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Generalized Adjoint Actions

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Abstract. The aim of this paper is to generalize the classical formula

$$e^x y e^{-x} = \sum_{k\geq 0} \frac{1}{k!}(\mathrm{ad}\ x)^k(y)$$

by replacing e^x with any formal power series

$$f(x) = 1 + \sum_{k\geq 1} a_k t^k.$$

We also obtain combinatorial applications to q-exponentials, q-binomials, and Hall-Littlewood polynomials.
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1. Notation and main results

One of the most fundamental tools in Lie theory, the adjoint action of Lie groups on their Lie algebras, is based on the following formula:

$$e^x y e^{-x} = e^{ad\ x}(y) = \sum_{k\geq 0} \frac{1}{k!}(ad\ x)^k(y)\ , \tag{1.1}$$

where $(ad\ x)^k(y) = [x, [x, \ldots, [x, y], \ldots]]$ and $[a, b] = ab - ba$.

The aim of this paper is to generalize (1.1) by replacing e^t with any formal power series

$$f = f(t) = 1 + \sum_{k\geq 1} a_k t^k \tag{1.2}$$

over a field $\Bbbk$.

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For any formal power series (1.2) over $\Bbbk$ define polynomials

$$P_k(t) = P_{f,k}(t) = (-1)^k \det \begin{pmatrix} 1 & a_1t & a_2t^2 & \dots & a_kt^k \\ 1 & a_1 & a_2 & \dots & a_k \\ 0 & 1 & a_1 & \dots & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

for $k = 0, 1, 2, \dots.$ (with the convention that $P_0(t) = 1$). Clearly, $P_k(1) = 0$ for $k \geq 1$. Using Cramer's rule with respect to the last column, one obtains a recursion

$$P_k(t) = a_kt^k - \sum_{i=1}^{k} a_iP_{k-i}(t) \ .$$

The following result is, probably, well-known (for readers' convenience, we prove it in Section 2).

Theorem 1.1. *For any power series $f(t)$ as in (1.2), one has*

$$\frac{f(tx)}{f(x)} = \sum_{k\geq 0} P_{f,k}(t) \cdot x^k \tag{1.3}$$

and

$$P_{f,k}(st) = \sum_{i=0}^{k} P_{f,i}(s)P_{f,k-i}(t)t^i \tag{1.4}$$

for all $k \geq 0$.

Furthermore, for any algebra $\mathcal{A}$ over $\Bbbk$, a subset $\mathbf{q} = \{q_1, \dots, q_k\} \subset \Bbbk$, $x, y \in \mathcal{A}$, and $k \geq 1$ define

$$(ad\ x)^{\mathbf{q}}(y) = [x, [x, \dots, [x, y]_{q_1}, \dots]_{q_{k-1}}]_{q_k}$$

where $[a, b]_q := ab - qba$. It is easy to see that

$$(ad\ x)^{\mathbf{q}}(y) = \sum_{j=0}^{k} (-1)^j e_j(q_1, \dots, q_k) \cdot x^{k-j}yx^j \ , \tag{1.5}$$

where $e_j(q_1, \dots, q_k)$ is the j-th elementary symmetric function.

Theorem 1.2. *Let $\mathcal{A}$ be $\Bbbk$-algebra and suppose that f is any power series (1.2) over $\Bbbk$ with $a_k \neq 0$ for $k \geq 1$. Then*

$$f(x)yf(x)^{-1} = y + \sum_{k\geq 1} a_k(ad\ x)^{\mathbf{q}_k}(y) \ , \tag{1.6}$$

for any $x, y \in \mathcal{A}$, where $\mathbf{q}_k = \{q_{1k}, \dots, q_{kk}\}$ is the set of roots of $P_{f,k}(t)$.

Remark 1.3. A formula for $f(x)yf(x)^{-1}$ without assumption that all $a_k \neq 0$ is given in Proposition 2.3.

Remark 1.4. Strictly speaking, the formula (1.6), similarly to (1.1) requires a completion of $\mathcal{A}$. One can bypass this by replacing x with $\tau \cdot x$ where τ is a purely transcendental element of $\Bbbk$ so that the right hand side of (1.6) becomes a power series in τ (and, maybe extending $\Bbbk$ if it lack such an element).

Remark 1.5. The subsets $\mathbf{q}_k$ may belong to an extension of $\Bbbk$, however, the operators $(ad\ x)^{\mathbf{q}_k}$ are defined over $\Bbbk$ due to (1.5) because all symmetric functions in $\mathbf{q}_k$ belong to $\Bbbk$.

It is easy to see that if $a_k = \frac{1}{k!}$ for all k, then $P_k(t) = \frac{(t-1)^k}{k!}$ which immediately recovers (1.1). Suppose now that $a_k = \frac{1}{[k]_q!}$ for all k, where $k_q! = [1]_q \cdots [k]_q$ is the q-factorial and $[\ell]_q = 1+q+\cdots+q^{\ell-1}$. We will show (Proposition 2.5) that $P_{f,k}(t) = \dfrac{(t-1)(t-q)\cdots(t-q^{k-1})}{[k]_q!}$ for $f(t) = e_q^t = \sum_{k\geq 0} \dfrac{t^k}{[k]_q!}$, therefore, recover the following famous result (see e.g., [3]).

Theorem 1.6. *Let $e_q^x = \sum_{k\geq 0} \dfrac{x^k}{[k]_q!}$ be the q-exponential. Then*

$$e_q^x \cdot y \cdot (e_q^x)^{-1} = \sum_{k\geq 0} \frac{1}{[k]_q!} (ad\ x)^{\{1,q,\ldots,q^{k-1}\}}(y)\ .$$

On the other hand, combining Theorem 1.1 and Proposition 2.5, we recover the following well-known properties of q-exponentials and q-binomials:

$$e_q^{q^n x} = e_q^x \left(1 + \sum_{k=1}^{n} \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{[k]_q!} x^k\right)$$

for $n \geq 0$, in particular,

$$e_q^{qx} = e_q^x \cdot (1 + (q-1)x)$$

and

$$1 + \sum_{k=1}^{n} \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{[k]_q!} x^k = \prod_{i=1}^{n} (1 + (q-1)q^{i-1}x)\ .$$

We conclude with a curious observation that the polynomials $P_{f,k}(t)$ are related to the Hall-Littlewood symmetric polynomials.

Proposition 1.7. *Suppose that $f(t) = \prod_{k\geq 1}(1 - x_k t)$. Then*

$$P_{f,k}(t) = Q_{(k)}(\mathbf{x}; t)$$

for all $k \geq 0$*, where* $\mathbf{x} = \{x_k, k \geq 0\}$ *is viewed as an infinite set of variables,* $Q_\lambda(\mathbf{x};t)$ *is Hall-Littlewood polynomial ([2, Section 3.2]), and* (k) *is a one-row Young diagram with* k *cells. In particular,*

$$Q_{(k)}(\mathbf{x};t) = (-1)^k \det \begin{pmatrix} 1 & -e_1 t & e_2 t^2 & \dots & (-1)^k e_k t^k \\ 1 & -e_1 & e_2 & \dots & (-1)^k e_k \\ 0 & 1 & -e_1 & \dots & (-1)^{k-1} e_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & -e_1 \end{pmatrix}$$

for all $k \geq 0$*, where* $e_k = e_k(\mathbf{x})$ *is the* k*-th elementary symmetric function.*

2. Proofs

Proof of Theorem 1.1. We need the following well-known fact (attributed to Wronski, see e.g., [1]).

Lemma 2.1. *Let* f *be any formal power series* (1.2)*. Then* $\frac{1}{f(t)} = 1 + \sum_{k\geq 1} D_k(f) t^k$*, where*

$$D_k(f) = (-1)^k \det \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_k \\ 1 & a_1 & a_2 & \dots & a_{k-1} \\ 0 & 1 & a_1 & \dots & a_{k-2} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

(with the convention $D_0(f) = 1$*).*

The following generalization of Lemma 2.1 is, apparently, well-known (for readers' convenience we prove it here).

Lemma 2.2. *Let* $f(t) = 1 + \sum_{k\geq 1} a_k t^k$*,* $g(t) = 1 + \sum_{k\geq 1} b_k t^k$ *be formal power series. Then*

$$\frac{g(t)}{f(t)} = \sum_{k\geq 0} D_k(g,f) t^k ,$$

where $D_k(g,f) = (-1)^k \det \begin{pmatrix} 1 & b_1 & \dots & b_{k-1} & b_k \\ 1 & a_1 & \dots & a_{k-1} & a_k \\ 0 & 1 & \dots & a_{k-2} & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$ *(with the convention* $D_0(g,f) = 1$*).*

Proof. Indeed, using Lemma 2.1, we obtain (with the convention $b_0 = 1$):

$$\frac{g(t)}{f(t)} = g(t) \cdot \frac{1}{f(t)} = \left(\sum_{i\geq 0} b_i t^i\right) \left(\sum_{j\geq 0} D_j(f) t^j\right) = \sum_{k\geq 0} d_k t^k$$

where

$$d_k = \sum_{i=0}^{k} b_i D_{k-i}(f) = (-1)^k \det \begin{pmatrix} b_0 & b_1 & \dots & b_{k-1} & b_k \\ 1 & a_1 & \dots & a_{k-1} & a_k \\ 0 & 1 & \dots & a_{k-2} & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

by Cramer rule because $b_i D_{k-i}(f) = (-1)^k \det \begin{pmatrix} 0 & 0 & \dots & b_i & \dots & 0 \\ 1 & a_1 & \dots & a_i & \dots & a_k \\ 0 & 1 & \dots & a_{i+1} & \dots & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 & a_2 \\ 0 & 0 & \dots & \dots & 1 & a_1 \end{pmatrix}$.

The lemma is proved. ■

Then taking $b_k = a_k t^k$ for $k \geq 1$ in Lemma 2.2, we obtain (1.3).

To prove (1.4), compute $\frac{f(stx)}{f(x)}$ in two ways, using the first assertion:

$$\frac{f(stx)}{f(x)} = \sum_{k \geq 0} P_{f,k}(st) \cdot x^k$$

and $\dfrac{f(stx)}{f(x)} = \dfrac{f(stx)}{f(tx)} \cdot \dfrac{f(tx)}{f(x)} = \left(\sum_{i \geq 0} P_{f,i}(s) \cdot (tx)^i \right) \left(\sum_{j \geq 0} P_{f,j}(t) \cdot x^j \right)$. Comparing the coefficients of x^k in both series, we obtain (1.4).

Theorem 1.1 is proved. ■

Proof of Theorem 1.2. We need the following result.

Proposition 2.3. *For any power series f as in* (1.2) *one has:*

(a) $P_{f,k}(t) = \sum_{j=0}^{k} a_{k-j} D_j(f) \cdot t^j$ *for all* $k \geq 0$.

(b) $f(x) y f(x)^{-1} = y + z_1 + z_2 + \cdots$, *where* $z_k = \sum_{i=0}^{k} a_i D_{k-i}(f) \cdot x^i y x^{k-i}$ *for all* $k \geq 1$.

Proof. Prove (a). Indeed, using Lemma 2.1, we obtain:

$$\frac{f(tx)}{f(x)} = \sum_{i,j \geq 0} \left(a_i t^i x^i \right) \left(D_j(f) x^j \right) = \sum_{k \geq 0} \left(\sum_{i=0}^{k} a_i D_{k-i}(f) \cdot t^i \right) .$$

This together with Theorem 1.1 proves (a).

Prove (b) now. Indeed,

$$f(x) y f(x)^{-1} = \sum_{i,j \geq 0} \left(a_i x^i \right) y \left(D_j(f) x^j \right) = \sum_{k \geq 0} \left(\sum_{j=0}^{k} a_i D_{k-i}(f) \cdot x^i y x^{k-i} \right)$$

This proves (b). ■

Now we can finish the proof of Theorem 1.2. Indeed, suppose that $P_{f,k}(t)$ is factored as

$$P_{f,k}(t) = a_k(t - q_{1k}) \cdots (t - q_{kk}) \ .$$

Then, by Proposition 2.3(a), $a_i D_{k-i} = a_k(-1)^{k-i} e_{k-i}(q_{1k}, \dots, q_{kk})$ for $i = 0, \dots, k$. Therefore, in the notation of Proposition 2.3(b),

$$z_k = \sum_{i=0}^{k} a_k(-1)^{k-i} e_{k-i}(q_{1k}, \dots, q_{kk}) \cdot x^i y x^{k-i} = a_k (ad\ x)^{\mathbf{q}_k}(y)$$

for all $k \geq 1$, which together with Proposition 2.3(b) verifies (1.6).

Theorem 1.2 is proved. ■

Proposition 2.4. $P_{e_q^t,k}(t) = \dfrac{(t-1)(t-q)\cdots(t-q^{k-1})}{[k]_q!}$ *for all* $k \geq 1$.

Proof. It suffices to show that $P_{e_q^t,k}(q^a) = 0$ for all $0 \leq a < k$. We proceed by induction in such pairs (a,k). If $a = 0$, then we have nothing to prove since $P_{f,k}(1) = 0$ for all f.

Using Theorem 1.1, we obtain:

$$P_{f,k}(q^a) = \sum_{i=0}^{k} P_{f,i}(q^b) P_{f,k-i}(q^{a-b}) q^{(a-b)i} \ .$$

Taking $f = e_q^t$, $1 \leq b \leq a < k$, and using the inductive hypothesis, this gives $P_{f,k}(q^a) = 0$ for any $1 \leq a < k$.

The proposition is proved. ■

Corollary 2.5.

$$\det \begin{pmatrix} 1 & \frac{t}{[1]_q} & \frac{t^2}{[2]_q} & \dots & \frac{t^k}{[k]_q} \\ 1 & \frac{1}{[1]_q} & \frac{1}{[2]_q} & \dots & \frac{1}{[k]_q} \\ 0 & 1 & \frac{1}{[1]_q} & \dots & \frac{1}{[k-1]_q} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \frac{1}{[1]_q} \end{pmatrix} = \frac{(1-t)(q-t)\cdots(q^{k-1}-t)}{[k]_q!}$$

for all $k \geq 1$.

Proof of Proposition 1.7. Indeed, if $f(t)$ is as in Proposition 1.7, then

$$\frac{f(tu)}{f(u)} = \prod_{k\geq 1} \frac{1 - x_k tu}{1 - x_k u} = \sum_{k\geq 0} Q_{(k)}(\mathbf{x}; t) u^k$$

by [2, Equations (2.10) and (2.13)]. This and Theorem 1.1 imply that $P_{f,k} = Q_{(k)}(\mathbf{x}; t)$ for all $k \geq 0$, which proves the first assertion of Proposition 1.7.

To prove the second assertion, note that $a_k = (-1)^k e_k(\mathbf{x})$ for all $k \geq 0$ because of the well-known formula (see e.g., [2, Section 1.2]):

$$\prod_{k\geq 1}(1 - x_k t) = \sum_{k\geq 0}(-1)^k e_k(\mathbf{x}) t^k \ .$$

This and the first assertion of Proposition 1.7 imply the second assertion of the proposition. ■

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