Journal of Lie Theory
Volume **26** (2016) 227–234

A Class of Novikov Superalgebras

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Communicated by E. B. Vinberg

Abstract. In this paper, we obtain a class of Novikov superalgebras which include some nonassociative superalgebras.
Mathematics Subject Classification 2010: 17B60, 17A30, 17D25.
Key Words and Phrases: Left symmetric algebra, Novikov superalgebra, linear function.

1. Introduction

Novikov superalgebras were introduced in [7] as a particular case of left symmetric superalgebras which were called by Gerstenhaber $\mathbb{Z}_2$-graded pre-Lie algebras [4]. Novikov superalgebras are related to quadratic conformal superalgebras [8].

A Novikov superalgebra A is a $\mathbb{Z}_2$-graded vector space $A = A_0 + A_1$ with a bilinear product $(x, y) \mapsto xy$ satisfying, for any $x \in A_i, y \in A_j,\ z \in A$,

$$(xy)z - x(yz) = (-1)^{ij}((yx)z - y(xz)),$$

$$(zx)y = (-1)^{ij}(zy)x.$$

The even part of a Novikov superalgebra is exactly a Novikov algebra. The supercommutator

$$[x, y] = xy - (-1)^{ij}yx, \quad \forall x \in A_i, y \in A_j$$

makes any Novikov superalgebra A a Lie superalgebra. The passage from a Novikov algebra A to a Lie algebra is analogous.

There are only a few results on Novikov superalgebras, such as Novikov superalgebras with associative, even, supersymmetric and nondegenerate bilinear forms are associative [1, 6], Novikov superalgebras in low dimensions [3, 5] and so on.

This paper is to obtain a class of new examples of Novikov superalgebras which include some nonassociative superalgebras. For a $\mathbb{Z}_2$-graded vector space

* This work was supported by NSF of China (No. 11301282) and Specialized Research Fund for the Doctoral Program of Higher Education (No. 20130031120004).

ISSN 0949–5932 / \$2.50

$A = A_0 + A_1$, a bilinear function $h : A \times A \to \mathbb{C}$ is said to be supersymmetric if

$$h(x, y) = (-1)^{ij} h(y, x), \quad \forall x \in A_i, y \in A_j.$$

Let $f, g : A \to \mathbb{C}$ be two linear functions. Define a product on A by

$$x * y = f(x)y + g(y)x + h(x, y)c_0 + h(x, y)c_1, \quad \forall x, y \in A, \tag{1}$$

where h is a supersymmetric bilinear function, $c_0 \in A_0$, $c_1 \in A_1$ and $c_0 + c_1 \neq 0$.

The main result in this paper is the following conclusion.

Theorem 1.1. *Assume that $A = A_0 + A_1$ is a Novikov superalgebra under the product defined by the equation (1), where $dim A_0 = n (n \geq 2)$ and $dim A_1 = m (m \geq 1)$, then A is isomorphic to one of the following algebras:*

1. N_0 *is a trivial algebra, i.e., $xy = 0$ for all $x, y \in N_0$;*
2. $N_1 = \langle e_i e_1 = e_i, f_j e_1 = f_j | 1 \leq i \leq n, 1 \leq j \leq m \rangle$;
3. $N_2^k = \langle f_{2j-1} f_{2j} = e_1 = -f_{2j} f_{2j-1} | 1 \leq j \leq k \rangle$ *for any $2k \leq m$;*
4. $N_3^k = \langle e_i f_i = f_1 = f_i e_i, e_1 f_j = h(e_1, f_j) f_1 = f_j e_1 | 2 \leq i \leq k, 2 \leq j \leq m \rangle$ *for any $k \leq \min\{m, n\}$;*
5. $N_4^{k_1, k_2} = \langle e_i e_i = e_1, f_{2j-1} f_{2j} = e_1 = -f_{2j} f_{2j-1} | 2 \leq i \leq k_1, j \leq k_2 \rangle$ *for any $1 \leq k_1 \leq n, 2k_2 \leq m$;*
6. $N_5 = \langle e_1 e_1 = e_1 \rangle$;
7. $N_6 = \langle e_1 e_i = e_i, e_1 f_j = f_j | 2 \leq i \leq n, 1 \leq j \leq m \rangle$;
8. $N_7 = \langle e_1 e_1 = e_1 + e_2, e_i e_1 = e_i, f_j e_1 = f_j | 2 \leq i \leq n, 1 \leq j \leq m \rangle$;
9. $N_8 = \langle e_1 e_i = e_i = e_i e_1, f_j e_1 = f_j = e_1 f_j | 1 \leq i \leq n, 1 \leq j \leq m \rangle$.

Here $\{e_1, \cdots, e_n\}$ is a basis of A_0 and $\{f_1, \cdots, f_m\}$ is a basis of A_1.

Remark 1.2.
1. When $dim A_0 = n (n \geq 2)$ and $dim A_1 = 0$, the classification of left symmetric algebras arising from the product defined by the equation (1) has been given in [2].
2. With conditions and notations in the above theorem, it is not difficult to see that A is not associative if and only if A is isomorphic to N_6 or N_7.

2. Proof of Theorem 1.1

In this section, we suppose that $A = A_0 + A_1$ is a Novikov superalgebra under the product defined by the equation (1), where $dim A_0 = n (n \geq 2)$ and $dim A_1 = m (m \geq 1)$. Throughout what follows, let $e_1, \cdots, e_n$ be a basis of A_0 and $f_1, \cdots, f_m$ be a basis of A_1.

Lemma 2.1. *With the above conditions, we have $f(x) = g(x) = 0$ for all $x \in A_1$.*

Proof. For any $x \in A_1$ and $y \in A_0$, we have

$$x * y = f(x)y + g(y)x + h(x,y)c_0 + h(x,y)c_1 \in A_1.$$

Thus $f(x)y + h(x,y)c_0 = 0$. Note that $dim A_0 \geq 2$, it follows that $f(x) = 0$ for all $x \in A_1$. Similarly we can prove that $g(x) = 0$ for all $x \in A_1$. ∎

Proposition 2.2. *If $c_0 \neq 0$ and $c_1 \neq 0$, then A is isomorphic to N_0 or N_1.*

Proof. For any $x, y \in A_1$, by Lemma 2.1, we have

$$x * y = f(x)y + g(y)x + h(x,y)c_0 + h(x,y)c_1 = h(x,y)c_0 + h(x,y)c_1 \in A_0.$$

Thus $h(x,y)c_1 = 0$ which implies $h(x,y) = 0$ for any $x, y \in A_1$. Similarly we can prove $h(x,y) = 0$ for all $x, y \in A$. Thus the product of A is given by

$$x * y = f(x)y + g(y)x.$$

Note that A_0 is a Novikov algebra, then by the discussion for Novikov algebras in [2], we have

1. $f(x) = g(x) = 0$ for all $x \in A_0$;
2. $f(x) = 0$ for all $x \in A_0$ and $g(x) \neq 0$ for some $x \in A_0$.

In the case 1, by Lemma 2.1, we have $f(x) = g(x) = 0$ for all $x \in A$, which implies that $x * y = 0$. Thus A is isomorphic to N_0.

In the case 2, by Lemma 2.1, we have $f(x) = 0$ for all $x \in A$ which implies that $x * y = g(y)x$. Without loss of generality, assume that $g(e_1) \neq 0, g(e_i) = 0, 2 \leq i \leq n$. Furthermore, we can normalize g by $g(e_1) = 1$. Note that $g(x) = 0$ for all $x \in A_1$. Hence A is isomorphic to N_1. ∎

Lemma 2.3. *If $c_0 \neq 0$ and $c_1 = 0$, then $h(x,y) = h(y,x) = 0$ for any $x \in A_0$ and $y \in A_1$,*

Proof. We only prove that $h(x,y) = 0, \forall x \in A_0, y \in A_1$. For any $x \in A_0$ and $y \in A_1$, by Lemma 2.1, we have

$$x * y = f(x)y + g(y)x + h(x,y)c_0 = f(x)y + h(x,y)c_0 \in A_1.$$

Thus $h(x,y)c_0 = 0$ which implies $h(x,y) = 0$ in this case. ∎

Proposition 2.4. *If $c_0 \neq 0$, $c_1 = 0$ and $h(x,y) = 0$ for all $x, y \in A_0$, then A is isomorphic to N_1 or N_2^k.*

Proof. Since $h(x,y)=0$ for all $x,y\in A_0$, we have

$$x*y=f(x)y+g(y)x, \forall x,y\in A_0.$$

Similar to Proposition 2.2, we only need to consider the following two cases:

Case 1 : $f(x)=g(x)=0$ *for all* $x,y\in A_0$.

By Lemma 2.1, we have $f(x)=g(x)=0$ for all $x,y\in A$ which implies that $x*y=h(x,y)c_0$. Then according to Lemma 2.3 together with the condition $h(x,y)=0$ for all $x,y\in A_0$, we deduce that the product of A (only consider the possible nonzero product in the following) is given by

$$x*y=h(x,y)c_0, \forall x,y\in A_1.$$

Note that $h(x,y)=-h(y,x)$ for any $x,y\in A_1$. Without loss of generality, we choose $c_0=e_1$ and assume that $h(f_{2j-1},f_{2j})=1=-h(f_{2j},f_{2j-1}), j\le k$ for any $2k\le m$. Thus we get that A is isomorphic to N_2^k.

Case 2 : $f(x)=0$ *for all* $x\in A_0$ *and* $g(x)\neq 0$ *for some* $x\in A_0$.

By Lemma 2.1, we have $f(x)=0$ for all $x\in A$ which implies that $x*y=g(y)x+h(x,y)c_0$. Without loss of generality, we assume that $g(e_1)=1, g(e_i)=0, 2\le i\le n$. Moreover, $g(f_j)=0, 1\le j\le m$. Similarly in the *Case* 1, we assume that $h(f_{2j-1},f_{2j})=1=-h(f_{2j},f_{2j-1}), j\le k$ for any $2k\le m$. We claim that $k=0$. Otherwise set $c_0=\sum_{i=1}^n a_ie_i$, we have

$$(e_1,f_{2j-1},f_{2j})=(e_1*f_{2j-1})*f_{2j}-e_1*(f_{2j-1}*f_{2j})=0-e_1c_0=-a_1e_1,$$

$$(f_{2j-1},e_1,f_{2j})=(f_{2j-1}*e_1)*f_{2j}-f_{2j-1}*(e_1*f_{2j})=c_0-0=c_0.$$

Thus $(e_1,f_{2j-1},f_{2j})=(f_{2j-1},e_1,f_{2j})$ implies $c_0=0$ which is a contradiction. Hence A is isomorphic to N_1. ■

If $c_0\neq 0$, $c_1=0$ and $h(x,y)\neq 0$ for some $x,y\in A_0$, the product in A is reduced to

$$x*y=f(x)y+g(y)x+h(x,y)c_0,\ \ \forall x,y\in A$$

where $h: A_0\times A_0\to\mathbb{C}$ is a nonzero symmetric bilinear function. Note that the even part of A is a Novikov algebra, then by [2], A_0 is isomorphic to one of the following algebras:

1. For every $k=0,\cdots,n-1$, $A_{(1)}^{(k)}=\langle e_je_j=e_1|2\le j\le k+1\rangle$;
2. $A_{(2)}=\langle e_1e_1=e_1\rangle$;
3. For every $k=0,\cdots,n-2$, $A_{(3)}^{(k),3}=\langle e_1e_2=e_1, e_2e_1=2e_1, e_2e_2=e_2, e_2e_i=e_i, e_je_j=e_1|3\le i\le n, 3\le j\le k+2\rangle$;
4. $A_{(4)}^0=\langle e_1e_j=e_j|2\le j\le n\rangle$;
5. $A_{(6)}=\langle e_1e_1=e_1+e_2, e_je_1=e_j|2\le j\le n\rangle$;
6. $A_{(7)}^\alpha=\langle e_1e_1=\alpha e_1, e_1e_j=e_j, e_je_1=\alpha e_j|2\le j\le n\rangle$, $\alpha\neq 0$.

Here $\{e_1, \cdots, e_n\}$ is a basis of A_0.

Proposition 2.5. *If $c_0 \neq 0$, $c_1 = 0$ and $h(x,y) \neq 0$ for some $x, y \in A_0$, then A is isomorphic to $N_4^{k_1,k_2}$, or N_5, or N_6, or N_7, or N_8.*

Proof. The proof can be divided as follows.

Case 1 : *The even part is $A_{(1)}^{(k)}$.* For this case, we know that $f(x) = g(x) = 0$, $h(x, c_0) = 0, \forall x \in A_0$ (see [2]). The product of A is given by

$$x * y = h(x,y)c_0, \forall x, y \in A_0; \quad x * y = h(x,y)c_0, \forall x, y \in A_1.$$

Thus, we can choose a basis $\{e_1, \cdots, e_n\}$ of A_0 and $\{f_1, \cdots, f_m\}$ of A_1 such that

$$e_i * e_i = e_1, 2 \leq i \leq k_1 \ for\ any\ 1 \leq k_1 \leq n;$$

$$f_{2l-1} * f_{2l} = e_1 = -f_{2l} * f_{2l-1}, l \leq k_2 \ for\ any\ 2k_2 \leq m.$$

This algebra is isomorphic to $N_4^{k_1,k_2}$.

Case 2 : *The even part is $A_{(2)}$.* Similar to the *Case* 1, the product of A is given by

$$e_1 * e_1 = e_1; \quad f_{2l-1} * f_{2l} = a_1^2 e_1 = -f_{2l} * f_{2l-1}, l \leq k \ for\ any\ 2k \leq m$$

where $a_1 \neq 0$. Since A is a Novikov superalgebra, we have

$$0 = (f_{2l-1} * e_1) * f_{2l} = (f_{2l-1} * f_{2l}) * e_1 = a_1^2 e_1 * e_1 = a_1^2 e_1.$$

Thus $k = 0$. Hence A is isomorphic to N_5.

Case 3 : *The even part is $A_{(3)}^{(k),3}$.* In this case, we have

$$x * y = f(x)y + h(x,y)c_0, \forall x, y \in A$$

where $f(x) = h(x, c_0)$ is a nonzero linear function on A_0 satisfying $f(c_0) = 0$. We choose some nonzero element $y \in A_1$. For any $x, z \in A_0$, by Lemma 2.3, we have

$$0 = (x * y) * z = (x * z) * y = f(x)f(z)y + h(x,z)f(c_0)y = f(x)f(z)y.$$

It follows that $f(x) = 0$ for all $x \in A_0$ which is a contradition.

Case 4 : *The even part is $A_{(4)}^{0}$.* According to Lemma 2.1 and Lemma 2.3, the product of A is given by

$$x * y = f(x)y, \forall x \in A_0, y \in A_1;$$

$$x * y = h(x,y)c_0, \forall x, y \in A_1;$$

$$x * y = f(x)y + h(x,y)c_0, \forall x, y \in A_0.$$

For any $x, y \in A_1$, we have $(y * x) * x = -(y * x) * x$, thus

$$0 = (y * x) * x = (h(y,x)c_0) * x = h(y,x)f(c_0)x$$

From $f(c_0) \neq 0$(see [2]), we have $h(y,x) = 0$ for any $x, y \in A_1$. Thus the case N_6 follows.

Case 5 : *The even part is* $A_{(6)}$. The product of A is given by

$$x * y = g(y)x, \forall x \in A_1, y \in A_0;$$

$$x * y = h(x, y)c_0, \forall x, y \in A_1;$$

$$x * y = g(y)x + h(x, y)c_0, \forall x, y \in A_0.$$

According to [2], there exists a basis $\{e_1, \cdots, e_n\}$ and $\alpha \in \mathbb{C}, \alpha \neq 0$ such that $c_0 = \sum_{i=2}^{n} a_i e_i,\ g(x) = h(x, \alpha e_1), h(e_1, e_1) = 1, h(e_i, e_j) = 0, 2 \leq i \leq n, 1 \leq j \leq n$. Without loss of generality, we suppose $a_2 \neq 0$, and set

$$e_1' = \frac{1}{\alpha} e_1;\ \ e_2' = \frac{1}{\alpha^2} c_0;\ \ e_j' = e_j,\ 3 \leq j \leq n,$$

then we can choose a basis $f_j, 1 \leq j \leq m$ of A_1 such that the product of A is given by as follows

$$e_1' * e_1' = e_1' + e_2', e_j' * e_1' = e_j', 2 \leq j \leq n;$$

$$f_{k'} * e_1' = f_{k'}, 1 \leq k' \leq m;\ f_{2l-1} * f_{2l} = e_2' = -f_{2l} * f_{2l-1}, l \leq k\ for\ any\ 2k \leq m.$$

However,

$$(e_1', f_{2l-1}, f_{2l}) = (e_1' * f_{2l-1}) * f_{2l} - e_1' * (f_{2l-1} * f_{2l}) = 0 - e_1' e_2' = 0,$$

$$(f_{2l-1}, e_1', f_{2l}) = (f_{2l-1} * e_1') * f_{2l} - f_{2l-1} * (e_1' * f_{2l}) = f_{2l-1} * f_{2l} - 0 = e_2'.$$

Thus $(e_1', f_{2l-1}, f_{2l}) = (f_{2l-1}, e_1', f_{2l})$ implies $k = 0$. Hence under the new basis, we can get the case N_7.

Case 6 : *The even part is* $A_{(7)}^{(\alpha)}$. According to Lemma 2.1 and Lemma 2.3, the product of A is given by

$$x * y = f(x)y, \forall x \in A_0, y \in A_1;$$

$$x * y = g(y)x, \forall x \in A_1, y \in A_0;$$

$$x * y = h(x, y)c_0, \forall x, y \in A_1;$$

$$x * y = f(x)y + g(y)x + h(x, y)c_0,\ \ \forall x, y \in A_0.$$

As the same proof as in *Case* 5, we obtain the product of A is given by as follows

$$e_1 * e_1 = \alpha e_1, e_1 * e_j = e_j, e_j * e_1 = \alpha e_j, 2 \leq j \leq n;$$

$$f_{2l-1} * f_{2l} = e_1 = -f_{2l} * f_{2l-1}, l \leq k\ for\ any\ 2k \leq m;$$

$$f_{k'} * e_1 = \alpha f_{k'} = \alpha e_1 * f_{k'}, \alpha \neq 0, 1 \leq k' \leq m.$$

However, $(1 - \alpha)e_1 = (e_1, f_{2l-1}, f_{2l}) = (f_{2l-1}, e_1, f_{2l}) = (\alpha - 1)e_1$ implies $\alpha = 1$. Furthermore $-2f_{2l} = (f_{2l}, f_{2l-1}, f_{2l}) = -(f_{2l-1}, f_{2l}, f_{2l}) = -f_{2l}$ implies $k = 0$. Thus A is isomorphic to the following algebra

$$\langle e_1 e_i = e_i = e_i e_1, f_j e_1 = f_j = e_1 f_j | 1 \leq i \leq n, 1 \leq j \leq m \rangle$$

which is the case N_8. ∎

Proposition 2.6. *If $c_0 = 0$ and $c_1 \neq 0$, then A is isomorphic to N_1 or N_3^k.*

Proof. For this case, the product of A is given by

$$x * y = f(x)y + g(y)x + h(x, y)c_1, \forall x, y \in A.$$

Moreover, the product of A_0 is reduced as follows

$$x * y = f(x)y + g(y)x, \forall x, y \in A_0.$$

Similar to Proposition 2.2, we will discuss the following two cases:

Case 1 : $f(x) = g(x) = 0$ *for all* $x, y \in A_0$.

By Lemma 2.1, $f(x) = g(x) = 0$ for all $x, y \in A$ which implies that $x * y = h(x, y)c_1$. Then

$$x * y = h(x, y)c_1 = y * x, \forall x \in A_0, y \in A_1.$$

For any $x, z \in A_0$ and $y \in A_1$, we have $(x * y) * z = (x * z) * y = 0$, thus

$$0 = (x * y) * z = (h(x, y)c_1)z = h(x, y)h(c_1, z)c_1.$$

Thus we obtain $h(c_1, z) = 0$. Without loss of generality, we assume that $f_1 = c_1$, it follows that

$$e_i * f_j = h(e_i, f_j)f_1 = f_j * e_i, 1 \leq i \leq n, 2 \leq j \leq m.$$

Then under a new basis, we can obtain that A is isomorphic to N_3^k.

Case 2 : $f(x) = 0$ *for all* $x \in A_0$ *and* $g(x) \neq 0$ *for some* $x \in A_0$.

By Lemma 2.1, we have $f(x) = 0$ for all $x \in A$ which implies that $x * y = g(y)x + h(x, y)c_1$. Then we have

$$x * y = h(x, y)c_1, \forall x \in A_0, y \in A_1;$$

$$y * x = g(x)y + h(x, y)c_1, \forall x \in A_0, y \in A_1;$$

$$x * y = g(y)x, \forall x, y \in A_0.$$

Without loss of generality, we assume that $f_1 = c_1$, $g(e_1) = 1$ and $g(e_i) = 0$ for $2 \leq i \leq n$, it follows that

$$f_j * e_i = h(e_i, f_j)f_1, \forall i \geq 2;\ f_j * e_1 = f_j + h(e_1, f_j)f_1;\ e_i * f_j = h(e_i, f_j)f_1;\ e_j * e_1 = e_j.$$

Note that $0 = (e_j * e_k) * f_i = (e_j * f_i) * e_k$ implies $h(f_1, e_k) = 0$ for $2 \leq k \leq n$, and $(e_j * e_1) * f_i = (e_j * f_i) * e_1$ implies $h(f_1, e_1) = 0$. Furthermore $(f_i, e_1, e_1) = (e_1, f_i, e_1)$ implies $h(e_1, f_i) = 0$. Now we can choose a new basis such that the product of A is given by

$$e_l * e_1 = e_l, 1 \leq l \leq n;$$

$$f_j * e_1 = f_j, 1 \leq j \leq m;\ e_i * f_i = f_1 = f_i * e_i, 2 \leq i \leq k \ for\ any\ k \leq min\{m, n\}.$$

If there exists some $2 \leq i \leq k, k \leq min\{m, n\}$ such that $e_i * f_i = f_1 = f_i * e_i$, then $f_1 = (f_i, e_1, e_i) = (e_1, f_i, e_i) = 0$ which is a contradiction. Hence A is isomorphic to N_1. ∎

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Received May 26, 2015
and in final form August 8, 2015