}_{\langle k\rangle}$ is an irreducible $\mathscr{K}$-submodule (cf. (3.12)). When $m_1 = m_2$, the subspace $\mathscr{B}_{\langle 0\rangle}$ is a direct sum of two irreducible $\mathscr{K}$-submodules.*

In fact, any pair of the irreducible submodules in the above are not isomorphic $\mathscr{K}$-modules because they have distinct weight sets of singular vectors with respect to the Lie subalgebra $\sum_{i,j=1}^m \mathbb{F}(E_{i,j} - E_{m+1+j,m+1+i}) \cong gl(m, \mathbb{F})$ (cf. [9]). When $m = m_1 = m_2 = 1$, $\mathscr{B} = \mathbb{F}[x_1, y_1]$ and

$$\pi_{c,S}(\mathscr{K}) = \mathbb{F}(x_1\partial_{x_1} + y_1\partial_{y_1} + 1) + \mathbb{F}x_1y_1 + \mathbb{F}\partial_{x_1}\partial_{y_1}. \tag{3.66}$$

So all $\mathscr{B}_{\langle k\rangle}$ with $k \in \mathbb{Z}$ are irreducible $\mathscr{K}$-submodules. Recall the representation $\pi_{c,S}^{\vec{0}}$ of $sp(2m + 2, \mathbb{F})$ is the representation $\pi_{c,S}$ (cf. (3.54)-(3.64)) on $\mathscr{A}$. The following is the third result in this section.

Theorem 3.5. *The representation $\pi_{c,S}^{\vec{0}}$ of $sp(2m+2, \mathbb{F})$ is irreducible if $c \notin \mathbb{Z}$.*

Proof. Let $\mathscr{M}$ be any nonzero $sp(2m + 2, \mathbb{F})$-submodule of $\mathscr{A}$. Repeatedly applying $E_{2m+2,m+1}$ to $\mathscr{M}$ by the first equation in (3.59), we get

$$\mathscr{M} \cap \mathscr{B} \neq \{0\}. \tag{3.67}$$

According to (3.64),

$$\mathscr{B}_{\langle k\rangle} = \{f \in \mathscr{B} \mid (E_{m+1,m+1} - E_{2m+2,2m+2})(f) = (k + \tilde{c})f\}. \tag{3.68}$$

Thus

$$\mathscr{M} = \bigoplus_{k\in\mathbb{Z}} \mathscr{M} \cap \mathscr{B}_{\langle k\rangle}. \tag{3.69}$$

If $\mathscr{M}\cap\mathscr{B}_{\langle 0\rangle}\neq\{0\}$, then (3.61) gives

$$(E_{2m+2,1}+E_{m+2,m+1})(\mathscr{M}\cap\mathscr{B}_{\langle 0\rangle})=x_1(\mathscr{M}\cap\mathscr{B}_{\langle 0\rangle})\subset\mathscr{M}\cap\mathscr{B}_{\langle -1\rangle}. \tag{3.70}$$

Thus we always have $\mathscr{M}\cap\mathscr{B}_{\langle k\rangle}\neq\{0\}$ for some $0\neq k\in\mathbb{Z}$. According to Lemma 3.4 and (3.66), $\mathscr{B}_{\langle k\rangle}$ is an irreducible $\mathscr{K}$-module. So

$$\mathscr{B}_{\langle k\rangle}\subset\mathscr{M}. \tag{3.71}$$

Next (3.60) yields

$$\mathscr{B}_{\langle k-r\rangle}=(\partial_{y_1})^r(\mathscr{B}_{\langle k\rangle})=(E_{1,m+1}-E_{2m+2,m+2})^r(\mathscr{B}_{\langle k\rangle})\subset\mathscr{M}\qquad\text{for } r\in\mathbb{N}. \tag{3.72}$$

On the other hand, if $\mathscr{B}_{\langle \ell\rangle}\subset\mathscr{M}$, then the assumption $c\not\in\mathbb{Z}$ and the second equation in (3.59) give

$$x_0^r\mathscr{B}_{\langle \ell\rangle}=(E_{2m+2,m+1})^r(\mathscr{B}_{\langle \ell\rangle})\subset\mathscr{M}\qquad\text{for } r\in\mathbb{N}. \tag{3.73}$$

Suppose that for some $s\in\mathbb{Z}$,

$$x_0^r\mathscr{B}_{\langle s\rangle},x_0^r\mathscr{B}_{\langle s-1\rangle}\subset\mathscr{M}\qquad\text{for } r\in\mathbb{N}. \tag{3.74}$$

For any $\ell\in\mathbb{N}$,

$$x_0^\ell\mathscr{B}_{\langle s+1\rangle}=(\tilde D+\tilde c-1)\partial_{x_1}(x_0^\ell\mathscr{B}_{\langle s\rangle})=[E_{m+1,m+2}+E_{1,2m+2}-x_0\partial_{y_1}](x_0^\ell\mathscr{B}_{\langle s\rangle}) \tag{3.75}$$

by (3.63). Note

$$x_0\partial_{y_1}(x_0^\ell\mathscr{B}_{\langle s\rangle})=x_0^{\ell+1}\mathscr{B}_{\langle s-1\rangle}\subset\mathscr{M}. \tag{3.76}$$

Thus (3.75) leads to

$$x_0^\ell\mathscr{B}_{\langle s+1\rangle}\subset\mathscr{M}. \tag{3.77}$$

By (3.72)-(3.77) and induction on s, we prove

$$x_0^r\mathscr{B}_{\langle k\rangle}\subset\mathscr{M}\qquad\text{for } x_0\in\mathbb{N},\ k\in\mathbb{Z}. \tag{3.78}$$

So $\mathscr{M}=\mathscr{A}$. Therefore, $\mathscr{A}$ is an irreducible $sp(2m+2,\mathbb{F})$-module. ■

Remark 3.6. The above irreducible representation depends on the three parameters $c\in\mathbb{F}$ and $m_1,m_2\in\overline{1,n}$. It is not highest-weight type because of the mixture of multiplication operators and differential operators in (3.57), (3.58) and (3.60)-(3.63). Since $\mathscr{B}$ is not completely reducible as a module of the Lie subalgebra $\sum_{i,j=1}^m\mathbb{F}(E_{i,j}-E_{m+1+j,m+1+i})$ by [9] when $m\geq 2$ and $m_1<m$, $\mathscr{A}$ is not a unitary $sp(2m+2,\mathbb{F})$-module. The constraints on the degrees of the monomials with a fixed weight via the operators $\pi_{c,S}(E_{i,i}-E_{m+1+i,m+1+i})$ in (3.54)-(3.56) with $i\in\overline{1,m}$ and $\pi_{c,S}(E_{m+1,m+1}-E_{2m+2,2m+2})$ in (3.64) show that the weight subspaces are finite-dimensional. Thus $\mathscr{A}$ is a weight $sp(2m+2,\mathbb{F})$-module with finite-dimensional weight subspaces.

Case 4. $S \neq \emptyset$, $\vec{a} \neq 0$, $a_{i_0} \neq 0$ *for some* $m+1+i_0 \in S \cap \overline{m+2,2m+1}$ *if* $S \cap \overline{m+2,2m+1} \neq \emptyset$, *and* $a_{m+1+j_0} \neq 0$ *for some* $j_0 \in S \cap \overline{1,m+1}$ *if* $S \cap \overline{1,m+1} \neq \emptyset$.

We take (3.51)-(3.64). By the above assumption, $b_{j_0} \neq 0$ for some $j_0 \in \overline{1,m_1}$, and $a_{i_0} \neq 0$ for some $i_0 \in \overline{m_2+1,m}$ if $m_2 < m$. Recall the representation $\pi^{\vec{a}}_{c,S}$ of $sp(2m+2,\mathbb{F})$ is the representation $\pi_{c,S}$ (cf. (3.54)-(3.64)) on $\mathscr{A}_{\vec{a}}$ (cf. (1.7)). Under the assumption, we have the following fourth result in this section:

Theorem 3.7. *The representation* $\pi^{\vec{a}}_{c,S}$ *of* $sp(2m+2,\mathbb{F})$ *is irreducible for any* $c \in \mathbb{F}$.

Proof. Let $\mathscr{M}$ be a nonzero $sp(2m+2,\mathbb{F})$-submodule of $\mathscr{A}_{\vec{a}}$. Take any $0 \neq fe^{\vec{a}\cdot\vec{x}} \in \mathscr{M}$ with $f \in \mathscr{A}$. By the assumption, $a_0 = 0$. Repeatedly applying the first equation in (3.59) to $fe^{\vec{a}\cdot\vec{x}}$ if necessary, we may assume $f \in \mathscr{B} = \mathbb{F}[x_1,...,x_m,y_1,...,y_m]$. Then (3.61) yields

$$(E_{2m+2,i}+E_{m+1+i,m+1}+a_i)(fe^{\vec{a}\cdot\vec{x}}) = -\partial_{x_i}(f)e^{\vec{a}\cdot\vec{x}} \in \mathscr{M} \quad \text{for } i \in \overline{m_1+1,m}. \tag{3.79}$$

Moreover, (3.60) yields

$$(E_{i,m+1}-E_{2m+2,m+1+i}-b_j)(fe^{\vec{a}\cdot\vec{x}}) = \partial_{y_j}(f)e^{\vec{a}\cdot\vec{x}} \in \mathscr{M} \quad \text{for } j \in \overline{1,m_2}. \tag{3.80}$$

Repeatedly applying (3.79) and (3.80) if necessary, we can assume

$$f \in \mathbb{F}[x_1,...,x_{m_1},y_{m_2+1},...,y_m]. \tag{3.81}$$

According to (3.57),

$$(E_{i,m+1+j_0}+E_{j_0,m+1+i}-a_{j_0}b_i-a_ib_{j_0})(fe^{\vec{a}\cdot\vec{x}}) = (b_{j_0}\partial_{x_i}+b_i\partial_{x_{j_0}})(f)e^{\vec{a}\cdot\vec{x}} \in \mathscr{M} \tag{3.82}$$

for $i \in \overline{1,m_1}$. Taking $i = j_0$ in (3.83), we get $\partial_{x_{j_0}}(f)e^{\vec{a}\cdot\vec{x}} \in \mathscr{M}$. Substituting it to (3.82) for general i, we obtain

$$\partial_{x_i}(f)e^{\vec{a}\cdot\vec{x}} \in \mathscr{M} \qquad \text{for } i \in \overline{1,m_1}. \tag{3.83}$$

Moreover, (3.58) yields

$$(E_{m+1+j,i_0}+E_{m+1+i_0,j}-a_jb_{i_0}-a_{i_0}b_j)(fe^{\vec{a}\cdot\vec{x}}) = (a_{i_0}\partial_{y_j}+a_j\partial_{y_{i_0}})(f)e^{\vec{a}\cdot\vec{x}} \in \mathscr{M} \tag{3.84}$$

for $j \in \overline{m_2+1,m}$. Letting $j = i_0$ in (3.87), we find $\partial_{y_{i_0}}(f)e^{\vec{a}\cdot\vec{x}} \in \mathscr{M}$. Substituting it to (3.84) for general j, we get

$$\partial_{y_j}(f)e^{\vec{a}\cdot\vec{x}} \in \mathscr{M} \qquad \text{for } j \in \overline{m_2+1,m}. \tag{3.85}$$

Repeatedly applying (3.83) and (3.85) if necessary, we obtain $e^{\vec{a}\cdot\vec{x}} \in \mathscr{M}$. Equivalently, $\mathscr{A}_{\vec{a},0} \subset \mathscr{M}$ (cf. (2.18)).

Suppose that for some $\ell \in \mathbb{N}$, $\mathscr{A}_{\vec{a},k} \subset \mathscr{M}$ whenever $\ell \geq k \in \mathbb{N}$. For any $ge^{\vec{a}\cdot\vec{x}} \in \mathscr{A}_{\vec{a},\ell}$, (3.61) implies

$$(E_{2m+2,i}+E_{m+1+i,m+1}-y_i\partial_{x_0})(ge^{\vec{a}\cdot\vec{x}}) = x_ige^{\vec{a}\cdot\vec{x}} \in \mathscr{M} \qquad \text{for } i \in \overline{1,m_1} \tag{3.86}$$

and (3.60) leads to

$$(E_{2m+2,m+1+i}-E_{i,m+1}+x_i\partial_{x_0})(ge^{\vec{a}\cdot\vec{x}})=y_jge^{\vec{a}\cdot\vec{x}}\in\mathscr{M} \tag{3.87}$$

for $j\in\overline{m_2+1,m}$. Moreover, (3.57) gives

$$\begin{aligned}&(E_{i,m+1+j_0}+E_{j_0,m+1+i})(ge^{\vec{a}\cdot\vec{x}})\\ =\;&[b_{j_0}x_i+x_i\partial_{y_{j_0}}+(\partial_{x_{j_0}}+a_{j_0})(\partial_{y_i}+b_i)](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\end{aligned} \tag{3.88}$$

if $i\in\overline{m_1+1,m_2}$, and

$$(E_{i,m+1+j_0}+E_{j_0,m+1+i})(ge^{\vec{a}\cdot\vec{x}})=[b_{j_0}x_i-a_{j_0}y_i+x_i\partial_{y_{j_0}}-y_i\partial_{x_{j_0}}](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M} \tag{3.89}$$

if $i\in\overline{m_2+1,m}$. Note that the inductional assumption imply

$$[x_i\partial_{y_{j_0}}+(\partial_{x_{j_0}}+a_{j_0})(\partial_{y_i}+b_i)](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M} \tag{3.90}$$

if $i\in\overline{m_1+1,m_2}$, and

$$[-a_{j_0}y_i+x_i\partial_{y_{j_0}}-y_i\partial_{x_{j_0}}](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M} \tag{3.91}$$

by (3.89) if $i\in\overline{m_2+1,m}$. Thus

$$x_ige^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\qquad\text{for }\; i\in\overline{m_1+1,m}. \tag{3.92}$$

On the other hand, (3.58) yields

$$(E_{m+1+j,i_0}+E_{m+1+i_0,j})(ge^{\vec{a}\cdot\vec{x}})=(a_{i_0}y_j-b_{i_0}x_j+y_j\partial_{x_{i_0}}-x_j\partial_{y_{i_0}})(g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M} \tag{3.93}$$

if $j\in\overline{1,m_1}$, and

$$\begin{aligned}&(E_{m+1+j,i_0}+E_{m+1+i_0,j})(ge^{\vec{a}\cdot\vec{x}})\\ =\;&[a_{i_0}y_j+y_j\partial_{x_{i_0}}+(\partial_{x_j}+a_j)(\partial_{y_{i_0}}+b_{i_0})](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\end{aligned} \tag{3.94}$$

if $j\in\overline{m_1+1,m_2}$. Observe that the inductional assumption imply

$$(-b_{i_0}x_j+y_j\partial_{x_{i_0}}-x_j\partial_{y_{i_0}})(g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M} \tag{3.95}$$

by (3.86) if $j\in\overline{1,m_1}$, and

$$[y_j\partial_{x_{i_0}}+(\partial_{x_j}+a_j)(\partial_{y_{i_0}}+b_{i_0})](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M} \tag{3.96}$$

if $j\in\overline{m_1+1,m_2}$. Hence

$$y_jge^{\vec{a}\cdot\vec{x}}\in\mathscr{M}\qquad\text{for }\; j\in\overline{1,m_2}. \tag{3.97}$$

Moreover, (3.63) yields

$$\begin{aligned}&(E_{m+1,m+1+j_0}+E_{j_0,2m+2})(ge^{\vec{a}\cdot\vec{x}})\\ =\;&[b_{j_0}x_0+x_0\partial_{y_{j_0}}+(\tilde{D}-\sum_{i=1}^{m_1}a_ix_i+\sum_{j=m_1+1}^{m}a_jx_j+\sum_{r=1}^{m_2}b_ry_r\\ &-\sum_{s=m_2+1}^{m}b_sy_s+\tilde{c}+1)(a_{j_0}+\partial_{x_{j_0}})](g)e^{\vec{a}\cdot\vec{x}}\in\mathscr{M}.\end{aligned} \tag{3.98}$$

Note that

$$x_0\partial_{y_{j_0}}(g)e^{\vec{a}\cdot\vec{x}} \in \mathscr{A}_{\vec{a},\ell} \subset \mathscr{M}; \quad (\tilde{D}+\tilde{c}+1)(\partial_{x_{j_0}}(g))e^{\vec{a}\cdot\vec{x}} \in \mathscr{A}_{\vec{a},\ell-1} \subset \mathscr{M}. \tag{3.99}$$

Now (3.86), (3.87), (3.92) and (3.97)-(3.99) imply $x_0ge^{\vec{a}\cdot\vec{x}} \in \mathscr{M}$. Therefore, $\mathscr{A}_{\vec{a},\ell+1} \subset \mathscr{M}$. By induction, $\mathscr{A}_{\vec{a},\ell} \subset \mathscr{M}$ for any $\ell \in \mathbb{N}$. So $\mathscr{A}_{\vec{a}} = \mathscr{M}$. Hence $\mathscr{A}_{\vec{a}}$ is an irreducible $sp(2m+2,\mathbb{F})$-module. ■

With respect to the restricted representation $\pi^{\vec{0}}_{c,S}$, $\mathscr{A}$ is an infinite-dimensional weight $sp(2m+2,\mathbb{F})$-module with finite-dimensional weight subspaces by (3.54)-(3.56) and (3.64). Now Theorem 2 follows from Theorems 3.2, 3.3. 3.5 and 3.7.

Acknowledgments. We would like to thank the referee for the helpful comments.

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Received November 19, 2014
and in final form May 27, 2015