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On the Continuity of the Centralizer Map of a Locally Compact Group

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Abstract. Let G be a locally compact group. We denote by $\mathcal{SUB}(G)$ the hyperspace of closed subgroups of G endowed with the Chabauty topology. In this article we study the continuity of the following map

$$\mathsf{Cent}_G\colon G \to \mathcal{SUB}(G),\ g \mapsto \mathsf{Cent}_G(g),$$

where $\mathsf{Cent}_G(g)$ is the centralizer of g in G.
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1. Introduction

Let G be a locally compact group and $\mathcal{SUB}(G)$ the space of all closed subgroups of G. The Chabauty topology on $\mathcal{SUB}(G)$ has the sets

$$\mathcal{O}_1(K) = \{H \in \mathcal{SUB}(G) \mid H \cap K = \varnothing\},$$
$$\mathcal{O}_2(V) = \{H \in \mathcal{SUB}(G) \mid H \cap V \neq \varnothing\},$$

as an open subbase, where V and K run respectively, over all open and compact subsets of G. The Chabauty space is named after Claude Chabauty, who introduced it in [5] to generalize Mahler's compactness criterion to lattices in locally compact groups. If G is locally compact, then $\mathcal{SUB}(G)$ is compact. We define the centralizer map $\mathsf{Cent}_G\colon G \to \mathcal{SUB}(G)$ by $g \mapsto \mathsf{Cent}_G(g)$, where $\mathsf{Cent}_G(g)$ is the centralizer of g in G. This paper is a contribution to the following problem, posed by K. H. Hofmann and G. A. Willis ([14, page 5]).

Problem 1.1. Study the continuity of the map Cent_G.

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2. The Chabauty topology

The purpose of this section is to collect several facts concerning the Chabauty topology and to present them in the manner we shall use them in the sequel.

2.1. Generalities.

Definition 2.1 (Chabauty topology). Given a topological group G, the Chabauty topology on the set $\mathcal{SUB}(G)$ of all closed subgroups of G has a subbase given by sets of the following form

$$\mathcal{O}_1(K) = \{H \in \mathcal{SUB}(G) \mid H \cap K = \varnothing\}, \qquad \text{where } K \text{ is compact}$$
$$\mathcal{O}_2(V) = \{H \in \mathcal{SUB}(G) \mid H \cap V \neq \varnothing\}, \qquad \text{where } V \text{ is open.}$$

Given a compact subset $K \subset G$ and open subsets $U_1, \dots, U_n \subset G$, define

$$\Omega(K; U_1, \dots, U_n) = \mathcal{O}_1(K) \cap \mathcal{O}_2(U_1) \cap \dots \cap \mathcal{O}_2(U_n). \tag{2.1}$$

Proposition 2.2. *Let G be a locally compact group. The set*

$$\mathcal{B} = \left\{ \Omega(K; U_1, \dots, U_n) \;\middle|\; \begin{array}{l} K \text{ is a compact subset of } G, \\ U_i \text{ is an open subset of } G. \end{array} \right\}$$

is a base for the Chabauty topology on $\mathcal{SUB}(G)$.

Proof. It suffices to observe that

$$\mathcal{O}_1(K_1) \cap \mathcal{O}_1(K_2) \cap \dots \cap \mathcal{O}_1(K_m) = \mathcal{O}_1(K_1 \cup k_2 \cup \dots \cup K_m). \qquad \blacksquare$$

For the following result we refer to [2, page 188].

Proposition 2.3. *Let G be a locally compact group whose neutral element is denoted by e. In $\mathcal{SUB}(G)$, a basis of neighborhoods of a closed subgroup Γ is given by*

$$\mathcal{U}(K, U)(\Gamma) = \{\Gamma' \in \mathcal{SUB}(G) \mid \Gamma' \cap K \subset \Gamma U, \ \Gamma \cap K \subset \Gamma' U\} \tag{2.2}$$

where K runs over the compact subsets of G and U runs over the neighborhoods of e.

Proposition 2.4. *Let $(H_\alpha)_{\alpha \in I}$ be a net of closed subgroups of a locally compact group G converging to a closed subgroup H and assume that there is a closed subset F of G such that $H_\alpha \subset F$ for every $\alpha \in I$. Then $H \subset F$.*

Proof. Let us argue by contradiction and so assume that $H \not\subset F$. Then $H \in \mathcal{O}_2(G \setminus F)$, and so, there exists $\alpha_0 \in I$ such that $H_\alpha \in \mathcal{O}_2(G \setminus F)$ whenever $\alpha \geq \alpha_0$. Hence $H_\alpha \cap (G \setminus F) \neq \varnothing$ by the definition of $\mathcal{O}_2(G \setminus F)$, a contradiction. $\blacksquare$

The following well-known result can be found in [2, Théorème 1, page 181] (see also [1, Lemma E.1.1]).

Proposition 2.5. *The Chabauty space $\mathcal{SUB}(G)$ of a locally compact group G is compact.*

The following proposition is often used in the sequel.

Proposition 2.6. *If a net $(H_\alpha)_{\alpha\in J}$ of closed subgroups of a locally compact group G converges to a closed subgroup H, then the following assertions are equivalent.*

(1) *The points $x_1, x_2, \dots, x_n$ belong to H.*

(2) *There exists a subnet $(H_{\alpha_\beta})_{\beta\in I}$ of the net $(H_\alpha)_{\alpha\in J}$ and there exist nets $\{x_1^\beta\}_{\beta\in I}, \dots, \{x_n^\beta\}_{\beta\in I}$ of G converging, respectively, to $x_1, \dots, x_n$ in the topology of G, with $x_1^\beta, \dots, x_n^\beta \in H_{\alpha_\beta}$ for all $\beta \in I$.*

Proof. This follows from Remark 1.1, page 628 and Lemma 1.2, page 629 of [17]. ∎

Proposition 2.7. *The mapping $\phi_{\mathbb{R}}\colon [0,\infty] \to \mathcal{SUB}(\mathbb{R})$ defined by*

$$\phi_{\mathbb{R}}(r) = \begin{cases} \frac{1}{r}\cdot\mathbb{Z} & \text{if } \ 0 < r < \infty, \\ \{0\} & \text{if } \ r = 0, \\ \mathbb{R} & \text{if } \ r = \infty, \end{cases}$$

is a homeomorphism.

Proof. See Proposition 1.7 of [10]. ∎

2.2. Limits of directed and filtered nets.

Recall the following definition from [9, Definition O-1.2].

Definition 2.8 (Directed and filtered nets)**.** A net $(x_j)_{j\in J}$ in a preordered set $(L, \le)$ is called directed (resp. filtered) net if for each fixed $i \in J$ there exists $j_0 \in J$ such that $x_i \le x_j$ (resp. $x_j \le x_i$) whenever $j_0 \le j$.

The following results were proved by I. V. Protasov for the case of nondecreasing and nonincreasing nets (see Lemma 1.3 and Lemma 1.4 of [17]).

Proposition 2.9 (Limits of filtered nets)**.** *Let G be a locally compact group. If $(H_j)_{j\in J}$ is a filtered net in $\mathcal{SUB}(G)$ then $\lim H_j = \bigcap_{j\in J} H_j$ in the Chabauty topology.*

Proof. Set $H = \bigcap_{j\in J} H_j$. Now let M be a cluster point of the net $(H_j)_{j\in J}$; $H \subset M$ should be clear (Proposition 2.6), we must understand the converse. There is a subnet $(j(i))_{i\in I}$ of J cofinal in J such that $M = \lim_i H_{j(i)}$. Let $x \in M$; then for some net $(i_k)_{k\in A}$, which is cofinal in I, there are elements $x_k \in H_{j(i_k)}$ and $x = \lim_k x_k$ (Proposition 2.6). Now for each $j \in J$ there is $k_0 \in A$ such that

$k_0 \leq k$ implies $j \leq j(i_k)$, and so $x_k \in H_{j(i_k)}$, since $(H_j)_{j\in J}$ is a filtered net. Therefore $x \in H_j$ for all $j \in J$ and thus $x \in H$. Hence $M \subset H$. ■

Proposition 2.10 (Limits of directed nets). *Let G be a locally compact group. If $(H_j)_{j\in J}$ is a directed net in $\mathcal{SUB}(G)$ then $\lim H_j = \overline{\bigcup_{j\in J} H_j}$ in the Chabauty topology.*

Proof. Set $H = \bigcup_{j\in J} H_j$. Let U be an open set of G such that $\overline{H} \in \mathcal{O}_2(U)$. Then $\overline{H} \cap U \neq \varnothing$ and so $H \cap U \neq \varnothing$. Let $i \in J$ such that $H_i \cap U \neq \varnothing$. As the net $(H_j)_{j\in J}$ is directed, there is $m \in J$ such that for every $j \geq m$ we have $H_i \subset H_j$ and hence $H_j \in \mathcal{O}_2(U)$. Now, let K be a compact set of G such that $\overline{H} \in \mathcal{O}_1(K)$. It is clear that for any $j \in J$, $H_j \in \mathcal{O}_1(K)$. Consequently, we have proved that for every open neighborhood set O of $\overline{H}$ in $\mathcal{SUB}(G)$ there is $j_0 \in J$ such that for each $j \geq j_0$, $H_j \in O$, that is $\lim H_j = \overline{H}$. ■

2.3. Lower Chabauty topology.

Definition 2.11 (Lower set). Let L be a set with preorder $\leq$. For $X \subset L$ we write $\downarrow X = \{y \in L \mid (\exists\, x \in X)\; y \leq x\}$. X is called a *lower set* if $X = \downarrow X$.

Notation 2.12. For a topological space X, we denote by $\mathcal{K}(X)$ the set of all compact subsets of X.

Example 2.13. In $(\mathcal{SUB}(G), \subset)$, all sets $\mathcal{O}_1(K)$, $K \in \mathcal{K}(G)$ are lower sets. Every union of sets $\mathcal{O}_1(K)$ is an open lower set in $\mathcal{SUB}(G)$.

Definition 2.14 (Lower Chabauty topology). The lower Chabauty topology λ on $\mathcal{SUB}(G)$ has as a base all sets of the form $\mathcal{O}_1(K)$, where $K \in \mathcal{K}(G)$.

Notation 2.15. If G is a locally compact group, denote by $\mathcal{SUB}^-(G)$ the set $\mathcal{SUB}(G)$ endowed with the lower Chabauty topology λ.

A topological space is called quasi-compact if every open cover of it has a finite subcover.

Proposition 2.16. *For every nontrivial locally compact group G, $\mathcal{SUB}^-(G)$ is a quasicompact non-Hausdorff T_0-space.*

Proof. As $\mathcal{SUB}(G)$ is the unique open set of $\mathcal{SUB}^-(G)$ containing G, $\mathcal{SUB}^-(G)$ is non-Hausdorff. The quasi-compactness of $\mathcal{SUB}^-(G)$ follows from the fact that the lower Chabauty topology is coarser than the Chabauty topology, and Proposition 2.5. Now, let A and B two distinct points of $\mathcal{SUB}(G)$. If there is $x \in A \setminus B$ then the open set $\mathcal{O}_1(\{x\})$ contains B and does not contain A. If not, there is $x \in B \setminus A$ and so the open set $\mathcal{O}_1(\{x\})$ contains A and does not contain B. ■

Let $H \in \mathcal{SUB}(G)$. As the set $\mathcal{B} = \{\mathcal{O}_1(K) \mid K \in \mathcal{K}(G)\}$ is a base for the lower Chabauty topology, then the members of the base $\mathcal{B}$ which contain H form a local base at H. Therefore we have

$$\mathbf{U}_H = \{\mathcal{O}_1(K) \mid K \in \mathcal{K}(G),\ K \cap H = \varnothing\}$$

is a local base at H. Let

$$\cap\mathbf{U}_H = \cap\{\mathcal{O}_1(K) \mid \mathcal{O}_1(K) \in \mathbf{U}_H\}.$$

Lemma 2.17. *With the notation introduced above we have*

$$\cap\mathbf{U}_H = \downarrow H.$$

Proof. Let $M \in \cap\mathbf{U}_H$ and suppose $M \not\subseteq H$. Then we find an $m \in M \setminus H$. Since H is closed, $V \stackrel{\text{def}}{=} G \setminus H$ is an open neighborhood of m, and so $m^{-1}V$ is an open neighborhood of e. Since G is locally compact group, there is a compact neighborhood W of e contained in $m^{-1}V$. Then $K \stackrel{\text{def}}{=} mW$ is a compact neighborhood of m contained in V and so $K \cap H = \varnothing$. Thus $\mathcal{O}_1(K) \in \mathbf{U}_H$ and therefore $M \in \mathcal{O}_1(K)$ by the selection of M. Hence $K \cap M = \varnothing$ by the definition of $\mathcal{O}_1(K)$. However, $m \in K \cap M$ and this is a contradiction to our supposition $M \not\subseteq H$. Hence $M \in \downarrow H$ and so $\cap\mathbf{U}_H \subset \downarrow H$. The converse inclusion is trivial. ∎

2.4. Lowerer semicontinuous functions.

Definition 2.18 (Lower semicontinuous map)**.** A function $f\colon X \to \mathcal{SUB}(G)$ is called *lower semicontinuous* at $x \in X$ if the function $f\colon X \to \mathcal{SUB}^{-}(G)$ is continuous at x. It is called *lower semicontinuous* if it is upper semicontinuous at all points $x \in X$.

Remark 2.19. Let G be a locally compact group, X a topological space with $x \in X$. Let $f\colon X \to \mathcal{SUB}(G)$. The following statements are equivalent:

(1) The function f is lower semicontinuous at x;

(2) for each $K \in \mathcal{K}(G)$ the relation $K \cap f(x) = \varnothing$ implies there is a neighborhood W of x such that for each $w \in W$ we have $K \cap f(w) = \varnothing$;

(3) for each $K \in \mathcal{K}(G)$ such that $f(x) \in \mathcal{O}_1(K)$, there is a neighborhood W of x such that $f[W] \subset \mathcal{O}_1(K)$.

Lemma 2.20. *Let G be a locally compact group, X a topological space with $x \in X$. Let $f\colon X \to \mathcal{SUB}(G)$ satisfying the following condition:*

(H1) *For each net $(x_j)_{j\in J}$ converging to x in X each cluster point M of the net $(f(x_j))_{j\in J}$ satisfies $M \subset f(x)$,*

Then

(R1) *the map f is lower semicontinuous at x.*

Proof. Let $K \in \mathcal{K}(G)$ such that $K \cap f(x) = \varnothing$. We suppose for every neighborhood W of x there exists $w \in W$ such that $K \cap f(w) \neq \varnothing$. Let $(W_\alpha)_{\alpha\in I}$ be a filter of neighborhoods of x in X. For every $\alpha \in I$, let $w_\alpha \in W_\alpha$ such that $f(w_\alpha) \cap K \neq \varnothing$. Then we find an $y_\alpha \in f(w_\alpha) \cap K$. Let y be a limit point of $(y_\alpha)_{\alpha\in I}$ and assume, without loss, that in fact $y_\alpha \to y \in K$. Now, let M be a cluster point of the net $(f(w_\alpha))_{\alpha\in I}$. By Proposition 2.6, $y \in M$ and hence $y \in M \cap K$. However, as $(w_\alpha)_{\alpha\in I}$ converges to x, $M \subset f(x)$ and so $M \cap K = \varnothing$ and this is a contradiction to $y \in M \cap K$. ■

The following example shows, the implication (R1) implies (H1) is not true in general.

Example 2.21. The map

$$f\colon \mathbb{R} \to \mathcal{SUB}(\mathbb{R}),\ x \mapsto \frac{2+\cos(\pi x)}{2-\cos(\pi x)}\mathbb{Z},$$

is continuous (see Proposition 2.7) and hence (R1) is satisfied. On the other hand, for $n \geq 0$, let $x_n = \frac{1}{2n+\frac{1}{2}}$. As $f(x_n) = \mathbb{Z}$ and $f(0) = 3\mathbb{Z}$, (H1) is not satisfied.

Definition 2.22. The equalizer of a pair of maps $f, g:\ A \to B$ is the largest subset $\mathrm{Eq}(f,g)$ of A on which f and g coincide, i.e.,

$$\mathrm{Eq}(f,g) = \{x \in A \mid f(x) = g(x)\}.$$

Clearly, if f and g are two continuous morphisms of a topological group G_1 into a Hausdorff topological group G_2, the set $\mathrm{Eq}(f,g)$ is a closed subgroup of G_1.

Let G be a locally compact group. We denote by $\mathrm{Aut}(G)$ the automorphism group of G equipped with the compact-open topology. For a pair of continuous functions $\alpha, \beta:\ G \to \mathrm{Aut}(G)$ we define

$$\mathbb{E}\mathrm{q}(\alpha,\beta):\ G \to \mathcal{SUB}(G),\quad g \mapsto \mathrm{Eq}(\alpha(g), \beta(g)).$$

The following example shows that the function $\mathbb{E}\mathrm{q}(\alpha,\beta)$ is not continuous in general.

Example 2.23. Let $G = \mathbb{R}$, $\alpha(g)(x) = (g^2+1)x$, and $\beta(g)(x) = (g^2-g+2)x$. We have

$$\mathbb{E}\mathrm{q}(\alpha,\beta)(g) = \begin{cases} \mathbb{R} & \text{if } g = 1, \\ \{0\} & \text{for } g \neq 1, \end{cases}$$

Proposition 2.24. *The function $\mathbb{E}\mathrm{q}(\alpha,\beta)$ is lower semi-continuous.*

Proof. Let $(g_j)_{j\in J}$ be a net converging to g in G. Let M be a cluster point of $(\mathbb{E}\mathrm{q}(\alpha,\beta)(g_j))_{j\in J}$. Then M is the limit of a subnet; by a change of notation we assume that $M = \lim \mathbb{E}\mathrm{q}(\alpha,\beta)(g_j)$. Let $m \in M$. Then there is a subnet $(g_{j_i})_{i\in I}$ and there are elements $x_i \in \mathbb{E}\mathrm{q}(\alpha,\beta)(g_{j_i})$ such that $m = \lim x_i$. Now

$\alpha(g_{j_i})(x_i) = \beta(g_{j_i})(x_i)$, and by the continuity of α and β and the joint continuity of evaluation for the compact open topology on $\mathrm{Aut}(G)$, passing to the limit through $i \in I$ we get $\alpha(g)(m) = \beta(g)(m)$, that is, $m \in \mathbb{E}\mathrm{q}(\alpha, \beta)(g)$. ■

We shall now give a sufficient condition for a lower semicontinuous map $f\colon X \to \mathcal{SUB}(G)$ to be continuous.

Proposition 2.25. *Let G be a locally compact group, X a topological space with $x \in X$. Let $f\colon X \to \mathcal{SUB}(G)$ be a map lower semicontinuous at x satisfying the following hypothesis:*

(H2) *For each net $(x_j)_{j\in J}$ which converges to x in X, each cluster point H of the net $(f(x_j))_{j\in J}$ satisfies $f(x) \subset H$.*

Then

(R2) *For each open set U of G the relation $U \cap f(x) \neq \varnothing$ implies there is a neighborhood W of x such that for each $w \in W$ we have $U \cap f(w) \neq \varnothing$.*

In particular, f is continuous at x.

Proof. Let U be an open set of G such that $U \cap f(x) \neq \varnothing$. We proceed by contradiction. Suppose that for every neighborhood W of x there exists $w \in W$ such that $U \cap f(w) = \varnothing$. Let $(W_\alpha)_{\alpha\in I}$ be a filter of neighborhoods of x in X. For each $\alpha \in I$, let $w_\alpha \in W_\alpha$ such that $f(w_\alpha) \cap K = \varnothing$. Let H be a cluster point of the net $(f(w_\alpha))_{\alpha\in I}$. As the net $(w_\alpha)_{\alpha\in I}$ converges to x, $f(x) \subset H$, and so $H \cap U \neq \varnothing$. On the other hand, as for each $\alpha \in I$, $f(w_\alpha)$ is contained in the closed set $G \setminus U$, by Proposition 2.4 $H \subset G \setminus U$ and hence $H \cap U = \varnothing$, a contradiction. ■

The following example shows that the implication (R2) implies (H2) is not true in general.

Example 2.26. The map

$$f\colon \mathbb{R} \to \mathcal{SUB}(\mathbb{R}),\ x \mapsto \frac{2-\cos(\pi x)}{2+\cos(\pi x)}\mathbb{Z},$$

is continuous (see Proposition 2.7). Then (R2) is satisfied. On the other hand, for $n \geq 0$, let $x_n = \frac{1}{2n+\frac{1}{2}}$. As $f(x_n) = \mathbb{Z}$ and $f(0) = \frac{1}{3}\mathbb{Z}$, (H2) is not satisfied.

2.5. Isolated subgroups.

The subgroup generated by a finite subset $X = \{x_1, \ldots, x_n\}$ of a group G is denoted by $\langle x_1, \ldots, x_n\rangle$. We begin by introducing the following definition.

Definition 2.27 (Strongly finitely generated groups)**.** A topological group G is called strongly finitely generated (abbreviated SFG) if there exist $U_1, \ldots, U_n$ open sets of G such that for every $(g_1, \ldots, g_n) \in U_1 \times \cdots \times U_n$, we have $\overline{\langle g_1, \ldots, g_n\rangle} = G$.

Example 2.28. Every discrete finitely generated group is strongly finitely generated.

Proposition 2.29. *The class* [SFG] *is stable under passage to quotient.*

Proof. Let G be a SFG-group and H a normal subgroup of G. Let $U_1, \dots, U_n$ be open subset of G such that $G = \overline{\langle g_1, \dots, g_n \rangle}$ for every $(g_1, \dots, g_n) \in U_1 \times \cdots \times U_n$. Let $\pi : G \to G/H$ be the canonical projection. It clear that $G/H = \pi(G) = \pi(\overline{\langle g_1, \dots, g_n \rangle}) \subset \overline{\pi(\langle g_1, \dots, g_n \rangle)} = \overline{\langle \pi(g_1), \dots, \pi(g_n) \rangle}$, because π is continuous morphism. then $G/H = \overline{\langle \pi(g_1), \dots, \pi(g_n) \rangle}$. On the other hard, as π is open, $\pi(U_1), \dots, \pi(U_n)$ are open subsets of G/H. ■

Recall the following definition from [8, Definition 5.1].

Definition 2.30. Let G be a locally compact group. A closed subgroup H of G is said to be *an isolated subgroup* of G if H is an isolated point of $\mathcal{SUB}(G)$.

The Definition 2.27 is motivated by the following.

Proposition 2.31. *Let G be a locally compact group. Then G is strongly finitely generated if and only if G is an isolated subgroup of G.*

Proof. Let $U_1, \dots, U_n$ be open subsets of G such that $G = \overline{\langle g_1, \dots, g_n \rangle}$ for every $(g_1, \dots, g_n) \in U_1 \times \cdots \times U_n$. Let $H \in \Omega(\varnothing; U_1, \dots, U_n)$. Then $H = G$ for $i \in \{1, \dots, n\}$, $H \cap U_i \neq \varnothing$. Hence $\Omega(\varnothing; U_1, \dots, U_n) = \{G\}$. Conversely, if G is isolated in $\mathcal{SUB}(G)$, then there is $U_1, \dots, U_n$ open set of G such that $\{G\} = \Omega(\varnothing; U_1, \dots, U_n)$. Let $(g_1, \dots, g_n) \in U_1 \times \cdots \times U_n$, it clear that $\overline{\langle g_1, \dots, g_n \rangle} \in \Omega(\varnothing; U_1, \dots, U_n)$ and hence $G = \overline{\langle g_1, \dots, g_n \rangle}$. ■

Recall that the Frattini subgroup $\Phi(G)$ of a profinite group G is the intersection of all maximal open subgroups.

Proposition 2.32. *Let G be a profinite group. Then G is strongly finitely generated if and only if its Frattini subgroup $\Phi(G)$ is open in G.*

Proof. An elegant proof of the proposition is given in [8, Theorem 5.6]. In this paper, we give another elementary proof of sufficiency. If $\Phi(G)$ is open then $G/\Phi(G)$ is a finite group. Thus there is a finite subset $\{g_1, \dots, g_m\}$ of G such that $G = \{g_1, \dots, g_m\}\Phi(G)$. For $i \in \{1, \dots, m\}$ let $U_i = g_i\Phi(G)$. Clearly, for each $(x_1, \dots, x_m) \in U_1 \times \cdots \times U_m$ we have $G = \{x_1, \dots, x_m\}\Phi(G)$. Then, by Proposition 1.9 of [8], $G = \overline{\langle \{x_1, \dots, x_m\} \rangle}$. ■

Remark 2.33. A characterization of profinite groups with open Frattini subgroup can be found in [8, Theorem 5.6] (see also [21, Proposition, page 148]).

For the following consequence of Proposition 2.32 recall that a pro-p group is a profinite group in which every open normal subgroup has index equal to some power of p.

Corollary 2.34. *A pro-p group is strongly finitely generated if and only if it is finitely generated.*

Proof. This follows from the fact that a pro-p group G is finitely generated if and only if $\Phi(G)$ is open in G (see Proposition 1.14 of [7] or Proposition 2.8.10 of [20]). ■

3. Quasidiscrete groups

The following definition was firstly introduced by Burger and Mozes ([3, page 116]).

Definition 3.1 (Quasicenter). The quasicenter of a topological group G is the subset $\mathrm{QZ}(G)$ consisting of all those elements possessing an open centralizer:

$$\mathrm{QZ}(G) = \{g \in G \mid \mathsf{Cent}_{\mathrm{G}}(g) \text{ is open in } G\}.$$

We summarize some of the properties of the quasi-center in the following proposition([23]).

Proposition 3.2. *Let G be a topological group. Then*

(1) $\mathrm{QZ}(G)$ *is a topologically characteristic subgroup in G.*

(2) *If U is an open subgroup of G, then* $\mathrm{QZ}(U) = \mathrm{QZ}(G) \cap U$.

(3) *For any closed normal subgroup N of G, the image of* $\mathrm{QZ}(G)$ *under the canonical projection to G/N is contained in* $\mathrm{QZ}(G/N)$.

(4) *Every discrete normal subgroup of G is contained in* $\mathrm{QZ}(G)$.

Recall the following definition from [4].

Definition 3.3 (Quasi-discrete group). A topological group is called quasi-discrete if its quasi-center is dense.

Notation 3.1. We note by $\mathscr{QD}$ the class of quasi-discrete topological groups.

Remark 3.4. (1) The class $\mathscr{QD}$ contains all discrete groups.

(2) The class $\mathscr{QD}$ contains all abelian groups.

(3) The class $\mathscr{QD}$ contains all topological groups with open center.

(4) The class $\mathscr{QD}$ is closed under forming open subgroups and passage to quotient groups (see (2) and (3) of Proposition 3.2).

Proposition 3.5. *Let* $(G_\alpha : \alpha \in I)$ *be a family of topological groups and set* $G = \prod_{\alpha\in I} G_\alpha$. *Denote by* I_F *the set of finite subsets of* I. *Then*

$$\mathrm{QZ}(G) = \{(x_\alpha) \in G \mid \exists J \in I_F : (x_\alpha \in \mathrm{QZ}(G_\alpha), \forall \alpha \in J), \\ (x_\alpha \in Z(G_\alpha), \forall \alpha \notin J)\}$$

Proof. Let $x = (x_\alpha) \in \mathrm{QZ}(G)$. Let $O = \prod_{\alpha\in I} O_\alpha$ be a non-empty elementary open set such that $O \subset \mathsf{Cent}_{\mathrm{G}}(x)$. We have $O_\alpha = G_\alpha$, except for indices α belonging to a finite subset J of I. As $\mathsf{Cent}_{\mathrm{G}}(x) = \prod_{\alpha\in I} \mathsf{Cent}_{G_\alpha}(x_\alpha)$, for each $\alpha \notin J$ we have $\mathsf{Cent}_{G_\alpha}(x_\alpha) = G_\alpha$. Hence for every $\alpha \notin J$ we have $x_\alpha \in \mathrm{Z}(G_\alpha)$. For $\alpha \in J$, it is clear that $x_\alpha \in \mathrm{QZ}(G_\alpha)$. This proves the first inclusion. The converse inclusion is trivial ■

The quasi-center $\mathrm{QZ}(G)$ need not be closed, as is seen in the following.

Example 3.6 ([3], Example 1.1.2)**.** Let $(G_n)_{n\in\mathbb{N}}$ be finite centerless groups, and let $G = \prod_{n\in\mathbb{N}} G_n$. Then $\mathrm{QZ}(G) = \bigoplus_{n\in\mathbb{N}} G_n$ is the direct sum of the groups $(G_n)_{n\in\mathbb{N}}$. Hence $\mathrm{QZ}(G)$ is dense in G and not closed.

Proposition 3.7. *Let* G *be a locally compact group, and* G_0 *the component connected of* e. *Then*

$$G_0 \subset \bigcap_{g\in\overline{\mathrm{QZ}(G)}} \mathsf{Cent}_{\mathrm{G}}(g). \tag{3.1}$$

Proof. We proceed by contradiction. So suppose that (3.1) doesn't hold. Let $g \in \overline{\mathrm{QZ}(G)}$, $x \in G_0$, such that $x \notin \mathsf{Cent}_{\mathrm{G}}(g)$. Let $(g_\alpha)_{\alpha\in I}$ be a net of $\mathrm{QZ}(G)$ which converges to g. As $g \in G \setminus \mathsf{Cent}_{\mathrm{G}}(x)$, there is $\alpha_0 \in I$, such that for every $\alpha \geq \alpha_0$, $x \notin \mathsf{Cent}_{\mathrm{G}}(g_\alpha)$, a contradiction. ■

As a consequence we have:

Corollary 3.8. *Let* G *be a quasi-discrete locally compact group. Then the identity component* G_0 *of* G *is central. In particular,* G_0 *is abelian.*

Corollary 3.9. *Every connected quasi-discrete group is abelian.*

To conclude this subsection, let us give some non-trivial examples of quasi-discrete groups.

Example 3.10. A group G is called FD-group if the derived subgroup $\mathcal{D}(G)$ is finite ([16]). For every FD-group G we have $\mathrm{QZ}(G) = G$. In fact, for $g \in G$, we set $C(g) = \{xgx^{-1} \mid x \in G\}$, then $C(g)g^{-1} \subset \mathcal{D}(G)$, whence $C(g) \subset \mathcal{D}(G)g$. Thus the finiteness of $\mathcal{D}(G)$ implies that of $C(g)$ for all $g \in G$. It follows that the centralizer $\mathsf{Cent}_{\mathrm{G}}(g)$ has finite index in G and thus is open (see also [12] and [13]).

Example 3.11 ([18]). A topological space X without isolated points is said to be maximal if X has isolated points in every stronger topology. A topological group is said to be maximal if its underlying space is maximal. For every maximal topological group G we have $\mathrm{QZ}(G) = G$ ([18, Theorem 6.1]).

Example 3.12. Let G be a locally compact group such that the mapping

$$\lambda_G \colon \mathcal{SUB}(G) \times \mathcal{SUB}(G) \to \mathcal{SUB}(G),\ (K, L) \mapsto K \cap L,$$

is continuous. By Lemma 5 of [19], G is totally disconnected periodic group and every topological p-element x of G is centralized by some neighborhood of the identity e. On the other hand, by Proposition 2.3 of [15], for every $x \in G$, the compact monothetic group $\overline{\langle x \rangle}$ contains a dense subgroup which is algebraically generated by topological p-elements. We deduce from the above that $\overline{\mathrm{QZ}(G)} = G$.

4. Continuity of the centralizer map

Let G be a topological group G, and let $g \in G$. The centralizer of g in G is defined by

$$\mathsf{Cent}_{\mathrm{G}}(g) = \{x \in G \mid gx = xg\}.$$

We have $\mathsf{Cent}_{\mathrm{G}}(g) = G$ if and only if g is central.

Remark 4.1. For a locally compact group G, the following conditions are equivalent:

(1) $\mathsf{Cent}_{\mathrm{G}} \colon G \to \mathcal{SUB}(G)$ is discontinuous at e;

(2) there is a net $(g_j)_{j\in J}$ converging to e such that $(\mathsf{Cent}_{\mathrm{G}}(g_j))_{j\in J}$ converges to a proper subgroup of G.

Example 4.2. Let $G = \mathrm{SL}(2, \mathbb{R})$, and let $K = \{\mathrm{diag}(a, \frac{1}{a}) \mid a \in \mathbb{R} \setminus \{0\}\}$, where $\mathrm{diag}(a, \frac{1}{a}) = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$. For $n \geq 1$, let $A_n = \mathrm{diag}(\frac{n}{n+1}, \frac{n+1}{n})$. Then $\lim_n A_n = \mathrm{I}_2 \overset{\text{def}}{=} \mathrm{diag}(1, 1)$ but $\mathsf{Cent}_{\mathrm{G}}(A_n) = K$ for all n and so $\lim_n \mathsf{Cent}_{\mathrm{G}}(A_n) = K$. Hence, the centralizer map $\mathsf{Cent}_{\mathrm{G}}$ is discontinuous at $e = \mathrm{I}_2$.

Proposition 4.3. *Let G be a locally compact group such that the centralizer map $\mathsf{Cent}_{\mathrm{G}}$ is continuous. Then for every open subgroup H of G, Cent_H is continuous.*

Proof. This follows from the commutativity of the following diagram

$$\begin{array}{ccc} H & \xrightarrow{\ i\ } & G \\ {\scriptstyle \mathsf{Cent}_H}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{Cent}_{\mathrm{G}}} \\ \mathcal{SUB}(H) & \xleftarrow[\mathcal{SUB}^*(i)]{} & \mathcal{SUB}(G) \end{array} \tag{4.1}$$

and the continuity of the map $\mathcal{SUB}^*(i)\colon \mathcal{SUB}(G) \to \mathcal{SUB}(H),\ A \mapsto A \cap H$ (see [10, Proposition 1.2]). ■

The proof of the following lemma is straightforward.

Lemma 4.4. *Let G be a locally compact topological group. Then the centralizer function Cent_G is continuous at e if and only if, for any compact set K of G and any open neighborhood U of e there exists a neighborhood V of e such that for each $h \in V$ we have*

$$K \subset U\,\mathsf{Cent}_G(h). \tag{4.2}$$

Proposition 4.5. *The class of locally compact groups G such that the centralizer map Cent_G is continuous at e is closed under passage to quotient groups modulo closed normal subgroups.*

Proof. Let N be a closed normal subgroup of G, $\pi\colon G \to G/N$ the canonical projection. Let $\widetilde{K}$ and $\widetilde{U}$ be respectively a compact subset and an open neighborhood of the identity in G/N. Let K and U be respectively a compact subset and an open neighborhood of the identity in G such that $\pi(K) = \widetilde{K}$ and $\pi(U) = \widetilde{U}$. By the continuity of Cent_G at e, there exists a neighborhood V of e such that, for each $h \in V$ we have $K \subset U\,\mathsf{Cent}_G(h)$. Then $\widetilde{K} \subset \widetilde{U}\pi(\mathsf{Cent}_G(h))$ and hence $\widetilde{K} \subset \widetilde{U}\,\mathsf{Cent}_{G/N}(\pi(h))$, because $\pi(\mathsf{Cent}_G(h)) \subset \mathsf{Cent}_{G/N}(\pi(h))$. We deduce from the above that for each $h \in V$, $\widetilde{K} \subset \widetilde{U}\,\mathsf{Cent}_{G/N}(\pi(h))$, which completes the proof (see Lemma 4.4). ■

Proposition 4.6. *For every locally compact group G, the centralizer map Cent_G is upper semicontinuous.*

Proof. This follows from Proposition 2.24 and the fact that $\mathsf{Cent}_G = \mathbb{E}\mathrm{q}(\alpha, \beta)$ where $\alpha(g)(x) = gxg^{-1}$ and $\beta(g)(x) = x$. ■

A subset of a topological space is called a *residual set* or a Baire set if it is the intersection of a countable number of dense open sets. A residual set in a locally compact space is dense.

Proposition 4.7 (Generic continuity from semicontinuity). *Let G be a locally compact group. The set of continuity points of the centralizer map Cent_G is a dense residual set in G.*

Proof. This follows from Proposition 4.6 and Théorème 1 page 95 of [6]. ■

Theorem 4.8. *Let G be a locally compact group which has an open abelian subgroup H. Then the centralizer map Cent_G is continuous on G if and only if it is continuous at e.*

Proof. Let g be a fixed element of G. Let K and U be respectively a compact subset and an open neighborhood of the identity in G. We will show

that $\mathsf{Cent}_{\mathrm{G}}{}^{-1}[\mathcal{U}(K,U)(\mathsf{Cent}_{\mathrm{G}}(g))]$ is a neighborhood of g in G.
Let V be a symmetric open relatively compact neighborhood of e in G such that $V^4 \subset U$ and let $F = V \cap H$. Since K is compact and is contained in FK, $K \subset \bigcup_{i=1}^{n} Fx_i$ for some finite elements $x_i \in K$, $1 \le i \le n$. Let $\widetilde{K} = \{x_1, \ldots, x_n\}$ and

$$\begin{aligned} K_1 &= \left\{x \in \widetilde{K} \mid \overline{F}x \cap \mathsf{Cent}_{\mathrm{G}}(g) \neq \varnothing\right\} = \widetilde{K} \cap \overline{F}\,\mathsf{Cent}_{\mathrm{G}}(g), \\ K_2 &= \left\{x \in \widetilde{K} \mid \overline{F}x \cap \mathsf{Cent}_{\mathrm{G}}(g) = \varnothing\right\}. \end{aligned}$$

It is clear that

$$K \subset \overline{F}K_1 \cup \overline{F}K_2. \tag{4.3}$$

As $\mathsf{Cent}_{\mathrm{G}}(g) \cap \overline{F}K_2 = \varnothing$, by Proposition 4.6, $V_g^{(1)} \stackrel{\text{def}}{=} \mathsf{Cent}_{\mathrm{G}}{}^{-1}[\mathcal{O}_1(\overline{F}K_2)]$ is an open set of G containing g. Thus, for $h \in V_g^{(1)}$ we have

$$\mathsf{Cent}_{\mathrm{G}}(h) \cap \overline{F}K_2 = \varnothing \tag{4.4}$$

On the other hand, as $K_1 \subset \overline{F}\,\mathsf{Cent}_{\mathrm{G}}(g)$ and $\overline{F} \subset \overline{V} \subset V^2$,

$$\overline{F}K_1 \subset U\,\mathsf{Cent}_{\mathrm{G}}(g). \tag{4.5}$$

From (4.3), (4.4) and (4.5) we deduce

$$K \cap \mathsf{Cent}_{\mathrm{G}}(h) \subset U\,\mathsf{Cent}_{\mathrm{G}}(g), \qquad \forall\, h \in V_g^{(1)}. \tag{4.6}$$

Next, we prove the other inclusion needed in (2.2). By Lemma 4.4 there exists an open neighborhood V_e of e such that for every $v \in V_e$, we have

$$K \subset H\,\mathsf{Cent}_{\mathrm{G}}(v). \tag{4.7}$$

In particular, for $v \in V_e \cap H$, we have

$$K \subset \mathsf{Cent}_{\mathrm{G}}(v), \tag{4.8}$$

because $H \subset \mathsf{Cent}_{\mathrm{G}}(v)$. Let $h = gv \in V_g^{(2)} \stackrel{\text{def}}{=} g(V_e \cap H)$. As $\mathsf{Cent}_{\mathrm{G}}(g) \cap \mathsf{Cent}_{\mathrm{G}}(v) \subset \mathsf{Cent}_{\mathrm{G}}(h)$,

$$K \cap \mathsf{Cent}_{\mathrm{G}}(g) \subset U\,\mathsf{Cent}_{\mathrm{G}}(h). \tag{4.9}$$

Finally, let $V_g = V_g^{(1)} \cap V_g^{(2)}$. It is clear from (4.6) and (4.9) that for every $h \in V_g$ we have $\mathsf{Cent}_{\mathrm{G}}(h) \in \mathcal{U}(K,U)(\mathsf{Cent}_{\mathrm{G}}(g))$. Hence there exists a neighborhood V_g of g in G such that

$$V_g \subset \mathsf{Cent}_{\mathrm{G}}{}^{-1}[\mathcal{U}(K,U)(\mathsf{Cent}_{\mathrm{G}}(g))].$$

This completes the proof. ∎

4.1. Case of connected locally compact groups.

We recall that for a topological group G a continuous morphism $X\colon \mathbb{R} \to G$ is called a *one-parameter subgroup* of G. Let $\mathrm{Z}(G)$ denote the center of G. We say that the one-parameter subgroup X is *central* if $X(\mathbb{R}) \subset \mathrm{Z}(G)$.

Lemma 4.9. *Let G be a locally compact group such that the centralizer map $\mathsf{Cent}_{\mathrm{G}}$ is continuous at e. Then every one-parameter subgroup X of G is central.*

Proof. We prove the lemma by contradiction, we suppose that X is not central. Then there is an $r > 0$ in $\mathbb{R}$ such that $X(r) \notin \mathrm{Z}(G)$. For $n \in \mathbb{N} = \{0, 1, \ldots\}$, we set

$$g_n = X(\frac{r}{2^n}).$$

By the continuity of X, $\lim g_n{=}e$. Moreover for each n we have $g_{n+1}^2{=}g_n$. Therefore $\mathsf{Cent}_{\mathrm{G}}(g_{n+1}) \subset \mathsf{Cent}_{\mathrm{G}}(g_n)$, that is the sequence $(\mathsf{Cent}_{\mathrm{G}}(g_n))_{n\in\mathbb{N}}$ is filtered. Let $H \overset{\mathrm{def}}{=} \bigcap_{n\in\mathbb{N}} \mathsf{Cent}_{\mathrm{G}}(g_n)$. By Proposition 2.9, $H = \lim \mathsf{Cent}_{\mathrm{G}}(g_n)$ in $\mathcal{SUB}(G)$. On the other hand, from the continuity of $\mathsf{Cent}_{\mathrm{G}}$ at e, and $\lim g_n = e$ we deduce that $H = G$. In particular $\mathsf{Cent}_{\mathrm{G}}(g_0) = G$. Which contradicts the fact that $g_0 = X(r)$ is not central. ■

As usual, for a topological group G, the identity component of G will be denoted G_0.

Theorem 4.10. *If for a locally compact group G the centralizer map $\mathsf{Cent}_{\mathrm{G}}$ is continuous at e, then the identity component G_0 of G is central.*

Proof. By Lemma 4.9, all one-parameter subgroups of G are central. The result follows from the fact that every pro-Lie group is topologically generated by its one-parameter subgroups (see [11, Corollary 4.22]), and every connected locally compact group is a pro-Lie group ([22]). The theorem is proven. ■

As a consequence we have

Corollary 4.11. *The centralizer map $\mathsf{Cent}_{\mathrm{G}}$ of a connected locally compact group G is continuous if and only if G is abelian.*

4.2. Case of quasidiscrete groups.

Proposition 4.12. *Let G be a locally compact group. Then the following statements are equivalent.*

(1) $\mathsf{Cent}_{\mathrm{G}}$ *is continuous at e;*

(2) $\mathsf{Cent}_{\mathrm{G}}$ *is continuous at all points of $\mathrm{QZ}(G)$.*

Proof. Since clearly (2) implies (1) we must prove that (1) implies (2). Let $g \in \mathrm{QZ}(G)$, U an open set of G such that $\mathsf{Cent}_{\mathrm{G}}(g) \in \mathcal{O}_2(U)$. As $\mathsf{Cent}_{\mathrm{G}}(g) \cap U$ is an open set of G, by the continuity of $\mathsf{Cent}_{\mathrm{G}}$ at e there exists a neighborhood V at e such that for any $h \in V$, $\mathsf{Cent}_{\mathrm{G}}(h) \in \mathcal{O}_2(\mathsf{Cent}_{\mathrm{G}}(g) \cap U)$. Using

$$\mathsf{Cent}_{\mathrm{G}}(h) \cap \mathsf{Cent}_{\mathrm{G}}(g) \subset \mathsf{Cent}_{\mathrm{G}}(x)$$

for every $x{=}gh{\in}gV$, we deduce that for every $x{\in}gV$ we have $\mathsf{Cent}_{\mathrm{G}}(x){\in}\mathcal{O}_2(U)$. ■

Recently, the following was shown by K. H. Hofmann and G. A. Willis ([14, Proposition 5]).

Proposition 4.13. *Let G be a totally disconnected locally compact quasi-discrete topological group. Then the centralizer function* $\mathsf{Cent}_G : G \to \mathcal{SUB}(G)$ *is continuous at all points in the quasi-center of G.*

The following generalizes Proposition 4.13.

Theorem 4.14. *Let G be a quasidiscrete locally compact group. Then the centralizer map* Cent_G *is continuous at all points in the quasi-center of G.*

Proof. Let $g \in \mathrm{QZ}(G)$. To establish the continuity of Cent_G at g, it remains, by Proposition 4.6, to show that if U is an open set of G with $\mathsf{Cent}_G(g) \in \mathcal{O}_2(U)$ then ${\mathsf{Cent}_G}^{-1}[\mathcal{O}_2(U)]$ is a neighborhood of g in G. As the quasi-center $\mathrm{QZ}(G)$ of G is dense, there exists $a \in \mathrm{QZ}(G) \cap \mathsf{Cent}_G(g) \cap U$. Since $\mathsf{Cent}_G(a)$ is an open subgroup, $g\,\mathsf{Cent}_G(a)$ is an open set containing g. For every $h \in g\,\mathsf{Cent}_G(a)$ we have $a \in \mathsf{Cent}_G(h) \cap U$ and hence $\mathsf{Cent}_G(h) \in \mathcal{O}_2(U)$. ∎

Proposition 4.15. *Let G be a compact group which contains an abelian open subgroup H. Then the following statements are equivalent.*

(1) Cent_G *is continuous on G;*

(2) *the center $\mathrm{Z}(G)$ of G is open in G.*

Proof. (2) $\Longrightarrow$ (1) As the center $Z(G)$ of G is open, $\mathrm{QZ}(G) = G$ and the result follows from Theorem 4.14.

(1) $\Longrightarrow$ (2) By Lemma 4.4 there exists an open neighborhood V of e such that for every $v \in V$, we have

$$G = H\,\mathsf{Cent}_G(v). \tag{4.10}$$

In particular, for any v in the open set $V \cap H$, we have

$$G = \mathsf{Cent}_G(v). \tag{4.11}$$

Hence $V \cap H \subset \mathrm{Z}(G)$ and therefore $\mathrm{Z}(G)$ is open. ∎

4.3. Case of profinite groups.

Proposition 4.16. *Let G be a strongly finitely generated group. Then the following statements are equivalent:*

(1) *The centralizer map* Cent_G *is continuous;*

(2) *the centralizer map* Cent_G *is continuous at e;*

(3) *the center $\mathrm{Z}(G)$ is open in G.*

Proof. (1) $\Longrightarrow$ (2) is trivial. (2) $\Longrightarrow$ (3) As $\{G\}$ is open and Cent_G is continuous at e, $Z(G) = \mathsf{Cent}_G{}^{-1}[\{G\}]$ is a neighborhood of e in G and so $Z(G)$ is open in G. (3) $\Longrightarrow$ (1) This follows from Theorem 4.14. ■

Combining Proposition 2.32 and Proposition 4.16 we get the following.

Corollary 4.17. *Let G be a profinite group. We suppose that the Frattini subgroup $\Phi(G)$ is open in G. Then the centralizer map Cent_G is continuous if and only if the center $Z(G)$ is open.*

Second proof. We apply Lemma 4.4 to the compact set G and the open set $\Phi(G)$, there exists a neighborhood V of e such that for each $x \in V$ we have

$$G \subset \Phi(G)\,\mathsf{Cent}_G(x).$$

Then $\Phi(G) \cup \mathsf{Cent}_G(x)$ is a generating set for G. By Proposition 1.9 of [7], $\mathsf{Cent}_G(x)$ generates G. It follows that for each $x \in V$, $\mathsf{Cent}_G(x) = G$. Consequently, $V \subset Z(G)$, and so $Z(G)$ is open in G. ■

Corollary 4.18. *For every finitely generated pro-p group G the centralizer map Cent_G is continuous if and only if the center $Z(G)$ is open.*

Proposition 4.19. *Let G be a p-adic Lie group such that the centralizer map Cent_G is continuous at e. Then the following hold.*

(1) Cent_G *is continuous on G.*

(2) *The Lie algebra $\mathcal{L}(G)$ of G is abelian.*

Proof. (1) By Theorem 8.32 of [7], the group G contains an open subgroup H which is a uniform pro-p group. On the other hand, by Proposition 4.3, Proposition 4.16 and Corollary 2.34, the center $Z(H)$ of H is open in H and so in G. Then, the group G contains an open abelian subgroup $Z(H)$. The result follows from Theorem 4.8.

(2) This follows from the fact that the Lie algebra $\mathcal{L}(G)$ is equal to $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_K$, where K is any uniform open subgroup of G (see Section 9.5 of [7], page 228). ■

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