

θ -Semisimple Classes of Type D in $\mathrm{PSL}_n(q)$

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Abstract. Let p be an odd prime, $m \in \mathbb{N}$ and set $q = p^m$, $\mathbf{G} = \mathrm{PSL}_n(q)$. Let θ be a standard graph automorphism of \mathbf{G} , d be a diagonal automorphism and Fr_q be the Frobenius endomorphism of $\mathrm{PSL}_n(\overline{\mathbb{F}}_q)$. We show that every $(d \circ \theta)$ -conjugacy class of a $(d \circ \theta, p)$ -regular element in \mathbf{G} is represented in some Fr_q -stable maximal torus of $\mathrm{PSL}_n(\overline{\mathbb{F}}_q)$ and that most of them are of type D. We write out the possible exceptions and show that, in particular, if $n \geq 5$ is either odd or a multiple of 4 and $q > 7$, then all such classes are of type D. We develop general arguments to deal with twisted classes in finite groups.

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1. Introduction

This paper belongs to the series started in [ACGa1], in which we intend to determine all racks related to (twisted) conjugacy classes in simple groups of Lie type which are of type D *cf.* (2.1), as proposed in [AFGaV2, Question 1]. This, although being mainly a group-theoretical question, is intimately related with the classification of finite-dimensional pointed Hopf algebras over non-abelian groups, see below. In this article we will focus on racks which arise as non-trivial twisted conjugacy classes in $\mathrm{PSL}_n(q)$ for $q = p^m$, p an odd prime.

Recall that a rack is a non-empty set X together with a binary operation \triangleright satisfying faithfulness and self-distributive axioms, see Section 2. The prototypical example of a rack is a twisted conjugacy class \mathcal{O}_x^ψ with respect to an automorphism $\psi \in \mathrm{Aut}(G)$ inside a finite group G , $x \in G$, with

$$y \triangleright z = y\psi(zy^{-1}), \quad y, z \in \mathcal{O}_x^\psi. \quad (1.1)$$

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This is in fact a quandle, as $y \triangleright y = y$, $\forall y \in \mathcal{O}_x^\psi$.

A rack X is said to be of *type D* when there exists a decomposable subrack $Y = R \sqcup S \subseteq X$ and elements $r \in R$, $s \in S$ such that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$, see Section 2. Their study is deeply connected with the classification problem of finite-dimensional pointed Hopf algebras, as follows.

Let H be a finite dimensional pointed Hopf algebra over an algebraically closed field \mathbb{k} and assume the coradical of H is $\mathbb{k}G$, for a finite non-abelian group G . Following [AG, Section 6.1], there exist a rack X and a 2-cocycle \mathbf{q} with values in $\text{GL}(n, \mathbb{k})$ such that $\text{gr } H$, the associated graded algebra with respect to the coradical filtration, contains as a subalgebra the bosonization $\mathfrak{B}(X, \mathbf{q}) \# \mathbb{k}G$. See *loc. cit.* for unexplained notation. Therefore, it is central for the classification of such Hopf algebras to know when $\dim \mathfrak{B}(X, \mathbf{q}) < \infty$ for given X , \mathbf{q} . A rack X is said to *collapse* when $\mathfrak{B}(X, \mathbf{q})$ is infinite dimensional for any \mathbf{q} . A remarkable result is that *if X is of type D, then it collapses*. This is the content of [AFGV1, Theorem 3.6], also [HS, Theorem 8.6], both of which follow from results in [AHS].

Now every rack X admits a rack epimorphism $\pi : X \rightarrow S$ with S simple and it follows that X is of type *D* if S is so. Hence, determining all simple racks of type *D* is a drastic reduction indeed for the classification problem, as many groups can be discarded and only a few conjugacy classes in simple groups remain. Only for such classes one needs to compute the possible cocycles that yield a finite dimensional Nichols algebra. Simple racks are classified into three classes [AG], also [J], namely *affine*, *twisted homogeneous* and that of *non-trivial twisted conjugacy classes on finite simple groups*, see [AG] for definitions. Most (twisted) conjugacy classes in sporadic groups are of type *D* [AFGV2], [FV]. This is also the case for non-semisimple classes in $\text{PSL}_n(q)$ [ACGa1], for unipotent classes in symplectic groups [ACGa2] and for (twisted) classes in alternating groups [AFGV1]. Similar results follow for twisted homogeneous racks [AFGaV1]. Affine racks seem to be not of type *D*.

In this article we begin the analysis of twisted classes of type *D* in $\text{PSL}_n(q)$, for q odd and automorphisms induced by algebraic group automorphisms of $\text{SL}_n(\overline{\mathbb{F}}_q)$. Recall that the automorphisms in $\text{PSL}_n(q)$ are compositions of automorphisms induced by conjugation in $\text{GL}_n(q)$ (diagonal and inner automorphisms), powers of a standard graph automorphism θ of the Dynkin diagram and powers of the Frobenius automorphism Fr_p . Inner automorphisms may be neglected [AFGaV1, §3.1]. Diagonal and graph automorphisms are induced by algebraic group automorphisms of $\text{SL}_n(\overline{\mathbb{F}}_q)$, whereas Fr_p is induced by an abstract group endomorphism. Their behaviour is therefore different [St, 10.13] and this is reflected in the structure of the twisted classes. In addition, if the $d \circ \theta^a$ -class of x in $\text{PSL}_n(p)$ is of type *D*, d a diagonal automorphism and $a = 0, 1$, then the $\text{Fr}_p^m \circ d \circ \theta^a$ -class of x in $\text{PSL}_n(q)$ is of type *D* for every m and every q . Thus, we will focus on twisted classes for automorphisms $\psi = d \circ \theta^a$. The analysis of standard conjugacy classes in simple groups of Lie type (corresponding to $a = 0$) has been started in [ACGa1, ACGa2]. For these reasons the first twisted classes to look at are the ψ -classes in $\text{PSL}_n(q)$, where ψ is a composition of a diagonal automorphism d with θ . In analogy to the case of standard conjugacy classes, it is possible to reduce most of the analysis to the study of classes whose behaviour

resembles that of semisimple or unipotent ones. However, in contrast to that case, the choices to be made depend on the gcd of $|\psi|$ and p cf. Definition 3.7. Therefore, the cases of p even and odd must be handled with different methods. The diagonal automorphisms always satisfy $(|\psi|, p) = 1$ so we restrict to the case $(|\psi|, p) = 1$ and we will require p to be odd.

Set $\mathbf{G} = \mathrm{PSL}_n(q)$, $\psi = d \circ \theta \in \mathrm{Aut}(\mathbf{G})$, for d a diagonal automorphism. The study of (ψ, p) -regular classes in \mathbf{G} , i. e., of those classes replacing semisimple ones, can be reduced to the study of (θ, p) -regular \mathbf{G} -orbits of elements in $\mathrm{PGL}_n(q)$. Such classes have a representative in a maximal torus $\overline{\mathrm{T}}_w^{\mathrm{Fr}_q}$ of $\mathrm{PGL}_n(q)$, for some $w \in W^\theta$, where we can take w up to conjugation cf. Theorem 5.1. It turns out that in most cases, the property of being of type D depends on n , q and the conjugacy class of w in W^θ . Such classes are parametrized by a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of $h = \lfloor \frac{n}{2} \rfloor$, with $r \in \mathbb{N}$, $\lambda_i > 0$, and a certain vector $\varepsilon \in \mathbb{Z}_2^r$. Hence our result depends on the number of cycles r of λ and on the vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r) \in \mathbb{Z}_2^r$. Let $\mathbf{1}$ stand for the partition $(1, \dots, 1)$.

Theorem 1.1. *Let q be as above. Let $x \in \overline{\mathrm{T}}_w^{\mathrm{Fr}_q}$. Then the class $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D, with the possible exceptions of classes fitting into the following table:*

w		n	q	x
$\lambda \neq \mathbf{1}$	$r = 2$	$\varepsilon = (0, \varepsilon_2)$	even	3,5 any
	$r = 1$	$\varepsilon = (0)$	4	3,7 any
			4	5,9 $\theta(x) \neq x^{-1}$
		$\varepsilon = (1)$	4	3,7 $\theta(x) \neq x^{-1}$
			$2 \times \text{odd}$	any
$\lambda = \mathbf{1}$		any*	3,5	any*
		3	7,13	any
		4	$\equiv 3(4)$	any
		4	9	any

Table 1: Possible exceptions; ν as in (5.5).

Actually, some of the classes listed on the table are of type D, for instance when $n \geq 6$, $n \neq 7$ and $\varepsilon = (0, \dots, 0)$, see Lemma 5.6. See also Remark 5.12. We present this result in the language of Nichols algebras, as a partial answer in this cases to [AFGaV2, Question 2], see also [AFGV1, Theorem 3.6], and *loc. cit.* for unexplained notation. Consider the classes $\mathcal{O}_x^{\theta, \mathbf{G}}$ in Theorem 1.1 as racks with the rack structure (1.1). These are simple racks.

Corollary 1.2. *Let $X = \mathcal{O}_x^{\theta, \mathbf{G}}$, $x \in \overline{\mathrm{T}}_w^{\mathrm{Fr}_q}$. Then $\dim \mathfrak{B}(\mathcal{O}_x^{\theta, \mathbf{G}}, \mathbf{q}) = \infty$ for any cocycle \mathbf{q} on X , with the possible exceptions of the classes in Table 1. ■*

Also, an extract of Theorem 1.1 can be rephrased as follows.

Theorem 1.1’. *Let p be an odd prime, $m \in \mathbb{N}$, $q = p^m$. Set $\mathbf{G} = \mathrm{PSL}_n(q)$, $\psi = d \circ \theta \in \mathrm{Aut}(\mathbf{G})$, for d a diagonal automorphism.*

If $n \geq 5$, $q \geq 7$, then any (ψ, p) -regular class \mathcal{O} is of type D with the possible exception $n = 2 \times \text{odd}$, $\mathcal{O} \simeq \mathcal{O}_1^{\mathrm{Ad}(\nu^{-1}) \circ \theta, \mathbf{G}}$, ν as in (5.5). ■

When $\psi = \theta$, we obtain the following for classes with trivial (θ, p) -regular part (also called θ -semisimple part) which is the content of Propositions 6.1 and 6.2:

Proposition 1.3. *Let \mathcal{O} be a θ -twisted conjugacy class with trivial θ -semisimple part. Then \mathcal{O} is of type D provided*

1. $n > 2$ is even, the unipotent part is nontrivial, and $q > 3$.
2. $n > 3$ is odd and the Jordan form of its p -part in \mathbf{G}^θ corresponds to the partition (n) . ■

The paper is organized as follows. In Section 2 we fix the notation and recall some generalities about racks and the group $\mathrm{PSL}_n(q)$. In Section 3 we discuss some general techniques to deal with twisted conjugacy classes in a finite group. In Section 4 we focus on $\mathrm{PSL}_n(q)$ and we begin a systematic approach to the study of its twisted classes, that includes an analysis of the Weyl group. In Section 5 we concentrate on θ -semisimple classes and obtain the main results of the article. In Section 6 we present some results on classes with trivial θ -semisimple part.

2. Preliminaries

Let H be a group, $\psi \in \mathrm{Aut}(H)$. A ψ -twisted conjugacy class, or simply a twisted conjugacy class, is an orbit for the action of H on itself by $h \cdot_\psi x = hx\psi(h)^{-1}$. We denote this class by \mathcal{O}_h^ψ . If $K < H$ is ψ -stable, we will write $\mathcal{O}_h^{\psi, K}$ to denote the orbit of h under the restriction of the \cdot_ψ -action to K . In particular, $\mathcal{O}_h = \mathcal{O}_h^{\mathrm{id}}$ denotes the (standard) conjugacy class of $h \in H$. The stabilizer in $K < H$ of an element $x \in H$ for the twisted action will be denoted by $K_\psi(x)$ so that $H_{\mathrm{id}}(x)$ is H_x , the usual centralizer of x . For any automorphism ψ of a group H , we write H^ψ for the set of ψ -invariants in H . The inner automorphism given by conjugation by $x \in H$ will be denoted by $\mathrm{Ad}(x)$. If $K \triangleleft H$ is normal, then we also denote by $\mathrm{Ad}(x)$ the automorphism induced from the conjugation by $x \in H$. $\mathcal{Z}(H)$ will denote the center of H . Recall that the group $\mu_n(\mathbb{F}_q)$ of roots of unity in a finite field \mathbb{F}_q is isomorphic to $\mathbb{Z}_{\mathbf{d}}$, for $\mathbf{d} := (n, q - 1)$.

We denote by \mathbb{S}_n , $n \in \mathbb{N}$, the symmetric group on n letters. We also set $\mathbb{I}_n := \{1, 2, \dots, n\}$ and $(b)_a = 1 + a + a^2 + \dots + a^{b-1}$, $a, b \in \mathbb{N}$.

A rack (X, \triangleright) is a non-empty finite set X together with a function $\triangleright : X \times X \rightarrow X$ such that $i \triangleright (\cdot) : X \rightarrow X$ is a bijection for all $i \in X$ and

$$i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k), \forall i, j, k \in X.$$

Recall that a rack (X, \triangleright) is a *quandle* when $i \triangleright i = i$, $\forall i \in X$.

We shall write simply X when the function \triangleright is clear from the context.

If H is a group, then the conjugacy class \mathcal{O}_h of any element $h \in H$ is a rack, with the function \triangleright given by conjugation. More generally, if $\psi \in \text{Aut}(H)$, any twisted conjugacy class in H is a rack with rack structure given by (1.1), see [AG, Theorem 3.12, (3.4)]. These are indeed examples of quandles.

A subrack Y of a rack X is a subset $Y \subseteq X$ such that $Y \triangleright Y \subseteq Y$. A rack is said to be *indecomposable* if it cannot be decomposed as the disjoint union of two subracks. A rack X is said to be *simple* if $\text{card } X > 1$ and for any surjective morphism of racks $\pi : X \rightarrow Y$, either π is a bijection or $\text{card } Y = 1$.

A rack X is of type D when there exists a decomposable subrack $Y = R \sqcup S$ of X and elements $r \in R, s \in S$ such that

$$r \triangleright (s \triangleright (r \triangleright s)) \neq s. \tag{2.1}$$

If a rack X has a subrack of type D, or if there is a rack epimorphism $X \twoheadrightarrow Z$ and Z is of type D, then X is again so. In particular, if X is decomposable and X has a component of type D, then X is of type D. On the other hand, if X is indecomposable, then it admits a projection $X \twoheadrightarrow Z$, with Z simple. Hence, in the quest of racks of type D it is enough to focus on simple racks. The classification of simple racks is given in [AG, Theorems 3.9, 3.12], see also [J]. A big class consists of twisted conjugacy classes in finite simple groups.

Remark 2.1. Let \mathcal{O} be a ψ -twisted conjugacy class. Then \mathcal{O} is of type D if there are $r, s \in \mathcal{O}$ such that $r \notin \mathcal{O}_s^{\psi, L}$, for L the ψ -stable closure of the subgroup generated by r and s , and

$$r\psi(s)\psi^2(r)\psi^3(s) \neq s\psi(r)\psi^2(s)\psi^3(r). \tag{2.2}$$

In fact, if the above conditions hold, we set $S = \mathcal{O}_s^{\psi, L}$ and $R = \mathcal{O}_r^{\psi, L}$ and then $Y = R \sqcup S$ is a decomposable subrack of \mathcal{O} which satisfies (2.1).

If $\psi = \text{id}$ then the condition is also necessary: if \mathcal{O} is of type D, then there are $r, s \in \mathcal{O}, r \notin \mathcal{O}_s^L$, satisfying (2.2) [ACGa1, Remark 2.3].

Fix $n \in \mathbb{N}$. Let $p \in \mathbb{N}$ be a prime number and let $\mathbb{k} = \overline{\mathbb{F}_p}$. Fix $m \in \mathbb{N}, q = p^m$. We assume throughout the paper that $n > 2$ or $q \neq 2, 3$.

We fix once and for all the following notation:

$$\mathbb{G} = \text{SL}_n(\mathbb{k}), \quad \overline{\mathbb{G}} = \text{PSL}_n(\mathbb{k}), \quad \mathbf{G} := \text{PSL}_n(q). \tag{2.3}$$

We also fix $\pi : \text{GL}_n(\mathbb{k}) \rightarrow \text{PGL}_n(\mathbb{k}) \simeq \overline{\mathbb{G}}$ the usual projection. We keep the name $\pi := \pi|_{\mathbb{G}} : \mathbb{G} \rightarrow \overline{\mathbb{G}}$ for the restriction of π . We fix the subgroups of diagonal matrices

$$\mathbb{T} \leq \text{GL}_n(\mathbb{k}), \quad \mathbb{T} \leq \mathbb{G}, \quad \overline{\mathbb{T}} := \pi(\mathbb{T}) \leq \overline{\mathbb{G}}. \tag{2.4}$$

Consider the exact sequence:

$$1 \longrightarrow \mathcal{Z}(\mathbb{G}) \longrightarrow \mathbb{G} \xrightarrow{\pi} \overline{\mathbb{G}} \longrightarrow 1 \tag{2.5}$$

and let $F = \text{Fr}_p^m$ be the endomorphism of $\text{GL}_n(\mathbb{k})$ raising every entry in $X \in \text{GL}_n(\mathbb{k})$ to the q -th power. Taking F -points, (2.5) yields:

$$1 \longrightarrow \mathcal{Z}(\text{SL}_n(q)) \longrightarrow \text{SL}_n(q) \longrightarrow \text{PGL}_n(q).$$

Then $\mathbf{G} \leq \text{PGL}_n(q)$ is the image of the last arrow:

$$\mathbf{G} = \text{PSL}_n(q) \simeq \text{SL}_n(q)/\mathcal{Z}(\text{SL}_n(q)) \simeq \text{SL}_n(q)/\mathbb{Z}_{\mathbf{d}},$$

for $\mathbf{d} = (n, q - 1)$. The group \mathbf{G} is simple¹.

We will denote by $\mathbb{B}, \mathbb{U}, \mathbb{U}^- \leq \mathbb{G}$ be the subgroups of \mathbb{G} of upper triangular, unipotent upper-triangular, unipotent lower-triangular matrices. Set

$$W := N_{\mathbb{G}}(\mathbb{T})/\mathbb{T} \simeq N_{\overline{\mathbb{G}}}(\overline{\mathbb{T}})/\overline{\mathbb{T}} \simeq \mathbb{S}_n.$$

Recall that $[\text{SL}_n(q), \text{SL}_n(q)] = \text{SL}_n(q)$ and $[\text{PGL}_n(q), \text{PGL}_n(q)] = \mathbf{G}$, for $n > 2$ or $q \neq 2, 3$. Also, we have the identifications:

$$\begin{aligned} \overline{\mathbb{G}}^F = \text{PGL}_n(q) &= \overline{\mathbb{T}}^F [\text{PGL}_n(q), \text{PGL}_n(q)] = \overline{\mathbb{T}}^F \mathbf{G} \\ &\simeq \text{GL}_n(q)/\mathcal{Z}(\text{GL}_n(q)) \simeq \text{GL}_n(q)/\mathbb{F}_q^\times. \end{aligned}$$

Recall that a *diagonal automorphism* of \mathbf{G} is an automorphism induced by conjugation by an element in $\overline{\mathbb{T}}^F$. The *graph automorphism* $\theta: \text{GL}_n(\mathbb{k}) \rightarrow \text{GL}_n(\mathbb{k})$ is given by $x \mapsto \mathbf{J}_n {}^t x^{-1} \mathbf{J}_n^{-1}$, for

$$\mathbf{J}_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1} & \cdots & 0 & 0 \end{pmatrix}. \tag{2.6}$$

It induces a non-trivial automorphism of \mathbb{G} for $n \geq 3$ and it is unique up to inner automorphisms². It also induces automorphisms of $\text{GL}_n(q), \text{SL}_n(q), \text{PGL}_n(q)$ and \mathbf{G} . We will drop the subscript n and write $\mathbf{J} = \mathbf{J}_n$ when it can be deduced from the context.

By [MT, Theorem 24.24] every automorphism of \mathbf{G} is the composition of an inner, a diagonal, a power of Fr_p and a power of θ , so the elements in group of outer automorphisms of \mathbf{G} have representatives in $\text{Out}(\mathbf{G}) := \langle \text{Fr}_p, \theta, \text{Ad}(t) : t \in \overline{\mathbb{T}}^F \rangle$.

3. General arguments

In this section we present some general techniques to deal with twisted conjugacy classes in finite groups. We start with a well-known lemma.

Lemma 3.1. *Let H be a finite group, $\varphi \in \text{Aut}(H)$. Let $K, N < H$ be φ -stable subgroups, with $N \triangleleft H$. Fix $x \in H$.*

¹Recall that $\text{PSL}_2(2) \simeq \mathbb{S}_3$, $\text{PSL}_2(3) \simeq \mathbb{A}_4 \leq \mathbb{S}_4$.

²Indeed, this is not the choice made in [ACGa1] but it is, however, more adequate for our setting.

1. The set $\mathcal{O}_x^{\varphi,K}$ is a subrack of $\mathcal{O}_x^{\varphi,H}$ if and only if for every $k \in K$ there is $t \in H_\varphi(x)$ such that $xkx^{-1}t \in K$.
2. [AFGaV1, §3.1] Assume $\varphi = \text{Ad}(x) \circ \psi$, for some $\psi \in \text{Aut}(H)$. Then for every $g \in H$ there are racks isomorphisms $\mathcal{O}_g^{\varphi,H} \simeq \mathcal{O}_{gx}^{\psi,H}$ and $\mathcal{O}_g^{\varphi,N} \simeq \mathcal{O}_{gx}^{\psi,N}$.
3. Let $y \in H$ with $y \in \mathcal{O}_x^{\varphi,H}$. Then $\mathcal{O}_x^{\varphi,N} \simeq \mathcal{O}_y^{\varphi,N}$.

Proof. 1. is straightforward. In 2., we have the equality of sets $\mathcal{O}_g^{\varphi,H} = \mathcal{O}_{gx}^{\psi,H}x^{-1}$ and right multiplication by x defines the rack isomorphism. The second isomorphism follows by restriction. As for 3., let $g \in H$ be such that $g \cdot_\varphi x = y$. Then the map $T : \mathcal{O}_x^{\varphi,N} \rightarrow \mathcal{O}_y^{\varphi,N}$ given by $T(z) = g \cdot_\varphi z$ is a rack isomorphism. Observe that if $z = h \cdot_\varphi x$ then $T(z) = (ghg^{-1}) \cdot_\varphi y$. ■

Remark 3.2. Notice that the assumption in 1. in Lemma 3.1 holds if $x \in N_H(K)$. In particular, it always holds for $K \triangleleft H$. Also, 2. allows us to neglect inner automorphisms of H .

The following slight generalization of [FV, Lemma 2.5] will be very useful.

Lemma 3.3. Let H be a finite group and let $K \triangleleft H$. Let $s \in H$ be a non-trivial involution. Then \mathcal{O}_s^K is a rack of type D if and only if there is r in \mathcal{O}_s^K such that $|rs|$ is even and greater than 4.

Proof. By Lemma 3.1, Remark 3.2, \mathcal{O}_s^K is a rack. Observe first that, if $r \in \mathcal{O}_s^K$, then the racks $\mathcal{O}_s^{\langle r,s \rangle}$ and $\mathcal{O}_r^{\langle r,s \rangle}$ are subracks of \mathcal{O}_s^K . Indeed, if $r = k \triangleright s = ksk^{-1}$, then a generic element of $\langle s, r \rangle$ has the form $y_{a,b} = s^a ksk^{-1} s \cdots ksk^{-1} s^b$ for $a, b \in \{0, 1\}$. Let $sk s = l \in K$. Then, if $a = 1$ we have

$$\begin{aligned} y_{1,b} \triangleright s &= y_{1,0} \triangleright s = lk^{-1} \cdots lk^{-1} \triangleright s \in \mathcal{O}_s^K, \\ y_{1,b} \triangleright r &= lk^{-1} \cdots lk^{-1} s^b k s^b \triangleright s \in \mathcal{O}_s^K, \end{aligned}$$

whereas if $a = 0$ we have

$$\begin{aligned} y_{0,b} \triangleright s &= y_{0,1} \triangleright s = kl^{-1} \cdots kl^{-1} \triangleright s \in \mathcal{O}_s^K, \\ y_{0,b} \triangleright r &= kl^{-1} \cdots kl^{-1} s^{b-1} k s^{b-1} \triangleright s \in \mathcal{O}_s^K, \end{aligned}$$

so the racks $\mathcal{O}_s^{\langle r,s \rangle}, \mathcal{O}_r^{\langle r,s \rangle} \subset \mathcal{O}_s^K$. Now, if an r as in the statement exists, then

$$r \triangleright (s \triangleright (r \triangleright s)) \neq s \text{ and } \mathcal{O}_s^{\langle r,s \rangle} \text{ and } \mathcal{O}_r^{\langle r,s \rangle} \text{ are disjoint,}$$

so \mathcal{O}_s^K is of type D by Remark 2.1 for $\psi = \text{id}$. Conversely, if there is no such an r , then for every $x \in \mathcal{O}_s^K$ either $|xs| \leq 4$ or it is odd, so either $(xs)^2 = (sx)^2$ or $\mathcal{O}_s^{\langle s,x \rangle} = \mathcal{O}_x^{\langle s,x \rangle}$ and Remark 2.1 for $\psi = \text{id}$ applies once more. ■

Remark 3.4. Let H be a finite group, $\phi \in \text{Aut}(H)$, $h \in H$.

1. Assume $K = H_h$ is ϕ -stable. If $k \in K$, then $\mathcal{O}_{kh}^{\phi,K} = \mathcal{O}_k^{\phi,K}h$ as sets and right multiplication by h^{-1} gives a rack isomorphism $\mathcal{O}_{kh}^{\phi,K} \simeq \mathcal{O}_k^{\phi,K}$.
2. Let $L = H \rtimes \langle \phi \rangle$. Then, for $x = g\phi$, we have the equality of sets: $\mathcal{O}_g^{\phi,H} = \mathcal{O}_x^L \phi^{-1}$ and $y \mapsto y\phi$ induces a rack isomorphism $\mathcal{O}_g^{\phi,H} \simeq \mathcal{O}_x^L$.

Remark 3.5. Let H be a finite group, $\phi \in \text{Aut}(H)$. Let A be a ϕ -stable abelian subgroup of H , $a \in A$.

1. By Remark 3.4.1, $\mathcal{O}_a^{\phi,A} \simeq \mathcal{O}_1^{\phi,A}$ as racks. Moreover $\gamma: A \rightarrow A, b \mapsto b\phi(b^{-1})$, is a group morphism and $\mathcal{O}_1^{\phi,A} = \text{Im}(\gamma) \simeq A/A^\phi$ as groups.
2. If ϕ is an involution, then $\mathcal{O}_a^{\phi,A}$ is of type D if and only if there is $b \in A/A^\phi$ such that $|b|$ is even, > 4 by Remark 3.4 2. and Lemma 3.3.
3. Let p be a prime number dividing $|H|$. Let $h = us = su \in H$ be the (unique) decomposition of h as a product of a p -element u and a p -regular element s . If $\mathcal{O}_u^{H_s}$ is of type D, then \mathcal{O}_h is again so, as $\mathcal{O}_u^{H_s}$ identifies with a subrack of \mathcal{O}_h^H .

Remark 3.6. Let H be a group, let $\phi, \psi \in \text{Aut}(H)$, with $\phi\psi = \psi\phi$, and let $N \triangleleft H$ be ϕ -stable and ψ -stable.

1. If $\mathcal{O}_h^{\phi,N} \cap H^\psi \neq \emptyset$, then $\psi(\mathcal{O}_h^{\phi,N}) = \mathcal{O}_h^{\phi,N}$. Indeed, let $x \in \mathcal{O}_t^{\phi,N}$ with $\psi(x) = x$. Now, if $y = kx\phi(k^{-1}) \in \mathcal{O}_x^{\phi,N} = \mathcal{O}_h^{\phi,N}$, $k \in N$, then $\psi(y) = \psi(k)x\phi(\psi(h)^{-1}) \in \mathcal{O}_h^{\phi,N}$.

2. Conversely, if $\psi(\mathcal{O}_h^{\phi,N}) = \mathcal{O}_h^{\phi,N}$ and the map $N \rightarrow N$, given by $x \mapsto x^{-1}\psi(x)$, $x \in N$, is surjective, then $\mathcal{O}_h^{\phi,N} \cap H^\psi \neq \emptyset$. To see this, fix $g \in N$ such that $\psi(h) = gh\phi(g^{-1})$ and let $x \in N$ be such that $g^{-1} = x^{-1}\psi(x)$. Then it follows that $x \cdot_\phi h \in H^\psi \cap \mathcal{O}_h^{\phi,N}$.

Let H be a finite group, p be a prime number dividing $|H|$ and let $\psi \in \text{Aut}(H)$, with $\ell := |\psi|$. Set $\widehat{H} = H \rtimes \langle \psi \rangle$.

Definition 3.7. An element $h \in H$ is called (ψ, p) -regular if $h\psi$ is p -regular in \widehat{H} , *i. e.* if $(|h\psi|, p) = 1$. An element $h \in H$ is called a (ψ, p) -element if $h\psi$ is a p -element in \widehat{H} , *i. e.* if $|h\psi| = p^a$ for some $a \in \mathbb{N}$.

Let $\psi = \psi_r \psi_p$ be the decomposition of ψ as a product of its usual p -regular part and its p -part in $\text{Aut}(H)$. Then for every $h\psi$ in \widehat{H} we have $h\psi = s\psi_r(u)\psi = u\psi_p(s)\psi$ where s is (ψ_r, p) -regular and u is a (ψ_p, p) -element in H .

In the quest of ψ -classes of type D, a first analysis can be done by looking at subracks given by the orbits with respect to H^{ψ_r} or H^{ψ_p} . For this reason, the analysis should begin with the cases in which either $\psi_p = 1$, *i. e.* when $(\ell, p) = 1$, or when $\psi_r = 1$, *i. e.* when ℓ is a power of p .

If $(\ell, p) = 1$, then for every $h \in H$ there is a unique decomposition $h = us = s\psi(u)$ with u a p -element in H and s a (ψ, p) -regular element. In this case s is (ψ, p) -regular if and only if the norm $\text{Norm}_\psi(s) := s\psi(s) \cdots \psi^{\ell-1}(s)$ is p -regular in H . Here, if $C = H_\psi(s)$ and $C' = \widehat{H}_{s\psi}$, then Remarks 3.4 2. and 3.5 (3) give the rack inclusions

$$\mathcal{O}_h^{\psi,H} \simeq \mathcal{O}_{h\psi}^{\widehat{H}} \supset \mathcal{O}_u^{C'} \supset \mathcal{O}_u^C. \tag{3.1}$$

So if \mathcal{O}_u^C is of type D, then $\mathcal{O}_h^{\psi,H}$ is again so. Hence the first classes to be attacked are either standard conjugacy classes of p -elements in C or twisted classes of (ψ, p) -regular elements in H . The latter are dealt with in Section 5.

Similarly, if $\ell = p^b$ for some $b > 0$, then for each $h \in H$ there is a unique decomposition $h = su = u\psi(s)$ with s a usual p -regular element in H and u a (ψ, p) -element. In this case u is a (ψ, p) -element if and only if $\text{Norm}_\psi(u)$ is a p -element in H . The first reduction is to look at classes of (ψ, p) -elements and the standard p -regular classes in $H_\psi(u)$. We will not pursue this analysis in this paper.

Notice that, when dealing with twisted classes in simple groups of Lie type, there is a privileged choice for p , namely, the defining characteristic.

4. Twisted classes and $\text{PSL}_n(q)$

In this section we collect some results that contribute to establish a systematic approach to twisted classes in $\text{PSL}_n(q)$. This in particular requires a detailed study of the conjugacy classes in the subgroup of θ -invariant elements of the Weyl group, and of the corresponding F -stable maximal tori in \mathbf{G} , that we develop ahead.

Recall the notation from §2, specially in (2.3), (2.4). Next proposition deals with diagonal automorphisms $d = \text{Ad}(t)$, $t \in \overline{\mathbb{T}}^F$.

Proposition 4.1. *Let $x \in \mathbf{G}$, $\varphi = \text{Ad}(t) \circ \psi \in \text{Aut}(\mathbf{G})$, $t \in \overline{\mathbb{T}}^F$. Let $y = t^{-1}x \in \overline{\mathbb{G}}^F$. Then $\mathcal{O}_x^{\varphi, \mathbf{G}} \simeq \mathcal{O}_y^{\psi, \mathbf{G}}$. If, in addition, $\psi \in \text{Out}(\mathbf{G})$ and $z \in \mathcal{O}_y^{\psi, \text{PGL}_n(q)}$, then $\mathcal{O}_x^{\varphi, \mathbf{G}} \simeq \mathcal{O}_z^{\psi, \mathbf{G}}$.*

Proof. In this case, $x = ty$ and $\mathcal{O}_x^{\varphi, \mathbf{G}} \simeq \mathcal{O}_y^{\psi, \mathbf{G}}$ by Lemma 3.1 2. The last assertion is Lemma 3.1 3. ■

Let $\psi = \text{Fr}_p^a \circ \theta^b \in \text{Aut}(\text{GL}_n(q))$ and let $\ell := |\psi|$. Then ψ induces an automorphism of $\text{SL}_n(q)$, $\text{PSL}_n(q)$ and $\text{PGL}_n(q)$ of the same order. Let H be either $\text{GL}_n(q)$, $\text{SL}_n(q)$, $\text{PSL}_n(q)$, or $\text{PGL}_n(q)$, $\widehat{H} = H \rtimes \langle \psi \rangle$.

If $(\ell, p) = 1$, then the (ψ, p) -elements in H are the unipotent elements in H . The (ψ, p) -regular elements are those $g \in H$ such that $\text{Norm}_\psi(g)$ is semisimple. If, instead, $\ell = p^b$ for some $b > 0$, then the (ψ, p) -regular elements in H are the semisimple elements in H , while the (ψ, p) -elements are those $g \in H$ such that $\text{Norm}_\psi(g)$ is a p -element.

We will concentrate on the case $(\ell, p) = 1$. We have the following equivalence.

Lemma 4.2. *Let $\psi \in \text{Aut}(\text{GL}_n(q))$ with $(|\psi|, p) = 1$. Then $\mathbf{x} \in \text{GL}_n(q)$ is (ψ, p) -regular if and only if $x = \pi(\mathbf{x}) \in \text{PGL}_n(q)$ is (ψ, p) -regular.*

Proof. $\text{Norm}_\psi(x)$ is semisimple if and only $\pi(\text{Norm}_\psi(x)) = \text{Norm}_\psi(\mathbf{x})$ is so. ■

We intend to study twisted classes for automorphisms induced from alge-

braic group automorphisms. By Remark 3.2 and Proposition 4.1, we may reduce to the case $\psi = \theta$. We will focus on the case of p odd and we shall investigate (ψ, p) -regular classes.

Remark 4.3. It was pointed to us by Prof. Vinberg that when the group is $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $\psi = \theta$, then the map $x \mapsto xJ$ allows to identify the θ -twisted conjugacy class of x with the equivalence classes of the non-degenerate bilinear form with associated matrix xJ . Thus, the classification of twisted classes in this case can be deduced from the classification of bilinear forms on $\overline{\mathbb{F}}_q^n$. The latter, in turn, goes over in odd characteristic, as the classification in characteristic zero which is to be found for instance in [HoP]. From this, $\mathrm{SL}_n(q)$ -orbits could be also classified. However, since the action of the center by twisted conjugation is non-trivial, the step to $\mathrm{PSL}_n(q)$ -orbits of elements in $\mathrm{PGL}_n(q)$ would need slight care. The main reason for our apparently less natural approach is related to the general problem of detecting twisted classes of type D in all finite simple groups. One of the aims in this paper is to propose a general systematic approach that could be applied, at least, to all finite simple groups of Lie type.

Lemma 4.4. *Let $x \in \mathrm{GL}_n(q)$.*

1. *x is θ -semisimple if and only if there is a $g \in \mathrm{GL}_n(\mathbb{k})$ such that $g \cdot_\theta x$ lies in a θ -stable torus T_0 in $\mathrm{GL}_n(\mathbb{k})$.*
2. *x is θ -semisimple if and only if there is a $g' \in \mathrm{SL}_n(\mathbb{k}) \subset \mathrm{GL}_n(\mathbb{k})$ such that $g' \cdot_\theta x \in T$.*

Proof. 1. is [Mo2, Proposition 3.4]. Following the construction in [Mo2, page 382] we can make sure that T_0 is F -stable and that it is contained in T . For 2., let $\mathcal{Z} := \mathcal{Z}(\mathrm{GL}_n(\mathbb{k}))$, hence $\mathrm{GL}_n(\mathbb{k}) = \mathcal{Z}\mathbb{G}$ and θ acts as inversion on \mathcal{Z} . Therefore, if $z \in \mathcal{Z}$, then $z \cdot_\theta x = xz^2$. Let $g = zg' \in \mathcal{Z}\mathbb{G}$ be such that $g \cdot_\theta x = t \in T$. Then $g' \cdot x = tz^{-2} \in T$, as \mathcal{Z} is contained in every maximal torus. ■

The lemma above motivates the following definition.

Definition 4.5. We say that an element $x \in \mathrm{PGL}_n(q)$ is θ -semisimple if it is (θ, p) -regular.

In this section we collect preparatory material in order to find suitable representatives of \mathbf{G} -classes of θ -semisimple elements in $\mathrm{PGL}_n(q)$. Unless otherwise stated, p is arbitrary.

Let H denote either \mathbb{G} , $\overline{\mathbb{G}}$ or $\mathrm{GL}_n(\mathbb{k})$ and, consequently, set $K = \mathbb{T}$, $\overline{\mathbb{T}}$ or T ($= \mathbb{T}\mathcal{Z}(\mathrm{GL}_n(\mathbb{k}))$). Let $w \in W$, $\dot{w} \in wK$ and $g = g_w \in H$ be such that $g^{-1}F(g) = \dot{w}$ (Lang-Steinberg's Theorem). We set

$$K_w := gKg^{-1}. \quad (4.1)$$

Then K_w is an F -stable maximal torus of H and all F -stable maximal tori in H are obtained this way [MT, Proposition 25.1]. Two tori K_w and K_σ are H^F -conjugate if and only if σ and w are W -conjugate. We will provide a θ -invariant

version of this fact in Lemma 4.7 for $K = \mathbb{T}$ and $\overline{\mathbb{T}}$. We set

$$F_w := \text{Ad}(\dot{w}) \circ F, \text{ so } (K_w)^F = gK^{F_w}g^{-1}. \tag{4.2}$$

The automorphisms θ and F preserve \mathbb{T} , hence they induce automorphisms on W which we denote by the same symbol. The action of F on W is trivial, whereas the action of θ is conjugation by the longest element $w_0 \in W$, so $W^\theta = W_{w_0}$. Observe that

$$w_0 = \begin{cases} (1, n)(2n-1) \dots (h, h+1) & \text{if } n = 2h, \\ (1, n)(2, n-1) \dots (h, h+2) & \text{if } n = 2h+1. \end{cases}$$

Any $\sigma \in W^\theta$ can be written as $\sigma = \omega\tau$ where ω permutes the 2-cycles in w_0 and τ is a product of transpositions occurring in the cyclic decomposition of w_0 . In fact, $W^\theta \simeq \mathbb{S}_h \rtimes \mathbb{Z}_2^h$, where $h = \lfloor \frac{n}{2} \rfloor$, the elements in \mathbb{S}_h correspond to products $c\theta(c)$ where c is a cycle in $\mathbb{S}_{\mathbb{I}_h} \leq \mathbb{S}_n$, $\theta(c) = w_0cw_0$ and the elements in \mathbb{Z}_2^h are products of transpositions of the form $(i, n+1-i)$.

Remark 4.6. There is a set of representatives $\{\dot{\sigma}\} \subset N_{\mathbb{G}}(\mathbb{T})$ of W such that $\dot{\sigma} \in N_{\mathbb{G}}(\mathbb{T})^\theta$ if $\sigma \in W^\theta$, [St, 8.2, 8.3 (b)]. In addition, $\mathbb{G}^\theta = \text{Sp}_n(\mathbb{k})$ if n is even, $\mathbb{G}^\theta = \text{SO}_n(\mathbb{k})$ if n is odd and W^θ is the corresponding Weyl group.

Lemma 4.7. *Let $w, \sigma \in W^\theta$. Then T_w and T_σ are $\text{SL}_n(q)^\theta$ -conjugate if and only if $\sigma \in \mathcal{O}_w^{W^\theta}$ if and only if \overline{T}_w and \overline{T}_σ are $\pi(\text{SL}_n(q)^\theta)$ -conjugate.*

Proof. Since $\text{Ker}(\pi)$ consists of central elements, it is enough to prove the first equivalence. By Remark 4.6 there are representatives $\dot{w}, \dot{\sigma}$ of w and $\sigma \in \mathbb{G}^\theta \cap N(\mathbb{T})$. By Lang-Steinberg’s Theorem applied to \mathbb{G}^θ we may find $y, z \in \mathbb{G}^\theta$ such that $y^{-1}F(y) = \dot{w}$, $z^{-1}F(z) = \dot{\sigma}$.

Assume there is $x \in \text{SL}_n(q)^\theta$ such that $xT_w x^{-1} = T_\sigma$. Then, $\dot{\tau} := z^{-1}xy \in N(\mathbb{T}) \cap \mathbb{G}^\theta$ and $\dot{\tau}\dot{w}\dot{\tau}^{-1}\mathbb{T} = \dot{\tau}\dot{w}F(\dot{\tau}^{-1})\mathbb{T} = \dot{\sigma}\mathbb{T}$.

Conversely, assume there is $\tau \in W^\theta$ such that $\tau w\tau^{-1} = \sigma$. Let $\dot{\tau} \in \mathbb{G}^\theta \cap \tau\mathbb{T}$. Then there exist $h, k \in \mathbb{G}^\theta \cap \mathbb{T} = \mathbb{T}^\theta = \mathbb{T}^{\theta, \circ}$ such that $F(\dot{\tau}) = \dot{\tau}h$ and $\dot{\sigma} = \dot{\tau}\dot{w}\dot{\tau}^{-1}k$. For $t \in \mathbb{T}^\theta$ we set $x_t = z\dot{\tau}ty^{-1} \in \mathbb{G}^\theta$. Now, $x_t T_w x_t^{-1} = T_\sigma$. In addition, $x_t \in \text{SL}_n(q)^\theta$ if and only if $t = \dot{w}(\dot{\tau}^{-1}k\dot{\tau})hF(t)\dot{w}^{-1}$. This happens if and only if $t^{-1}(\text{Ad}(\dot{w}) \circ F)(t) = \dot{w}h^{-1}(\dot{\tau}^{-1}k^{-1}\dot{\tau})\dot{w}^{-1}$. By Lang-Steinberg’s Theorem applied to the Steinberg endomorphism $\text{Ad}(\dot{w}) \circ F$ on \mathbb{T}^θ , there is $t \in \mathbb{T}^\theta$ satisfying this condition. ■

Lemma 4.8. *Let $w \in W^\theta$, $v \in W$, $\dot{w} \in N_{\mathbb{G}^\theta}(\mathbb{T})$ and $\dot{v} \in N_{\mathbb{G}}(\mathbb{T})$ be representatives of w and v , respectively. Let $y \in \mathbb{G}^\theta$ such that $y^{-1}F(y) = \dot{w}$. Then*

1. $\dot{v}\mathbb{T} \cap \mathbb{G}^\theta \cap \mathbb{G}^{F_w} \neq \emptyset$ if and only if $v \in W_w^\theta$.
2. An element v in $W = N_{\overline{\mathbb{G}}}(\overline{\mathbb{T}})/\overline{\mathbb{T}}$ has a representative in $\overline{\mathbb{G}}^{\theta, \circ} \cap \pi(\mathbb{G}^{F_w})$ if and only if $v \in W_w^\theta$.

Proof. 1. If $\dot{v}\mathbb{T} \cap \mathbb{G}^\theta \neq \emptyset$, then, $\theta(\dot{v}) \in \dot{v}\mathbb{T}$, so $v \in W^\theta$ and we may assume $\dot{v} \in \mathbb{G}^\theta$. If $\dot{v}\mathbb{T} \cap \mathbb{G}^{F_w} \neq \emptyset$, then $F_w(\dot{v}) \in \dot{v}\mathbb{T}$, that is $\text{Ad}(\dot{w})(\dot{v}) \in \dot{v}\mathbb{T}$, *i. e.* $wv = vw$. Conversely, assume $v \in W_w^\theta$. Now W^θ is the Weyl group of \mathbb{G}^θ and F_w is a Steinberg endomorphism of \mathbb{G}^θ preserving its maximal torus \mathbb{T}^θ . By [MT, Proposition 23.2 ff],

$$(W^\theta)^{F_w} = (N_{\mathbb{G}^\theta}(\mathbb{T}^\theta)/\mathbb{T}^\theta)^{F_w} \simeq N_{\mathbb{G}^\theta \cap \mathbb{G}^{F_w}}(\mathbb{T}^\theta)/(\mathbb{T}^\theta \cap \mathbb{T}^{F_w})$$

so any $v \in W_w^\theta = (W^\theta)^{F_w}$ has a representative in

$$N_{\mathbb{G}^\theta \cap \mathbb{G}^{F_w}}(\mathbb{T}^\theta) = N_{\mathbb{G}^\theta}(\mathbb{T}^\theta) \cap \mathbb{G}^{F_w} = N_{\mathbb{G}^\theta}(\mathbb{T}) \cap \mathbb{G}^{F_w} = N_{\mathbb{G}}(\mathbb{T}) \cap \mathbb{G}^\theta \cap \mathbb{G}^{F_w}.$$

2. Follows from 1. recalling that $\pi(\mathbb{G}^\theta) = \overline{\mathbb{G}}^{\theta, \circ}$. ■

We end the section with a lemma that shows how some of the results on the Weyl group apply to the quest of preferred representatives in a twisted class.

Lemma 4.9. *Let $t \in \mathbb{T}$ be such that $\mathcal{O}_t^{\theta, \mathbb{G}} \cap \text{GL}_n(q) \neq \emptyset$. Then*

1. *There are $\sigma \in W^\theta$ and $\dot{\sigma} \in \sigma\mathbb{T} \cap \mathbb{G}^\theta$ such that $\mathcal{O}_t^{\theta, \mathbb{G}} \cap \mathbb{T}_\sigma^F \neq \emptyset$.*
2. *Let σ be as in 1. Then $\mathcal{O}_t^{\theta, \mathbb{G}} \cap \mathbb{T}_w^F \neq \emptyset$ for every $w \in \mathcal{O}_\sigma^{W^\theta}$.*
3. *Fix p odd and $x \in \mathcal{O}_t^{\theta, \mathbb{G}} \cap \text{GL}_n(q)$. Then $\mathcal{O}_t^{\theta, \mathbb{G}} \cap \text{GL}_n(q) = \mathcal{O}_x^{\theta, \mathbb{G}^F}$.*

Proof. 1. Pick a set of representatives $\{\dot{\tau}, \tau \in W\} \subset N_{\mathbb{G}}(\mathbb{T})$ as in Remark 4.6. Let $g \in \mathbb{G}$ be such that $F(t) = gt\theta(g^{-1})$, see Remark 3.6 1. Let $u \in \mathbb{U} \cap \tau^{-1}\mathbb{U}^{-}\tau$, $\dot{\tau} \in N_{\mathbb{G}}(\mathbb{T}) \cap \tau\mathbb{T}$, $s \in \mathbb{T}$, $v \in \mathbb{U}$ such that $g = u\dot{\tau}sv$. Then

$$F(t)\theta(g) = (F(t)\theta(u)F(t^{-1})) \cdot (F(t)\theta(\dot{\tau})\theta(s)) \cdot \theta(v) \in \mathbb{B}\tau\mathbb{B}.$$

On the other hand, $F(t)\theta(g) = gt = u\dot{\tau}svt = u(\dot{\tau}st)(t^{-1}vt) \in \mathbb{B}\tau\mathbb{B}$, which gives, by the uniqueness of the Bruhat decomposition, $\theta(\tau) = \tau \in W$ and, by construction, $\theta(\dot{\tau}) = \dot{\tau}$. Also this yields $F(t)\theta(\dot{\tau})\theta(s) = \dot{\tau}st$, that is $F(t) = (\dot{\tau}s) \cdot_\theta t \in \mathcal{O}_t^{\theta, N_{\mathbb{G}}(\mathbb{T})}$. Let $\dot{\sigma} := \dot{\tau}^{-1} \in N_{\mathbb{G}^\theta}(\mathbb{T})$. Then $F_\sigma = \text{Ad}(\dot{\sigma}) \circ F$ is again a Steinberg endomorphism for \mathbb{T} and $F_\sigma(t) = ts\theta(s^{-1}) \in \mathcal{O}_t^{\theta, \mathbb{T}}$. Let $r \in \mathbb{T}$ be such that $r^{-1}F_\sigma(r) = s$. Then $x = r^{-1} \cdot_\theta t \in \mathcal{O}_t^{\theta, \mathbb{T}} \cap \mathbb{T}^{F_\sigma}$. Indeed,

$$F_\sigma(x) = F_\sigma(r^{-1})F_\sigma(t)\theta(F_\sigma(r)) = F_\sigma(r^{-1})st\theta(s^{-1}F_\sigma(r)) = r^{-1}t\theta(r) = x.$$

Let $y \in \mathbb{G}^\theta$ be such that $y^{-1}F(y) = \dot{\sigma}$ and set $z = y \cdot_\theta x = yxy^{-1}$. Then $z \in y\mathbb{T}^{F_\sigma}y^{-1} = (y\mathbb{T}y^{-1})^F \cap \mathcal{O}_t^{\theta, \mathbb{G}}$, by (4.2) and 1. follows.

2. By Lemma 4.7 there is $g \in \text{SL}_n(q)^\theta$ such that $g\mathbb{T}_\sigma^Fg^{-1} = \mathbb{T}_w^F$. Hence, for $x \in \mathcal{O}_t^{\theta, \mathbb{G}} \cap \mathbb{T}_\sigma^F$ we have $g \cdot_\theta x \in \mathcal{O}_t^{\theta, \mathbb{G}} \cap \mathbb{T}_w^F$.

3. The group $\mathbb{G}_\theta(t) = \mathbb{G}^{\text{Ad}(t^{-1})\circ\theta}$ is connected by [St, Theorem 8.1] since $\text{Ad}(t^{-1}) \circ \theta$ is a semisimple automorphism as defined in [St, p. 51]. The result follows from [MT, Theorem 21.11]. ■

5. Twisted classes of θ -semisimple elements

We assume from now on that p is odd. Recall the notation from §2, (2.3), (2.4).

Next theorem is the first main result of the paper and a key step to apply our strategy.

Theorem 5.1. *Let $x \in \text{PGL}_n(q)$ be θ -semisimple. Then there are $w \in W^\theta$ and $z \in \overline{\mathbb{T}}_w^F$ such that $\mathcal{O}_x^{\theta, \mathbb{G}} = \mathcal{O}_z^{\theta, \mathbb{G}}$.*

Proof. Let $\mathbf{x} \in \text{GL}_n(q)$ be such that $x = \pi(\mathbf{x})$. By Lemma 4.4 3., there is $g \in \mathbb{G}$ such that $g \cdot_\theta \mathbf{x} = t \in \mathbb{T}$. Then there is $w \in W^\theta$ and $\mathbf{z} \in \mathcal{O}_t^{\theta, \mathbb{G}} \cap \mathbb{T}_w^F$ such that $\mathcal{O}_x^{\theta, \mathbb{G}} = \mathcal{O}_z^{\theta, \mathbb{G}}$, by Lemma 4.9 1. On the other hand, we have that $\mathcal{O}_z^{\theta, \mathbb{G}} \cap \text{GL}_n(q) = \mathcal{O}_z^{\theta, \text{SL}_n(q)}$, by Lemma 4.9 3. The statement now follows applying π , for $z = \pi(\mathbf{z})$, as $\pi(\mathbb{T}_w^F) \subset \overline{\mathbb{T}}_w^F$ and $\pi(\mathcal{O}_z^{\theta, \text{SL}_n(q)}) = \mathcal{O}_z^{\theta, \mathbb{G}}$. ■

Let x be a θ -semisimple element in $\text{PGL}_n(q)$. By Theorem 5.1 we may assume $x \in \overline{\mathbb{T}}_w^F$ for some $w \in W^\theta$. We have the following inclusions of subracks:

$$\mathcal{O}_x^{\psi, \mathbb{G}} \supseteq \mathcal{O}_x^{\psi, \mathbb{G}} \cap \overline{\mathbb{T}}_w^F \supseteq \mathcal{O}_{x, \pi(\mathbb{T}_w^F)}^{\psi, \mathbb{G}} \simeq \mathcal{O}_1^{\psi, \pi(\mathbb{T}_w^F)}. \tag{5.1}$$

We will establish sufficient conditions ensuring $\mathcal{O}_1^{\psi, \pi(\mathbb{T}_w^F)}$ is of type D. If the conditions are not satisfied and $\mathcal{O}_x^{\psi, \mathbb{G}} \cap \overline{\mathbb{T}}_w^F \neq \mathcal{O}_{x, \pi(\mathbb{T}_w^F)}^{\psi, \mathbb{G}}$, we will establish sufficient conditions ensuring $\mathcal{O}_x^{\psi, \mathbb{G}}$ is of type D.

We look at the subracks $\mathcal{O}_{x, \pi(\mathbb{T}_w^F)}^{\theta, \mathbb{G}} \simeq \mathcal{O}_1^{\theta, \pi(\mathbb{T}_w^F)}$ as in (5.1). Thus we investigate the abelian subgroups $\pi(\mathbb{T}_w^F)$ and $\pi(\mathbb{T}_w^F) \cap \mathbb{G}^\theta$. Let $\dot{w} \in w\mathbb{T} \cap \mathbb{G}^\theta$ and let $y \in \mathbb{G}^\theta$ be such that $y^{-1}F(y) = \dot{w}$. We have

$$\mathcal{O}_1^{\theta, \pi(\mathbb{T}_w^F)} \simeq \pi(\mathbb{T}_w^F) / (\pi(\mathbb{T}_w^F) \cap \mathbb{G}^\theta) \simeq \mathbb{T}_w^F / K$$

for $K = \{t \in \mathbb{T}_w^F \mid \theta(t) \in t\mathcal{Z}(\text{SL}_n(q))\}$. Let us set

$$K_w = \{s \in \mathbb{T}^{Fw} \mid \theta(s) \in s\mathcal{Z}(\text{SL}_n(q))\}.$$

Lemma 5.2. *Assume $\mathcal{O}_x^{\theta, \mathbb{G}} \cap \overline{\mathbb{T}}_w^F \neq \emptyset$. If there is $s \in \mathbb{T}^{Fw} / K_w$ such that $|s|$ is even and > 4 , then $\mathcal{O}_x^{\theta, \mathbb{G}}$ is of type D.*

Proof. It follows from Remark 3.4 (2) and Lemma 3.3, as conjugation by y gives the group isomorphism $\mathbb{T}^{Fw} / K_w \simeq \mathcal{O}_1^{\theta, \pi(\mathbb{T}_w^F)}$. ■

When conditions in Lemma 5.2 do not hold, we will use the following lemma.

Lemma 5.3. *Let $x \in \overline{\mathbb{T}}_w^F$ for some $w \in W^\theta$, and assume $\mathcal{O}_x^{\theta, \mathbb{G}} \cap \overline{\mathbb{T}}_w^F \neq \mathcal{O}_x^{\theta, \pi(\mathbb{T}_w^F)}$. If there is $z \in \mathcal{O}_1^{\theta, \pi(\mathbb{T}_w^F)} \simeq \mathbb{T}^{Fw} / K_w$ such that $z^4 \neq 1$, then $\mathcal{O}_x^{\psi, \mathbb{G}}$ is of type D.*

Proof. The subrack $X = \mathcal{O}_x^{\theta, \mathbb{G}} \cap \overline{\mathbb{T}}_w^F$ is a disjoint union of orbits under the θ -conjugation by $\pi(\mathbb{T}_w^F)$, one of which is $R = \mathcal{O}_x^{\theta, \pi(\mathbb{T}_w^F)} = x\mathcal{O}_1^{\theta, \pi(\mathbb{T}_w^F)}$. Let $S =$

$\mathcal{O}_t^{\theta, \pi(\mathbb{T}_w^F)} = t\mathcal{O}_1^{\theta, \pi(\mathbb{T}_w^F)} \subset X$, $S \neq R$. As $\overline{\mathbb{T}}_w^F$ is abelian and $\theta^2 = 1$, (2.2) becomes

$$(r\theta(r)^{-1})^2 \neq (s\theta(s^{-1}))^2. \tag{5.2}$$

If (5.2) holds for $r := x$, $s := t$, we are done. Otherwise, we replace s by $s' = sz \in S$, obtaining the desired inequality. \blacksquare

We need to describe \mathbf{S}_w and \mathbf{K}_w , $w \in W^\theta$. We will use the identification of W^θ with $\mathbb{S}_h \rtimes \mathbb{Z}_2^h$, for $h = \lfloor \frac{n}{2} \rfloor$. Set $\{\mathbf{e}_i : 1 \leq i \leq h\}$ the canonical \mathbb{Z}_2 -basis of \mathbb{Z}_2^h . Also, for $\lambda = (\lambda_1, \dots, \lambda_r)$ $\lambda_j \geq \lambda_{j+1}$ a partition of h , consider the set $\mathcal{E}(\lambda)$ consisting of all vectors $\varepsilon \in \mathbb{Z}_2^r$ such that if $\lambda_j = \lambda_{j+1}$, then $\varepsilon_j = 0$ implies $\varepsilon_{j+1} = 0$.

By Lemma 4.9 2. it is enough to look at a set representatives of each W^θ -conjugacy class. According to [Ca1, Proposition 24] such a set is given by all

$$\sigma_{\lambda, \varepsilon} := (1, 2, \dots, i_1)\mathbf{e}_{i_1}^{\varepsilon_1}(i_1 + 1, i_1 + 2, \dots, i_2)\mathbf{e}_{i_2}^{\varepsilon_2} \cdots (i_{r-1}, i_{r-1} + 1, \dots, h)\mathbf{e}_h^{\varepsilon_r}.$$

with $i_j = \sum_{l \leq j} \lambda_l$ and $\varepsilon \in \mathcal{E}(\lambda)$.

To simplify the exposition, let $\vartheta : \mathbb{I}_n \rightarrow \mathbb{I}_n$ be the permutation $i \mapsto n+1-i$. Let us denote by $\mathbf{s}_{p,q}$ the permutation (p, q) . As an element in \mathbb{S}_n , w becomes a product of cycles as follows:

$$w = \left(\mathbf{c}_1 \theta(\mathbf{c}_1) \mathbf{s}_{i_1, \vartheta(i_1)}^{\varepsilon_1} \right) \cdots \left(\mathbf{c}_h \theta(\mathbf{c}_h) \mathbf{s}_{i_h, \vartheta(i_h)}^{\varepsilon_h} \right), \tag{5.3}$$

$\mathbf{c}_j = (i_{j-1}, i_{j-1} + 1, \dots, i_j)$, $1 \leq j \leq h$, $i_{-1} = 0$. We set $w_j := \mathbf{c}_j \theta(\mathbf{c}_j) \mathbf{s}_{i_j, \vartheta(i_j)}^{\varepsilon_j}$.

We analyze cases n odd and even separately and apply the results in Lemma 5.4.

Let $n = 2h + 1$ and $w = \sigma_{\lambda, \varepsilon}$. Let, for $j = 1, \dots, r$:

$$\mathbb{F}(j) := \begin{cases} \mathbb{F}_q^{\times \lambda_j} \times \mathbb{F}_q^{\times \lambda_j}, & \text{if } \varepsilon_j = 0, \\ \mathbb{F}_q^{\times 2\lambda_j}, & \text{if } \varepsilon_j = 1. \end{cases}$$

Direct computation shows that $\mathbb{T}^{F_w} \simeq \mathbb{F}_q^{\times} \times \prod_{j=1}^r \mathbb{F}(j)$. For $j \in \mathbb{I}_r$, $z_j \in \mathbb{F}(j)$, we set:

$$\bar{z}_j := \begin{cases} x_j y_j, & \text{if } \varepsilon_j = 0 \text{ and } z_j = (x_j, y_j), \\ z_j, & \text{if } \varepsilon_j = 1, \end{cases} \quad \text{and} \quad \mathbf{z}_j := \bar{z}_j^{1+\varepsilon_j q^{\lambda_j}} \in \mathbb{F}_q^{\times \lambda_j}.$$

Observe that as z_j runs in $\mathbb{F}(j)$ then \mathbf{z}_j covers $\mathbb{F}_q^{\times \lambda_j}$ and $\mathbf{z}_j^{(\lambda_j)_q}$ covers \mathbb{F}_q^{\times} . We have

$$\mathbb{T}^{F_w} := \{(z, z_1, \dots, z_r) \in \mathbb{F}_q^{\times} \times \prod_{j=1}^r \mathbb{F}(j) \mid z \prod_j \mathbf{z}_j^{(\lambda_j)_q} = 1\} \simeq \prod_{j=1}^r \mathbb{F}(j).$$

It follows from direct computation that

$$\mathbf{K}_w \simeq \{(z_1, \dots, z_r) \in \prod_{j=1}^r \mathbb{F}(j) \mid \mathbf{z}_j = \zeta, 1 \leq j \leq r, \zeta \in \mu_n(\mathbb{F}_q)\}.$$

Hence, if $\gamma : \mathbb{T}^{F_w} \rightarrow \mathbb{F}_{q^{\lambda_1}}^\times \times \mathbb{F}_{q^{\lambda_1\lambda_2}}^\times \times \cdots \times \mathbb{F}_{q^{\lambda_{r-1}\lambda_r}}^\times$ is given by

$$(z_1, \dots, z_r) \mapsto (\mathbf{z}_1^{\mathbf{d}}, \mathbf{z}_1\mathbf{z}_2^{-1}, \dots, \mathbf{z}_{r-1}\mathbf{z}_r^{-1}), \tag{5.4}$$

then $\mathbb{T}^{F_w}/\mathbb{K}_w \simeq \text{Im } \gamma$.

Let $n = 2h$, $w = \sigma_{\lambda, \varepsilon}$. With notation as above odd we have:

$$\mathbb{T}^{F_w} = \{(z_1, \dots, z_r) \in \prod_{j=1}^r \mathbb{F}(j) \mid \prod_j z_j^{(\lambda_j)q} = 1\}.$$

It follows from direct computation that

$$\mathbb{K}_w \simeq \{(z_1, \dots, z_r) \in \mathbb{T}^{F_w} \mid z_j = \zeta, 1 \leq j \leq r, \zeta \in \mu_n(\mathbb{F}_q)\},$$

hence $\mathbb{T}^{F_w}/\mathbb{K}_w \simeq \text{Im } \gamma$, for $\gamma : \mathbb{T}^{F_w} \rightarrow \mathbb{F}_{q^{\lambda_1}}^\times \times \mathbb{F}_{q^{\lambda_1\lambda_2}}^\times \times \cdots \times \mathbb{F}_{q^{\lambda_{r-1}\lambda_r}}^\times$ as in (5.4).

We will deal with classes $\mathcal{O}_x^{\theta, \pi(\mathbb{T}_w^F)}$ for $x \in \overline{\mathbb{T}}_w^F$. Observe that as $\overline{\mathbb{T}}_w = \mathbf{y}\overline{\mathbb{T}}\mathbf{y}^{-1}$, x is represented by an element in \mathbb{T}^{F_w} up to multiplication by matrices in $\mathcal{Z}(\text{GL}_n(q))$, *i. e.*, up to a scalar factor in \mathbb{F}_q^\times . We apply Lemma 5.2 and the description of $\mathbb{T}^{F_w}/\mathbb{K}_w$ from Section 5 on each case to detect classes of type D. Let $\mathbf{1}$ denote the partition $(1, \dots, 1)$.

Lemma 5.4. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of h , $\varepsilon \in \mathcal{E}(\lambda)$ and let $w = \sigma_{\lambda, \varepsilon} \in W^\theta$. Let $x \in \overline{\mathbb{T}}_w^F$. Then $\mathcal{O}_x^{\theta, G}$ is of type D provided any of the following conditions hold.*

1. n is odd, $\lambda \neq \mathbf{1}$.
2. n is even, $\lambda \neq \mathbf{1}$, and $r > 2$.
3. $\lambda = \mathbf{1}$, $n \neq 3, 4$ and $q > 5$.
4. If $\lambda = \mathbf{1}$, $n = 3$ and $q = 9, 11$ or $q > 13$.
5. If $\lambda = \mathbf{1}$, $n = 4$ and $q > 9$ and $q \equiv 1 \pmod{4}$.

Proof. In all cases we will provide a suitable element in the image of the map γ from (5.4) and apply Lemma 5.2.

1. Assume $r > 1$. If j is such that $\varepsilon_j = 0$ and $\lambda_j > 1$, consider $\tilde{z}_j = (x_j, 1)$, for a generator $x_j \in \mathbb{F}_{q^{\lambda_j}}^\times$. If

$$\gamma_j := \gamma(1, \dots, \tilde{z}_j, \dots, 1) = (x_j^{\delta_{1,j}\mathbf{d}}, \dots, x_j, x_j^{-1}, \dots, 1),$$

then $|\gamma_j| = |x_j| = q^{\lambda_j} - 1 > 4$ and even. Similarly, if $r > 1$ and j is such that $\varepsilon_j = 1$ and $\lambda_j > 1$, then it follows that if

$$\gamma_j := \gamma(1, \dots, z_j, \dots, 1) = (z_j^{\delta_{1,j}\mathbf{d}(1+q^{\lambda_j})}, \dots, z_j^{1+q^{\lambda_j}}, z_j^{-1-q^{\lambda_j}}, \dots, 1)$$

for a generator z_j of $\mathbb{F}_{q^{2\lambda_j}}^\times$, then

$$|\gamma_j| = |z_j^{1+q^{\lambda_j}}| = \frac{q^{2\lambda_j} - 1}{(q^{2\lambda_j} - 1, 1 + q^{\lambda_j})} = q^{\lambda_j} - 1 > 4.$$

Now, if $r = 1$, then $\lambda = (h)$, $h > 1$. Pick \bar{z} such that \mathbf{z} is a generator of $\mathbb{F}_{q^h}^\times$. Then

$$|\gamma(\mathbf{z})| = |\mathbf{z}^{\mathbf{d}}| = \frac{q^h - 1}{(\mathbf{d}, q^h - 1)} = \frac{q - 1}{\mathbf{d}}(h)_q > (h)_q \geq 4.$$

Observe that $\frac{q-1}{\mathbf{d}}$ is always even, whence the first inequality. Moreover, $(h)_q = 4$ only if $q = 3$, $n = 5$ in which case $\frac{q-1}{\mathbf{d}}(h)_q = 2(h)_q > 4$.

2. Assume now that n is even. We distinguish the following cases:

Case $r > 2$, $\lambda \neq \mathbf{1}$.

Let us choose \bar{z}_1 such that \mathbf{z}_1 is a generator of $\mathbb{F}_{q^{\lambda_1}}^\times$. Choose $\mathbf{z}_2 = \dots = \mathbf{z}_{r-1} = 1$ and z_r such that $\mathbf{z}_1^{(\lambda_1)_q} \mathbf{z}_r^{(\lambda_r)_q} = 1$. Then $(z_1, \dots, z_r) \in \mathbb{T}^{F_w}$ and

$$|\gamma(z_1, \dots, z_r)| \geq |\mathbf{z}_1| = q^{\lambda_1} - 1 > 4 \text{ and even.}$$

3., 4., 5. If n is odd, $n \neq 3$ the computation in 1. shows that we can find $x \in \text{Im } \gamma$ with $|x| = q - 1 > 4$ for $q > 5$. If $n = 3$, then $\text{Im } \gamma$ is cyclic of order $\frac{q-1}{\mathbf{d}} > 4$ for $q \geq 9$, $q \neq 13$ and always even.

If n is even, then $h = r \geq 2$. If $r > 2$ we may choose \mathbf{z}_1 as a generator of \mathbb{F}_q^\times , $\mathbf{z}_2 = \mathbf{z}_1^{-1}$ and $\mathbf{z}_j = 1$ for $j \geq 3$ and proceed as before. If $r = 2$ then $n = 4$. We need $\mathbf{z}_2 = \mathbf{z}_1^{-1}$ and, choosing \mathbf{z}_1 as above we have $|(\mathbf{z}_1^{\mathbf{d}}, \mathbf{z}_1^2)| = \frac{q-1}{2}$. ■

Lemma 5.5. *Let $w \in W^\theta$ and $x \in \bar{\mathbb{T}}_w^{F_w}$. If $\mathcal{O}_{w_0}^{W_w}$ is of type D, then $\mathcal{O}_x^{\theta, \mathbf{G}}$ is so.*

Proof. Let \dot{w} be a representative of w in $\bar{\mathbb{G}}^\theta \cap N(\bar{\mathbb{T}})$, see Remark 4.6, and let $y \in (\bar{\mathbb{G}}^\theta)^\circ = \pi(\mathbb{G}^\theta)$ be such that $y^{-1}F(y) = \dot{w}$, so $x = yty^{-1}$ for some $t \in \bar{\mathbb{T}}^{F_w}$. Since $\mathbf{G} = [\bar{\mathbb{G}}^F, \bar{\mathbb{G}}^F] = y[\bar{\mathbb{G}}^{F_w}, \bar{\mathbb{G}}^{F_w}]y^{-1}$ we have $\mathcal{O}_x^{\theta, \mathbf{G}} \simeq \mathcal{O}_t^{\theta, [\bar{\mathbb{G}}^{F_w}, \bar{\mathbb{G}}^{F_w}]}$. Now, F_w is again a Steinberg endomorphism of $\bar{\mathbb{G}}$, and $\bar{\mathbb{T}}$ is F_w -stable. Hence, [MT, Proposition 23.2] applies and by [MT, Exercise 30.13] there is a group epimorphism

$$N_{\bar{\mathbb{G}}}(\bar{\mathbb{T}}) \cap [\bar{\mathbb{G}}^{F_w}, \bar{\mathbb{G}}^{F_w}] \twoheadrightarrow W^{F_w} = W_w$$

inducing a rack epimorphism $\mathcal{O}_x^{\theta, \mathbf{G}} \twoheadrightarrow \mathcal{O}_1^{\theta, W_w}$. The statement follows from Lemma 3.1 2. ■

For $\lambda = \mathbf{1}$ and $j = 0, \dots, h$ we set $\varepsilon^j := (\underbrace{1, \dots, 1}_{(h-j) \text{ times}}, \underbrace{0, \dots, 0}_{j \text{ times}}) \in \mathcal{E}(\lambda)$.

Lemma 5.6. *Let $w = \sigma_{\mathbf{1}, \varepsilon^j}$ and let $x \in \bar{\mathbb{T}}_w^{F_w}$. If n is even and $j \geq 3$, or if n is odd and $j > 3$, then $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D. In particular, if $x \in \bar{\mathbb{T}}^F$, then $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D provided $n \geq 6$, $n \neq 7$.*

Proof. By Lemma 5.5 it is enough to prove that $\mathcal{O}_{w_0}^{W_w}$ is of type D. Now $w \in W$ is the permutation $(1, n) \cdots (h-j, n+1-h+j) \in W' \times 1 \leq W' \times W''$ where

$$W' \times W'' = \mathbb{S}_{\{1, \dots, h-j, n+1-h+j, \dots, n\}} \times \mathbb{S}_{\{h-j+1, \dots, n-h+j\}} \simeq \mathbb{S}_{2(h-j)} \times \mathbb{S}_{n-2(h-j)}$$

and $W_w = W'_w \times W''_w$, so $\mathcal{O}_{w_0}^{W_w} \simeq \mathcal{O}_w^{W'_w} \times \mathcal{O}_{ww_0}^{W''_w} \simeq \mathcal{O}_{ww_0}^{W''_w}$. The latter is of type D by [AFGaV1, Theorem 4.1]. ■

Lemma 5.7. *Assume $n = 2h$ and let $\lambda = (\lambda_1, \dots, \lambda_h)$ be a partition of h .*

1. *If $w = \sigma_{\lambda, \varepsilon} = w_1 \dots w_j \in W^\theta$ as in (5.3), then there is a block matrix $y = \text{Diag}(y_1, \dots, y_h) \in \mathbb{G}^\theta$ such that $\dot{w} = y^{-1}F(y)$ is a representative of w in $N_{\mathbb{G}^\theta}(\mathbb{T}) \cap \text{SL}_n(q)$, each block $y_j \in \text{Sp}_{2\lambda_j}(\mathbb{k})$ and $\dot{w}_j = y_j^{-1}F(y_j) \in w_j\mathbb{T}$.*
2. *If $\lambda = (\lambda_1)$ and $w = \sigma_{\lambda, 0}$, there are $y_1 \in \text{SL}_{\lambda_1}(\mathbb{k})$ and $\dot{w} \in w\mathbb{T} \cap N_{\mathbb{G}^\theta}(\mathbb{T}) \cap \text{SL}_n(q)$ such that $\dot{w} = y^{-1}F(y)$, $y = \text{Diag}(y_1, J_{\lambda_1} {}^t y_1^{-1} J_{\lambda_1}^{-1}) \in \mathbb{G}^\theta$.*

Proof. 1. Set $i_j = \sum_{l \leq j} \lambda_l$, $i_{-1} := 0$, $\Lambda_j = \{i_{j-1} + 1, \dots, i_j\}$, $1 \leq j \leq h$. Recall from (5.3) that $w \in \mathbb{S}_{2h}$ can be viewed as an element in $\mathbb{S}_{2\lambda_1} \times \dots \times \mathbb{S}_{2\lambda_h}$, if we identify $\mathbb{S}_{2\lambda_j}$ with the permutation group of $\Lambda_j \cup \vartheta(\Lambda_j)$, for $1 \leq j \leq h$. Notice that $w_j = \mathbf{c}_j \theta(\mathbf{c}_j) \mathbf{s}_{i_j, \vartheta(i_j)} \in \mathbb{S}_{2\lambda_j}^{\theta_j}$ for each $1 \leq j \leq h$. Hence each w_j lies in the Weyl group of a θ -invariant subgroup $\mathbb{G}_j \simeq \text{Sp}_{2\lambda_j}(\mathbb{k})$ of \mathbb{G} , namely the subgroup of matrices:

$$\begin{pmatrix} \text{Id} & & & & \\ & A & & B & \\ & C & \text{Id} & D & \\ & & & & \text{Id} \end{pmatrix}, \quad \text{for } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2\lambda_j}(\mathbb{k})$$

and the non-zero entries outside the diagonal are indexed by integers in $\Lambda_j \cup \vartheta(\Lambda_j)$. Let us denote by θ_j the graph automorphism for \mathbb{G}_j . There exists a representative \dot{w}_j of w_j in $\mathbb{G}_j^{\theta_j} \simeq \text{Sp}_{2\lambda_j}(\mathbb{k})$, as n is even. Therefore, there exists $y_j \in \mathbb{G}_j^{\theta_j} \simeq \text{Sp}_{2\lambda_j}(\mathbb{k})$ such that $y_j^{-1}F(y_j) = \dot{w}_j$. We remark that $[\mathbb{G}_i, \mathbb{G}_j] = 1$ for $i \neq j$ and thus y can be chosen as $y = y_1 \dots y_h$.

2. If $\varepsilon = 0$ then w lies in \mathbb{S}_{λ_1} and it is represented by block matrices of the form $\dot{w} = \text{Diag}(A, J_{\lambda_1} {}^t A^{-1} J_{\lambda_1}^{-1}) \in \mathbb{G}^\theta$. We can assume that $A \in \text{SL}_{\lambda_1}(q)$ [MT, Proposition 23.2], we can apply Lang-Steinberg's Theorem to the connected group $\text{SL}_{\lambda_1}(\mathbb{k})$. ■

Lemma 5.8. *Let $n = 2h$ for $h > 1$, $\lambda = (h)$, $\varepsilon = (0)$ and $w = \sigma_{\lambda, \varepsilon}$. Let $x \in \overline{\mathbb{T}}_w^F$. Then $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D provided one of the following holds:*

1. *$x\theta$ is not an involution and $\mathbf{G} \neq \text{PSL}_4(3), \text{PSL}_4(7)$.*
2. *$n \geq 6$.*
3. *$x\theta$ is an involution, $n = 4$ and $q \equiv 1(4)$, $q \neq 5, 9$.*

Proof. We have $w = (1, 2, \dots, h)(n, n-1, \dots, h+1)$. Let $y \in \text{Sp}_n(\mathbb{k})$ satisfy $\dot{w} = \pi(y^{-1}F(y))$. Set $\mathbf{y} = \pi(y)$. Thus we may assume

$$x = \mathbf{y}\pi(t)\mathbf{y}^{-1}, \quad \text{for } t = \text{diag}(a, a^q, \dots, a^{q^{h-1}}, b^{q^{h-1}}, \dots, b),$$

for some $a, b \in \mathbb{F}_{q^h}^\times$. We set, for $\xi \in \mathbb{k}$, $\xi^{(h)q} = 1$:

$$t_\xi = \text{diag}(a\xi, (a\xi)^q, \dots, (a\xi)^{q^{h-1}}, (b\xi)^{q^{h-1}}, \dots, b\xi) \in \overline{\mathbb{T}}_w^F.$$

It follows that $\mathcal{O}_x^{\theta, \pi(\mathbb{T}_w^F)} = \mathbf{y}\{\pi(t_\xi) : \xi \in \mathbb{k}, \xi^{(h)_q} = 1\}\mathbf{y}^{-1}$.

Set $\kappa = \kappa_n := \pi(\text{diag}(-\text{id}_h, \text{id}_h)) \in \text{PGL}_n(q)$. Notice that $x\theta$ is an involution if and only if $\theta x = x^{-1}$ which happens only if $x \in \mathcal{O}_1^{\theta, \overline{\mathbb{T}}_w^F} \cup \mathcal{O}_\kappa^{\theta, \overline{\mathbb{T}}_w^F}$. We claim that if $x \notin \mathcal{O}_1^{\theta, \overline{\mathbb{T}}_w^F} \cup \mathcal{O}_\kappa^{\theta, \overline{\mathbb{T}}_w^F}$, then $\mathcal{O}_x^{\theta, \mathbf{G}} \cap \overline{\mathbb{T}}_w^F \neq \mathcal{O}_x^{\theta, \pi(\mathbb{T}_w^F)}$.

Let us compute the (twisted) action of $w_0 \in W_w^\theta = \langle w, w_0 \rangle$ on $\pi(t)$. We have

$$w_0 \cdot_\theta \pi(t)t = \text{diag}(b, b^q, \dots, b^{q^{h-1}}, a^{q^{h-1}}, \dots, a).$$

Hence, $w_0 \cdot_\theta x \in \mathcal{O}_x^{\theta, \pi(\mathbb{T}_w^F)}$ only if $ab^{-1} = ba^{-1}$. This gives the claim.

1. We apply Lemma 5.3: we search for $z \in \mathcal{O}_1^{\theta, \pi(\mathbb{T}_w^F)}$ such that $|z| \neq 1, 2, 4$. According to the discussion for the case n even above, $\mathcal{O}_1^{\theta, \pi(\mathbb{T}_w^F)}$ is a cyclic group of order ℓ , for $\ell = \frac{(h)_q}{(\mathbf{d}, (h)_q)} = \frac{(h)_q}{(q-1, h)}$, as $\mathbf{d} = (q-1, 2h)$ and $(q-1, (h)_q) = (q-1, h)$. If $h = 2$, so $n = 4$, we have $\ell = \frac{1+q}{2}$, so $q \neq 3, 7$ is enough. If h is odd, then ℓ is odd and $\ell > 1$ since $\ell > \frac{1+q}{q-1}$. Then we can find such a z . From now we shall assume that $h \geq 4$ is even. We distinguish three cases, according to $h > q-1$, $h = q-1$ or $h < q-1$. If $h > q-1$ then $\ell > \frac{1+q(h-1)}{q-1} = h + \frac{h-(q-1)}{q-1} > 4$ and we are done. The same computation proves the claim if $h = q-1 > 4$. If $h = q-1 = 4$ a direct computation gives the claim. Finally, if $h < q-1$, then $\ell > \frac{(h)_{h+1}}{h} \geq \frac{h+h(h-2)h}{h} > 6$.

2. If $x\theta$ is not an involution, then we apply 1. If $x\theta$ is an involution, then we have that either $\mathcal{O}_x^{\theta, \mathbf{G}} \simeq \mathcal{O}_1^{\theta, \mathbf{G}}$ or $\mathcal{O}_x^{\theta, \mathbf{G}} \simeq \mathcal{O}_\kappa^{\theta, \mathbf{G}}$, by Lemma 3.1 3. Now, $1, \kappa \in \overline{\mathbb{T}}^F$ by Lemma 5.7 2., as we may assume $y = \text{Diag}(A, J^t A^{-1} J^{-1})$ for some there $A \in \text{GL}_h(\mathbb{k})$. Thus, by Lemma 5.6, $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D if $h \geq 3$.

3. Since $1, \kappa \in \overline{\mathbb{T}}^F$, we apply Lemma 5.4 5. ■

Proposition 5.9. *Let $q \equiv 3(4)$, $q \neq 3, 7$, $\mathbf{G} = \text{PSL}_4(q)$. Let t be either 1 or $\kappa = \begin{pmatrix} \text{id}_2 & 0 \\ 0 & -\text{id}_2 \end{pmatrix}$. Then $\mathcal{O}_t^{\theta, \mathbf{G}}$ is of type D.*

Proof. We will apply Lemma 3.3. It is enough to find $x \in \mathbf{G}$ such that the order of $xt\theta(x)^{-1}t$ in \mathbf{G} is even and > 4 . Set $\mathbf{u}_t : \mathbf{G} \rightarrow \mathbf{G}$, $\mathbf{u}_t(x) = -xt\theta(x)^{-1}t = xtJ^t x Jt$. For each $e, f \in \mathbb{F}_q$, $A, E, F \in \mathbb{F}_q^{2 \times 2}$, with E, F traceless, let us set

$$\mathbf{m}(A, e, f) = \begin{pmatrix} A & e \text{id}_2 \\ f \text{id}_2 & J_2^t A J_2^{-1} \end{pmatrix}, \quad \mathbf{n}(A, E, F) = \begin{pmatrix} A & E \\ F & J_2^t A J_2^{-1} \end{pmatrix}.$$

We have that $\mathbf{u}_1(\mathbf{m}(A, e, f)) = \mathbf{m}(A, e, f)^2$, $\mathbf{u}_\kappa(\mathbf{n}(A, E, F)) = \mathbf{n}(A, E, F)^2$.

Moreover, for any $x \in \mathbf{G}$ there are $e, f \in \mathbb{F}_q$, $A, E, F \in \mathbb{F}_q^{2 \times 2}$, E, F traceless such that $\mathbf{u}_1(x) = \mathbf{m}(A, e, f)$ and $\mathbf{u}_\kappa(x) = \mathbf{n}(A, E, F)$.

We shall exhibit a matrix $\mathbf{m}(A, 0, 0) = \mathbf{n}(A, 0, 0)$ whose projective order is a multiple of 4 and it is bigger than 8. This will prove the statement for both $t = 1, \kappa$.

Let $\mathbb{F}_{q^2}^\times = \langle \xi \rangle$ and consider the matrix $z = \text{diag}(\xi^{\frac{q-1}{2}}, -\xi^{\frac{1-q}{2}}, -\xi^{\frac{1-q}{2}}, \xi^{\frac{q-1}{2}})$ in $\text{SL}_4(\mathbb{F}_{q^2})$. The order of z is $2(q+1)$ and $z^{\frac{q+1}{2}} = \text{diag}(\omega, \omega^{-1}, \omega^{-1}, \omega)$ for ω a primitive fourth root of 1, hence the projective order of z is $q+1$.

We claim that z is $\mathrm{PGL}_4(\overline{\mathbb{F}}_q)$ -conjugate to $x = \mathbf{m}\left(\begin{pmatrix} \mathrm{Tr}(z)/2 & 1 \\ & 0 \end{pmatrix}, 0, 0\right)$ and that $\mathrm{Tr}(z) \in \mathbb{F}_q^\times$. If this is the case, $\mathbf{u}_1(x) = x^2$ and its projective order is $\frac{q+1}{2}$ which is even as $q \equiv 3(4)$ and bigger than 4 since $q \geq 11$.

The claim is proved if the following conditions hold, namely

$$\det z = 1; \quad \mathrm{Tr} z = 2(\xi^{\frac{q-1}{2}} - \xi^{\frac{1-q}{2}}) \in \mathbb{F}_q; \quad \xi^{\frac{q-1}{2}} \neq -\xi^{\frac{1-q}{2}}.$$

Indeed, in this case, the matrix $\mathbf{m}\left(\begin{pmatrix} \mathrm{Tr}(z)/2 & 1 \\ & 0 \end{pmatrix}, 0, 0\right)$ is diagonalizable and it is necessarily $\mathrm{GL}_4(\overline{\mathbb{F}}_q)$ -conjugate to the matrix z . The first and third conditions are immediate. For the second, let σ be the (involutive) generator of the Galois group $\mathrm{Gal}(\mathbb{F}_{q^2}, \mathbb{F}_q)$ of the extension $\mathbb{F}_q \subset \mathbb{F}_{q^2}$. We need $\sigma(\mathrm{Tr} x) = \mathrm{Tr} x$. But σ coincides with Fr_p^m , that is $\sigma(\xi) = \xi^q$ and thus the equality holds. ■

Lemma 5.10. *Let $n = 2h$, $h > 1$, $\lambda = (h)$ and $\varepsilon = (1)$, $w = \sigma_{\lambda, \varepsilon}$. Let $x \in \overline{\mathbb{T}}_w^F$. Then $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D provided that one of the following holds:*

1. $x\theta$ is not an involution and $\mathbf{G} \neq \mathrm{PSL}_4(3), \mathrm{PSL}_4(7)$.
2. $x \in \mathcal{O}_1^{\theta, \mathrm{PGL}_n(q)}$, $n \geq 6$.
3. $x \in \mathcal{O}_1^{\theta, \mathrm{PGL}_n(q)}$, $n = 4$, $q > 9$.
4. $x\theta$ is an involution, $x \notin \mathcal{O}_1^{\theta, \mathrm{PGL}_n(q)}$ and h is even.

Proof. In this case $w = (1, 2, \dots, h-1, h, n, n-1, \dots, h+2, h+1)$ as a permutation in \mathbb{S}_n . Arguing as in Lemma 5.8 we may assume that, for some $a \in \mathbb{F}_{q^n}^\times$,

$$x = \mathbf{y}\pi(t)\mathbf{y}^{-1}, \quad \text{for } t = t_a = \mathrm{diag}(a, a^q, \dots, a^{q^{h-1}}, a^{q^{2h-1}}, \dots, a^{q^h})$$

and \mathbf{y} such that $\mathbf{y}^{-1}F(\mathbf{y}) = \dot{w}$.

1. Notice that $x\theta$ is an involution if and only if a^2 lies in $\mathbb{F}_{q^h}^\times$. Now, set, for $\xi \in \mathbb{F}_{q^n}^\times$ such that $\xi^{(n)q} = 1$ and $z = \xi^{1+q^h}$ in $C_{(h)q} \subset \mathbb{F}_{q^h}^\times$:

$$t_{az} = \mathrm{diag}(az, (az)^q, \dots, (az)^{q^{h-1}}, \dots, (az)^{q^h}) \in \mathbb{T}^{F_w}.$$

It follows that $\mathcal{O}_x^{\theta, \pi(\mathbb{T}_w^F)} = \mathbf{y}\{\pi(t_{az}) : \xi \in \mathbb{k}, \xi^{(n)q} = 1\}\mathbf{y}^{-1}$. Observe that

$$(\mathbf{y}w_0^{-1}\mathbf{y}^{-1}) \cdot_\theta x = \mathbf{y}\pi(t^{q^h})\mathbf{y}^{-1} \text{ lies in } \mathcal{O}_x^{\theta, \pi(\mathbb{T}_w^F)}$$

if and only if $a^2 \in \mathbb{F}_{q^h}^\times$. In other words, if $w_0 \cdot_\theta t$ lies in $\mathcal{O}_t^{\theta, \pi(\mathbb{T}^{F_w})}$ only if $x\theta$ is an involution. If this is not the case, we can proceed as in Lemma 5.8 and obtain that if $\mathbf{G} \neq \mathrm{PSL}_4(3), \mathrm{PSL}_4(7)$, then $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D.

2., 3. Assume $a \in \mathbb{F}_{q^h}^\times$. Then $\mathcal{O}_x^{\theta, \overline{\mathbb{T}}_w^F} = \mathcal{O}_1^{\theta, \overline{\mathbb{T}}_w^F}$ and $\mathcal{O}_x^{\theta, \pi(\mathbb{T}_w^F)} \cap \overline{\mathbb{T}}_w^F = \mathcal{O}_x^{\theta, \pi(\mathbb{T}_w^F)}$.

In this case $x \in \mathcal{O}_1^{\theta, (\mathbf{y}\overline{\mathbb{T}}_w^F)^F} \subset \mathcal{O}_1^{\theta, \mathrm{PGL}_n(q)}$ and we may assume $a = 1$, $t = \mathrm{id}$ by Proposition 4.1. If $n \geq 6$ this class is of type D by Lemma 5.6. Assume $n = 4$. If $q \equiv 1(4)$, $q > 9$, then we may apply Lemma 5.4 5. For $q \equiv 3(4)$, $q > 7$ we apply Proposition 5.9.

4. Assume $a^{q^h} = -a$ and moreover that h is even. We apply Lemma 3.3: We search for $r \in \mathcal{O}_{x\theta}^{\mathbf{G}^{\times(\theta)}}$ such that $|rx\theta|$ is even and bigger than 4. Equivalently, we look for $z \in \mathbf{G}^{F_w}$ such that the order of $(z \cdot_{\theta} t)\theta t\theta = (z \cdot_{\theta} t)t^{-1}$ is even and bigger than 4. If h is even, then this is achieved by taking $z = t$, as $t \cdot_{\theta} t = t^3$ and thus $|t^2| = (h)_q$ which is even and bigger than 4, as $h > 1$. ■

We are missing the case $r = 2$, $\lambda = (\lambda_1, \lambda_2) \neq \mathbf{1}$. That is, the case in which $n = 2(\lambda_1 + \lambda_2)$, with $\lambda_2 \geq 1$, $\lambda_1 \geq 2$. This is the content of Lemmas 5.11 (when $\varepsilon_1 = 0$) and 5.13 (when $\varepsilon_1 = 1$). Let us set

$$w = c_1\theta(c_1)s_{\lambda_1, n-\lambda_1+1}^{\varepsilon_1}c_2\theta(c_2)s_{\lambda_1+\lambda_2, \lambda_1+\lambda_2+1}^{\varepsilon_2}$$

where $c_1 = (1, 2, \dots, \lambda_1)$ and $c_2 = (\lambda_1 + 1, \dots, \lambda_1 + \lambda_2)$. We write

$$w_1 := c_1\theta(c_1)s_{\lambda_1, n-\lambda_1+1}^{\varepsilon_1}, \quad w_2 = c_2\theta(c_2)s_{\lambda_1+\lambda_2, \lambda_1+\lambda_2+1}^{\varepsilon_2}.$$

The group W_w^{θ} always contains w_1 , w_2 and the elements

$$w_0^{(1)} = (1n) \cdots (\lambda_1, n - \lambda_1 + 1), \quad w_0^{(2)} = (\lambda_1 + 1, n - \lambda_1) \cdots (\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + 1),$$

which correspond to the longest elements in each block.

Lemma 5.11. *Let $n = 2(\lambda_1 + \lambda_2)$, $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \geq 2$, $\lambda_2 \geq 1$, and $\varepsilon_1 = 0$. Let $x \in \overline{\mathbb{T}}_w^F$. If $q > 5$, then $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D.*

Proof. We have $x = \pi(\mathbf{y})\pi(t)\pi(\mathbf{y})^{-1}$ where $\mathbf{y}^{-1}F(\mathbf{y}) = \dot{w}$, $t = \text{diag}(t_{11}, t_2, t_{12})$. Here t_{11} and t_{12} are diagonal matrices of size λ_1 and $t_2 \in \text{GL}_{2\lambda_2}(\mathbb{k})^{F_{w_2}}$ is also diagonal. Also, $t_1 := \text{diag}(t_{11}, t_{12}) \in \text{GL}_{2\lambda_1}(\mathbb{k})^{F_{w_1}}$.

Assume first $W_w^{\theta} \cdot \mathcal{O}_{\pi(t)}^{\theta, \pi(\mathbb{T}^{F_w})} \neq \mathcal{O}_{\pi(t)}^{\theta, \pi(\mathbb{T}^{F_w})}$. With notation as above, the abelian group $\mathcal{O}_1^{\theta, \pi(\mathbb{T}^{F_w})}$ is isomorphic to the image of the map γ . Since $\lambda_1 > 1$ we may choose z_1 so that $|\gamma(z_1, 1)| = |(\mathbf{z}_1^{\mathbf{d}}, \mathbf{z}_1)| = (\lambda_1)_q > 4$ and Lemma 5.3 applies.

We now determine when $W_w^{\theta} \cdot \mathcal{O}_{\pi(t)}^{\theta, \pi(\mathbb{T}^{F_w})} = \mathcal{O}_{\pi(t)}^{\theta, \pi(\mathbb{T}^{F_w})}$, acting by $w_0^{(1)}$. Arguing as in Lemma 5.8, we see that $w_0^{(1)} \cdot_{\theta} \pi(t) \in \mathcal{O}_{\pi(t)}^{\theta, \pi(\mathbb{T}^{F_w})}$ only if $\mathcal{O}_x^{\theta, \mathbf{G}} \simeq \mathcal{O}_{x'}^{\theta, \mathbf{G}}$ where x' has the form $x' = \pi(\mathbf{y} \text{diag}(\text{id}_{\lambda_1}, t'_2, \pm \text{id}_{\lambda_1})\mathbf{y}^{-1})$ with t'_2 a diagonal element in $\text{GL}_{2\lambda_2}(\mathbb{k})^{F_{w_2}}$. By Lemma 5.7 2., x' lies in $\overline{\mathbb{T}}_{w_2}^F$. So, if $\lambda_2 = 1$, then the partition associated with w_2 is $\lambda' = \mathbf{1}$ and Lemma 5.4 3. applies. If $\lambda_2 > 1$, then the partition associated with w_2 is $\lambda' \neq \mathbf{1}$ so $r = \lambda_1 + 1 > 2$ and Lemma 5.4 2. applies. ■

Remark 5.12. It follows from the proof of Lemma 5.11 that even in the case $q = 5$ the class $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D, provided $\lambda_2 > 1$. Also, if $q = 3$ then this class is of type D provided $\lambda_1 > 1$ and $\lambda_2 > 2$.

Lemma 5.13. *Let $n = 2(\lambda_1 + \lambda_2)$, $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \geq 2$, $\lambda_2 \geq 1$, and $\varepsilon_1 = 1$. Let $x \in \overline{\mathbb{T}}_w^F$. Then $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D.*

Proof. We follow the strategy and notation from Lemma 5.11. In this case $x = \pi(y)\pi(t)\pi(y)^{-1}$ for $t = \text{diag}(t_{11}, t_2, t_{12})$, and

$$t_1 = \text{diag}(t_{11}, t_{12}) = (a, a^q, \dots, a^{q^{\lambda_1-1}}, a^{q^{2\lambda_1-1}}, \dots, a^{q^{\lambda_1}}) \in \text{GL}_{2\lambda_1}(\mathbb{k})^{F_{w_1}}.$$

Applying $w_0^{(1)}$ and arguing as in the proof of Lemma 5.10, we see that $w_0^{(1)} \cdot \theta \mathcal{O}_{\pi(t)}^{\theta, \pi(\mathbb{T}^{F_w})} \neq \mathcal{O}_{\pi(t)}^{\theta, \pi(\mathbb{T}^{F_w})}$ with the possible exception of the case in which $a^2 \in \mathbb{F}_{q_1^\lambda}^\times$. If $a \in \mathbb{F}_{q_1^\lambda}^\times$, then there are diagonal elements d_{11}, d_{12} in $\text{GL}_{\lambda_1}(\mathbb{k})$ and $d_2 \in \text{GL}_{2\lambda_2}(\mathbb{k})^{F_{w_2}}$ such that $d_1 := \text{diag}(d_{11}, d_{12}) \in \text{GL}_{2\lambda_1}(\mathbb{k})^{F_{w_1}}$, $\det d = 1$ for $d := \text{diag}(d_{11}, d_2, d_{12})$ and $d \cdot_\theta t = (\text{id}_{\lambda_1}, t_2', \text{id}_{\lambda_1})$. By Lemma 5.7 the latter lies in the F -stable maximal torus associated with w_2 . In this case, $r > 2$ and we apply Lemma 5.4 2. We assume from now on that $a^2 \in \mathbb{F}_{q_1^\lambda}^\times$ and $a \notin \mathbb{F}_{q_1^\lambda}^\times$, *i. e.*, $a^{q^{\lambda_1-1}} = -1$. Hence if $\varepsilon_2 = 0$, then

$$t = t_{a,b,c} = (a, a^q, \dots, a^{q^{\lambda_1-1}}, b, b^q, \dots, b^{q^{\lambda_2-1}}, c^{q^{\lambda_2-1}}, \dots, c, -a^{q^{\lambda_1-1}}, \dots, -a),$$

for $b, c \in \mathbb{F}_{q^{\lambda_2}}^\times$, while if $\varepsilon_2 = 1$, then

$$t = t_{a,b} = (a, a^q, \dots, a^{q^{\lambda_1-1}}, b, \dots, b^{q^{\lambda_2-1}}, b^{q^{2\lambda_2-1}}, \dots, b^{q^\lambda}, -a^{q^{\lambda_1-1}}, \dots, -a),$$

for $b \in \mathbb{F}_{q^{2\lambda_2}}^\times$. The \mathbb{T}^{F_w} -orbit consists of elements of the form t_{az_1, bz_2, cz_2} (t_{az_1, bz_2} , respectively) for $\mathbf{z}_1 \in \mathbb{F}_{q^{\lambda_1}}^\times$ and $\mathbf{z}_2 \in \mathbb{F}_{q^{\lambda_2}}^\times$ satisfying $\mathbf{z}_1^{(\lambda_1)_q} \mathbf{z}_2^{(\lambda_2)_q} = 1$. Since all elements in $\mathbb{F}_{q^{2\lambda_1}}^\times$ satisfying $a^{q^{\lambda_1-1}} = -1$ lie in the same $\mathbb{F}_{q^{\lambda_1}}^\times$ -coset, we may assume that $|a| = 2(q^{\lambda_1} - 1)$. We consider $w_1^{-1} \cdot \pi(t_{a,b})$ ($w_1^{-1} \cdot \pi(t_{a,b,c})$, respectively). If it lies in a different $\pi(\mathbb{T}^{F_w})$ -orbit than $\pi(t_{a,b})$ ($\pi(t_{a,b,c})$, respectively), we apply Lemma 5.3. Otherwise, there are $\mathbf{z}_1 \in \mathbb{F}_{q^{\lambda_1}}^\times$, $\ell \in \mathbb{F}_q^\times$, and $\mathbf{z}_2 \in \mathbb{F}_{q^{\lambda_2}}^\times$ such that $\ell \mathbf{z}_2 = 1$, $a^{q-1} = \ell \mathbf{z}_1$, and $\mathbf{z}_1^{(\lambda_1)_q} \mathbf{z}_2^{(\lambda_2)_q} = 1$. If this is the case, then $|a^{q-1}| = |\mathbf{z}_2^{-1} \mathbf{z}_1| = 2(\lambda_1)_q$. Thus, there is an element in the image of γ in (5.4) of even order > 4 and Lemma 5.2 applies. ■

Remark 5.14. Lemma 5.10 does not cover the case h odd, $W_w^\theta \cdot_\theta \mathcal{O}_{\pi(t)}^{\theta, \pi(\mathbb{T}^{F_w})} = \mathcal{O}_{\pi(t)}^{\theta, \pi(\mathbb{T}^{F_w})}$, and $x \neq 1$. This actually amounts to at most a single class for each group, up to rack isomorphism: Keep the notation from the lemma, let ζ be a generator of $\mathbb{F}_{q^n}^\times$, $\eta = \zeta^{\frac{1+q^h}{2}}$ and set

$$\nu = y\pi(t_\eta)y^{-1}. \tag{5.5}$$

Then $x \in \mathcal{O}_\nu^{\theta, \text{PGL}_n(q)}$ and $\mathcal{O}_x^{\theta, \mathbf{G}} \simeq \mathcal{O}_\nu^{\theta, \mathbf{G}}$, by Proposition 4.1.

1. This class seems to be difficult. For instance, the subrack

$$\mathcal{O}_\nu^{\theta, \pi(\mathbb{T}_w^F)} \simeq \mathcal{O}_{\pi(t_\eta)}^{\theta, \pi(\mathbb{T}^{F_w})} = \mathcal{O}_{\pi(t_\eta)}^{\theta, N_{\pi(\mathbb{G}^{F_w, \theta})}(\mathbb{T})}$$

is not of type D. Indeed, $|\pi(t_\eta)\theta| = 2$ and $\mathcal{O}_{t_\eta}^{\theta, \pi(\mathbb{T}^{F_w})} = \{\pi(t_{az}) : \xi \in \mathbb{k}, \xi^{(n)_q} = 1, z = \xi^{1+q^h}\}$. Then for every $r \in \mathcal{O}_{t_\eta}^{\theta, \pi(\mathbb{T}^{F_w})}$ the order $|r\theta\pi(t_\eta)\theta| = |r\pi(t_\eta^{-1})|$ divides $(h)_q$ and hence it is odd.

2. This class does not occur when dealing with θ -conjugacy classes in \mathbf{G} instead of $\mathrm{PGL}_n(q)$, if $q \equiv 1(4)$. Let $s = t_a$ as in the proof of Lemma 5.10. Then s lies in $\mathrm{SL}_n(\mathbb{k})$ only if $q \equiv 3(4)$. Indeed $\det(s) = a^{(n)_q} = a^{(1+q^h)(h)_q} = -a^{2(h)_q} = 1$ gives $a^{2(h)_q} = -1$. Also, $-1 = a^{(1-q)(h)_q} = a^{\frac{q-1}{2}2(h)_q} = (-1)^{\frac{q-1}{2}}$ so $\frac{q-1}{2}$ is odd.

In group-theoretical terms, the class of ν is of type D if and only if the following question has an affirmative answer:

Problem 5.15. Let η and $s = t_\eta$ be as above, recall the matrix J from (2.6). Is there a matrix in $A \in \mathrm{SL}_n(\overline{\mathbb{F}}_q)^{F_w}$ such that the projective order of

$$m(A) := JAJ^tAs \tag{5.6}$$

is even and bigger than 4?

So far, we have seen that most θ -semisimple classes in $\mathrm{PSL}_n(\mathbb{F}_q)$ are of type D. Next proposition shows that there exist classes that are not of type D. This shows that the condition $q \neq 3$ in Proposition 5.9 is necessary.

Proposition 5.16. *The class $\mathcal{O}_1^{\theta, \mathrm{PSL}_4(3)}$ is not of type D.*

Proof. We adopt the notation from the proof of Proposition 5.9 to show that the projective order of any matrix Y of the form $\mathbf{m}(A, e, f)$ is at most equal to 4. For such Y , we verify that $\det Y = (\det A - ef)^2$, that $\mathrm{Tr}(Y) = 2 \mathrm{Tr}(A)$, and that the matrix Y annihilates the polynomial $X^2 - \mathrm{Tr}(A)X + (\det A - ef)$. Since $Y \in \mathrm{SL}_4(\mathbb{F}_3)$ we have $\delta := \det A - ef = \pm 1$. Therefore, whenever $\mathrm{Tr}(A) = 0$, the matrix Y is an involution in $\mathrm{PSL}_4(3)$. Let us assume $\mathrm{Tr}(A) \neq 0$. We have $Y^2 = \mathrm{Tr}(A)Y - \delta$ so

$$Y^3 = \mathrm{Tr}(A)Y^2 - \delta Y = \mathrm{Tr}(A)^2Y - \delta \mathrm{Tr}(A) - \delta Y = (1 - \delta)Y - \delta \mathrm{Tr}(A).$$

If $\delta = 1$, then $Y^3 = \pm 1$. If $\delta = -1$, then $Y^4 = -Y^2 + \mathrm{Tr}(A)Y = -1$. ■

We proceed now to prove Theorem 1.1.

Proof. We cite the rows in the table according to their position in the last column. We make the convention that the head row is the 0th one. Let us consider the class of an element x that might possibly be not of type D. If $\lambda \neq 1$ then by Lemma 5.4 1. and 2., n has to be even and $r \leq 2$. By Lemmata 5.11 and 5.13, the case $r = 2$ and either $\varepsilon_1 = 1$ or $q > 5$ is ruled out, yielding the first row. If $r = 1$ and $\varepsilon = (0)$, then by Lemma 5.8 and Proposition 5.9, the classes that are not of type D may occur only for $n = 4$ and either $q = 3, 7$, or $\theta(x) = x^{-1}$ and $q = 5, 9$. This gives the second and the third row. If $r = 1$ and $\varepsilon = (1)$, then by Lemma 5.10, the classes that are not of type D may occur only in the following situations:

- $n = 4$; $q = 3, 7$ and $\theta(x) \neq x^{-1}$, which is the fourth row;

- $n = 4, q = 3, 5, 7, 9$ and $x \in \mathcal{O}_1^{\theta, \text{PGL}_n(q)}$. This case is considered in the last row, as, up to rack isomorphism, this class is represented by an element in an F -stable maximal torus with associated partition **1**.
- n twice an odd number, $x \in \mathcal{O}_\nu^{\theta, \text{PGL}_n(q)}$, which is the fifth row.

Assume now $\lambda = \mathbf{1}$. Then, by Lemma 5.4 the only classes that could occur in the table are those for $q = 3, 5$, or for $n = 3$ and $q = 7, 13$, or else $n = 4$ and $q \equiv 3(4)$, or $q = 9$. This gives the sixth and the seventh row. ■

6. Twisted classes of elements with trivial θ -semisimple part

Recall that for $x \in \text{PGL}_n(q)$ there is a unique decomposition $x = us$ with u unipotent, s a θ -semisimple element and $us = s\theta(u)$. Then we have the rack inclusions, see (3.1):

$$\mathcal{O}_x^{\theta, \mathbf{G}} \supset \mathcal{O}_u^{\mathbf{G}_\theta(s)}.$$

Assume $s = \pi(s'), s' \in \mathbf{T}$. According to [St, 8.1], $\mathbf{G}_\theta(s')$ is a connected reductive group. By [Mo1, Theorem 1.1, Proposition 3.1], any simple component of $\mathbf{G}_\theta(s')$ is isomorphic either to $Sp_{2a}(\mathbb{k})$ or to $SL_a(\mathbb{k})$, $a \in \mathbb{N}$, if n is even; and either to $Sp_{2a}(\mathbb{k})$, $SL_a(\mathbb{k})$ or $SO_{2a+1}(\mathbb{k})$, $a \in \mathbb{N}$, if n is odd. Taking F -invariants and arguing as in [Ca2, §3], one sees that $\mathbf{G}_\theta(s')^F$ contains a product of finite classical groups: unitary, special linear and symplectic if n is even, and orthogonal, unitary, special linear and symplectic if n is odd. Then $\mathcal{O}_x^{\theta, \mathbf{G}}$ is of type D whenever the conjugacy class of some component of u in one of these factors is so.

The group $\mathbf{G}_\theta(s) = \pi(\mathbb{G}^F)^{\text{Ad}(s) \circ \theta}$ might properly contain $\pi((\mathbb{G}^{\text{Ad}(s') \circ \theta})^F) = \pi(\mathbf{G}_\theta(s')^F)$ although the latter already contains all unipotent elements. So, even if $\mathcal{O}_u^{\theta, \pi(\mathbf{G}_\theta(s')^F)}$ fails to be of type D, it is still possible that $\mathcal{O}_u^{\theta, \mathbf{G}_\theta(s)}$ is so.

The unipotent classes in $\text{PSL}_n(q)$ and $\text{Sp}_{2n}(q)$ are studied in [ACGa1, ACGa2], whereas a similar analysis for unitary groups and orthogonal groups is in preparation. This enables us to draw conclusions in case $s = 1$.

Proposition 6.1. *Assume n is even and $q > 3$. Let \mathcal{O} be a θ -twisted conjugacy class with trivial θ -semisimple part and non-trivial unipotent part. Then \mathcal{O} is of type D.*

Proof. By the discussion in Section 3, a representative of the class is a unipotent element $u \in \mathbf{G}^\theta$ and \mathcal{O} has a subrack isomorphic to $\mathcal{O}_u^{\text{id}, (\mathbf{G}^\theta)^\circ} \simeq \mathcal{O}_u^{\text{id}, Sp_n(q)}$ (see the isogeny argument in [ACGa1, 1.2]). This rack is of type D with the exception of the classes with Jordan form corresponding to the partition $(2, 1, \dots, 1)$, for q either 9 or not a square. Observe that the form used for defining $\text{Sp}_n(\mathbb{k})$ in [ACGa2] differs from the one we consider. Explicitly, if $n = 2h$, then they are related by the change of basis:

$$e_i \mapsto \begin{cases} e_i & i \leq h \text{ odd or } i > h \text{ even;} \\ e_{n-i+1} & i \leq h \text{ even or } i > h \text{ odd.} \end{cases}$$

Hence, if M is the matrix that gives this basis change, elements in \mathbb{G}^θ are obtained from matrices therein conjugating by M .

Let us consider the partition $(2, 1, \dots, 1)$. There are two unipotent classes in $\mathrm{Sp}_n(q)$ associated with it. They are represented by $u_1 = 1 + \eta_1 e_{1,n}$ and by $u_2 = 1 + \eta_2 e_{1,n}$ for η_1 a square and η_2 not a square in \mathbb{F}_q^\times , respectively. Consider $g = \pi(\mathrm{diag}(-\mathrm{id}_2, \mathrm{id}_{n-2})) \in \mathbf{G}$ and set

$$v_i := g \cdot_\theta u_i = \pi \left(\sum_{\substack{j=1,2, \\ n-1,n}} e_{jj} - \sum_{j=3}^{n-3} e_{jj} - \eta_i e_{1n} \right) \in \mathbf{G}^\theta, \quad i = 1, 2.$$

Let us consider the matrix $\sigma = \begin{pmatrix} 0 & \mathrm{id}_{n-2} & 1 \\ -1 & & 0 \end{pmatrix}$; in particular $\pi(\sigma) \in \mathbf{G}^\theta$.

Now, set $r_i = u_i$, $s_i = \pi(\sigma) \cdot_\theta v_i$ and $R_i = \mathcal{O}_{u_i}^{\mathrm{id}, \mathbf{G}^\theta}$, $S_i = \mathcal{O}_{s_i}^{\mathrm{id}, \mathbf{G}^\theta}$, $i = 1, 2$. It follows that the elements r_i, s_i satisfy (2.1). Unless $q \equiv 1(4)$ and $n = 4$, the subbracks R and S are disjoint as one is a unipotent class and the other is not.

Assume now $n = 4$, $q \equiv 1(4)$. Then, for $i = 1, 2$ there is $\xi_i \in \mathbb{F}_q^\times$ such that $\xi_i^2 \neq 1$ and $(\xi_i \eta_i)^2 \neq 2$. Indeed, this excludes at most 4 elements, hence the case of $q > 5$ follows, whereas if $q = 5$, then 2 is not a square and we can take $\xi_i = 2 \in \mathbb{F}_5$. In this case we take $g = \pi(\mathrm{diag}(1, \xi_i, 1, \xi_i^{-1})) \in \mathbf{G}$ and

$$v_i := g \cdot_\theta u_i = \pi \begin{pmatrix} 1 & \xi_i \eta_i \\ \xi_i^2 \mathrm{id}_2 & \\ & 1 \end{pmatrix}, \quad i = 1, 2.$$

Then we consider $r = u_i$, $s = \pi(s) \cdot_\theta v_i$ with s_i as above, $R = \mathcal{O}_{u_i}^{\mathrm{id}, \mathbf{G}^\theta}$ and $S = \mathcal{O}_{s_i}^{\mathrm{id}, \mathbf{G}^\theta}$ and the proposition follows. ■

If n is odd, less is known about unipotent conjugacy classes in \mathbf{G}^θ . We can still obtain the following. Recall that in this case $\mathbb{G}^\theta \simeq \mathrm{SO}_n(\mathbb{k})$.

Proposition 6.2. *Assume $n > 3$ is odd. Let \mathcal{O} be a θ -twisted conjugacy class with trivial θ -semisimple part. If the Jordan form of its p -part in \mathbf{G}^θ corresponds to the partition (n) , then \mathcal{O} is of type D.*

Proof. As above, a representative of the class is $u \in \mathbf{G}^\theta$ and \mathcal{O} has a subbrack isomorphic to $\mathcal{O}_u^{\mathrm{id}, \mathbf{G}^\theta} \simeq \mathcal{O}_u^{\mathrm{id}, \mathrm{SO}_n(q)}$. We apply [ACGa2, 3.7]. ■

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