Chevalley's Theorem for Affine Nash Groups

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Abstract.   We formulate and prove Chevalley's theorem in the setting of affine Nash groups. As a consequence, we show that the semi-direct product of two almost linear Nash groups is also an almost linear Nash group.
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1.   Introduction

The reader is referred to [BCR98, Sh87] for basic notions concerning Nash manifolds and Nash maps. See also [Sun15, Section 2]. Recall that a Nash group is a group which is simultaneously a Nash manifold so that all group operations are Nash maps. A Nash manifold is said to be affine if it is Nash diffeomorphic to a Nash submanifold of some finite dimensional real vector space. It is known that every affine Nash manifold is actually Nash diffeomorphic to a closed Nash submanifold of some finite dimensional real vector space (*see* [Sh96, Section 2.22]). A Nash group is said to be affine if it is affine as a Nash manifold. Thanks to the work of E. Hrushovski and A. Pillay (see Theorem 3.3), we know that affine Nash groups are closely related to real algebraic groups.

Chevalley's theorem (see Theorem 2.2) is a fundamental result in the structure theory of algebraic groups. In this note, we will formulate and prove an analogue of Chevalley's theorem in the setting of affine Nash groups.

Recall from [Sun15] that a Nash group is said to be almost linear if there exists a Nash homomorphism with finite kernel from it to the Nash group $\mathrm{GL}_n(\mathbb{R})$, for some $n \geq 0$. Using [Sh87, Proposition III.1.7], we know that every almost linear Nash group is affine. Almost linear Nash groups provide a very convenient setting for the study of infinite dimensional smooth representations (*see* [AGKL, AGS09, Clo91, SZ12]). The structure theory of almost linear Nash groups is systematically studied in [Sun15]. On the other hand, we introduce the following definition.

Definition 1.1.   An affine Nash group is said to be complete if it has no non-

trivial connected almost linear Nash subgroup. An abelian Nash manifold is a connected, complete affine Nash group.

Similar to abelian varieties, every abelian Nash manifold is compact and commutative as a Lie group (see Corollary 3.9). We describe all one-dimensional abelian Nash manifolds in the following example.

Example 1.2. For each nonzero real number α, there is a unique Nash structure on the Lie group $\mathbb{R}/\mathbb{Z}$ such that it makes $\mathbb{R}/\mathbb{Z}$ a Nash group, and that the Weierstrass elliptic function

$$\wp(x) := \frac{1}{x^2} + \sum_{\omega \in \mathbb{Z} + \alpha\sqrt{-1}\,\mathbb{Z},\, \omega \neq 0} \left(\frac{1}{(x-\omega)^2} - \frac{1}{\omega^2} \right)$$

is a Nash function on $(\mathbb{R}/\mathbb{Z}) \setminus \{0\}$ (*see* [MS92, Section 4]). We write $(\mathbb{R}/\mathbb{Z})_\alpha$ for the resulting Nash group. It follows from Lemma 3.7 of Section 3 that $(\mathbb{R}/\mathbb{Z})_\alpha$ is a one-dimensional abelian Nash manifold, and conversely, every one-dimensional abelian Nash manifold is of this form.

But a connected, compact, abelian, affine Nash group is not necessarily an abelian Nash manifold, as in the following example.

Example 1.3. The group $\mathbb{S}$ of complex numbers of modulus one is naturally a Nash group which is connected, compact, abelian and affine. It is almost linear, and is not an abelian Nash manifold.

In Proposition 4.1, we will prove that the quotients of affine Nash groups by their Nash subgroups are naturally affine Nash manifolds, and when the Nash subgroups are normal, the quotients are naturally affine Nash groups. The following theorem is an analogue of Chevalley's theorem for affine Nash groups.

Theorem 1.4. *Let G be a connected affine Nash group. Then there exists a unique connected normal almost linear Nash subgroup H of G such that G/H is an abelian Nash manifold.*

On the other hand, as we shall show in the following example, for some connected affine Nash group G, there is no normal Nash subgroup H of G such that H is an abelian Nash manifold and G/H is an almost linear Nash group.

Example 1.5. Let α be a nonzero real number. Write $\mathbb{R}_\alpha$ for the locally Nash group such that $\mathbb{R}_\alpha = \mathbb{R}$ as a Lie group, and that the quotient map $\mathbb{R}_\alpha \to (\mathbb{R}/\mathbb{Z})_\alpha$ is a locally Nash homomorphism (for the notion of locally Nash groups, see [Sh96, Section 6], for example). Let $t \in \mathbb{S}$. Put

$$G := \frac{\mathbb{R}_\alpha \times \mathbb{S}}{\{(n, t^n) \mid n \in \mathbb{Z}\}},$$

which is a Nash group. Using the theory of complex elliptic curves, we know that

G is affine. It fits to an exact sequence

$$1 \to \mathbb{S} \to G \to (\mathbb{R}/\mathbb{Z})_\alpha \to 1$$

of affine Nash groups. Therefore $\mathbb{S}$ is a connected normal almost linear Nash subgroup of G such that $G/\mathbb{S}$ is an abelian Nash manifold. It is easy to see that G contains no non-trivial abelian Nash manifold as a Nash subgroup, provided that $t \in \mathbb{S}$ is torsion free.

The semi-direct product of two affine Nash groups is clearly an affine Nash group. We will also prove the following theorem, which asserts that the semi-direct product of two almost linear Nash groups is also an almost linear Nash group.

Theorem 1.6.   *Let G and H be almost linear Nash groups, with a Nash action $G \times H \to H$ of G as Nash automorphisms of H. Then the semi-direct product $G \ltimes H$ is an almost linear Nash group.*

Theorem 1.6 is basic to the structure theory of almost linear Nash groups, as developed in [Sun15]. The authors' original motivation of this note is just to provide a proof of Theorem 1.6. As an analogue of Theorem 1.6 for linear algebraic groups, we know that the semi-direct product of two linear algebraic groups is also a linear algebraic group. This is proved by the reason that an algebraic group is linear if and only if it is affine as an algebraic variety. But affine Nash groups are not necessarily almost linear. Therefore the aforementioned simple reason for linear algebraic groups does not prove Theorem 1.6. It seems to the authors that Theorem 1.6 is not easy to prove within the framework of [Sun15] for almost linear Nash groups, and it is necessary to go to the broader setting of affine Nash groups.

2.   Preliminaries on algebraic groups

For later use, we recall some well-known facts concerning algebraic groups in this section. The reader is referred to [Mil15] for more details. Let k be a field. As usual, an algebraic variety over k is defined to be a separated, geometrically reduced scheme over k of finite type. Let G be an algebraic group over k, namely a group object in the category of algebraic varieties over k. Then G is automatically smooth over k. Recall that when k has characteristic zero, every group scheme over k is geometrically reduced (*see* [Per75, Section V.3, Corollary 3.9]). Let H be an algebraic subgroup of G, namely, a geometrically reduced closed subgroup scheme of G.

Proposition 2.1.   *The fppf quotient G/H is represented by a smooth algebraic variety over k.*

Proof.   This is proved in [DG70, Exp VIA, Thm 3.2]   ∎

Recall that a fppf quotient is a universal geometric quotient, and a universal geometric quotient is a universal categorical quotient (*see* [GM, Chapter IV]).

Specifically, the quotient G/H of Proposition 2.1 is a universal categorical quotient as well as a universal geometric quotient. When H is normal in G, the quotient G/H is naturally an algebraic group over k.

Recall that G is said to be linear if it is isomorphic to a closed algebraic subgroup of the general linear group $\mathrm{GL}(n)_{/\mathrm{k}}$, for some $n \geq 0$. It is well know that G is linear if and only if it is affine (*see* [Wat79, Theorem 3.4]). We say that G is an abelian variety if it is complete and connected. When this is the case, G is commutative, and is projective as a variety. Note that every connected algebraic group over k is geometrically connected (*see* Lemma 32.5.14 of the stacks project).

The following fundamental result is due to Chevalley and is known as Chevalley's theorem.

Theorem 2.2. (*see* [Con02, Theorem 1.1]) *Assume that* k *is a perfect field, and* G *is connected. Then there exists a unique connected, normal, linear, algebraic subgroup* L *of* G *such that the quotient* G/L *is an abelian variety.*

Chevalley's theorem has the following consequence.

Corollary 2.3. *The smooth algebraic variety* G/H *is quasi-projective.*

Proof. It is proved in [Con02, Corollary 1.2] that G is quasi-projective. The same proof (with a slight modification) shows that G/H is also quasi-projective. ∎

Recall the following well-known result (*see* [Con02, Lemma 2.3]).

Lemma 2.4. *Let* G_1 *be a connected linear algebraic group, and let* G_2 *be an abelian variety, both defined over* k. *Then there is no non-trivial algebraic homomorphism from* G_1 *to* G_2, *and no non-trivial algebraic homomorphism from* G_2 *to* G_1.

The following Proposition is also well known.

Proposition 2.5. *Let*

$$1 \to \mathsf{G}' \overset{i}{\to} \mathsf{G} \overset{\pi}{\to} \mathsf{G}'' \to 1$$

be a sequence of algebraic homomorphisms of algebraic groups over k. *Assume that it is exact, namely,* π *is faithfully flat, and* i *induces an isomorphism from* G' *onto the scheme-theoretic kernel of* π. *Then the following statments hold.*
  (a) *The algebraic group* G *is linear if and only if both* G' *and* G'' *are so.*
  (b) *The algebraic group* G *is complete if and only if both* G' *and* G'' *are so.*

Proof. The reader is referred to [BSU13, Lemma 2.1.1] for a proof of part (a) of the proposition. We sketch a proof of part (b) for the convenience of the reader.

If G is complete, then G' is obviously complete, and G'' is also complete since π is surjective (*see* [Mil14, Property 7.6]).

Note that the diagram

$$
\begin{array}{ccc}
\mathsf{G}' \times_{\mathrm{Speck}} \mathsf{G} & \xrightarrow{(x,y)\mapsto xy} & \mathsf{G} \\
{\scriptstyle (x,y)\mapsto y}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\mathsf{G} & \xrightarrow{\quad\pi\quad} & \mathsf{G}''
\end{array}
$$

is Cartesian. Since being proper is local on the base for the fppf topology, we know that if G' is complete, then π is proper. Therefore, if both G' and G'' are complete, then so is G. ∎

We remark that an algebraic homomorphism of algebraic groups is fully faithful if and only if it is surjective (*see* [Mil15, Fact 6.30]). Moreover, with the notation as in Proposition 2.5, π induces an isomorphism form G/G' onto G'' (*see* [Mil15, Theorem 6.27]).

We will also need the following elementary lemma.

Lemma 2.6. *Let G be an algebraic group over k. Let G_0 be an open algebaic subgroup of G. Then G is linear (or complete) if and only if G_0 is so.*

Proof. The lemma is obvious when k is algebraically closed. The general case is easily reduced to this case. ∎

3. Algebraizations

For every equi-dimensional smooth algebraic variety X over $\mathbb{R}$, $\mathsf{X}(\mathbb{R})$ and $\mathsf{X}(\mathbb{C})$ are naturally Nash manifolds. Likewise, for every algebraic group G over $\mathbb{R}$, $\mathsf{G}(\mathbb{R})$ and $\mathsf{G}(\mathbb{C})$ are naturally Nash groups. We introduce the following definition.

Definition 3.1. Let G be a Nash group. An algebraization of G is an algebraic group G over $\mathbb{R}$, together with a Nash homomorphism $G \to \mathsf{G}(\mathbb{R})$ which has a finite kernel and whose image is Zariski dense in G.

Here $\mathsf{G}(\mathbb{R})$ is naturally identified with a subset of the underlying topological space of the scheme G. Similar identifications will be used later on without further explanation. We call the homomorphism $G \to \mathsf{G}(\mathbb{R})$ of Definition 3.1 the algebraization homomorphism of the algebraization.

Lemma 3.2. *Let G be an algebraic group over $\mathbb{R}$. Let G be a Nash subgroup of $\mathsf{G}(\mathbb{R})$ which is Zariski dense in G. Then G is open in the Nash group $\mathsf{G}(\mathbb{R})$.*

Proof. Note that G is Zariski dense in $\mathsf{G}(\mathbb{R})$. Therefore by [Cos02, Theorem 3.20], we know that the Nash groups G and $\mathsf{G}(\mathbb{R})$ has the same dimension. Hence G is open in $\mathsf{G}(\mathbb{R})$. ∎

Lemma 3.2 implies that the Lie algebra of a Nash group is identical to the Lie algebra of every algebraization of it.

Recall that every real projective space is affine as a Nash manifold (see [BCR98, Theorem 3.4.4]). Therefore, by Proposition 2.1, for every algebraic group G over $\mathbb{R}$, $\mathsf{G}(\mathbb{R})$ is an affine Nash group. The following result is crucial to this note.

Theorem 3.3. *A Nash group is affine if and only if it has an algebraization.*

Proof. The "if" part follows from the fact that a finite fold cover of an affine Nash manifold is an affine Nash manifold (*see* [Sh87, Proposition III.1.7] and [Sun15, Proposition 2.4]). The "only if" part is proved by E. Hrushovski and A. Pillay (see [HP94, Theorem B] and [HP11, Proposition 3.1]). ∎

The following lemma asserts that algebraizations of an affine Nash group is unique up to coverings.

Lemma 3.4. *Let G be an affine Nash group. Let $\mathsf{G}_1, \mathsf{G}_2$ be two algebraizations of G. Then there exist an algebarization G_3 of G, and two surjective algebraic homomorphisms $\mathsf{G}_3 \to \mathsf{G}_1$ and $\mathsf{G}_3 \to \mathsf{G}_2$ with finite kernels such that the the diagram*

$$
\begin{array}{ccc}
G & \longrightarrow & \mathsf{G}_1(\mathbb{R}) \\
\downarrow & \searrow & \uparrow \\
\mathsf{G}_2(\mathbb{R}) & \longleftarrow & \mathsf{G}_3(\mathbb{R})
\end{array}
$$

commutes. Here the three arrows starting from G are the algebraization homomorphisms.

Proof. Take G_3 to be the Zariski closure of $\{(\phi_1(x), \phi_2(x)) \mid x \in G\}$ in $\mathsf{G}_1 \times_{\mathrm{Spec}\,\mathbb{R}} \mathsf{G}_2$, where $\phi_1 : G \to \mathsf{G}_1(\mathbb{R})$ and $\phi_2 : G \to \mathsf{G}_2(\mathbb{R})$ denotes the algebraization homomorphisms. Then the lemma easily follows. ∎

Lemma 3.5. *Let G be an algebraization of an affine Nash group G. Let G' be an affine Nash group with a Nash homomorphism $\varphi : G' \to G$. Then there exists an algebraization G' of G', together with an algebraic homomorphism $\underline{\varphi} : \mathsf{G}' \to \mathsf{G}$ such that the diagram*

$$
\begin{array}{ccc}
G' & \xrightarrow{\ \varphi\ } & G \\
\phi' \downarrow & & \downarrow \phi \\
\mathsf{G}'(\mathbb{R}) & \xrightarrow{\ \underline{\varphi}\ } & \mathsf{G}(\mathbb{R})
\end{array}
$$

commutes. Here ϕ' and ϕ denote the algebraization homomorphisms.

Proof. Let G'_0 be an algebraization of G'. Take G' to be the Zariski closure in $\mathsf{G}'_0 \times_{\mathrm{Spec}\,\mathbb{R}} \mathsf{G}$ of the image of the homomorphism

$$G' \to \mathsf{G}'_0(\mathbb{R}) \times \mathsf{G}(\mathbb{R}), \quad x \mapsto (\phi'_0(x), \phi(\varphi(x))),$$

where $\phi'_0 : G' \to \mathsf{G}'_0(\mathbb{R})$ denotes the algebraization homomorphisms. Let $\underline{\varphi}$ be the restriction to G' of the projection homomorphism $\mathsf{G}'_0 \times_{\mathrm{Spec}\,\mathbb{R}} \mathsf{G} \to \mathsf{G}$. Then the lemma follows. ∎

Lemma 3.6. *Let* G *be an algebraization of an affine Nash group* G. *Then* G *is almost linear if and only if* G *is linear as an algebraic group.*

Proof. The "if" part is trivial. To prove the "only if" part of the lemma, we assume that G is almost linear. Take a Nash homomorphism $\varphi : G \to \mathrm{GL}_n(\mathbb{R})$ with finite kernel. Denote by G_1 the Zariski closure of $\varphi(G)$ in the algebraic group $\mathrm{GL}(n)_{/\mathbb{R}}$. Put $\mathsf{G}_2 := \mathsf{G}$, and let G_3 be as in Lemma 3.4. Since G_1 is linear, part (a) of Proposition 2.5 implies that G_3 is linear, which further implies that G_2 is linear. ∎

Lemma 3.7. *Let* G *be an algebraization of a connected affine Nash group* G. *Then* G *is an abelian Nash manifold if and only if* G *is an abelian variety.*

Proof. Let $\phi : G \to \mathsf{G}(\mathbb{R})$ denote the algebraization homomorphism. First note that G is connected since it is the closure of a connected subset.

Let L be the connected linear algebraic subgroup of G as in Theorem 2.2. Then $\phi^{-1}(\mathsf{L}(\mathbb{R}))$ is an almost linear Nash group in G. If G is an abelian Nash manifold, then $\phi^{-1}(\mathsf{L}(\mathbb{R}))$ is finite. Hence L is trivial and $\mathsf{G} = \mathsf{G}/\mathsf{L}$ is an abelian variety.

Let H be a connected almost linear Nash subgroup of G. Denote by H the Zariski closure of $\phi(H)$ in G, which is a connected algebraic subgroup of G. By Lemma 3.6, H is linear. If G is an abelian variety, then Lemma 2.4 implies that H is trivial. Hence H is trivial. This proves that G is an abelian Nash manifold. ∎

Lemma 3.8. *Let* G *be an algebraization of an affine Nash group* G. *Then* G *is complete if and only if* G *is complete as an algebraic variety.*

Proof. Note that G is complete if and only if its identity connected component G° is an abelian Nash manifold. Similarly, by Lemma 2.6, G is complete if and only if its identity connected component G° is an abelian variety. Since G° is an algebraization of G°, the lemma is a direct consequence of Lemma 3.7. ∎

Corollary 3.9. *Every abelian Nash manifold is commutative and compact as a Lie group.*

Proof. Let G be an abelian Nash manifold. Let G be an algebraization of G, which is an abelian variety by Lemma 3.7. Then $\mathsf{G}(\mathbb{R})$ is a commutative compact Lie group. Since G is connected and there is a Lie group homomorphism with finite kernel from G onto the identity connected component of $\mathsf{G}(\mathbb{R})$, we know that G is commutative and compact. ∎

Corollary 3.10. *Let* G_1 *be a connected almost linear Nash group, and let* G_2 *be an abelian Nash manifold. Then there is no non-trivial Nash homomorphism from* G_1 *to* G_2, *and no non-trivial Nash homomorphism from* G_2 *to* G_1.

Proof.   In view of Lemmas 3.6 and 3.7, the corollary easily follows from Lemma 3.5 and Lemma 2.4. ∎

4.   Chevalley's theorem for affine Nash groups

We begin with the following proposition, which defines the quotients of affine Nash groups.

Proposition 4.1.   *Let G be an affine Nash group, and let H be a Nash subgroup of it. Then there exists a unique Nash structure on the quotient topological space G/H which makes the quotient map $G \to G/H$ a submersive Nash map. With this Nash structure, G/H becomes an affine Nash manifold, and the left translation map $G \times G/H \to G/H$ is a Nash map. Furthermore, if H is a normal Nash subgroup of G, then the topological group G/H becomes an affine Nash group under this Nash structure.*

Proof.   Let G be an algebraization of G, with the algebraization homomorphism $\phi : G \to \mathsf{G}(\mathbb{R})$. Let H denote the Zariski closure of $\phi(H)$ in G. Then by Corollary 2.3, $\mathsf{G}(\mathbb{C})/\mathsf{H}(\mathbb{C}) = (\mathsf{G}/\mathsf{H})(\mathbb{C})$ is the set of $\mathbb{C}$-points of a smooth quasi-projective algebraic variety over $\mathbb{R}$. Therefore $\mathsf{G}(\mathbb{C})/\mathsf{H}(\mathbb{C})$ is naturally an affine Nash manifold. Note that all arrows in

$$G \xrightarrow{\phi} \mathsf{G}(\mathbb{R}) \to \mathsf{G}(\mathbb{C}) \to \mathsf{G}(\mathbb{C})/\mathsf{H}(\mathbb{C})$$

are Nash maps. Therefore, the composition, which is denoted by ϕ', is also a Nash map. By [BOR03, Proposition 5.53], the homogeneity (under the action of G) of $\phi'(G)$ implies that $\phi'(G)$ is a Nash submanifold of $\mathsf{G}(\mathbb{C})/\mathsf{H}(\mathbb{C})$. Note that ϕ' induces a finite-fold covering map

$$G/H \to \phi'(G). \tag{1}$$

By [Sun15, Proposition 2.4], there is a unique Nash structure on the topological space G/H which makes the map (1) a submersive Nash map. By [Sun15, Lemma 2.7], we know that the quotient map $G \to G/H$ is a Nash map, with the aforementioned Nash structure on G/H. It is a consequence of [War83, Theorem 3.62] that the quotient map $G \to G/H$ is submersive, and it is implied by [Sh87, Proposition III.1.7] that the Nash manifold G/H is affine. This proves the existence of the desired Nash structure on G/H. All other assertions of the proposition easily follows from [Sun15, Lemma 2.3]. ∎

When G is almost linear, Proposition 4.1 is formulated and proved in [Sun15, Proposition 1.2]. By Proposition 4.1, the quotients of an affine Nash group by a Nash subgroup of it is naturally an affine Nash manifold, and is naturally an affine Nash group if the Nash subgroup is normal.

Now we are prepared to formulate and prove Chevalley's theorem for affine Nash groups.

Theorem 4.2.   *Let G be a connected affine Nash group. Then there exists a unique connected normal almost linear Nash subgroup L of G such that G/L is an abelian Nash manifold.*

Proof. Let G be an algebraization of G, with the algebraization homomorphism $\phi : G \to \mathsf{G}(\mathbb{R})$. Using Chevalley's theorem, let L be the unique connected normal linear algebraic subgroup of G such that G/L is an abelian variety. Let L be the identity connected component of $\phi^{-1}(\mathsf{L}(\mathbb{R}))$, which is a connected normal Nash subgroup of G. Note that L is an algebraization of L and G/L is an algebraization of G/L. Therefore Lemma 3.6 and Lemma 3.7 imply that L is an almost linear Nash group, and G/L is an abelian Nash manifold. This proves the "existence" part of the theorem.

To prove the uniqueness, let L' be a connected normal almost linear Nash subgroup of G such that G/L' is an abelian Nash manifold. Let L' denote the Zariski closure of $\phi(L')$ in G, which is a connected normal algebraic subgroup of G. Note that L' is an algebraization of L' and G/L' is an algebraization of G/L'. Lemma 3.6 and Lemma 3.7 again imply that L' is a linear algebraic group, and G/L' is an abelian variety. Then Chevalley's theorem implies that $\mathsf{L}' = \mathsf{L}$. Therefore $L' = L$ since they have the same Lie algebra. ∎

The following result is an analogue of Proposition 2.5 in the setting of affine Nash groups.

Theorem 4.3. *Let*
$$1 \to G' \overset{i}{\to} G \overset{\pi}{\to} G'' \to 1 \tag{2}$$
be a sequence of Nash homomorphisms of affine Nash groups. Assume that it is exact as a sequence of abstract groups. Then the followings hold true.

(a) The affine Nash group G is almost linear if and only if both G' and G'' are so.

(b) The affine Nash group G is complete if and only if both G' and G'' are so.

Proof. Let G'' be an algebraization of G''. By Lemma 3.5, we have an algebraization G of G, together with an algebraic homomorphism $\underline{\pi} : \mathsf{G} \to \mathsf{G}''$ such that the diagram

$$
\begin{array}{ccc}
G & \overset{\pi}{\longrightarrow} & G'' \\
\downarrow & & \downarrow \\
\mathsf{G}(\mathbb{R}) & \overset{\underline{\pi}}{\longrightarrow} & \mathsf{G}''(\mathbb{R})
\end{array}
$$

commutes, where the vertical arrows are the algebraization homomorphisms. Denote by G' the kernel of $\underline{\pi}$, which is an algebraic group over $\mathbb{R}$ and fits to an exact sequence
$$1 \to \mathsf{G}' \overset{i}{\to} \mathsf{G} \overset{\pi}{\to} \mathsf{G}'' \to 1 \tag{3}$$
of algebraic groups. Note that an open algebraic subgroup of G' is an algebraization of G'. In view of Lemma 2.6, Lemmas 3.6 and 3.7 imply that G' is almost linear or complete if and only if G' is respectively linear or complete. Lemmas 3.6 and 3.7 also imply that G'' (or G) is almost linear or complete if and only if G'' (or G, respectively) is respectively linear or complete. Therefore, the proposition is a consequence of Proposition 2.5. ∎

Theorem 1.6 is clearly a special case of part (a) of Theorem 4.3.

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