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# Representing Lie Algebras Using Approximations with Nilpotent Ideals

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**Abstract.** We prove a refinement of Ado's theorem: a $d$-dimensional nilpotent Lie algebra over an algebraically closed field of characteristic zero with an ideal of class $\varepsilon_1$ and codimension $\varepsilon_2$ admits a faithful representation of degree $\binom{d+\varepsilon_1}{\varepsilon_1} \cdot \binom{d+\varepsilon_2}{\varepsilon_2}$. We then apply the theory of almost-algebraic hulls to generalise this result to the representation of arbitrary finite-dimensional Lie algebras and of Lie algebras graded by an abelian, finitely-generated, torsion-free group.
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## 1. Introduction

The classical theorem of Ado and Iwasawa states that every finite-dimensional Lie algebra $\mathfrak{g}$ admits a finite-dimensional, faithful representation. But it is in general quite difficult to determine whether a given Lie algebra admits a faithful representation of a given degree. The problem is easily seen to be equivalent to the computation of the minimal degree $\mu(\mathfrak{g})$ of a faithful representation of $\mathfrak{g}$. The study of this invariant was crucial in finding (filiform) counter-examples to a conjecture by Milnor on the existence of affine structures on manifolds, [19], [4] and [9]. Lower and upper bounds for $\mu(\mathfrak{g})$ in terms of other natural invariants of $\mathfrak{g}$ were given for various families of Lie algebras by Benoist, Birkhoff, Burde, de Graaf, Eick, Grunewald and the author, [5], [6], [8], [10], [11], and [13]. This paper aims to refine the current upper bounds.

Recall that a nilpotent Lie algebra $\mathfrak{n}$ acts faithfully on its universal enveloping algebra $\mathcal{U}(\mathfrak{n})$. Birkhoff observed that the subspace $S$ of all sufficiently large elements (made precise in the next sections), yields a faithful quotient module $\mathcal{U}(\mathfrak{n})/S$ of dimension $\frac{d^{c+1}-1}{d-1}$, where $d$ is the dimension of $\mathfrak{n}$ and $c = c(\mathfrak{n})$ is its class (the length of its lower central series). Malcev later observed that Ado's

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theorem is a natural consequence of the existence of almost-algebraic hulls (also called splittings) and Birkhoff's construction. A constructive approach to Malcev's theorem by Neretin made use of elementary expansions and it allowed Burde and the author to find explicit upper bounds for $\mu(\mathfrak{g})$ for all finite-dimensional complex Lie algebras $\mathfrak{g}$: $\mu(\mathfrak{g}) = O(2^d)$.

Several examples in the literature then suggested that Lie algebras which have an abelian ideal of small codimension in the solvable radical also have a small, faithful representation (see for example [10] and propositions 2.12, 2.15, 4.5 and remark 4.8 of [11]). This was made precise and proven by Burde and the author to obtain bounds of the form $\mu(\mathfrak{n}) = O(d^{\gamma+1})$, where $\mathfrak{n}$ is a $d$-dimensional nilpotent Lie algebra and $\gamma$ is the minimal codimension of an abelian ideal. We prove that this result can be generalised to nilpotent ideals of arbitrary class:

**Theorem 1.1.** *Consider a Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic zero. Let $d$ be its dimension, let $r$ be the dimension of the solvable radical and let $n$ be the dimension of the nilradical. Suppose $\mathfrak{g}$ has a nilpotent ideal of class $\varepsilon_1$ and codimension $\varepsilon_2$ in $\mathrm{rad}(\mathfrak{g})$. Then*

$$\mu(\mathfrak{g}) \le d - n + \binom{r+\varepsilon_1}{\varepsilon_1} \cdot \binom{r+\varepsilon_2}{\varepsilon_2}.$$

We note that de Graaf's theorem, [13], corresponds with the special case where the Lie algebra is itself nilpotent and the ideal is chosen to be the whole Lie algebra: $d := n$, $\varepsilon_1 := c(\mathfrak{g})$ and $\varepsilon_2 := 0$. We define the nil-defect $\varepsilon = \varepsilon(\mathfrak{g})$ of $\mathfrak{g}$ to be the minimal value $\varepsilon_1 + \varepsilon_2$ as $\mathfrak{h}$ runs over all nilpotent ideals of $\mathfrak{g}$.

**Corollary 1.2.** *Consider a Lie algebra $\mathfrak{g}$ of dimension $d$ and nil-defect $\varepsilon$. Then $\mathfrak{g}$ has a faithful representation of degree*

$$P_\varepsilon(d) := d + \frac{(d+\varepsilon)\cdots(d+1)}{\lfloor \frac{\varepsilon}{2} \rfloor! \cdot \lceil \frac{\varepsilon}{2} \rceil!}.$$

We can also apply the construction to graded Lie algebras; there exists a function $E : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that the following holds.

**Corollary 1.3.** *Consider a Lie algebra $\mathfrak{g} = \oplus_{a\in A}\mathfrak{g}_a$ graded by a group of the form $(\mathbb{Z}^k, +)$, for $k \in \mathbb{N}$. Let $\sigma := |\{a \in A | \mathfrak{g}_a \ne \{0\}\}|$ be the cardinality of the support and let $\delta := \dim(\mathfrak{g}_0)$ be the dimension of the homogeneous component corresponding with the neutral element. Then $\mathfrak{g}$ admits a faithful representation of degree $P_{E(\sigma,\delta)}(d)$.*

Convention: we will only consider finite-dimensional Lie algebras over an algebraically closed field of characteristic zero.

## 2. Preliminaries

Let us introduce some of the concepts used in our construction of faithful representations. We first define the nil-defect $\varepsilon(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. We then show that $\mathfrak{g}$ can (almost) be embedded into an almost-algebraic Lie algebra $\widehat{\mathfrak{g}}$ for which the nil-defect $\varepsilon(\widehat{\mathfrak{g}})$ is at most $\varepsilon(\mathfrak{g})$.

**The nil-defect of a Lie algebra** The following definition is justified by the existence of a (unique) solvable radical, that is: the theorem of Levi-Malcev.

**Definition 2.1** (Nil-defect). Let $\mathfrak{r}$ be a solvable Lie algebra and $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{r}$. The nil-defect $\varepsilon(\mathfrak{r},\mathfrak{n})$ of $\mathfrak{n}$ in $\mathfrak{r}$ is $\dim(\mathfrak{r}/\mathfrak{n})+c(\mathfrak{n})$. The nil-defect $\varepsilon(\mathfrak{r})$ of $\mathfrak{r}$ is

$$\min_{\mathfrak{n}}\{\varepsilon(\mathfrak{r},\mathfrak{n})\},$$

where $\mathfrak{n}$ runs over all nilpotent ideals of $\mathfrak{r}$. The nil-defect of an arbitrary Lie algebra is the nil-defect of its solvable radical.

Let us consider a few special cases.

**Example 2.2.** The nil-defect of a semisimple Lie algebra is $0$. The nil-defect of a nilpotent Lie algebra $\mathfrak{n}$ is bounded by the nilpotency class: $\varepsilon(\mathfrak{n})\leq\varepsilon(\mathfrak{n},\mathfrak{n})=c(\mathfrak{n})$. In particular: the family of all Lie algebras of nil-defect at most $\varepsilon$ contains the family of all nilpotent Lie algebras of class at most $\varepsilon$.

We note that Lie algebras of a given nil-defect can have arbitrarily high nilpotency class.

**Example 2.3.** Recall that a nilpotent Lie algebra $\mathfrak{f}$ is called filiform iff $\mathfrak{f}$ has maximal class, that is: $c(\mathfrak{f})+1=\dim(\mathfrak{f})$. There are filiform Lie algebras of nil-defect two and arbitrarily high nilpotency class. More generally: for filiform Lie algebras $\mathfrak{f}$, we have $\varepsilon(\mathfrak{f})\leq 2\sqrt{\dim(\mathfrak{f})}+1$. This is a direct consequence of the fact that any ideal of $\mathfrak{f}$ with codimension $a>1$ is nilpotent of class at most $\lceil(c(\mathfrak{f})+1)/a\rceil-1$, cf. [22].

The following result, not strictly necessary for the rest of the paper, is due to B. Kostant and allows us to approximate solvable Lie algebras with nilpotent *subalgebras*, rather than ideals (personal communication with G. Glauberman and N. Wallach; see for example [16] and [2]. See also [7].)

**Theorem 2.4.** *Consider a finite-dimensional complex,* solvable *Lie algebra* $\mathfrak{r}$ *and a nilpotent subalgebra* $\mathfrak{n}$*. Then* $\mathfrak{r}$ *has an ideal* $\mathfrak{h}$ *of class at most* $c(\mathfrak{n})$ *and dimension equal to* $\dim(\mathfrak{n})$*. In particular:*

$$\varepsilon(\mathfrak{r})=\min_{\mathfrak{m}}\{c(\mathfrak{m})+\operatorname{codim}_{\mathfrak{r}}(\mathfrak{m})\},$$

*where* $\mathfrak{m}$ *runs over the nilpotent subalgebras of* $\mathfrak{r}$*.*

**Almost-algebraic Lie algebras** In this paragraph we are given an arbitrary Lie algebra $\mathfrak{g}$ and we show that there is an embedding $\iota : \mathfrak{g} \longrightarrow \widehat{\mathfrak{g}}$ of $\mathfrak{g}$ into a slightly larger Lie algebra $\widehat{\mathfrak{g}}$ that admits a "nice" decomposition. In order to do this, we first recall the definition of almost-algebraic Lie algebras:

**Definition 2.5** (Almost-algebraic)**.** A Lie algebra $\mathfrak{g}$ is almost-algebraic if it admits a decomposition of the form $\mathfrak{g} = \mathfrak{p} \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is the nilradical of $\mathfrak{g}$ and $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}$ that acts fully reducibly on $\mathfrak{g}$ (by the adjoint representation).

A theorem of Malcev, later generalized by Auslander and Brezin, states that every finite-dimensional Lie algebra over an algebraically closed field of characteristic zero can be embedded into an almost-algebraic Lie algebra, and that there is a minimal such algebra: the *almost-algebraic hull*, [18], [3], [21]. Neretin later gave an explicit construction, as a succession of finitely many elementary expansions, of such an embedding, [20]. This construction was used by Burde and the author to obtain explicit upper bounds for $\mu(\mathfrak{g})$, [11]. Auslander and Brezin observed that ideals are compatible with elementary expansions:

**Lemma 2.6.** *Let $\iota_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ be an elementary expansion of $\mathfrak{g}_1$. Then every ideal $\mathfrak{i}$ of $\mathfrak{g}_1$ maps onto an ideal $\iota_1(\mathfrak{i})$ of $\mathfrak{g}_2$.*

In particular: if $(\iota_j : \mathfrak{g}_j \longrightarrow \mathfrak{g}_{j+1})_{1\le j\le u}$ is a finite sequence of elementary expansions, and $i =: \iota_u \circ \iota_{u-1} \circ \cdots \circ \iota_1$, then every ideal $\mathfrak{i}$ of $\mathfrak{g}_1$ maps onto an ideal $\iota(\mathfrak{i})$ of $\mathfrak{g}_{u+1}$. We may thus combine the lemma with proposition 4.1 of [11] to obtain the following theorem.

**Proposition 2.7.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over the complex numbers. Let $\mathfrak{r}$ be its solvable radical and let $\mathfrak{n}$ be its nilradical. Then there exists an embedding $\iota : \mathfrak{g} \longrightarrow \widehat{\mathfrak{g}}$ of $\mathfrak{g}$ into the Lie algebra $\widehat{\mathfrak{g}}$ such that:*

1. *(Decomposition): $\widehat{\mathfrak{g}}$ decomposes as $\mathfrak{p} \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is nilpotent and $\mathfrak{p}$ acts fully reducibly on $\widehat{\mathfrak{g}}$,*
2. *(Control of dimensions):* $\dim(\mathfrak{m}) = \dim(\mathfrak{r})$ *and* $\dim(\mathfrak{p}) = \dim(\mathfrak{g}/\mathfrak{n})$,
3. *(Preservation of ideals): if $\mathfrak{h}$ is a nilpotent ideal of $\mathfrak{g}$, then $\iota(\mathfrak{h})$ is a nilpotent ideal of $\mathfrak{p} \ltimes \mathfrak{m}$ contained in $\mathfrak{m}$.*

**Proof.** Points (1) and (2) can be obtained by expanding $\dim(\mathfrak{r}/\mathfrak{n})$ times with respect to the nilradical, cf. 4.1 of [11]. Point (3) follows from the lemma. ∎

**Remark 2.8.** In the above decomposition $\mathfrak{p} \ltimes \mathfrak{m}$, the nilpotent ideal $\mathfrak{m}$ need not be the whole nilradical. However, if we let $\mathfrak{p}_0$ be the kernel of the action of $\mathfrak{p}$ on $\mathfrak{m}$, then $\mathfrak{p} \ltimes \mathfrak{m} \cong \mathfrak{p}_0 \oplus (\mathfrak{p}/\mathfrak{p}_0 \ltimes \mathfrak{m})$ and $\mathfrak{m}$ will be the nilradical of the almost-algebraic algebra $(\mathfrak{p}/\mathfrak{p}_0 \ltimes \mathfrak{m})$.

In order to construct faithful representations of $\mathfrak{g}$ it therefore suffices to construct (sufficiently small) faithful representations of $\widehat{\mathfrak{g}}$.

## 3. Quotients of the universal enveloping algebra

In this paragraph we will construct faithful representations of almost-algebraic Lie algebras $\mathfrak{p} \ltimes \mathfrak{m}$. In order to do this we first introduce weight functions $\omega : \mathcal{U}(\mathfrak{m}) \longrightarrow \mathbb{N} \cup \{\infty\}$ on the universal enveloping algebra $\mathcal{U}(\mathfrak{m})$ of the nilpotent Lie algebra $\mathfrak{m}$. We shall then see that the elements of $\mathcal{U}(\mathfrak{m})$ that are sufficiently large with respect to such a weight function form a $(\mathfrak{p} \ltimes \mathfrak{m})$-submodule $\mathcal{S}'$ of $\mathcal{U}(\mathfrak{m})$. A good choice of weight functions will allow us to construct a quotient $\mathcal{U}(\mathfrak{m})/\mathcal{S}$ that is faithful and finite-dimensional.

**Filtrations.** In the following sections we will be working with pairs of filtrations of a given nilpotent Lie algebra. It will be convenient to have a basis that is compatible with those filtrations.

**Definition 3.1.** A *filtration* of a Lie algebra $\mathfrak{g}$ is a flag $(\mathfrak{g}(t))_{t\in\mathbb{N}}$ of subspaces of $\mathfrak{g}$ of the form, $\mathfrak{g} = \mathfrak{g}(0) \supseteq \mathfrak{g}(1) \supseteq \mathfrak{g}(2) \supseteq \cdots$ such that for all $\mathfrak{g}(i), \mathfrak{g}(j)$ we have $[\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j)$. It is a *positive* filtration if $\mathfrak{g}(0) = \mathfrak{g}(1)$.

Note that each element of a filtration is an ideal of the Lie algebra. We will consider the following example in the next paragraphs.

**Example 3.2.** Let $\mathfrak{h}$ be an ideal of a Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}(0) := \mathfrak{g}$, $\mathfrak{h}(1) := \mathfrak{h}$ and $\mathfrak{g}(i) := \mathfrak{h}_i := [\mathfrak{h}, \mathfrak{h}_{i-1}]$ for $i \geq 2$ defines a filtration of $\mathfrak{g}$. Let us call this the $(\mathfrak{g}, \mathfrak{h})$-filtration. The $(\mathfrak{g}, \mathfrak{h})$-filtration is clearly positive if $\mathfrak{g} = \mathfrak{h}$ is nilpotent.

**Definition 3.3.** Let $V$ be a vector space and consider a flag $(V_j)_{j\in\mathbb{N}}$ of $V$. We say that a basis $\mathcal{B}$ of $V$ is *weakly adapted* to the flag, iff for each $V_j$ there exists a subset $\mathcal{B}_j$ of $\mathcal{B}$ that is a basis of $V_j$.

Some elementary observations in linear algebra lead to the following.

**Lemma 3.4.** *Consider a finite-dimensional vector space $V$ and a pair of flags of $V$. Then $V$ has a basis that is weakly adapted to both filtrations.*

The corresponding statement for triples of filtrations fails trivially.

**Notation.** Let us fix some notation. We let $\widehat{\mathfrak{g}}$ be a fixed Lie algebra that decomposes as $\mathfrak{p} \ltimes \mathfrak{m}$, with $\mathfrak{m}$ nilpotent. We let $\mathfrak{h}$ be a nilpotent ideal of $\widehat{\mathfrak{g}}$, contained in $\mathfrak{m}$. Let us also fix the following pair of filtrations of $\mathfrak{m}$: the $(\mathfrak{m}, \mathfrak{m})$-filtration and the $(\mathfrak{m}, \mathfrak{h})$-filtration of $\mathfrak{m}$. The lemma above then allows us to choose a basis for $\mathfrak{m}$ that is weakly adapted to both filtrations. Let $\{x_1, \ldots, x_r\}$ be such a basis.

The Poincaré-Birkhoff-Witt theorem states that the standard (non-commutative, ordered) monomials in the $x_i$ form a basis for the universal enveloping algebra $\mathcal{U}(\mathfrak{m})$ of $\mathfrak{m}$. We also recall that $\mathfrak{p} \ltimes \mathfrak{m}$ acts naturally on $\mathcal{U}(\mathfrak{m})$: $\mathfrak{p}$ acts by derivations and $\mathfrak{m}$ acts by left multiplication. To be precise:

$$\delta * (x_{i_1} \cdots x_{i_t}) := \sum_{1 \leq j \leq t} x_{i_1} \cdots x_{i_{j-1}} \cdot [\delta, x_{i_j}] \cdot x_{i_{j+1}} \cdots x_{i_t}$$

and $x * (x_{i_1} \cdots x_{i_t}) := x \cdot x_{i_1} \cdots x_{i_t}$ for all $x \in \mathfrak{m}$, $\delta \in \mathfrak{p}$, and monomials $x_{i_1} \cdots x_{i_t}$. The $\mathfrak{p} \ltimes \mathfrak{m}$-module $\mathcal{U}(\mathfrak{m})$ is faithful but infinite-dimensional.

**From filtrations to weights and submodules** Let us first define weight functions on $\mathcal{U}(\mathfrak{m})$ and then show how they can be constructed from positive filtrations on $\mathfrak{m}$. Set $\overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$.

**Definition 3.5.** A map $\omega : \mathcal{U}(\mathfrak{m}) \longrightarrow \overline{\mathbb{N}}$ is a *weight* on $\mathcal{U}(\mathfrak{m})$ iff it satisfies the following conditions:

1. $\omega(X) = +\infty \Leftrightarrow X = 0$
2. $\omega(X + Y) \geq \min\{\omega(X), \omega(Y)\}$

for all $X$ and $Y$ in $\mathcal{U}(\mathfrak{m})$. The weight is *compatible with the action* of $\mathfrak{p} \ltimes \mathfrak{m}$ on $\mathcal{U}(\mathfrak{m})$ iff in addition the following conditions hold:

3. $\omega(x * X) \geq \omega(x) + \omega(X)$
4. $\omega(\delta * X) \geq \omega(X)$

for all $X, Y$ in $\mathcal{U}(\mathfrak{m})$, $x$ in $\mathfrak{m}$ and $\delta$ in $\mathfrak{p}$.

We may then define subsets all elements of which are sufficiently large.

**Definition 3.6.** Let $\omega : \mathcal{U}(\mathfrak{m}) \longrightarrow \overline{\mathbb{N}}$ be a map and let $k$ be a natural number. Then we define the set

$$\mathcal{U}^k(\mathfrak{m}, \omega) := \{X \in \mathcal{U}(\mathfrak{m}) | \omega(X) \geq k\}.$$

Let us consider a filtration of $\mathfrak{m}$ and show how it can be used to define a weight on $\mathcal{U}(\mathfrak{m})$. Let $\mathfrak{m}(0) \supseteq \mathfrak{m}(1) \supseteq \mathfrak{m}(2) \supseteq \cdots$ be the filtration. For $x \in \mathfrak{m}$ we define

$$\omega(x) := \sup\{t \in \overline{\mathbb{N}} | x \in \mathfrak{m}(t)\}.$$

We obtain a map $\omega : \mathfrak{m} \longrightarrow \overline{\mathbb{N}}$. We may now extend $\omega$ to all of $\mathcal{U}(\mathfrak{m})$ as follows. For a standard monomial $X^\alpha := x_1^{\alpha_1} \cdots x_r^{\alpha_r}$, with $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ we define

$$\omega(X^\alpha) := \sum_{1 \leq j \leq r} \alpha_j \cdot \omega(x_j),$$

and for a non-redundant linear combination $\sum_j \varphi_j X^{\alpha_j}$ of standard monomials $X^{\alpha_j}$ with linear coefficients $\varphi_j$ we set

$$\omega(\sum_j \varphi_j X^{\alpha_j}) := \min_j \{\omega(X^{\alpha_j})\}.$$

In particular, we may consider special cases:

**Example 3.7.** Suppose $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{m}$, for example $\mathfrak{m}$ itself or $\mathfrak{h}$. We then let $\omega_{(\mathfrak{m},\mathfrak{n})}$ be the weight on $\mathcal{U}(\mathfrak{m})$ obtained from the $(\mathfrak{m},\mathfrak{n})$-filtration. We let $\lambda$ be the weight on $\mathcal{U}(\mathfrak{m})$ obtained from the trivial, positive filtration $\mathfrak{m} := \mathfrak{m}(0) := \mathfrak{m}(1)$ and $\mathfrak{m}(i) := \{0\}$ for $i \geq 2$.

This $\lambda$ can be considered a length function (on the standard monomials).

**Lemma 3.8.** *If $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{p} \ltimes \mathfrak{m}$ contained in $\mathfrak{m}$, then $\omega_{(\mathfrak{m},\mathfrak{n})}$ is a weight on $\mathcal{U}(\mathfrak{m})$ that is compatible with the action of $\mathfrak{p} \ltimes \mathfrak{m}$.*

**Proof.** Property (1) holds since $\mathfrak{n}$ is nilpotent (and only if the ideal is nilpotent). Since $\mathcal{U}^k(\mathfrak{m}, \omega_{(\mathfrak{m},\mathfrak{n})})$ is spanned by the standard monomials of degree at least $k$, we also have the second property. If $x_i$ and $x_j$ are basis vectors with $j < i$, then we have $\omega(x_i \cdot x_j) = \omega(x_j \cdot x_i + [x_i, x_j]) \geq \min(\omega(x_j \cdot x_i), \omega([x_i, x_j])) \geq \omega(x_i) + \omega(x_j)$, by using the definition of $\omega$ on standard monomials and the property of the filtration on $\mathfrak{m}$. By using induction on the length of a standard monomials (and (2)), we then obtain property (3). Property (4) follows from (3) and the fact that $\delta$ stabilises the flag (since $\mathfrak{n}$ is an ideal of $\mathfrak{p} \ltimes \mathfrak{m}$). ∎

We note that if a weight $\omega$ is compatible with the action of $\mathfrak{p} \ltimes \mathfrak{m}$, then each such subspace $\mathcal{U}^k(\mathfrak{m}, \omega)$ is a $(\mathfrak{p} \ltimes \mathfrak{m})$-submodule of $\mathcal{U}(\mathfrak{m})$. In particular: we conclude that for each $k_1, k_2$ in $\mathbb{N}$,

$$\mathcal{S}(\mathfrak{m}, \omega_{(\mathfrak{m},\mathfrak{m})}, k_1, \omega_{(\mathfrak{m},\mathfrak{h})}, k_2) := \mathcal{U}^{k_1}(\mathfrak{m}, \omega_{(\mathfrak{m},\mathfrak{m})}) + \mathcal{U}^{k_2}(\mathfrak{m}, \omega_{(\mathfrak{m},\mathfrak{h})})$$

is a $(\mathfrak{p} \ltimes \mathfrak{m})$-stable subspace of $\mathcal{U}(\mathfrak{m})$ and we may consider the quotient module.

**Properties of the quotient module** We now need to determine two things: the dimension of the quotient and for which choices the quotient will be faithful.

**Proposition 3.9.** *Suppose $\mathfrak{p}$ acts faithfully on $\mathfrak{m}$. If $k_1 > c(\mathfrak{m})$ and $k_2 > c(\mathfrak{h})$, then the quotient of $\mathcal{U}(\mathfrak{m})$ by $\mathcal{S}(\mathfrak{m}, \omega_{(\mathfrak{m},\mathfrak{m})}, k_1, \omega_{(\mathfrak{m},\mathfrak{h})}, k_2)$ is a faithful $(\mathfrak{p} \ltimes \mathfrak{m})$-module.*

**Proof.** Note that $\mathfrak{m}$ is the vector space spanned by all the standard monomials of length one. Also note that the submodule $\mathcal{S}$ is contained in the subspace $\mathcal{U}^2(\mathfrak{m}, \lambda)$ of $\mathcal{U}(\mathfrak{m})$ that is spanned by all standard monomials of length at least two. Clearly, $\mathfrak{m} \cap \mathcal{U}^2(\mathfrak{m}, \lambda) = \{0\}$. Now suppose that $(\delta, x) \in \mathfrak{p} \ltimes \mathfrak{m}$ maps $\mathcal{U}(\mathfrak{m})$ into $\mathcal{S}$. Then $x = (\delta, x) * 1 = \delta(1) + x * 1 = 0 + x = x \in \mathcal{S} \subseteq \mathcal{U}^2(\mathfrak{m}, \lambda)$. We conclude that $x \in \mathcal{U}^2(\mathfrak{m}, \lambda) \cap \mathfrak{m} = \{0\}$. Similarly, $\mathfrak{p}$ maps $\mathfrak{m}$ into $\mathcal{U}^2(\mathfrak{m}, \lambda) \cap \mathfrak{m} = \{0\}$. Since $\mathfrak{p}$ is assumed to act faithfully on $\mathfrak{m}$, we have $(\delta, x) = (0, 0)$. ∎

We recall the following well-known result about Sylvester denumerants.

**Lemma 3.10.** *Consider a finite multiset $M = \{m_1, \ldots, m_p\}$ of positive integers and $t \in \mathbb{N}$. Then the number $\Delta(t; M)$ of $M$-partitions of $t$ is bounded from above by $\binom{p+t-1}{t-1}$.*

In particular: the number of $M$-partitions of $0 \leq t \leq T$ is at most $\binom{p+T}{T}$. The proposition now suggests the choice $k_1 := c(\mathfrak{m})+1$ and $k_2 := c(\mathfrak{h})+1$. We then get:

**Proposition 3.11.** *The* $(\mathfrak{p} \ltimes \mathfrak{m})$*-module*

$$\mathcal{Q} := \mathcal{U}(\mathfrak{m})/\mathcal{S}(\mathfrak{m}, \omega_{(\mathfrak{m},\mathfrak{m})}, c(\mathfrak{m})+1, \omega_{(\mathfrak{m},\mathfrak{h})}, c(\mathfrak{h})+1)$$

*has dimension at most*

$$\binom{\dim(\mathfrak{m}) + \dim(\mathfrak{m}/\mathfrak{h})}{\dim(\mathfrak{m}/\mathfrak{h})} \cdot \binom{\dim(\mathfrak{m}) + c(\mathfrak{h})}{c(\mathfrak{h})}.$$

**Proof.** Since $\mathcal{S}$ is spanned by all standard monomials $X$ satisfying $\omega_{(\mathfrak{m},\mathfrak{m})}(X) \geq c(\mathfrak{m})+1$ or $\omega_{(\mathfrak{m},\mathfrak{h})}(X) \geq c(\mathfrak{h})+1$, the dimension of $\mathcal{Q}$ is bounded from above by the number of standard monomials $Y$ satisfying $\omega_{(\mathfrak{m},\mathfrak{m})}(Y) \leq c(\mathfrak{m})$ and $\omega_{(\mathfrak{m},\mathfrak{h})}(Y) \leq c(\mathfrak{h})$. The lemma above then gives the crude upper bound

$$\dim(\mathcal{Q}) \leq \binom{\dim(\mathfrak{m}/\mathfrak{h}) + c(\mathfrak{m})}{c(\mathfrak{m})} \cdot \binom{\dim(\mathfrak{h}) + c(\mathfrak{h})}{c(\mathfrak{h})}$$

and the identity $\binom{a+b}{b} = \binom{a+b}{a}$ finishes the proof. ■

We note that if a weight $\omega$ is given explicitly, it makes sense to compute the corresponding Sylvester denumerant directly.

**Example 3.12.** If $\mathfrak{f}$ is filiform, then we can find a decomposition $\mathfrak{p} \ltimes \mathfrak{m}$ of $\mathfrak{f}$ (cf. [22] and [11]) such that the number of standard monomials $X$ of $\mathcal{U}(\mathfrak{m})$ satisfying $\omega_{(\mathfrak{m},\mathfrak{m})}(X) = t$ is given by the usual partition function

$$p(t) \sim \frac{e^{\pi\sqrt{\frac{2t}{3}}}}{4t\sqrt{3}}.$$

## 4. Proof of the main theorem

Recall that $\mu(\mathfrak{g})$ is the minimal degree of a faithful representation of $\mathfrak{g}$ and that we wish to prove the inequality

$$\mu(\mathfrak{g}) \leq d - n + \binom{r+\varepsilon_1}{\varepsilon_1} \cdot \binom{r+\varepsilon_2}{\varepsilon_2}.$$

**Proof.** (Main theorem) Let $\iota : \mathfrak{g} \longrightarrow \mathfrak{p} \ltimes \mathfrak{m}$ be the embedding of proposition 1. The monotonicity of the $\mu$-invariant yields $\mu(\mathfrak{g}) \leq \mu(\mathfrak{p} \ltimes \mathfrak{m})$, [11]. We may decompose $\mathfrak{p} \ltimes \mathfrak{m}$ as in remark 1: $\mathfrak{p}_0 \oplus (\mathfrak{p}/\mathfrak{p}_0 \ltimes \mathfrak{m})$. The subadditivity of the $\mu$-invariant gives $\mu(\mathfrak{p} \ltimes \mathfrak{m}) \leq \mu(\mathfrak{p}_0) + \mu(\mathfrak{p}/\mathfrak{p}_0 \ltimes \mathfrak{m})$, [11]. Since $\mathfrak{p}$ acts reductively on itself, it is itself reductive and its reductive subalgebra $\mathfrak{p}_0$ satisfies $\mu(\mathfrak{p}_0) \leq \dim(\mathfrak{p}_0) \leq \dim(\mathfrak{p})$, [11]. Proposition 1 gives $\dim(\mathfrak{p}) \leq \dim(\mathfrak{g}/\mathfrak{n})$ so that $\mu(\mathfrak{p}_0) \leq d - n$.

Proposition 1 guarantees that $\iota(\mathfrak{h})$ is an ideal of $\mathfrak{p}/\mathfrak{p}_0 \ltimes \mathfrak{m}$ of codimension $\dim(\mathfrak{r}/\mathfrak{h})$ in $\mathfrak{m}$. The proposition also gives $\dim(\mathfrak{m}) = \dim(\mathfrak{r})$. Since $\mathfrak{p}/\mathfrak{p}_0$ acts faithfully on $\mathfrak{m}$, we may apply propositions 2 and 3. This finishes the proof. ■

Note that the upper bound is a polynomial in $d$ of degree $\varepsilon_1 + \varepsilon_2$.

**Proof.** (Corollary 1) It suffices to make the following two observations. For natural $\varepsilon_1$ and $\varepsilon_2$ we have the inequality $\lfloor \frac{\varepsilon_1+\varepsilon_2}{2} \rfloor ! \cdot \lceil \frac{\varepsilon_1+\varepsilon_2}{2} \rceil ! \leq \varepsilon_1! \cdot \varepsilon_2!$. Similarly: $(d+\varepsilon_1)!/d! \cdot (d+\varepsilon_2)!/d! \leq (d+\varepsilon_1+\varepsilon_2)!/d!$. ■

## 5. Application: representations of graded Lie algebras

A well-known theorem of Jacobson, about weakly closed sets of nilpotent operators, states that a Lie algebra $\mathfrak{g}$ is nilpotent if it admits a regular derivation, [14]. (For Lie algebras admitting such a transformation, it is known that $\mu(\mathfrak{g}) = O(\dim(\mathfrak{g}))$.) The theorem was later generalized and refined in many different ways, see for example [17] and [12]. Khukhro, Makarenko and Shumyatsky showed in [15] that the existence of almost-regular derivations implies the almost-nilpotency of the Lie algebra. To be precise: they proved that there exist functions $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ and $g : \mathbb{N} \longrightarrow \mathbb{N}$ for which the following is true.

**Theorem 5.1.** *Consider Lie algebra $\mathfrak{g} = \oplus_{a \in A} \mathfrak{g}_a$ graded by a group of the form $(\mathbb{Z}^k, +)$, for $k \in \mathbb{N}$. Let $\sigma$ be the cardinality of the support and let $\delta$ be the dimension of the trivial component. Then $\mathfrak{g}$ admits a nilpotent ideal $\mathfrak{i}$ satisfying $\dim(\mathfrak{g}/\mathfrak{i}) \leq f(\sigma)$ and $c(\mathfrak{i}) \leq g(\sigma, \delta)$.*

The nil-defect of $\mathfrak{g}$ is thus bounded from above by $\varepsilon(\mathrm{rad}(\mathfrak{g}), \mathfrak{i}) \leq E(\sigma, \delta) := f(\sigma, \delta) + g(\sigma)$. Note that the functions $f$ and $g$ may grow very quickly as $\sigma$ and $\delta$ increase. This proves corollary 2.

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