Journal of Lie Theory
Volume **26** (2016) 181–192

Lie Algebras of Maximal Class with Polynomial Multiplication

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Communicated by A. Fialowski

Abstract. We prove that over an algebraically closed field $\mathbb{F}$ of zero characteristic an $\mathbb{N}$-graded Lie algebra of maximal class with polynomial multiplication is isomorphic to W^+, the positive part of the Witt algebra, or to a certain subalgebra of W^+.
Mathematics Subject Classification 2010: Primary 17B50. Secondary 17B05, 17B70.
Key Words and Phrases: Lie algebras of maximal class, $\mathbb{N}$-graded Lie algebras, polynomial multiplication, filiform Lie algebras.

1. Introduction

Filiform Lie algebras of dimension n constitute a subclass of nilpotent Lie algebras of the same dimension. More precisely, a filiform Lie algebra is a nilpotent Lie algebra of maximal class of nilpotency. They form a generic set of points of the affine variety consisting of all n-dimensional nilpotent Lie algebras. Filiform Lie algebras were considered by M. Vergne [14] in the 60's in her studies of the variety of all nilpotent Lie algebras of dimension n. Since that time many attempts have been made in order to classify them. In [10] D. Millionschikov gave a complete list of filiform Lie algebras $\mathrm{g} = \oplus_{i=1}^{n}\mathrm{g}_i$, $\dim \mathrm{g}_i = 1$, $i = 1, \ldots, n$ and $[\mathrm{g}_1, \mathrm{g}_i] = \mathrm{g}_{i+1}$, $i < n$.

A generalization of a filiform Lie algebra to the infinite-dimensional case is a Lie algebra of *maximal class*. Namely, a Lie algebra g is called *residually nilpotent* if $\bigcap_{i=1}^{\infty} \mathrm{g}^i = \{0\}$ where $\mathrm{g}^1 = \mathrm{g}$ and $\{\mathrm{g}^i\}$ is the lower central series of g. A residually nilpotent Lie algebra g is called a Lie algebra of *maximal class* if $\sum_i(\dim \mathrm{g}^i/\mathrm{g}^{i+1} - 1) = 1$. Let us consider an $\mathbb{N}$-graded Lie algebra $L = \sum_{i=1}^{\infty} L_i$. The classification of such algebras L with one-dimensional homogeneous components L_i, and two generators, was obtained by A. Fialowski in [6]. In particular, she showed that there are only three isomorphism types of $\mathbb{N}$-graded Lie algebras of maximal class generated by L_1 and L_2. If L is an $\mathbb{N}$-graded Lie algebra of maximal class generated by the first homogeneous component, then Vergne proved that $L \cong \mathrm{m}_0$ where $\mathrm{m}_0 = \mathrm{span}\{e_i \,|\, i \geq 1\}$, $[e_1, e_i] = e_{i+1}$, $i \geq 2$

ISSN 0949–5932 / $2.50

and the remaining products are all zero. In [1] the case of $\mathbb{N}$-graded Lie algebras of maximal class $L = \langle L_1, L_q \rangle$, $q > 2$, over an algebraically closed field of zero characteristic was studied. Under some technical condition there can only be one isomorphism type of such algebras. The classification of positively graded Lie algebras of maximal class in positive characteristic, which are generated by their first homogeneous component, is obtained in [2, 3, 7]. If positively graded Lie algebras over a field F with positive characteristic $p \geq 3$ are not generated by the first homogeneous component, then their classification was obtained in [4, 5]. Notice that our definition of an $\mathbb{N}$-graded Lie algebra of maximal class is different from one given in [2]. Namely, we do not assume that the first homogeneous component of the grading must be of dimension 2.

In [12, 13] Shalev and Zelmanov suggested to look at the classification problem from a different angle: consider the classification of objects not up to an isomorphism but up to being *commensurable*. Two algebras A and A' are *commensurable* if there exist two ideals I and I' of finite co-dimensions such that $I \cong I'$. They also proposed the following general conjecture:

Let $L = L_1 \supset L_2 \supset \dots$ be a just-infinite filtered Lie algebra over an algebraically closed field of zero characteristic. Suppose that dimensions $\dim L_i/L_{i+1}$, $i \geq 1$ are uniformly bounded. Then one of the following holds:

1. L is solvable;

2. The completion of L is commensurable with the completion of the positive part of a (possibly twisted) loop algebra.

3. The completion of L is commensurable with the completion of the positive part of the Witt algebra W^+.

If L is an $\mathbb{N}$-graded just-infinite Lie algebra, then conjecturally L is either solvable or commensurable with the positive part of some (twisted) loop algebra or of W^+.

The following theorem partially confirms the above conjecture.

Theorem 1.1. *[12] Let L be an $\mathbb{N}$-graded just-infinite Lie algebra with polynomial multiplication. Then L is commensurable with the positive part of the Witt algebra or with the positive part of some loop algebra.*

Let L be a Lie algebra of maximal class over an algebraically closed field of zero characteristic. Then L possesses a canonical filtration: $L = L^1 \supset L^2 \supset \dots$ and all $\dim L^i/L^{i+1}$ are uniformly bounded by 2. Since any Lie algebra of maximal class is just-infinite (see Lemma 4) this possible classification can be applied to Lie algebras of maximal class.

Our goal in this paper is to prove that an $\mathbb{N}$-graded Lie algebra of maximal class with polynomial multiplication is isomorphic to either W^+ or W^q, a subalgebra of W^+ where $q > 2$. Notice that W^+ and W^q are commensurable to each other since they possess isomorphic ideals of finite co-dimensions. In terms of commensurability, W^+ and W^q are indistinguishable.

2. Basic concepts and definitions

One of the main examples of filiform Lie algebras is a *model* one $\mathrm{m}_0(n)$ defined by its basis $e_1, \ldots, e_n$ and nontrivial Lie products: $[e_1, e_i] = e_{i+1},\ i = 2, \ldots, n-1$. In her classical work [14] M. Vergne demonstrated the significance of the model filiform Lie algebra by proving that an arbitrary n-dimensional filiform Lie algebra is isomorphic to some deformation of $\mathrm{m}_0(n)$. An infinite-dimensional analogue of $\mathrm{m}_0(n)$ is a Lie algebra of maximal class m_0 with an infinite basis $\{e_1, e_2, \ldots\}$ such that $[e_1, e_i] = e_{i+1},\ i > 1$, and all other products are zero.

In the case of an $\mathbb{N}$-graded Lie algebras of maximal class, Vergne proved the following:

Theorem 2.1. *Let $L = \bigoplus_{i\in\mathbb{N}} L_i$ be an infinite-dimensional $\mathbb{N}$-graded Lie algebra of maximal class and suppose $L = \langle L_1\rangle$. Then $L \cong \mathrm{m}_0$.*

Let us consider important examples of infinite-dimensional Lie algebras of maximal class.

Let $\mathbb{F}[x]$ be an algebra of polynomials in one variable over $\mathbb{F}$. Then the Witt algebra $W = Der(\mathbb{F}[x]) = \left\{f(x)\frac{\partial}{\partial x} \mid f(x) \in \mathbb{F}[x]\right\}$ is a full algebra of derivations of $\mathbb{F}[x]$. A $\mathbb{Z}$-grading is given by the following decomposition:

$$W = W_{-1} \oplus W_0 \oplus W_1 \oplus \ldots$$

where $W_k = \mathrm{span}_{\mathbb{F}}\left\{x^{k+1}\frac{\partial}{\partial x}\right\},\ k \geq -1$. Denote $e_k = x^{k+1}\frac{\partial}{\partial x}$. Then

$$[e_i, e_j] = (j-i)e_{i+j},\ i, j \geq -1. \tag{1}$$

A subalgebra $W^+ = \bigoplus_{i>0} W_i$ is called the positive part of W. Condition (1) also implies that W^+ is an $\mathbb{N}$-graded algebra with the ith component of the grading defined by $(W^+)_i = \mathrm{span}\{e_i\}$ with

$$[(W^+)_1, (W^+)_i] = (W^+)_{i+1}. \tag{2}$$

The Lie algebra W^q, $q > 2$ is a subalgebra of W^+ spanned by $\{e_1, e_q, \ldots\}$. This is an $\mathbb{N}$-graded Lie algebra with lacunas in the grading from 2 to $q-1$. It is easy to see that W^q is a graded subalgebra of W^+, and graded components of W^q also satisfy (2). In fact, W^q is an ideal of W^+ of a finite co-dimension. By setting $y_1 = e_1$ and $y_i = 60(i-2)!e_i,\ i \geq q$ we have that $[y_1, y_i] = y_{i+1}$, $i \geq q$ and $[y_i, y_j] = \beta_{ij}\, y_{i+j}$, $i, j \geq q$ where

$$\beta_{ij} = \frac{60(i-2)!(j-2)!(j-i)}{(i+j-2)!}. \tag{3}$$

Any n-dimensional filiform Lie algebra g admits an *adapted basis*:

$$f_1, f_2, \ldots, f_n$$

such that

$$[f_1, f_s] = f_{s+1}, \qquad [f_i, f_j] \in \mathrm{span}\{f_k \mid k \geq i+j\}$$

where $2 \leq s \leq n-1$. Considering a Lie algebra L of maximal class as the direct limit of nested filiform Lie algebras we obtain that L also possesses an infinite-dimensional adapted basis.

Recall that an algebra is said to be *just-infinite* if it has no ideals of infinite co-dimension. This is a generalization of the concept of being simple.

Let L be an $\mathbb{N}$-graded Lie algebra, i.e. $L =$ $bigoplus_{i\in\mathbb{N}} L_i$ with all $\dim L_i = d$. Choose a basis for each L_i: $e_1^i, \dots, e_d^i$. We say that multiplication of L is *polynomial* (with respect to the above basis) if there are polynomials $p_{kq}^r(i,j)$, $k, q, r = 1, \dots, d$, such that

$$[e_k^i, e_q^j] = \sum_{r=1}^{d} p_{kq}^r(i,j) e_r^{i+j}. \tag{4}$$

Clearly, multiplication of W^q is polynomial and defined by $p(i,j) = j - i$. Throughout the paper we assume that the base field is an algebraically closed of zero characteristic.

3. Proof of the main theorem

Lemma 3.1. *Let L be a Lie algebra of maximal class. Then the minimum possible number of generators for L equals to two. If L is additionally $\mathbb{N}$-graded, then the minimum possible number of homogeneous generators for L is also two.*

Proof. As mentioned in Section 2, one can choose a basis $\{f_i\}_{i=1}^{\infty}$ with

$$[f_1, f_i] = f_{i+1},\ i \geq 2 \text{ and } [f_i, f_j] \in \operatorname{span}\{f_k \mid k \geq i+j\}.$$

In particular, this means that $L = \langle f_1, f_2 \rangle$, i.e. L is generated by f_1 and f_2.

Let us now assume that L is $\mathbb{N}$-graded, i.e. $L = \sum_{i\in\mathbb{N}} L_i$. Choose an $\mathbb{N}$-graded basis $\{e_i\}$, and write

$$f_1 = \alpha_1 e_{i_1} + \ldots + \alpha_s e_{i_s},\ f_2 = \beta_1 e_{i_1} + \ldots + \beta_s e_{i_s},$$

where $\alpha_i, \beta_j \in \mathbb{F}$. Thus, L is generated by the homogeneous set $\{e_{i_1}, \dots, e_{i_s}\}$. We next show that the number of homogeneous generators can be reduced to two. If some e_{i_j} from the above set can be expressed as a linear combination of products of (lower degree) basis elements, i.e.

$$e_{i_j} = \sum_{\bar{\mu}\in\mathbb{N}^l} \alpha_{\bar{\mu}} [\ldots [e_{\mu_1}, e_{\mu_2}], \ldots, e_{\mu_l}] \tag{5}$$

where $\bar{\mu} = (\mu_1, \dots, \mu_l)$, $\alpha_{\bar{\mu}}$ are scalars, then we replace e_{i_j} in the generating set with $\bigcup_{\bar{\mu}} \{e_{\mu_1}, e_{\mu_2}, \dots, e_{\mu_l}\}$. Of course, we do not add repeating basis elements or basis elements that were already in the generating set. The resulting set still generates L. After the finite number of such replacements the procedure stops. Consequently, we obtain a generating set in which no element admits representation (5). We denote these generators by $e_{r_1}, e_{r_2}, \dots, e_{r_m}$. By Corollary 1.12 [11], the graded algebra of L with respect to the canonical filtration

$$gr_C L = \bigoplus_{i\geq 1} L^i / L^{i+1} \cong \mathrm{m}_0.$$

Hence, $\dim L/L^2 = 2$ where $L^2 = [L, L]$. By our choice of generators, no e_{r_i}, $i = 1, \ldots, m$ belongs to L^2. Otherwise, it could be expressed as a linear combination of products of basis elements of lower degrees. Therefore,

$$2 = \dim L/L^2 \geq m, \text{ hence } m = 1 \text{ or } 2.$$

If $m = 1$, then L is a trivial one-dimensional Lie algebra and cannot be of maximal class. Hence, $m = 2$, as required. The proof is complete. ∎

It is easy to see that the analogous result holds for a finite-dimensional *filiform* Lie algebra L, that is, the minimal number of homogeneous generators is also two.

Lemma 3.2. *Let $L = \bigoplus_{i=1}^{n} L_i$ be an $\mathbb{N}$-graded filiform Lie algebra. Assume that L cannot be generated by the first nonzero homogeneous component. Then each nonzero component is one-dimensional.*

Proof. Let us write L as

$$L = L_{i_1} + L_{i_2} + \ldots + L_{i_k}$$

where each L_{i_j}, $j = 1, 2, \ldots, k$, is a nonzero homogeneous component. Clearly, $\dim L \geq k$. Recall that L is filiform. Hence, its nil-index $m = \dim L - 1 \geq k - 1$. Directly computing components of the lower central series of L we obtain the following:

$$\begin{aligned} &L^2 \subseteq L_{i_3} + \ldots + L_{i_k}, \\ &L^3 \subseteq L_{i_4} + \ldots + L_{i_k}, \\ &\ldots \\ &L^{k-1} \subseteq L_{i_k}, \\ &L^k = \{0\}. \end{aligned}$$

This means that nil-index $m \leq k - 1$. Therefore, $m = \dim L - 1 = k - 1$, and $\dim L = k$. Since there are exactly k nonzero graded components, each component must be one-dimensional. The proof is complete. ∎

Corollary 3.3. *Let $L = \bigoplus_{i=1}^{\infty} L_i$ be an $\mathbb{N}$-graded Lie algebra of maximal class. Assume that L cannot be generated by the first nonzero component of the grading. Then each non-zero component is one-dimensional.*

Proof. This follows from the fact that L is a direct limit of filiform Lie algebras $L_{(n)} = L/\bigoplus_{i \geq n} L_i$ where $L_{(n+1)}$ is a one-dimensional central extension of $L(n)$. ∎

Lemma 3.4. *Let $L = L_1 + L_q + L_{q+1} + \ldots$ be a Lie algebra of maximal class generated by both L_1 and L_q. Then $L_{2+q} \neq \{0\}$.*

Proof. By Corollary 1 all nonzero graded components are one-dimensional. Choose e_1, e_q such that $L_1 = \text{span}\{e_1\}$ and $L_q = \text{span}\{e_q\}$. Since L_1, L_q generate L, $[L_1, L_q] \neq \{0\}$ and $[L_1, L_q] = L_{q+1} \neq \{0\}$. Assume that $L_{q+2} = \{0\}$. Equivalently, $[L_1, [L_1, L_q]] = \{0\}$. Choose a basis element e_{q+1} of L_{q+1} such that $e_{q+1} = [e_1, e_q]$. Then we have that $[e_1, e_{q+1}] = 0$, i.e. $[e_1, [e_1, e_q]] = 0$. Choose an adapted basis for L:

$$f_1, f_2, f_3, \dots$$

such that $[f_1, f_i] = f_{i+1}$ and $[f_i, f_j] \in \text{span}\{f_k \,|\, k \geq i + j\}$. Since $\dim L/L^2 = 2$,

$$L/L^2 = \text{span}\{f_1, f_2\} + L^2 = \text{span}\{e_1, e_q\} + L^2$$

where $L^2 = [L, L] = \text{span}\{f_k \,|\, k \geq 3\}$, and $L^2/L^3 = f_3 + L^3 = e_{q+1} + L^3$ where $L^3 = [[L, L], L] = \text{span}\{f_k \,|\, k \geq 4\}$. Write $e_1 = \lambda_1 f_1 + \lambda_2 f_2 + h$, $e_q = \mu_1 f_1 + \mu_2 f_2 + h'$, $h, h' \in \text{span}\{f_k \,|\, k \geq 3\}$, and $e_{q+1} = \gamma_3 f_3 + h''$ where $\gamma_3 \neq 0$, $h'' \in \text{span}\{f_k \,|\, k \geq 4\}$. Therefore,

$$[e_1, e_{q+1}] = [\lambda_1 f_1 + \lambda_2 f_2 + h, \gamma_3 f_3 + h''] = \lambda_1 \gamma_3 f_4 + h''' = 0$$

where $h''' \in \text{span}\{f_k \,|\, k \geq 5\}$. Thus, $\lambda_1 \gamma_3 = 0$. Since $\gamma_3 \neq 0$, $\lambda_1 = 0$. This means that $e_1 = \lambda_2 f_2 + h$, $\lambda_2 \neq 0$ and $e_q = \mu_1 f_1 + \mu_2 f_2 + h'$, $\mu_1 \neq 0$. Then $e_{q+1} = [e_1, e_q] = -\mu_1 \lambda_2 f_3 + \tilde{h}$ where $\tilde{h} \in \text{span}\{f_k \,|\, k \geq 4\}$. Therefore,

$$0 = [e_1, e_{q+1}] = [\lambda_2 f_2 + h, -\lambda_2 \mu_1 f_3 + \tilde{h}] = -\lambda_2^2 \mu_1 f_5 + \hat{h} = 0$$

where $\hat{h} \in \text{span}\{f_k \,|\, k \geq 6\}$. Therefore, $\lambda_2^2 \mu_1 = 0$ which contradicts to the fact $\lambda_2 \neq 0$ and $\mu_1 \neq 0$. Hence, $L_{q+2} \neq \{0\}$. The lemma is proved. ∎

It follows from the previous lemma that the more general version of Corollary 3.13 from [1] (without restriction $L_{q+2} \neq \{0\}$) holds true.

Corollary 3.5. *Let L be an $\mathbb{N}$-graded Lie algebra of maximal class not generated by the first homogeneous component. Then L is generated by L_1 and L_q where L_q is a nonzero homogeneous component following L_1. Moreover, each graded component is one-dimensional, and there exists a basis $e_1, e_q, e_{q+1}, \dots$ for L such that*

$$L_i = \text{span}\{e_i\} \text{ and } [e_1, e_i] = e_{i+1},\ i \geq q.$$

Lemma 3.6. *Let L be an $\mathbb{N}$-graded Lie algebra of maximal class. Then L is just-infinite.*

Proof. If $L = \langle L_1 \rangle$, then by Theorem 2 $L \cong \mathrm{m}_0$. It is obvious that m_0 is just-infinite, as required. If $L \neq \langle L_1 \rangle$, then by the previous corollary, $L = \langle L_1, L_q \rangle$ with a homogeneous basis: $e_1, e_q, e_{q+1}, \dots$ and $[e_1, e_i] = e_{i+1}$, $i \geq q$. Consider a nontrivial ideal $J \lhd L$. Notice that if $e_i \in J$, then

$$[e_1, e_i] = e_{i+1} \in J,$$

$$[e_1, e_{i+1}] = e_{i+2} \in J,$$

$$\dots .$$

This means that each basis element starting from e_i belongs to J. Thus, J is of finite codimension, as required. Let no e_i be an element of J, i.e. $\bar{e}_i = e_i + J \neq \bar{0}$ in L/J, $i \geq 1$. Choose any nonzero $x \in J$. Let

$$x = \alpha_1 e_1 + \alpha_q e_q + \ldots + \alpha_k e_k$$

where $\alpha_k \neq 0$. Hence,

$$\alpha_1 \bar{e}_1 + \alpha_q \bar{e}_q + \ldots + \alpha_k \bar{e}_k = \bar{0} \text{ in } L/J.$$

This means that $\bar{e}_k \in \operatorname{span}\{\bar{e}_1, \bar{e}_q, \ldots, \bar{e}_{k-1}\}$. Multiplying x by e_1 we obtain $[e_1, x] \in J$, and

$$\begin{aligned}[e_1, x] &= \alpha_q [e_1, e_q] + \ldots + \alpha_{k-1}[e_1, e_{k-1}] + \alpha_k [e_1, e_k] \\ &= \alpha_q e_{q+1} + \ldots + \alpha_{k-1} e_k + \alpha_k e_{k+1} \in J.\end{aligned}$$

Therefore,

$$\bar{0} = \alpha_q \bar{e}_{q+1} + \ldots + \alpha_{k-1}\bar{e}_k + \alpha_k \bar{e}_{k+1}, \ \alpha_k \neq 0.$$

Thus,

$$\bar{e}_{k+1} \in \operatorname{span}\{\bar{e}_{q+1}, \ldots, \bar{e}_k\} \subseteq \operatorname{span}\{\bar{e}_1, \bar{e}_q, \ldots, \bar{e}_{k-1}\},$$

and so on. Consequently, $L/J \subseteq \operatorname{span}\{\bar{e}_1, \bar{e}_q, \ldots, \bar{e}_{k-1}\}$. This means that

$$\dim L/J < \infty,$$

as required. ■

Corollary 3.7. *Let $L = \bigoplus_{i \geq 1} L_i$ be an $\mathbb{N}$-graded Lie algebra of maximal class with polynomial multiplication. Then L is nonsolvable.*

Proof. By Corollary 1, all nonzero components of the grading are one-dimensional. The multiplication is defined by a polynomial $p(i,j)$ as follows

$$[e_i, e_j] = p(i,j) e_{i+j}.$$

Assume that L is solvable. By Theorem 2.2 [9], L is a solvable Lie algebra of maximal class if and only if $[L, L]$ is Abelian. Since L is just-infinite, $[L, L]$ must be of finite co-dimension in L. Hence, $[L, L]$ contains an infinite tail $\sum_{i \geq M} L_i$ for a sufficiently large M. Consequently, $[e_i, e_j] = 0$ if $i, j \geq M$. Hence, $p(i,j) = 0$ if $i, j \geq M$. This means that p is a zero polynomial, and L has a trivial multiplication. This contradicts to the fact that L is of maximal class. Therefore, L is nonsolvable. The proof is complete. ■

Lemma 3.8. *Let $L = \bigoplus_{i=1,m}^{\infty} L_i$ be an $\mathbb{N}$-graded Lie algebra of maximal class generated by L_1, L_m, $m \geq 2$. Let S be a graded subalgebra of L isomorphic to W^l. If $\dim L/S = 1$, then either $L \cong W^+$ for $m = 2$ and $L \cong W^m$ for $m > 2$.*

Proof. By Corollary 2,

$$L = L_1 \oplus L_m \oplus L_{m+1} \oplus L_{m+2} \oplus \ \ldots$$

where all components are one-dimensional. Given that S is a graded subalgebra of L. Since $S \cong W^l$ we have that

$$S = L_{i_1} \oplus \ldots \oplus L_{i_k} \oplus L_{i_{k+1}} \oplus \ldots$$

and the first component of the grading of S acts on the other components according to (2), i.e. $[L_{i_1}, L_{i_k}] = L_{i_{k+1}}$. Since S is of finite co-dimension in L, S must contain $\sum_{i \geq r} L_i = L_r + L_{r+1} + \ldots$ for a sufficiently large r. This means that there exists $s \geq 1$ such that $i_s = r$ and $i_{s+1} = r + 1$. Hence, $[L_{i_1}, L_{i_s}] = L_{i_{s+1}}$ implies that $L_{r+1} = [L_{i_1}, L_r] = L_{i_1+r}$, i.e. $i_1 = 1$. This proves that $L_1 \subseteq S$, and

$$S = L_1 \oplus L_{i_2} \oplus L_{i_2+1} \ldots \oplus L_{i_2+k} \oplus \ldots.$$

Since $\dim L/S = 1$ we have that $i_2 = m + 1$. Thus, $S \cong W^{m+1}$.

We next want to show that $L \cong W^+$ for $m = 2$ and $L \cong W^m$ for $m > 2$. It is sufficient to prove that multiplication of L is uniquely determined by multiplication of S. Since $S \cong W^{m+1}$ we can choose a basis for S:

$$y_1, y_{m+1}, y_{m+2}, \ldots$$

with structure constants given by (3). Choose $y_m \in L_m$ such that $[y_1, y_m] = y_{m+1}$. Let $[y_m, y_i] = \alpha_{mi} y_{m+i}$ where $i \geq m + 1$, and α_{mi} are non-zero scalars. To show that multiplication of S defines that of L, we need to express α_{mi} in terms of β_{ij}. For convenience, denote $\alpha_{m,m+1}$ by α. Hence, $[y_m, y_{m+1}] = \alpha y_{2m+1}$ where α is unknown.

Since $J(y_1, y_m, y_{m+1}) = 0$ we have that

$$[[y_1, y_m], y_{m+1}] + [[y_m, y_{m+1}], y_1] + [[y_{m+1}, y_1], y_m] = 0,$$

$$\alpha[y_{2m+1}, y_1] - [y_{m+2}, y_m] = 0,$$
$$-\alpha y_{2m+2} + \alpha_{m,m+2} y_{2m+2} = 0,$$
$$\alpha_{m,m+2} = \alpha$$

Since $J(y_1, y_m, y_{m+2}) = 0$ we have that

$$[[y_1, y_m], y_{m+2}] + [[y_m, y_{m+2}], y_1] + [[y_{m+2}, y_1], y_m] = 0,$$

$$[y_{m+1}, y_{m+2}] + \alpha_{m,m+2}[y_{2m+2}, y_1] - [y_{m+3}, y_m] = 0,$$
$$\beta_{m+1,m+2} - \alpha_{m,m+2} + \alpha_{m,m+3} = 0,$$
$$\alpha_{m,m+3} = \alpha_{m,m+2} - \beta_{m+1,m+2},$$

$$\alpha_{m,m+3} = \alpha - \beta_{m+1,m+2}$$

Since $J(y_1, y_m, y_{m+3}) = 0$ we have that

$$[[y_1, y_m], y_{m+3}] + [[y_m, y_{m+3}], y_1] + [[y_{m+3}, y_1], y_m] = 0$$
$$[y_{m+1}, y_{m+3}] + \alpha_{m,m+3}[y_{2m+3}, y_1] - [y_{m+4}, y_m] = 0,$$
$$\beta_{m+1,m+3} - \alpha_{m,m+3} + \alpha_{m,m+4} = 0,$$
$$\alpha_{m,m+4} = \alpha_{m,m+3} - \beta_{m+1,m+3} = \alpha - \beta_{m+1,m+2} - \beta_{m+1,m+3}.$$

Therefore, by induction, we can show that

$$\alpha_{m,m+k} = \alpha - \beta_{m+1,m+2} - \ldots - \beta_{m+1,m+k-1},\ k \geq 2.$$

Finally, it follows from $J(y_m, y_{m+1}, y_{m+2}) = 0$ that

$$[[y_m, y_{m+1}], y_{m+2}] + [[y_{m+1}, y_{m+2}], y_m] + [[y_{m+2}, y_m], y_{m+1}] = 0,$$
$$\alpha[y_{2m+1}, y_{m+2}] + \beta_{m+1,m+2}[y_{2m+3}, y_m] - \alpha_{m,m+2}[y_{2m+2}, y_{m+1}] = 0,$$
$$\alpha\beta_{2m+1,m+2} - \alpha_{m,2m+3}\beta_{m+1,m+2} - \alpha_{m,m+2}\beta_{2m+2,m+1} = 0,$$
$$\alpha\beta_{2m+1,m+2} - (\alpha - \beta_{m+1,m+2} - \ldots - \beta_{m+1,2m+2})\beta_{m+1,m+2} - \alpha\beta_{2m+2,m+1} = 0,$$
$$\alpha(\beta_{2m+1,m+2} - \beta_{m+1,m+2} - \beta_{2m+2,m+1}) = -\beta_{m+1,m+2}(\beta_{m+1,m+2} + \ldots + \beta_{m+1,2m+2}).$$

Since

$$\beta_{2m+1,m+2} - \beta_{m+1,m+2} - \beta_{2m+2,m+1} = \frac{30\Gamma(m)(-\Gamma(m)\Gamma(3m+2) + 4m^3\Gamma(2m)^2 + 14m^2\Gamma(2m)^2 + 6m\Gamma(2m)^2)}{(2m+1)\Gamma(2m)\Gamma(3m+2)} \neq 0$$

for $m \geq 2$ ($\Gamma(\ )$ denotes a gamma function). This means that α is uniquely determined by the above equation. Therefore, L has a basis $\{y_i\}$ with structure constants given by (2). The lemma is proved. ■

Lemma 3.9. *Let $L = \bigoplus_{i=1,m}^{\infty} L_i$ be an $\mathbb{N}$-graded Lie algebra of maximal class generated by L_1 and L_m, $m \geq 2$. If S is a graded subalgebra of type W^l, then either $L \cong W^+$ for $m = 2$ or $L \cong W^m$ for $m > 2$.*

Proof. By Lemma 4, L is just-infinite. Since $S \lhd L$, S must be of a finite co-dimension. Iterated application of Lemma 5 completes the proof. ■

Before proving the main result of the paper let us recall one construction from [12]. Assume that $L = \bigoplus_{i\in\mathbb{N}} L_i$ is an $\mathbb{N}$-graded Lie algebra with polynomial multiplication defined by polynomials $p_{kq}^r(i,j)$ where $k, q, r = 1, \ldots, d$. Then L can be turned into a $\mathbb{Z}$-graded Lie algebra $\tilde{L}$ by adding graded components of the same dimension d, i.e. $\tilde{L}_i = \mathrm{span}\{e_1^i, \ldots, e_d^i\}$, $i \in \mathbb{Z}$, with multiplication defined by

$$[e_k^i, e_q^j] = \sum_{r=1}^{d} p_{kq}^r(i,j) e_r^{i+j},\ k, q = 1, \ldots, d,\ i, j \in \mathbb{Z}$$

Clearly, L is a graded subalgebra of $\tilde{L}$. As proved in [12] if L is just-infinite, then the following set

$$I = \{x \in \tilde{L} \,|\, [x, \sum_{i\geq k} \tilde{L}_i] = 0\} \text{ for a sufficiently large } k = k(x)$$

is a graded ideal of $\tilde{L}$ such that $L \bigcap I = \{0\}$. Moreover, $K = \tilde{L}/I$ is a $\mathbb{Z}$-graded algebra, i.e. $K = \bigoplus_{i\in\mathbb{Z}} K_i$. Besides, L is isomorphically embedded into K, and can be considered a graded subalgebra of K. By Lemma 2.8 [12] $[K,K]$ is a $\mathbb{Z}$-graded simple Lie algebra that were classified in [8].

We are now ready to prove the main result of this paper.

Theorem 3.10. *Let $L = \bigoplus_{i\in\mathbb{N}} L_i$ be an $\mathbb{N}$-graded Lie algebra of maximal class with polynomial multiplication. Assume that L_1 is nonzero. If L_2 is also nonzero, then $L \cong W^+$. Otherwise, $L \cong W^q$ for some natural $q > 2$.*

Proof. First, we note that under the above conditions L cannot be generated by L_1. Otherwise, by Theorem 2 $L \cong \mathrm{m}_0$ and, therefore, L must be solvable. This contradicts to Corollary 3. By Corollary 2 the first two nonzero graded components L_1 and L_q generates L, i.e. $L = \langle L_1, L_q\rangle$. If $q = 2$, then $L = \langle L_1, L_2\rangle$. By Corollary 3 L is nonsolvable. Applying Theorem 1.7 [12] we have that $L \cong W^+$.

Let $q > 2$. In other words, there is a *lacunas* in grading of L from 2 to $q-1$. By Corollary 2 we can choose a graded basis:

$$e_1, e_q, e_{q+1}, \ldots$$

such that $L_i = \mathrm{span}\{e_i\}$ and $[e_1, e_i] = e_{i+1}$, $i \geq q$. Thus,

$$L = L_1 + \sum_{i\geq q} L_i.$$

By Lemma 4 L is just-infinite. Therefore, we can apply results from the paper [12]. In particular, we can consider L as a graded subalgebra of a $\mathbb{Z}$-graded algebra $K = \sum_{i\in\mathbb{Z}} K_i$ where K was defined above. Let us write L as follows

$$L = K_1 + \sum_{i\geq q} K_i$$

where K_1 is the 1-component of the grading of K. As shown in the proof of Theorem 2.4 [12], the positive part of $[K,K]$ is either a positive part of a loop algebra or W^+ (the case of twisted loop algebra is not possible). Notice that $[L,L] \lhd L$ and $[L,L] \subseteq [K,K]$. Since L is just-infinite, the ideal $[L,L]$ has a finite codimension in L. Therefore, it must contain $\sum_{i\geq r} K_i$ for sufficiently large r. Let S stand for $\sum_{i\geq r} K_i$. Clearly, $S \subseteq [L,L]$. Hence, $S \subseteq [K,K]$, i.e. S is a graded subalgebra of $[K,K]$. Notice that homogeneous components of $[K,K]$ are all of dimension 1. If $[K,K]$ is a loop algebra, i.e. $[K,K] = \mathrm{g}[t,t^{-1}]$ where g is a finite-dimensional simple Lie algebra, then the dimensions of homogeneous components are equal to $\dim \mathrm{g} > 1$, a contradiction.

Let us consider the other case when the positive part $[K,K]^+$ of $[K,K]$ is isomorphic to W^+. Then

$$[K,K]^+ = \bigoplus_{j=1}^{\infty} K_{i_j}, \text{ and all } i_j > 0.$$

Due to $[K,K]^+ \cong W^+$, the first component of the grading of $[K,K]^+$ should act on the remaining components according to (2). This means that $[K_{i_1}, K_{i_j}] = K_{i_{j+1}}$. Since

$$S = \sum_{i \geq r} K_i = K_r + K_{r+1} + \ldots \subseteq [K,K] = \bigoplus_{j=1}^{\infty} K_{i_j},$$

there exists an index s such that $i_s = r$, and $i_{s+1} = r+1$. Then $[K_{i_1}, K_{i_s}] = K_{i_{s+1}}$ implies that $K_{r+1} = [K_{i_1}, K_r] = K_{i_1+r}$. Thus, $i_1 + r = r+1$, $i_1 = 1$. Hence, $K_1 \subseteq [K,K]$. Consider

$$\tilde{S} = K_1 \oplus S = K_1 + \sum_{i \geq r} K_i, \tag{6}$$

which is a subalgebra of $[K,K]^+$. Since $[K,K]^+ \cong W^+$, it is easy to see that $\tilde{S}$ as in (6) must be of type W^r. Since $K_1 \subseteq L$ and $S \subseteq L$ we have that $\tilde{S} \subseteq L$. Clearly, $\tilde{S}$ is of finite codimension in L. Applying Lemma 6 we can show that L is of type W^q, as required. The proof is complete. ■

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Received November 11, 2014
and in final form March 16, 2015