Journal of Lie Theory
Volume **26** (2016) 371–382

Deformation of discontinuous groups acting on $(H_{2n+1} \times H_{2n+1})/\Delta$

Sami Dhieb

Communicated by T. Kobayashi

Abstract. Let H_{2n+1} be the $(2n+1)$-dimensional Heisenberg group and Δ the diagonal subgroup of the product $P := H_{2n+1} \times H_{2n+1}$. Given any discontinuous group Γ for P/Δ, we study some local geometric and topological features of the associated deformation space $\mathscr{T}(\Gamma, P; P/\Delta)$ such as rigidity, stability and Hausdorffness. In particular, we show that $\mathscr{T}(\Gamma, P; P/\Delta)$ is a Hausdorff space if and only if Γ is a cocompact abelian discontinuous group for P/Δ.
Mathematics Subject Classification 2010: Primary 22E27; Secondary 32G 05.
Key Words and Phrases: Heisenberg group, proper action, free action, rigidity, stability.

1. Introduction

Let G be a nilpotent Lie group, $P := G \times G$ the direct product Lie group and $\Delta := \Delta(G)$ the diagonal Lie subgroup of P. In this paper, we study some geometric features of the deformation space of discontinuous groups acting on the homogeneous space P/Δ, for which the group G is the $(2n+1)$-dimensional Heisenberg group for an integer $n \geq 1$. One motivation to seek such a setting, comes up from the semisimple case which was studied earlier. In fact, for a semisimple group G, T. Kobayashi proved in [11] that any discontinuous group for the group manifold $G \times G/\Delta(G)$ is of the form of a graph up to a finite index subgroup if and only if G is of real rank one. In [13], T. Kobayashi initiated the general theory of deformation of discontinuous groups for the general non-Riemannian settings (see [14, Problem C] for further perspectives and basic examples). In particular, he proved a stability theorem (see Section 4.2 for the definition) in a certain semisimple setting. For $G = \mathrm{SL}(2, \mathbb{R})$, the deformation space of cocompact discontinuous groups for P/Δ is nothing but the deformation of anti-de Sitter structures of compact three-dimensional manifolds, which has been extensively studied recently, see [8] and the references therein. On the other hand, discontinuous groups for the group manifold P/Δ for nilpotent groups have not been studied except for a recent work by S. Barmeier [7] on the three dimensional

ISSN 0949–5932 / \$2.50

Heisenberg group H_3. In this article, we consider the deformation of discontinuous groups for P/Δ when G is a higher dimensional Heigensberg group H_n. The deformation space $\mathscr{T}(\Gamma, G; G/H)$ of discontinuous groups Γ in the general setting was first introduced by T. Kobayashi [13, (5.3.1)], who gave in a subsequent paper with S. Nasrin (see [15]) a complete and explicit description of such a space for $\Gamma \simeq \mathbb{Z}^k$ acting properly discontinuously on $G/H \simeq \mathbb{R}^{k+1}$ by affine transformations. In [6], A. Baklouti, I. Kedim and T. Yoshino obtained a complete description of that space when the transformation group in question is the Heisenberg group in the case of compact Clifford-Klein forms.

The purpose of this paper is to give a concrete description of the parameter space $R(\Gamma, P; P/\Delta)$ up to a P-equivariant homeomorphism for $P = H_{2n+1} \times H_{2n+1}$. By using this description, we discuss stability and rigidity of discontinuous groups and the Hausdorfness of deformation space of discontinuous groups.

The paper is organized as follows. The next section recalls some basic properties of Clifford-Klein forms and reviews the parameter and deformation spaces of the action of a discontinuous group on a 2-step homogeneous space G/H. In the third section, we provide an explicit description of the parameter space of a discontinuous group acting on P/Δ. The fourth section aims to study the topological features of deformations making use of the results of the section before. Namely, the rigidity (Section 4.1), the stability (Section 4.2) and the Hausdorffness of the deformation space (Section 4.3). We show that the space $\mathscr{T}(\Gamma, P; P/\Delta)$ is a Hausdorff space if and only if Γ is a cocompact abelian discontinuous group for P/Δ.

2. Preliminaries

2.1. Proper and free actions. Let G be a locally compact group and H a closed subgroup of G. The action of a closed subgroup L of G on the homogeneous space $X = G/H$ is said to be:

(1) proper if for any compact set S in G, $SHS^{-1} \cap L$ is compact. Here the product SHS^{-1} is the subset $\{abc, a \in S, b \in H, c \in S^{-1}\}$.

(2) free if for any $g \in G$, $gHg^{-1} \cap L = \{e\}$.

In this context, the subgroup L is said to be a discontinuous group for the homogeneous space G/H, if L is a discrete subgroup of G and L acts properly and freely on G/H.

2.2. Parameter and deformation spaces. We review from [14] the deformation space of discontinuous groups for homogeneous spaces. For any given discontinuous group Γ for G/H, the quotient $\Gamma \backslash G/H$ is said to be a Clifford-Klein form for G/H. It is endowed through the action of Γ with a manifold structure such that the canonical projection:

$$\pi : G/H \to \Gamma \backslash G/H$$

is a covering map. We designate by $\mathrm{Hom}(\Gamma, G)$ the set of group homomorphisms from Γ to G. Suppose that Γ is a finitely generated group with generators $\{\gamma_1, ..., \gamma_k\}$. Then we give a topology on $\mathrm{Hom}(\Gamma, G)$ by the embedding:

$$\mathrm{Hom}(\Gamma, G) \to G \times \cdots \times G, \ \varphi \mapsto (\varphi(\gamma_1), ..., \varphi(\gamma_k))$$

to equip $\mathrm{Hom}(\Gamma, G)$ with the relative topology induced from the product $G\times\cdots\times G$. The parameter space $R(\Gamma, G; G/H)$ was introduced by T. Kobayashi in [11, page 137] as the set:

$$R(\Gamma, G; G/H) := \left\{ \varphi \in \mathrm{Hom}(\Gamma, G) \;\middle|\; \begin{array}{l} \varphi \text{ is injective and } \varphi(\Gamma) \\ \text{acts properly discontinuously} \\ \text{on } G/H \end{array} \right\}. \quad (1)$$

For $\varphi \in \mathrm{Hom}(\Gamma, G)$ and $g \in G$, we define the element φ^g of $\mathrm{Hom}(\Gamma, G)$ defined by:

$$\varphi^g(\gamma) = g\varphi(\gamma)g^{-1}, \gamma \in \Gamma.$$

Then, it is clear that $R(\Gamma, G; G/H)$ is invariant under this action. In [14], T. Kobayashi defined the deformation space of discontinuous groups for G/H as

$$\mathscr{T}(\Gamma, G; G/H) = R(\Gamma, G; G/H)/G.$$

2.3. On 2-step nilpotent Lie groups. Let $G = \exp\mathfrak{g}$ be a 2-step connected and simply connected nilpotent Lie group, which means that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g})$, the center of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ acts on itself by the adjoint representation ad and the group G acts on its Lie algebra $\mathfrak{g}$ by the adjoint representation Ad. Let L be the syndetic hull of Γ which is the smallest connected subgroup of G containing Γ co-compactly, (see [16]). We have the following:

Fact 1. (Nasrin [17]) Let $G = \exp\mathfrak{g}$ be a connected, simply connected 2-step nilpotent Lie group, $H = \exp\mathfrak{h}$ a closed connected subgroup of G and Γ a discrete subgroup of G. We denote by $L = \exp\, \mathfrak{l}$ the syndetic hull of Γ. Then the following five conditions on the triple (G, H, Γ) are equivalent:

(i) Γ acts on G/H properly.
(ii) Γ acts on G/H freely.
(iii) L acts on G/H properly.
(iv) L acts on G/H freely.
(v) $\mathfrak{l} \cap \mathrm{Ad}_g\, \mathfrak{h} = \{0\}$ for any $g \in G$.

We now prove the following lemma which is a counterpart in the nilpotent setting of Kobayashi's criterion [9, Theorem 4.7] in the semisimple setting:

Lemma 2.1. *Let Γ be a discontinuous group for the homogeneous space G/H and L its syndetic hull. Then Γ is cocompact for G/H if and only if*

$$\dim G = \dim L + \dim H.$$

Proof. As Γ acts properly on G/H, L also does as in Fact 1 and then $L\cap H = \{e\}$. Now, $L\backslash G/H$ is compact and then $G = LH$ as in the proof of [19, Lemma 17]. Therefore, $\dim G = \dim L + \dim H$. Conversely, if $\dim G = \dim L + \dim H$ and L acts on G/H freely as in Fact 1, then L acts on G/H transitively and hence $\Gamma\backslash G/H$ is compact. ∎

The following useful result was given in [15]:

Theorem 2.2. *Let $G = \exp \mathfrak{g}$ be a connected, simply-connected 2-step nilpotent Lie group, $H = \exp \mathfrak{h}$ a closed connected subgroup of G and Γ a discrete subgroup of G. We denote by $L = \exp \mathfrak{l}$ the syndetic hull of Γ. Then, the parameter space $R(\Gamma, G; G/H)$ has a bijection onto:*

$$R(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \{\psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) : \dim \psi(\mathfrak{l}) = \dim \mathfrak{l} \text{ and } \mathrm{Ad}_g(\mathfrak{l}) \cap \mathfrak{h} = \{0\} \text{ for any } g \in G\}.$$

The deformation space $\mathscr{T}(\Gamma, G; G/H)$ is likewise homeomorphic to the space

$$\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = R(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) / \mathrm{Ad}.$$

3. Parameter space for the group $H_{2n+1} \times H_{2n+1}$

3.1. Generalities. Let $\mathfrak{h}_{2n+1}$ be the $(2n+1)$-dimensional Heisenberg group spanned by the vectors $(X_1, ..., X_n, Y_1, ..., Y_n, Z)$ with the Lie brackets relations:

$$[X_i, Y_j] = \delta_{ij} Z, \ 1 \leq i, j \leq n,$$

where δ_{ij} is the Kronecker symbol and Z is a generator of the center. Let $P = H_{2n+1} \times H_{2n+1}$ and $\mathfrak{p}$ its Lie algebra. Let exp be the exponential of P defined on $\mathfrak{p}$ by

$$\exp(X, Y) = (\exp X, \exp Y), \ (X, Y) \in \mathfrak{p}.$$

The center $\mathfrak{z}$ of the Lie algebra $\mathfrak{p}$ is 2-dimensional and is generated by the vectors $(Z, 0)$ and (Z, Z). We denote by $\Delta = \{(x, x) : \ x \in H_{2n+1}\}$ the diagonal subgroup of P, which is a $(2n+1)$-dimensional subgroup of P. Its Lie algebra $\mathfrak{D} = \{(X, X) : \ X \in \mathfrak{h}_{2n+1}\}$ is generated by $\{(Z, Z), (X_i, X_i), (Y_i, Y_i,) : \ 1 \leq i \leq n\}$ so that $\mathfrak{p}$ is decomposed as:

$$\mathfrak{p} = \mathbb{R}(Z, 0) \oplus \mathfrak{D} \oplus \mathfrak{k},$$

where $\mathfrak{k}$ is the vector space spanned by $\{(X_i, 0), (Y_i, 0) : \ 1 \leq i \leq n\}$. The group P acts on $\mathfrak{p}$ by the adjoint action Ad_P such that for all $(X, X'), (Y, Y') \in \mathfrak{p}$,

$$\begin{aligned}\mathrm{Ad}_{\exp(X,X')}(Y, Y') &= e^{\mathrm{ad}_{(X,X')}}(Y, Y') \\ &= (Y, Y') + \mathrm{ad}_{(X,X')}(Y, Y') \\ &= (Y, Y') + ([X, Y], [X', Y']).\end{aligned}$$

From now on, we fix the following basis of $\mathfrak{p}$:

$$\mathscr{B} = \{(Z, 0), (Z, Z), (X_i, X_i), (Y_i, Y_i), (X_i, 0), (Y_i, 0) : \ 1 \leq i \leq n\}.$$

For $(X, Y), (X', Y') \in \mathfrak{p}$, one has:

$$\begin{aligned}[(X, Y), (X', Y')] &= ([X, X'], [Y, Y']) \\ &= b_1\big((X, Y), (X', Y')\big)(Z, 0) + b_2\big((X, Y), (X', Y')\big)(Z, Z).\end{aligned}$$

where b_1 and b_2 are the skew symmetric bilinear forms on $\mathfrak{p}$ defining its Lie brackets. By a routine computation, the matrices J_{b_1} and J_{b_2} of b_1 and b_2, written through the basis $\mathscr{B}$, are given by:

$$J_{b_1} = \begin{pmatrix} 0_{\mathscr{M}_2(\mathbb{R})} & 0 & 0 \\ 0 & 0 & J \\ 0 & J & J \end{pmatrix}, \quad J_{b_2} = \begin{pmatrix} 0_{\mathscr{M}_2(\mathbb{R})} & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Let now Γ be a discontinuous group for P/Δ and $L = \exp \mathfrak{l}$ its syndetic hull. Since $\mathfrak{l} \cap \mathfrak{D} = \{0\}$, then $(Z, Z) \notin \mathfrak{l}$. Therefore, L is abelian or 2-step nilpotent with $\dim[\mathfrak{l}, \mathfrak{l}] = \dim\ (\mathfrak{l} \cap \mathfrak{z}) = 1$. We fix then a basis $\mathscr{B}_0 = \{e_1, ..., e_k\}$ of $\mathfrak{l}$ such that e_1 is a generator of $[\mathfrak{l}, \mathfrak{l}]$ whenever this space is not trivial. Then, for all $1 \leq i, j \leq k$, $[e_i, e_j] = \alpha_{ij} e_1$. Denote by K the matrix $(\alpha_{ij})_{i,j=1,...,k}$. This matrix equals zero when $\mathfrak{l}$ is abelian. Otherwise, K has the form:

$$K = \begin{pmatrix} 0 & 0 \\ 0 & K_0 \end{pmatrix}, \tag{2}$$

where $K_0 \in \mathscr{M}_{k-1}(\mathbb{R})$ is a skew symmetric matrix. Let ψ be a linear map from $\mathfrak{l}$ to $\mathfrak{p}$. Its matrix written through the bases $\mathscr{B}_0$ and $\mathscr{B}$ is

$$M = M(x, y, A, B) = \begin{pmatrix} x \\ y \\ A \\ B \end{pmatrix}, x, y \in \mathscr{M}_{1,k}(\mathbb{R}), \ A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \ B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{3}$$

where A_1, A_2, B_1, B_2 are in $\mathscr{M}_{n,k}(\mathbb{R})$. If $\psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{p})$, we have:

$$[\psi(e_i), \psi(e_j)] = \psi([e_i, e_j]) \text{ for all } \ 1 \leq i, j \leq k. \tag{4}$$

When $\mathfrak{l}$ is non-abelian, there exist $V_1, V_2 \in \mathfrak{l}$ such that $[V_1, V_2] = e_1$. So,

$$\psi(e_1) = \psi([V_1, V_2]) = [\psi(V_1), \psi(V_2)] \in \mathfrak{z}.$$

There exist therefore x_0 and y_0 in $\mathbb{R}$ such that $\psi(e_1) = x_0(Z, 0) + y_0(Z, Z)$. Hence,

$$\begin{aligned} \psi([e_i, e_j]) &= \alpha_{ij} \psi(e_1) \\ &= \alpha_{ij}(x_0(Z, 0) + y_0(Z, Z)) \\ &= x_0 \alpha_{ij}(Z, 0) + y_0 \alpha_{ij}(Z, Z). \end{aligned}$$

Then, the equations (4) read in term of matrices:

$${}^t M J_{b_1} M = x_0 K \text{ and } {}^t M J_{b_2} M = y_0 K,$$

which is equivalent to:

$${}^t A J B + {}^t B J A + {}^t B J B = x_0 K \text{ and } {}^t A J A = y_0 K,$$

or also

$$\begin{cases} {}^tB_2A_1 - {}^tA_1B_2 + {}^tA_2B_1 - {}^tB_1A_2 + {}^tB_2B_1 - {}^tB_1B_2 = x_0K \\ {}^tA_2A_1 - {}^tA_1A_2 = y_0K. \end{cases} \tag{5}$$

If $\mathfrak{l}$ is abelian, we get the same equations as in (5) but with $K = 0$ as it was mentioned above. Let

$$\mathscr{E} = \{M(x, y, A, B) \in \mathscr{M}_{4n+2,k}(\mathbb{R}) \text{ satisfying (5)}\}.$$

Since the spaces $\mathrm{Hom}(\Gamma, P)$ and $\mathrm{Hom}(\mathfrak{l}, \mathfrak{p})$ are homeomorphic (see [3, Proposition 3.6 and Section 4.2]), the following proposition is straightforward:

Proposition 3.1. *The space* $\mathrm{Hom}(\Gamma, P)$ *is homeomorphic to* $\mathscr{E}$*.*

Recall that P acts on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{p})$ by $\psi^g = \mathrm{Ad}_{g^{-1}} \circ \psi$, $g \in P$, $\psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{p})$ and P acts on $\mathscr{E}$ by $M^g = \mathrm{Ad}_{g^{-1}} \cdot M$, $g \in P$, and $M \in \mathscr{E}$. The identification $\Phi : \psi \longmapsto M(\psi, \mathscr{B}_0, \mathscr{B})$ is a homeomorphism which is P-equivariant. That is $(\Phi(\psi))^g = \Phi(\psi^g), g \in P$. Let $X \in \mathfrak{p}$ with coordinates ${}^t(a, b, \alpha, \beta, \gamma, \delta)$ through the basis $\mathscr{B}$, where $a, b \in \mathbb{R}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}^{2n}$. For a matrix $M = M(x, y, A, B)$ as in (3), we have:

$$\mathrm{Ad}_{\exp X} \cdot M = M\Big(x - \delta A_1 + \gamma A_2 - (\beta + \delta)B_1 + (\alpha + \gamma)B_2, y - \beta A_1 + \alpha A_2, A, B\Big).$$

Putting $u = (-(\beta + \delta), \alpha + \gamma)$ and $v = (-\beta, \alpha)$, we get:

$$\mathrm{Ad}_{\exp X} \cdot M = M\Big(x + u(A + B) - vA, y + vA, A, B\Big) := \mathrm{Ad}_{(u,v)} \cdot M. \tag{6}$$

This allows us to give the following description of the parameter space:

Proposition 3.2. *The parameter space* $R(\Gamma, P; P/\Delta)$ *is homeomorphic to the space*

$$\{M \in \mathscr{E} : rk \begin{pmatrix} x - \omega A \\ B \end{pmatrix} = k, \text{ for all } \omega \in \mathbb{R}^{2n}\}. \tag{7}$$

Proof. Let $\psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{p})$ and $M = M(x, y, A, B)$ its associated matrix. By Fact 1, the action of $\exp \psi(\mathfrak{l})$ on P/Δ is proper if and only if for all $X \in \mathfrak{p}$, $\mathrm{Ad}_{\exp X}(\psi(\mathfrak{l})) \cap \mathfrak{D} = \{0\}$. That is, $\mathrm{Ad}_{\exp X} M(Y) \notin \mathfrak{D}$ for all $Y \in \mathfrak{l} \backslash \{0\}$ and for all $X \in \mathfrak{p}$. According to (6), this means that for all $u, v \in \mathbb{R}^{2n}$ and $Y \in \mathfrak{l} \backslash \{0\}$,

$$\begin{pmatrix} x + (u - v)A + uB \\ B \end{pmatrix} Y \neq 0,$$

which is in turn equivalent to

$$\mathrm{rk} \begin{pmatrix} x + (u - v)A + uB \\ B \end{pmatrix} = \mathrm{rk} \begin{pmatrix} x + (u - v)A \\ B \end{pmatrix} = k \text{ for all } u, v \in \mathbb{R}^{2n}.$$

■

Recall that the homogeneous space P/Δ admits a cocompact abelian discontinuous group $\Gamma(\simeq \mathbb{Z}^{2n+1})$ for P/Δ. Indeed, if we take $x = (0, ..., 0, 1)$, $y = 0$ and $A, B \in \mathscr{M}_{2n,2n+1}(\mathbb{R})$ such that $A_1 = -B_1 = (I_n\ 0)$, $A_2 = 0$ and $B_2 = (0_n\ I_n\ 0)$, then $M(x, y, A, B)$ corresponds to an element in $R(\mathbb{Z}^{2n+1}, P; P/\Delta)$.

Assume now that the subgroup Γ is non-abelian, which means that $\mathfrak{l} \cap \mathfrak{z} = \mathbb{R}e_1$. Note that if $\psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{p})$, $\psi(e_1)$ belongs to $\mathfrak{z}$ and therefore its associated matrix $M = M(x, y, A, B)$ is such that $x = (x_0, \tilde{x})$, $y = (y_0, \tilde{y})$, $A = (0\ A_0)$ and $B = (0\ B_0)$ where $x_0, y_0 \in \mathbb{R}$, $\tilde{x}, \tilde{y} \in \mathbb{R}^{k-1}$ and $A_0, B_0 \in \mathscr{M}_{2n,k-1}(\mathbb{R})$. Moreover,

$$\mathrm{rk}\begin{pmatrix} x - \omega A \\ B \end{pmatrix} = \mathrm{rk}\begin{pmatrix} x_0 & \tilde{x} - \omega A_0 \\ 0 & B_0 \end{pmatrix} = k \text{ for all } \omega \in \mathbb{R}^{2n}$$

if and only if $x_0 \neq 0$ and $\mathrm{rk}(B_0) = k - 1$. Then, we have:

Corollary 3.3. *If the subgroup Γ is non-abelian,the parameter space $R(\Gamma, P; P/\Delta)$ is homeomorphic to the space*

$$\{M\Big((x_0, \tilde{x}), (y_0, \tilde{y}), (0, A_0), (0, B_0)\Big) \in \mathscr{E} :\ x_0 \neq 0 \textit{ and } rk(B_0) = k - 1\}. \qquad (8)$$

4. Local geometric and topological features of deformations

4.1. The concept of rigidity. We keep the same notation and assumptions. Generalizing Weil's notion of local rigidity of discontinuous groups for Riemannian symmetric spaces, T. Kobayashi introduced the notion of local rigidity and rigidity of discontinuous for non-Riemannian homogeneous spaces. Notably, he proved in [13] that for the reductive case, the local rigidity may fail even for irreducible symmetric space of high dimensions. We briefly recall here some details. For comprehensible information, we refer the readers to the references [9, 10, 11, 12, 13, 14]. For $\varphi \in R(\Gamma, G; G/H)$, the discontinuous group $\varphi(\Gamma)$ for the homogeneous space G/H is said to be *locally rigid* (see [11]) as a discontinuous group for G/H, if the orbit of φ under the inner conjugation is open in $R(\Gamma, G; G/H)$. This means equivalently that any point sufficiently close to φ should be conjugate to φ under an inner automorphism of G. The homomorphisms which are locally rigid are those which correspond to isolated points in the deformation space $\mathscr{T}(\Gamma, G; G/H)$. When every point in $R(\Gamma, G; G/H)$ is locally rigid, the deformation space turns out to be discrete and the Clifford-Klein form $\Gamma\backslash G/H$ does not admit *continuous deformations*. If a given $\varphi \in R(\Gamma, G; G/H)$ is not locally rigid, it admits continuous deformations and the related Clifford-Klein form is continuously deformable. The following result was proved in [4] for general 2-step nilpotent connected and simply connected Lie groups. We give here a simpler proof in our context.

Theorem 4.1. *Let Γ be a non-trivial discontinuous group for the homogenous space P/Δ. Then, the local rigidity fails to hold.*

Proof. We consider the action of $\mathbb{R}_+^*$ on $R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ defined by:

$$\mathbb{R}_+^* \times R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D}) \longrightarrow R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$$

$$\Big(t, M = M(x, y, A, B)\Big) \longmapsto t \star M := M(t^2x, t^2y, tA, tB).$$

This action factors to an action on $\mathscr{T}(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ through the following:

$$t \cdot [M] = [t \star M],\ M \in R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D}), t \in \mathbb{R}_+^*,$$

where $[M]$ designates the class of M in $\mathscr{T}(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$. According to formula (6), for $u, v \in \mathbb{R}^{2n}$, we get

$$t\star(\mathrm{Ad}_{(u,v)} \cdot M) = M\Big(t^2x+t^2(u-v)A+t^2uB, t^2y+t^2vA, tA, tB\Big) = \mathrm{Ad}_{(tu,tv)} \cdot (t\star M).$$

That is, the $\mathbb{R}_+^*$-action on $\mathscr{T}(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ is well-defined. Moreover, for $s, t \in \mathbb{R}_+^*$, $t \cdot [M] = s \cdot [M]$ if and only if $t \star M = G \cdot (s \star M)$. Hence, $s = t$ by formula (6). This shows that $[M]$ lies in a one dimensional curve and therefore it can not be an open point inside $\mathscr{T}(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$. ■

4.2. The concept of stability. A homomorphism $\varphi \in R(\Gamma, G; G/H)$ is said to be *stable* in the sense of Kobayashi-Nasrin [15], if there is an open set in $\mathrm{Hom}(\Gamma, G)$ which contains φ and is contained in $R(\Gamma, G; G/H)$. When the set $R(\Gamma, G; G/H)$ is an open subset of $\mathrm{Hom}(\Gamma, G)$, then, obviously each of its elements is stable which is the case for any irreducible Riemannian symmetric space under the assumption that Γ is a torsion free uniform lattice of G ([14] and [18]). Furthermore, the concept of stability may be fundamental to understand the local structure of the deformation space. We give here a necessary and sufficient condition of stability for the space $R(\Gamma, P; P/\Delta)$.

Theorem 4.2. *Let Γ be a discontinuous group for the homogeneous space P/Δ with the syndetic hull $L = \exp \mathfrak{l}$. Then, the following assertions are equivalent:*

(i) $\mathfrak{z}(\mathfrak{p}) \subset \mathfrak{D} \oplus \psi(\mathfrak{l})$ *for any* $\psi \in R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$.

(ii) Γ *is non-abelian or* Γ *is a cocompact abelian discontinuous group for* P/Δ.

(iii) *The stability holds for all* $\psi \in R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$*; namely,* $R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ *is open in* $\mathrm{Hom}(\mathfrak{l}, \mathfrak{p})$.

Proof. Let ψ be an element of $R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ and $M = M(x, y, A, B)$ its matrix as above. Then, $\mathfrak{z}(\mathfrak{p}) \subset \mathfrak{D} \oplus \psi(\mathfrak{l})$ if and only if there exist two vectors $S \in \mathfrak{l}$ and $T \in \mathfrak{D}$ satisfying

$$MS = (Z, 0) + T. \tag{9}$$

We prove first that (i) $\Leftrightarrow$ (ii). If Γ is non-abelian, then, the matrix M takes the form as in Corollary 3.3. Equation (9) holds by taking $S = \frac{1}{x_0}e_1$ and $T = \frac{y_0}{x_0}(Z, Z)$. Suppose that Γ is a cocompact abelian discontinuous group for P/Δ. Then by Lemma 2.1, $k = 2n + 1$. According to Proposition 3.2, it follows that the linear map ${}^t(x\ B) : \mathfrak{l} \longrightarrow \mathrm{Span}\{(Z, 0), (X_i, 0), (Y_i, 0) : 1 \leq i \leq n\}$ is invertible. Equation (9) holds for $S = {}^t(x\ B)^{-1}(Z, 0)$ and $T = -{}^t(y\ A)S$. Conversely, if Γ is abelian and not cocompact for P/Δ, then $k \leq 2n$. We take $\psi \in R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ of matrix

$M(x,y,0,B)$ with $B = {}^t(I_k\ 0)$. Equation (9) gives that $BS = 0$ and therefore $S = 0$ which is impossible as $(Z,0) \notin \mathfrak{D}$.
We now prove that (ii) $\Leftrightarrow$ (iii). Notice that the parameter space is open in $\mathrm{Hom}(\mathfrak{l},\mathfrak{p})$ in both cases where $\mathfrak{l}$ is non-abelian (from Corollary 3.3) or abelian and the Clifford-Klein form $\Gamma\backslash G/H$ is compact (see [1, Proposition 4.1]). Let then $\mathfrak{l}$ be abelian and $k \le 2n$. Suppose first that $k \le n+1$. To see that $R(\mathfrak{l},\mathfrak{p},\mathfrak{D})$ is not open in $\mathrm{Hom}(\mathfrak{l},\mathfrak{p})$, choose $M = M(x,0,0,B)$ such that ${}^t(x\ B) = {}^t(I_k\ 0)$. Then, according to Proposition 3.2, $M \in R(\mathfrak{l},\mathfrak{p},\mathfrak{D})$. For $\varepsilon > 0$, let $M^\varepsilon = M(x,0,A^\varepsilon,B)$ where $A^\varepsilon = (a_{i,j}) \in \mathscr{M}_{2n,k}(\mathbb{R})$ such that $a_{1,1} = \varepsilon$ and $a_{i,j} = 0$ elsewhere. Then, M^ε satisfies the homomorphism conditions (5) as $K = 0$ here and hence M^ε is in $\mathscr{E}$. For $w^\varepsilon = (\frac{1}{\varepsilon},0,...,0) \in \mathbb{R}^{2n}$, we have ${}^t(x - w^\varepsilon A^\varepsilon\ B)$ is of rank $k-1$. It follows that for all $\varepsilon > 0$, $M^\varepsilon \notin R(\mathfrak{l},\mathfrak{p},\mathfrak{D})$. Suppose now that $n+2 \le k \le 2n$. Let $x_0 = (0,...,0,1) \in \mathbb{R}^k$ and

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_{k-n-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathscr{M}_{2n,k}(\mathbb{R}),\ A = \begin{pmatrix} 0 \\ -B_2 \end{pmatrix}. \tag{10}$$

Then, for any $w \in \mathbb{R}^{2n}$, $\mathrm{rk}\ {}^t(x - wA\ B) = k$. Therefore, $M(x_0,0,A,B)$ is in $R(\mathfrak{l},\mathfrak{p},\mathfrak{D})$. For any $\varepsilon > 0$, let $A_1^\varepsilon = (a_{i,j}) \in \mathscr{M}_{n,k}(\mathbb{R})$ such that $a_{nk} = \varepsilon$ and $a_{ij} = 0$ elsewhere and $A^\varepsilon = {}^t(A_1^\varepsilon\ \ -B_2)$. Then, $M^\varepsilon = M(x_0,0,A^\varepsilon,B)$ belongs to $\mathscr{E}$ and the same conclusion holds as in the previous case by taking $w^\varepsilon = (0,...,0,w_n^\varepsilon = \frac{1}{\varepsilon},0,...,0)$. ■

4.3. Hausdorffness of the deformation space. This section aims to study the Hausdorffness property of the deformation space $\mathscr{T}(\Gamma,P;P/\Delta)$. Our result is the following:

Theorem 4.3. *Let Γ be a non-trivial discontinuous group for the homogeneous space P/Δ. Then, the deformation space $\mathscr{T}(\Gamma,P;P/\Delta)$ is a Hausdorff space if and only if Γ is abelian and cocompact for P/Δ.*

Proof. As $\mathfrak{l} \cap \mathfrak{D} = \{0\}$, then as earlier, $\dim \mathfrak{l} = k \le 2n+1$. The following lemma is proved in [5, Lemma 5.2] for $k = n+1$ but still holds for any $k \ge n$.

Lemma 4.4. *Let $k \ge n$ and $A_1, A_2 \in \mathscr{M}_{n,k}(\mathbb{R})$ such that ${}^tA_2A_1 - {}^tA_1A_2 = 0$. Then $\mathrm{rk}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \le n$.*

We first treat the case where Γ is a cocompact abelian discontinuous group for P/Δ. Then, by Lemma 2.1 $k = 2n+1$. We will prove that in this case the P-orbits in $R(\mathfrak{l},\mathfrak{p},\mathfrak{D})$ have a common dimension and therefore, according to [6, Theorem 6.2], the deformation space is a Hausdorff space. By formula (6), this amounts to show that for any $M(x,y,A,B)$ in $R(\mathfrak{l},\mathfrak{p},\mathfrak{D})$, A and $A+B$ have a common rank. Indeed, recall that the homomorphism conditions (5) in this case are equivalent to

$$\begin{cases} {}^tA_2A_1 - {}^tA_1A_2 = 0 \\ {}^t(A_2+B_2)(A_1+B_1) - {}^t(A_1+B_1)(A_2+B_2) = 0. \end{cases}$$

By Lemma 4.4, we have $\mathrm{rk}(A) \leq n$ and $\mathrm{rk}(A+B) \leq n$. In addition, as $k = 2n+1$, Proposition 3.2 gives that $\mathrm{rk}(B) = 2n$. It follows that $\mathrm{rk}(A) = \mathrm{rk}(A+B) = n$. Assume now Γ be an abelian discontinuous group for P/Δ and where the Clifford Klein form $\Gamma\backslash P/\Delta$ is not compact. Then, $k \leq 2n$. For $i = 1,2$ let $M_i = M(x, y^i, A, B)$ such that $x = (0, ..., 0, 1)$, $A = {}^t(A_1\ A_2)$ and $B = {}^t(B_1\ B_2)$. First assume that $k \leq n$. Let $y^1 = 0$, $y^2 = (1, 0, ..., 0)$, $A = 0$ and $B = \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}$. Then M_1 and M_2 belong to $R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ and $[M_1] \neq [M_2]$. Let $\varepsilon > 0$. For $i = 1,2$, the matrices $M_i^\varepsilon = M(x, y^i, A^\varepsilon, B)$ such that $A^\varepsilon = (a_{i,j})$ where $a_{2n,1} = \varepsilon$ and $a_{i,j} = 0$ elsewhere are in $R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$. Moreover, for $u^\varepsilon = (1, 0, ..., 0)$ and $v^\varepsilon = (0, ..., 0, \frac{1}{\varepsilon})$,

$$\mathrm{Ad}_{(u^\varepsilon, v^\varepsilon)} \cdot M_1^\varepsilon = M_2^\varepsilon, \tag{11}$$

which is enough to conclude in this case. Suppose now that $n+1 \leq k \leq 2n$. Let $y^1 = 0$, $y^2 = (1, 0, ..., 0)$ and A, B in $\mathscr{M}_{2n,k}(\mathbb{R})$ be as in (10). For $i = 1,2$, let $M_i^\varepsilon = M(x, y^i, A^\varepsilon, B^\varepsilon)$ be such that

$$A_1^\varepsilon = 0,\ B_1^\varepsilon = B_1 \text{ and } B_2^\varepsilon = -A_2^\varepsilon = \begin{pmatrix} 0 & I_{k-n-1} & 0 \\ \nabla^\varepsilon & 0 & 0 \end{pmatrix},$$

where $\nabla^\varepsilon = (\nu_{i,j}) \in \mathscr{M}_{2n+1-k,n}(\mathbb{R})$ with $\nu_{1,1} = \varepsilon$ and $\nu_{i,j} = 0$ elsewhere. Equation (11) holds for $u^\varepsilon = (1, 0, ..., 0)$ and $v^\varepsilon = (0, ..., 0, v_k^\varepsilon = \frac{1}{\varepsilon}, 0, ..., 0)$. Finally, let $\mathfrak{l}$ be a non-abelian subalgebra of $\mathfrak{p}$ spanned by $\{e_1, ..., e_k\}$ such that e_1 is a generator of $[\mathfrak{l}, \mathfrak{l}]$. For $i = 1,2$, according to Corollary 3.3, let us take

$$M_i = M\Big((1, \tilde{x}), (0, \tilde{y}^i), (0, A), (0, B)\Big), \tag{12}$$

where $\tilde{x},\ \tilde{y}^i \in \mathbb{R}^{k-1}$, $\tilde{y}^1 = 0$, $\tilde{y}^2 = (1, 0, ..., 0)$. Suppose first that $k \leq n+1$. Let $A = 0$ and $B = {}^t(B_1\ B_2) \in \mathscr{M}_{2n,k-1}(\mathbb{R})$ such that $B_1 = {}^t(I_{k-1}\ 0)$ and $B_2 = -\frac{1}{2}{}^t(K_0\ 0)$. Then, M_1 and M_2 are in $R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ and $[M_1] \neq [M_2]$. Moreover, for $\varepsilon > 0$, let

$$M_i^\varepsilon = M\Big((1+\varepsilon, \tilde{x}), (0, \tilde{y}^i), (0, A^\varepsilon), (0, B)\Big),\ 1 \leq i \leq 2$$

such that $A^\varepsilon = {}^t(\varepsilon I_{k-1}\ 0)$. It follows that M_i^ε, $i = 1,2$ are in $R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ and as the matrices A^ε, B and $A^\varepsilon + B$ are of rank $k-1$, equation (11) holds for some $(u^\varepsilon, v^\varepsilon) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. Assume now that $\mathfrak{l}$ is non-abelian and $n+2 \leq k \leq 2n+1$. We can and do assume that $[e_i, e_j] = 0$ for all $i, j \geq n+1$. In these circumstances, the matrix K_0 takes the form:

$$K_0 = \begin{pmatrix} K_1 & K_2 \\ -{}^tK_2 & 0_{\mathscr{M}_{k-1-n}(\mathbb{R})} \end{pmatrix},$$

where $K_1 \in \mathscr{M}_n(\mathbb{R})$ is a skew symmetric matrix and K_2 is in $\mathscr{M}_{n,k-1-n}(\mathbb{R})$. Choose then $B = {}^t(B_1\ B_2) = {}^t(I_{k-1}\ 0) \in \mathscr{M}_{2n,k-1}(\mathbb{R})$ such that $B_1 = {}^t(I_n\ 0)$ and $A = {}^t(0\ A_2)$, $A_2 \in \mathscr{M}_{n,k-1}(\mathbb{R})$. Let $E = K_0 - {}^tB_2B_1 + {}^tB_1B_2$ and write

$$E = \begin{pmatrix} E_1 & E_2 \\ -{}^tE_2 & 0_{\mathscr{M}_{n,k-1-n}(\mathbb{R})} \end{pmatrix}$$

for some skew symmetric matrix $E_1 \in \mathscr{M}_n(\mathbb{R})$ and E_2 in $\mathscr{M}_{n,k-1-n}(\mathbb{R})$. Then the homomorphism conditions (5) hold when ${}^tA_2B_1 - {}^tB_1A_2 = E$. Choose then $A_2 = \begin{pmatrix} A' & A'' \end{pmatrix} \in \mathscr{M}_n(\mathbb{R}) \times \mathscr{M}_{n,k-1-n}(\mathbb{R})$, the last equation holds when $A'' = E_2$ and A' is an upper triangular matrix such that ${}^tA' - A' = E_1$. Hence, M_1 and M_2 are in $R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ and as the first column of A vanishes, $[M_1] \neq [M_2]$. Moreover, for $\varepsilon > 0$, let in (12) $A^\varepsilon = {}^t(A_1^\varepsilon\ A_2)$ where $A_1^\varepsilon = (a_{ij}) \in \mathscr{M}_{n,k-1}(\mathbb{R})$ such that $a_{n1} = \varepsilon$ and $a_{ij} = 0$ elsewhere. Then, M_i^ε, $i = 1, 2$ are in $R(\mathfrak{l}, \mathfrak{p}, \mathfrak{D})$ and $[M_1^\varepsilon] = [M_2^\varepsilon]$, which shows that the deformation space fails to be a Hausdorff space. ■

Remarks 4.5. (i) In [7], S. Barmeier studied the case where $n = 1$ and Γ is the discrete Heisenberg group where the parameter space is defined modulo the kernel $G \to \operatorname{Diff}(G/H)$. He proved that in these circumstances the deformation space is homeomorphic to $\mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^\times \times \mathbb{R}^3$.
(ii) In [3], it is shown that in the case of the Heisenberg group H_{2n+1} and H and Γ are arbitrary, the equivalence between the fact that $\mathscr{T}(\Gamma, H_{2n+1}; H_{2n+1}/H)$ is a Hausdorff space and the fact that the stability holds. We remark that here it is no longer the case $P = H_{2n+1} \times H_{2n+1}$.

Acknowledgments. The author is deeply indebted to the referee for the valuable comments and the careful reading of the paper. Great thanks to Professor Ali Baklouti for his careful reading of the paper and for many suggestions.

References

[1] Abdelmoula, L., A. Baklouti, I. Kédim, *The Selberg-Weil-Kobayashi local rigidity theorem for exponential Lie groups*, Int. Math. Res. Not. IMRN 2012, **17**, 4062–4084.

[2] Baklouti, A., *On discontinuous subgroups acting on solvable homogeneous spaces*, Proc. Japan Acad., **87** (2011), 173–177.

[3] Baklouti, A., S. Dhieb, and K. Tounsi, *When is the deformation space $\mathscr{T}(\Gamma, H_{2n+1}, H)$ a smooth manifold?*, Int. J. Math. **22** (2011), 1661–1681.

[4] Baklouti, A., N. ElAloui, and I. Kédim, *A rigidity theorem and a stability theorem for two-step nilpotent Lie groups*, J. Math. Sci. Univ. Tokyo **19** (2012), 281–307.

[5] Baklouti, A., and I. Kédim, *On non-abelian discontinuous subgroups acting on exponential solvable homogeneous spaces*, Int. Math. Res. Not. IMRN 2010, **7**, 1315–1345.

[6] Baklouti, A., I. Kédim, and T. Yoshino, *On the deformation space of Clifford-Klein forms of Heisenberg groups*, Int. Math. Res. Not. IMRN 2008, **16**, doi:10.1093/imrn/rnn066.

[7] Barmeier, S., *Deformations of the discrete Heisenberg groups*, Proc. Japan Acad. Ser. A, Math. Sci. **89** (2013), 55–59.

[8] Kassel, F., and T. Kobayashi, *Poincaré series for non-Riemannian locally symmetric spaces*, to appear in Advances in Mathematics, (available at arXiv:1209.4075).

[9] Kobayashi, T., *Proper action on homogeneous space of reductive type*, Math. Ann. **285** (1989), 249–263.

[10] —, *Discontinuous groups acting on homogeneous spaces of reductive type*, Proceeding of the Conference on Representation Theories of Lie Groups and Lie Algebras held in August-September of 1990 at Fuji-Kawaguchiko (ICM-90 Satellite Conference). Word Scientific (1992), 59–75.

[11] —, *On discontinuous groups on homogeneous space with noncompact isotropy subgroups*, J. Geom. Phys. **12** (1993), 133–144.

[12] —, *Criterion of proper action on homogeneous space of reductive type*, J. Lie Theory **6** (1996), 147–163.

[13] —, *Deformation of compact Clifford-Klein forms of indefinite Riemannian homogeneous manifolds*, Math. Ann. **310** (1998), 394–408.

[14] —, *Discontinuous groups for non-Riemannian homogeneous spaces*, Mathematics Unlimited-2001 and Beyond, Springer (2001), 723–747.

[15] Kobayashi, T., and S. Nasrin, *Deformation of properly discontinuous action of $\mathbb{Z}^k$ on $\mathbb{R}^{k+1}$*, Int. J. Math. **17** (2006), 1175–119.

[16] Kobayashi, T., and T. Yoshino, *Compact Clifford-Klein forms of symmetric spaces revisited*, Pure and Applied Math. Quarterly, **1** (2005), 603–684.

[17] Nasrin, S., *Criterion of proper actions for 2-step nilpotent Lie groups*, Tokyo J. Math. **24** (2001), 535–543

[18] Weil, A., *Remarks on the cohomology of groups*, Ann. Math. **80** (1964), 149–157.

[19] Yoshino, T., *Deformation spaces of compact Clifford-Klein forms of homogeneous spaces of Heisenberg groups*, RIMS Kôkyûroku Bessatsu **B7** (2008), 45–55.

Sami Dhieb
Department of Mathematics
Faculty of Sciences of Sfax
Route de Soukra, 3038, Sfax, Tunisia
Sami.Dhieb@fss.rnu.tn

Received July 16, 2013
and in final form May 30, 2015