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Multivariate Meixner, Charlier and Krawtchouk Polynomials According to Analysis on Symmetric Cones

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Abstract. We introduce some multivariate analogues of Meixner, Charlier and Krawtchouk polynomials, and establish their main properties by using analysis on symmetric cones, that is, duality, degenerate limits, generating functions, orthogonality relations, difference equations, recurrence formulas and determinant expressions. A particularly important and interesting result is that "the generating function of the generating functions" for the Meixner polynomials coincides with the generating function of the Laguerre polynomials, which has previously not been known even for the one variable case. Actually, main properties for the multivariate Meixner, Charlier and Krawtchouk polynomials are derived from some properties of the multivariate Laguerre polynomials by using this key result.
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1. Introduction

The standard Meixner, Charlier and Krawtchouk polynomials of a single discrete variable are defined by

$$M_m(x;\alpha,c) := {}_2F_1\left(\begin{matrix}-m,-x\\ \alpha\end{matrix};1-\frac{1}{c}\right) = \sum_{k=0}^{m}\frac{k!}{(\alpha)_k}\binom{m}{k}\binom{x}{k}\left(1-\frac{1}{c}\right)^k,$$

$$C_m(x;a) := {}_2F_0\left(\begin{matrix}-m,-x\\ -\end{matrix};-\frac{1}{a}\right) = \sum_{k=0}^{m}k!\binom{m}{k}\binom{x}{k}\left(-\frac{1}{a}\right)^k,$$

$$K_m(x;p,N) := {}_2F_1\left(\begin{matrix}-m,-x\\ -N\end{matrix};\frac{1}{p}\right) = \sum_{k=0}^{m}\frac{k!}{(-N)_k}\binom{m}{k}\binom{x}{k}\left(\frac{1}{p}\right)^k,$$

respectively. These polynomials have been generalized to the multivariate case [2], [10], [11], and [13]. Although these multivariate discrete orthogonal polynomials are written by the Aomoto-Gelfand hypergeometric series, we introduce other

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types of multivariate Meixner, Charlier and Krawtchouk polynomials in this article, which are defined by the generalized binomial coefficients. Moreover, we provide their fundamental properties by using analysis on symmetric cones, that is, duality, degenerate limits, generating functions, orthogonality relations, difference equations and recurrence formulas. The most basic result in these properties is Theorem 3.5, which states that "the generating function of the generating functions" for the multivariate Meixner polynomials

$$\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \left\{ \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) \right\} \Phi_{\mathbf{x}}(w)$$

coincides with the generating function for the multivariate Laguerre polynomials

$$\sum_{\mathbf{m}\in\mathscr{P}} e^{\operatorname{tr} w} L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}\left(\left(\frac{1}{c}-1\right)w\right)\Phi_{\mathbf{m}}(z).$$

Even though this result has not been known even for one variable, many properties for our multivariate discrete special orthogonal polynomials follow from this and the unitary picture (2.43) according to analysis on the symmetric cones.

Let us describe our scheme in the one variable case more precisely. We put $\alpha > 1$, $(\alpha)_m := \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+m-1)$, $\binom{m}{k} = (-1)^k \frac{(-m)_k}{k!}$, $\mathcal{D} := \{w \in \mathbb{C} \mid |w| < 1\}$, $T := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$, m is the Lebesgue measure on $\mathbb{C}$. Further, we introduce the following function spaces and their complete orthogonal bases.

(1) $\psi_m^{(\alpha)}$; exponential multiplied by the Laguerre polynomials

$$L^2_\alpha(\mathbb{R}_{>0}) := \{\psi : \mathbb{R}_{>0} \longrightarrow \mathbb{C} \mid \|\psi\|^2_{\alpha,\mathbb{R}_{>0}} < \infty\},$$
$$\|\psi\|^2_{\alpha,\mathbb{R}_{>0}} := \frac{2^\alpha}{\Gamma(\alpha)} \int_0^\infty |\psi(u)|^2 u^{\alpha-1}\, du,$$
$$\psi_m^{(\alpha)}(u) := e^{-u} L_m^{(\alpha-1)}(2u) = \frac{(\alpha)_m}{m!} e^{-u} \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{1}{(\alpha)_k}(2u)^k.$$

(2) $F_m^{(\alpha)}$; Cayley transform of the polynomials

$$\mathcal{H}^2_\alpha(T) := \{F : T \longrightarrow \mathbb{C} \mid F \text{ is analytic in } T \text{ and } \|F\|^2_{\alpha,T} < \infty\},$$
$$\|F\|^2_{\alpha,T} := \frac{\alpha-1}{4\pi} \int_T |F(z)|^2 x^{\alpha-2}\, m(dz),$$
$$F_m^{(\alpha)}(z) := \frac{(\alpha)_m}{m!} \left(\frac{1+z}{2}\right)^{-\alpha} \left(\frac{z-1}{z+1}\right)^m.$$

(3) $f_m^{(\alpha)}$; monomials

$$\mathcal{H}^2_\alpha(\mathcal{D}) := \{f : \mathcal{D} \longrightarrow \mathbb{C} \mid f \text{ is analytic in } \mathcal{D} \text{ and } \|f\|^2_{\alpha,\mathcal{D}} < \infty\},$$
$$\|f\|^2_{\alpha,\mathcal{D}} := \frac{\alpha-1}{\pi} \int_{\mathcal{D}} |f(w)|^2 (1-|w|^2)^{\alpha-2}\, m(dw),$$
$$f_m^{(\alpha)}(w) := \frac{(\alpha)_m}{m!} w^m.$$

We remark that

$$\|\psi_m^{(\alpha)}\|^2_{\alpha,\mathbb{R}_{>0}} = \|F_m^{(\alpha)}\|^2_{\alpha,T} = \|f_m^{(\alpha)}\|^2_{\alpha,\mathcal{D}} = \frac{(\alpha)_m}{m!}.$$

Furthermore, the following unitary isomorphisms are known.
Modified Laplace transform

$$\mathcal{L}_\alpha : L^2_\alpha(\mathbb{R}_{>0}) \overset{\simeq}{\to} \mathcal{H}^2_\alpha(T), \ \ (\mathcal{L}_\alpha\psi)(z) := \frac{2^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-zu}u^{\alpha-1}\psi(u)\,du.$$

Modified Cayley transform

$$C_\alpha^{-1} : \mathcal{H}^2_\alpha(T) \overset{\simeq}{\to} \mathcal{H}^2_\alpha(\mathcal{D}), \ \ (C_\alpha^{-1}F)(w) := (1-w)^{-\alpha}F\left(\frac{1+w}{1-w}\right).$$

To summarize, we obtain the following picture given by the unitary transformations.

$$\begin{array}{ccccc} L^2_\alpha(\mathbb{R}_{>0}) & \underset{\mathcal{L}_\alpha}{\overset{\simeq}{\longrightarrow}} & \mathcal{H}^2_\alpha(T) & \underset{C_\alpha^{-1}}{\overset{\simeq}{\longrightarrow}} & \mathcal{H}^2_\alpha(\mathcal{D}). \\ \in & & \in & & \in \\ \psi_m^{(\alpha)} & \longmapsto & F_m^{(\alpha)} & \longmapsto & f_m^{(\alpha)} \\ (\mathbf{1}) & & (\mathbf{2}) & & (\mathbf{3}) \end{array} \tag{1.1}$$

On the other hand, by elementary calculation, we have

$$\begin{aligned} e^{-\frac{1+c}{1-c}u}\sum_{x\geq 0}\frac{1}{x!}\left(\frac{2c}{1-c}\right)^x\left\{\sum_{m\geq 0}\frac{(\alpha)_m}{m!}M_m(x;\alpha,c)z^m\right\}u^x &= \sum_{m\geq 0}\psi_m^{(\alpha)}(u)z^m \\ &= (1-z)^{-\alpha}e^{-u\frac{1+z}{1-z}}. \end{aligned} \tag{1.2}$$

It is interesting to note that there is a correspondence between Laguerre and Meixner polynomials. The former orthogonality is defined by the integral on $\mathbb{R}_{\geq 0}$ and the latter is defined by the summation on non negative integers.

From (1.1) and (1.2), for the Meixner polynomials, we derive **(a)** generating function, **(b)** orthogonality relation, **(c)** difference equation and recurrence formula as follows.

(a) By comparing the coefficients of u on the first equality of (1.2),

$$(1-z)^{-\alpha}\left(\frac{1-\frac{1}{c}z}{1-z}\right)^x = \sum_{m\geq 0}\frac{(\alpha)_m}{m!}M_m(x;\alpha,c)z^m.$$

(b) By applying the unitary transformations $C_\alpha^{-1}\circ\mathcal{L}_\alpha$ in (1.1) to (1.2), we have

$$\begin{aligned} &(1-c)^\alpha\sum_{x\geq 0}\frac{(\alpha)_x}{x!}c^x\left\{\sum_{m\geq 0}\frac{(\alpha)_m}{m!}M_m(x;\alpha,c)z^m\right\}(1-cw)^{-\alpha}\left(\frac{1-w}{1-cw}\right)^x \\ &= \sum_{m\geq 0}\frac{(\alpha)_m}{m!}w^m z^m = (1-wz)^{-\alpha}. \end{aligned} \tag{1.3}$$

We remark that the generating functions of the Meixner polynomials appear in the top left hand side of (1.3). Hence, by using the generating functions of the Meixner polynomials and comparing the coefficients of w and z in (1.3), we have the orthogonal relation for the Meixner polynomials.

$$\sum_{x\geq 0}\frac{(\alpha)_x}{x!}c^x M_m(x;\alpha,c)M_n(x;\alpha,c)=\frac{c^{-m}}{(1-c)^\alpha}\frac{m!}{(\alpha)_m}\delta_{m,n}.$$

(c) We recall the differential operator $D_\alpha^{(1)}=-u\partial_u^2-\alpha\partial_u+u-\alpha$ which satisfies $D_\alpha^{(1)}\psi_m^{(\alpha)}(u)=2m\psi_m^{(\alpha)}(u)$. Therefore, by applying $\frac{c-1}{2}e^{\frac{1+c}{1-c}u}D_\alpha^{(1)}$ to (1.3) and comparing the coefficients of z and u, we obtain the following difference equation which is equivalent to a recurrence formula.

$$\begin{aligned}(c-1)mM_m(x;\alpha,c)&=(x+\alpha)cM_m(x+1;\alpha,c)\\&\quad-(x+(x+\alpha)c)M_m(x;\alpha,c)\\&\quad+xM_m(x-1;\alpha,c).\end{aligned}$$

The purpose of this article is to provide a multivariate analogue of this scheme which has previously not been known even for the one variable case. Let us now describe the content in this paper. The basic definitions and fundamental properties of Jordan algebras and symmetric cones, and lemmas for analysis on symmetric cones and tube domains have been presented in the first subsection of Section 2, so that they can be referred to later. The next subsection presents a compilation of basic facts for the multivariate Laguerre polynomials and their unitary picture. Section 3 which is the main part of this papers provides a multivariate analogue of the above results for Meixner, Charlier and Krawtchouk polynomials. Finally, in Section 4, we present a conjecture and some problems for a further generalization of the multivariate Meixner, Charlier and Krawtchouk polynomials.

2. Preliminaries

Throughout the paper, we denote the ring of rational integers by $\mathbb{Z}$, the field of real numbers by $\mathbb{R}$, the field of complex numbers by $\mathbb{C}$. Further, we fix a positive integer r and denote the partition set of length r by

$$\mathscr{P}:=\{\mathbf{m}=(m_1,\dots,m_r)\in\mathbb{Z}_{\geq 0}^r \mid m_1\geq\cdots\geq m_r\}. \tag{2.1}$$

For any vector $\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{C}^r$, we put

$$\operatorname{Re}\mathbf{s}:=(\operatorname{Re}s_1,\dots,\operatorname{Re}s_r), \tag{2.2}$$
$$|\mathbf{s}|:=s_1+\cdots+s_r, \tag{2.3}$$
$$\|\mathbf{s}\|:=(|s_1|,\dots,|s_r|). \tag{2.4}$$

Moreover, for $\mathbf{m}\in\mathscr{P}$

$$\mathbf{m}!:=m_1!\cdots m_r!$$

and we set $\delta:=(r-1,r-2,\dots,1,0)$. Refer to Faraut and Koranyi [6] for the details in this section.

2.1. Analysis on symmetric cones. Let Ω be an irreducible symmetric cone in V which is a finite dimensional simple Euclidean Jordan algebra of dimension n as a real vector space and rank r. The classification of irreducible symmetric cones is well-known. Namely, there are four families of classical irreducible symmetric cones $\Pi_r(\mathbb{R}), \Pi_r(\mathbb{C}), \Pi_r(\mathbb{H})$, the cones of all $r \times r$ positive definite matrices over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, the Lorentz cones Λ_r and an exceptional cone $\Pi_3(\mathbb{O})$ (see [6] p. 97). Also, let $V^{\mathbb{C}}$ be the complexification of V. For $w, z \in V^{\mathbb{C}}$, we define

$$\begin{aligned} L(w)z &:= wz, \\ w\Box z &:= L(wz) + [L(w), L(z)], \\ P(w,z) &:= L(w)L(z) + L(z)L(w) - L(wz), \\ P(w) &:= P(w,w) = 2L(w)^2 - L(w^2). \end{aligned}$$

We denote the Jordan trace and determinant of the complex Jordan algebra $V^{\mathbb{C}}$ by $\operatorname{tr} x$ and by $\Delta(x)$ respectively.

Fix a Jordan frame $\{c_1, \ldots, c_r\}$ that is a complete system of orthogonal primitive idempotents in V and define the following subspaces:

$$\begin{aligned} V_j &:= \{x \in V \mid L(c_j)x = x\}, \\ V_{jk} &:= \left\{ x \in V \,\middle|\, L(c_j)x = \frac{1}{2}x \text{ and } L(c_k)x = \frac{1}{2}x \right\}. \end{aligned}$$

Then, $V_j = \mathbb{R}e_j$ for $j = 1, \ldots, r$ are 1-dimensional subalgebras of V, while the subspaces V_{jk} for $j, k = 1, \ldots, r$ with $j < k$ all have a common dimension $d = \dim_{\mathbb{R}} V_{jk}$. Then, V has the Peirce decomposition

$$V = \left(\bigoplus_{j=1}^{r} V_j \right) \oplus \left(\bigoplus_{j<k} V_{jk} \right),$$

which is the orthogonal direct sum. It follows that $n = r + \frac{d}{2}r(r-1)$. Let $G(\Omega)$ denote the automorphism group of Ω and let G be the identity component in $G(\Omega)$. Then, G acts transitively on Ω and $\Omega \cong G/K$ where $K \in G$ is the isotropy subgroup of the unit element, $e \in V$. K is also the identity component in $\operatorname{Aut}(V)$.

For any $x \in V$, there exist $k \in K$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ such that

$$x = k \sum_{j=1}^{r} \lambda_j c_j, \quad (\lambda_1 \geq \cdots \geq \lambda_r).$$

As in the case of V, we also have the following spectral decomposition for $V^{\mathbb{C}}$. Every z in $V^{\mathbb{C}}$ can be written

$$z = u \sum_{j=1}^{r} \lambda_j c_j,$$

with u in U which is the identity component of $Str(V^{\mathbb{C}}) \cap U(V^{\mathbb{C}})$, $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Moreover, we define the spectral norm of $z \in V^{\mathbb{C}}$ by $|z| = \lambda_1$ and introduce the open unit ball $\mathcal{D} \in V^{\mathbb{C}}$ as follows.

$$\mathcal{D} = \{z \in V^{\mathbb{C}} \mid |z| < 1\}.$$

For $j = 1, \ldots, r$, let $e_j := c_1 + \cdots + c_j$, and set

$$V^{(j)} := \{x \in V \mid L(e_j)x = x\}.$$

Denote the orthogonal projection of V onto the subalgebra $V^{(j)}$ by P_j, and define

$$\Delta_j(x) := \delta_j(P_j x)$$

for $x \in V$, where δ_j denotes the determinant with respect to $V^{(j)}$. In particular, $\delta_r = \Delta$. Then, Δ_j is a polynomial on V that is homogeneous of degree j. Let $\mathbf{s} := (s_1, \ldots, s_r) \in \mathbb{C}^r$ and define the function $\Delta_{\mathbf{s}}$ on V by

$$\Delta_{\mathbf{s}}(x) := \Delta(x)^{s_r} \prod_{j=1}^{r-1} \Delta_j(x)^{s_j - s_{j+1}}. \tag{2.5}$$

That is the generalized power function on V. Furthermore, for $\mathbf{m} \in \mathscr{P}$, $\Delta_{\mathbf{m}}$ becomes a polynomial function on V, which is homogeneous of degree $|\mathbf{m}|$.

The gamma function Γ_Ω for the symmetric cone Ω is defined, for $\mathbf{s} \in \mathbb{C}^r$, with $\operatorname{Re} s_j > \frac{d}{2}(j-1)$ $(j = 1, \ldots, r)$ by

$$\Gamma_\Omega(\mathbf{s}) := \int_\Omega e^{-\operatorname{tr}(x)} \Delta_{\mathbf{s}}(x) \Delta(x)^{-\frac{n}{r}}\, dx. \tag{2.6}$$

Its evaluation gives

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(s_j - \frac{d}{2}(j-1)\right). \tag{2.7}$$

Hence, Γ_Ω extends analytically as a meromorphic function on $\mathbb{C}^r$.

For $\mathbf{s} \in \mathbb{C}^r$ and $\mathbf{m} \in \mathscr{P}$, we define the generalized shifted factorial by

$$(\mathbf{s})_{\mathbf{m}} := \frac{\Gamma_\Omega(\mathbf{s} + \mathbf{m})}{\Gamma_\Omega(\mathbf{s})}. \tag{2.8}$$

It follows from (2.7) that

$$(\mathbf{s})_{\mathbf{m}} = \prod_{j=1}^{r} \left(s_j - \frac{d}{2}(j-1)\right)_{m_j}. \tag{2.9}$$

Lemma 2.1. *If* $\mathbf{s} \in \mathbb{C}^r, \mathbf{m}, \mathbf{k} \in \mathscr{P}$ *and* $\mathbf{m} \supset \mathbf{k}$*, then*

$$\left|\frac{(\mathbf{s})_{\mathbf{m}}}{(\mathbf{s})_{\mathbf{k}}}\right| \leq \frac{(\|\mathbf{s}\| + d(r-1))_{\mathbf{m}}}{(\|\mathbf{s}\| + d(r-1))_{\mathbf{k}}}. \tag{2.10}$$

Proof. We remark that for any $s \in \mathbb{C}, N \in \mathbb{Z}_{\geq 0}$ and $j = 1, \ldots, r$, the following is satisfied.

$$\left|s + N - \frac{d}{2}(j-1)\right| \leq |s| + N + d(r-1) - \frac{d}{2}(j-1) = |s| + N + \frac{d}{2}(2r - j - 1).$$

Hence,

$$\begin{aligned}
\left|\frac{(\mathbf{s})_{\mathbf{m}}}{(\mathbf{s})_{\mathbf{k}}}\right| &= \prod_{j=1}^{r}\left|\left(s_j + k_j - \frac{d}{2}(j-1)\right)_{m_j-k_j}\right| \\
&\le \prod_{j=1}^{r}\left(|s_j| + k_j + d(r-1) - \frac{d}{2}(j-1)\right)_{m_j-k_j} \\
&= \frac{(\|\mathbf{s}\| + d(r-1))_{\mathbf{m}}}{(\|\mathbf{s}\| + d(r-1))_{\mathbf{k}}}.
\end{aligned}$$

■

Corollary 2.2. *If* $\mathbf{s} \in \mathbb{C}^r, \mathbf{m} \in \mathscr{P}$*, then*

$$|(\mathbf{s})_{\mathbf{m}}| \le (\|\mathbf{s}\| + d(r-1))_{\mathbf{m}} \le \prod_{j=1}^{r}(|s_j| + d(r-1))_{m_j}. \tag{2.11}$$

The space $\mathcal{P}(V)$ of polynomials on V has the following decomposition.

$$\mathcal{P}(V) = \bigoplus_{\mathbf{m}\in\mathscr{P}} \mathcal{P}_{\mathbf{m}},$$

where the subspaces $\mathcal{P}_{\mathbf{m}}$ are mutually inequivalent, and finite dimensional irreducible G-modules. Further, their dimensions are denoted by $d_{\mathbf{m}}$. For $d_{\mathbf{m}}$, the following formula is known (see, [21] Lemma 2.6 or [6] p. 315).

Lemma 2.3. *For any* $\mathbf{m} \in \mathscr{P}$*,*

$$\begin{aligned}
d_{\mathbf{m}} &= \frac{c(-\rho)}{c(\rho-\mathbf{m})c(\mathbf{m}-\rho)} && (2.12)\\
&= \prod_{1\le p<q\le r} \frac{m_p - m_q + \frac{d}{2}(q-p)}{\frac{d}{2}(q-p)} \frac{B\left(m_p - m_q, \frac{d}{2}(q-p-1)+1\right)}{B\left(m_p-m_q, \frac{d}{2}(q-p+1)\right)} && (2.13)\\
&= \prod_{j=1}^{r} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}j\right)\Gamma\left(\frac{d}{2}(j-1)+1\right)} \\
&\quad \cdot \prod_{1\le p<q\le r}\left(m_p - m_q + \frac{d}{2}(q-p)\right)\frac{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p+1)\right)}{\Gamma\left(m_p-m_q+\frac{d}{2}(q-p-1)+1\right)}. && (2.14)
\end{aligned}$$

Here, $\rho = (\rho_1, \ldots, \rho_r)$*,* $\rho_j := \frac{d}{4}(2j - r - 1)$*, and* c *is the Harish-Chandra function:*

$$c(\mathbf{s}) = \prod_{1\le p<q\le r} \frac{B\left(s_q - s_p, \frac{d}{2}\right)}{B\left(\frac{d}{2}(q-p), \frac{d}{2}\right)}.$$

In particular, for $d = 2$

$$d_{\mathbf{m}} = \prod_{1\le p<q\le r}\left(\frac{m_p - m_q + q - p}{q-p}\right)^2 = s_{\mathbf{m}}(1, \ldots, 1)^2. \tag{2.15}$$

Here, $s_{\mathbf{m}}$ is the Schur polynomial corresponding to $\mathbf{m} \in \mathscr{P}$ defined by

$$s_{\mathbf{m}}(\lambda_1, \dots, \lambda_r) := \frac{\det(\lambda_j^{m_k+r-k})}{\det(\lambda_j^{r-k})}.$$

The following lemma is necessary to evaluate the Laplace transform of the multivariate Laguerre polynomial.

Lemma 2.4 ([6] Theorem XI. 2.3). *For $p \in \mathcal{P}_{\mathbf{m}}$, $\operatorname{Re}\alpha > (r-1)\frac{d}{2}$, and $y \in \Omega + iV$,*

$$\int_\Omega e^{-(y|x)} p(x) \Delta(x)^{\alpha - \frac{n}{r}} \, dx = \Gamma_\Omega(\mathbf{m} + \alpha) \Delta(y)^{-\alpha} p(y^{-1}). \tag{2.16}$$

Here, α is regarded as $(\alpha, \dots, \alpha) \in \mathbb{C}^r$.

For each $\mathbf{m} \in \mathscr{P}$, the spherical polynomial of weight $|\mathbf{m}|$ on Ω is defined by

$$\Phi_{\mathbf{m}}^{(d)}(x) := \int_K \Delta_{\mathbf{m}}(kx) \, dk. \tag{2.17}$$

We will often omit the multiplicity d and simply write $\Phi_{\mathbf{m}}$. The algebra of all K-invariant polynomials on V, denoted by $\mathcal{P}(V)^K$, decomposes as

$$\mathcal{P}(V)^K = \bigoplus_{\mathbf{m} \in \mathscr{P}} \mathbb{C}\Phi_{\mathbf{m}}.$$

By analytic continuation to the complexification $V^{\mathbb{C}}$ of V, we can extend $\operatorname{tr}, \Delta$ and $\Phi_{\mathbf{m}}$ to polynomial functions on $V^{\mathbb{C}}$.

Remark 2.5. (1) Since $\Phi_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}^K$, for $x = k\sum_{j=1}^r \lambda_j c_j$, $\Phi_{\mathbf{m}}(x)$ can be expressed by

$$\Phi_{\mathbf{m}}(\lambda_1, \dots, \lambda_r) := \Phi_{\mathbf{m}}\left(\sum_{j=1}^r \lambda_j c_j\right) (= \Phi_{\mathbf{m}}(x)).$$

$\Phi_{\mathbf{m}}(x)$ also has the following expression (see [5]).

$$\Phi_{\mathbf{k}}^{(d)}(\lambda_1, \dots, \lambda_r) = \frac{P_{\mathbf{k}}^{(\frac{2}{d})}(\lambda_1, \dots, \lambda_r)}{P_{\mathbf{k}}^{(\frac{2}{d})}(1, \dots, 1)}. \tag{2.18}$$

Here, $P_{\mathbf{k}}^{(\frac{2}{d})}(\lambda_1, \dots, \lambda_r)$ is an r-variable Jack polynomial (see [17], Chapter VI.10). In particular, since $P_{\mathbf{k}}^{(1)}(\lambda_1, \dots, \lambda_r) = s_{\mathbf{m}}(\lambda_1, \dots, \lambda_r)$, $\Phi_{\mathbf{m}}^{(2)}$ becomes the Schur polynomial.

$$\Phi_{\mathbf{m}}^{(2)}(\lambda_1, \dots, \lambda_r) = \frac{s_{\mathbf{m}}(\lambda_1, \dots, \lambda_r)}{s_{\mathbf{m}}(1, \dots, 1)} = \frac{\delta!}{\prod_{p<q}(m_p - m_q + q - p)} s_{\mathbf{m}}(\lambda_1, \dots, \lambda_r). \tag{2.19}$$

(2) When $r = 2$, $\Phi^{(d)}_{\mathbf{m}}$ has the following hypergeometric expression (see [20]).

$$\Phi^{(d)}_{m_1,m_2}(\lambda_1,\lambda_2) = \lambda_1^{m_1}\lambda_2^{m_2}{}_2F_1\left(\begin{matrix}-(m_1-m_2),\frac{d}{2}\\ d\end{matrix};\frac{\lambda_1-\lambda_2}{\lambda_1}\right)$$
$$= \lambda_1^{m_1}\lambda_2^{m_2}\frac{\left(\frac{d}{2}\right)_{m_1-m_2}}{(d)_{m_1-m_2}}{}_2F_1\left(\begin{matrix}-(m_1-m_2),\frac{d}{2}\\ -(m_1-m_2)-\frac{d}{2}+1\end{matrix};\frac{\lambda_2}{\lambda_1}\right).$$

We remark that the function $\Phi_{\mathbf{m}}(e+x)$ is a K-invariant polynomial of degree $|\mathbf{m}|$ and define the generalized binomial coefficients $\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}}$ by using the following expansion.

$$\Phi^{(d)}_{\mathbf{m}}(e+x) = \sum_{|\mathbf{k}|\le|\mathbf{m}|}\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}}\Phi^{(d)}_{\mathbf{k}}(x). \tag{2.20}$$

For $\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}}$, we also often omit $\frac{d}{2}$. The fact that if $\mathbf{k}\not\subset\mathbf{m}$, then $\binom{\mathbf{m}}{\mathbf{k}} = 0$, is well known. Hence, we have

$$\Phi_{\mathbf{m}}(e+x) = \sum_{\mathbf{k}\subset\mathbf{m}}\binom{\mathbf{m}}{\mathbf{k}}\Phi_{\mathbf{k}}(x). \tag{2.21}$$

Moreover, for the spherical polynomials, we refer two Lemmas in [6].

Lemma 2.6 ([6] Theorem XII. 1.1 (i)). *For $z = u\sum_{j=1}^r\lambda_j c_j$ with $u\in U$, $\lambda_1\ge\cdots\ge\lambda_r\ge 0$ and $\mathbf{m}\in\mathscr{P}$, we have*

$$|\Phi_{\mathbf{m}}(z)| \le \lambda_1^{m_1}\cdots\lambda_r^{m_r} \le \lambda_1^{|\mathbf{m}|} = \Phi_{\mathbf{m}}(\lambda_1). \tag{2.22}$$

Lemma 2.7 ([6] Chapter XV. Exercise 3 (a)). *For any $\alpha\in\mathbb{C}, z\in\overline{\mathcal{D}}, w\in\mathcal{D}$, we have*

$$\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(z)\Phi_{\mathbf{m}}(w) = \Delta(w)^{-\alpha}\int_K\Delta(kw^{-1}-z)^{-\alpha}\,dk. \tag{2.23}$$

The spherical function, $\varphi_{\mathbf{s}}$, on Ω for $\mathbf{s}\in\mathbb{C}^r$ is defined by

$$\varphi_{\mathbf{s}}(x) := \int_K\Delta_{\mathbf{s}+\rho}(kx)\,dk. \tag{2.24}$$

We remark that for $x\in\Omega$

$$\varphi_{\mathbf{s}}(x^{-1}) = \varphi_{-\mathbf{s}}(x) \tag{2.25}$$

and for $x\in\Omega, \mathbf{m}\in\mathscr{P}$

$$\Phi_{\mathbf{m}}(x) = \varphi_{\mathbf{m}-\rho}(x). \tag{2.26}$$

Let $\mathbb{D}(\Omega)$ be the algebra of G-invariant differential operators on Ω, $\mathcal{P}(V)^K$ be the space of K-invariant polynomials on V, and $\mathcal{P}(V\times V)^G$ be the space of polynomials on $V\times V$, which are invariant in the sense that

$$p(gx,\xi) = p(x,g^*\xi), \quad (g\in G).$$

Here, we write g^* for the adjoint of an element g (i.e., $(gx|y) = (x|g^*y)$ for all $x, y \in V$). The spherical function $\varphi_{\mathbf{s}}$ is an eigenfunction of every $D \in \mathbb{D}(\Omega)$. Thus, we denote its eigenvalues by $\gamma(D)(\mathbf{s})$, that is, $D\varphi_{\mathbf{s}} = \gamma(D)(\mathbf{s})\varphi_{\mathbf{s}}$.

The symbol σ_D of a partial differential operator D which acts on the variable $x \in V$ is defined by

$$De^{(x|\xi)} = \sigma_D(x, \xi)e^{(x|\xi)} \quad (x, \xi \in V).$$

A differential operator D on Ω is invariant under G if and only if its symbol σ_D belongs to $\mathcal{P}(V \times V)^G$. In addition, the map $D \mapsto \sigma_D$ establishes a linear isomorphism from $\mathbb{D}(\Omega)$ onto $\mathcal{P}(V \times V)^G$. Moreover, the map $D \mapsto \sigma_D(e, u)$ is a vector space isomorphism from $\mathbb{D}(\Omega)$ onto $\mathcal{P}(V)^K$. In particular, for $\mathbf{k} \in \mathscr{P}, \mathbf{s} \in \mathbb{C}^r$, we put

$$\gamma_{\mathbf{k}}(\mathbf{s}) := \gamma(\Phi_{\mathbf{k}}(\partial_x))(\mathbf{s}) = \Phi_{\mathbf{k}}(\partial_x)\varphi_{\mathbf{s}}(x)|_{x=e}. \tag{2.27}$$

Here, $\Phi_{\mathbf{k}}(\partial_x)$ is the unique G-invariant differential operator satisfying

$$\sigma_{\Phi_{\mathbf{k}}(\partial_x)}(e, \xi) = \Phi_{\mathbf{k}}(\xi) \in \mathcal{P}(V)^K, \quad \text{i.e., } \Phi_{\mathbf{k}}(\partial_x)e^{(x|\xi)}|_{x=e} = \Phi_{\mathbf{k}}(\xi)e^{\operatorname{tr} \xi}.$$

We remark that $\Phi_k(\partial_x) = \partial_x^k$ and $\gamma_k(s) = s(s-1)\cdots(s-k+1)$ in the $r = 1$ case, and for any $\alpha \in \mathbb{C}$, $\mathbf{k} \in \mathscr{P}$, we have

$$\gamma_{\mathbf{k}}(\alpha - \rho) = (-1)^{|\mathbf{k}|}(-\alpha)_{\mathbf{k}}. \tag{2.28}$$

The function γ_D is an r variable symmetric polynomial and map $D \mapsto \gamma_D$ is an algebra isomorphism from $\mathbb{D}(\Omega)$ onto the algebra $\mathcal{P}(\mathbb{R}^r)^{\mathfrak{S}_r}$, which is a special case of the Harish-Chandra isomorphism. If a K-invariant function ψ is analytic in the neighborhood of e, it admits a spherical Taylor expansion near e:

$$\psi(e + x) = \sum_{\mathbf{k} \in \mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \{\Phi_{\mathbf{k}}(\partial_x)\psi(x)|_{x=e}\}\Phi_{\mathbf{k}}(x).$$

By the definition of $\gamma_{\mathbf{k}}$, we have

$$\varphi_{\mathbf{s}}(e + x) = \sum_{\mathbf{k} \in \mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \gamma_{\mathbf{k}}(\mathbf{s})\Phi_{\mathbf{k}}(x).$$

Since $\Phi_{\mathbf{m}} = \varphi_{\mathbf{m}-\rho}$,

$$\binom{\mathbf{m}}{\mathbf{k}} = d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \gamma_{\mathbf{k}}(\mathbf{m} - \rho).$$

For a complex number α, we define the following differential operator on Ω:

$$D_\alpha = \Delta(x)^{1+\alpha}\Delta(\partial_x)\Delta(x)^{-\alpha}.$$

For this operator, we have

$$\gamma(D_\alpha)(\mathbf{s}) = \prod_{j=1}^{r} \left(s_j - \alpha + \frac{d}{4}(r-1)\right). \tag{2.29}$$

The operators $D_{j\frac{d}{2}}$, $j = 0, \ldots, r-1$ generate the algebra $\mathbb{D}(\Omega)$.

Lemma 2.8. *For all* $\mathbf{k} \in \mathscr{P}$*, there exist some constant* $C > 0$ *and integer* N *such that for any* $\mathbf{s} \in \mathbb{C}^r$

$$|\gamma_{\mathbf{k}}(\mathbf{s})| \leq C \prod_{l=1}^{r} \left(|s_l| + \frac{d}{4}(r-1) \right)^N . \tag{2.30}$$

Proof. Since the algebra $\mathbb{D}(\Omega)$ is generated by $D_{j\frac{d}{2}}$, $j = 0, \ldots, r-1$, for $\Phi_{\mathbf{k}}(\partial_x) \in \mathbb{D}(\Omega)$,

$$\Phi_{\mathbf{k}}(\partial_x) = \sum_{l_0, \ldots, l_{r-1}; \text{finite}} a_{l_0, \ldots, l_{r-1}} D_{0\frac{d}{2}}^{l_0} \cdots D_{(r-1)\frac{d}{2}}^{l_{r-1}}.$$

Here, we remark that for $j = 0, \ldots, r-1$

$$|\gamma(D_{\frac{d}{2}(j-1)})(\mathbf{s})| = \left| \prod_{l=1}^{r} \left(s_l + \frac{d}{4}(r-1) - \frac{d}{2}(j-1) \right) \right| \leq \prod_{l=1}^{r} \left(|s_l| + \frac{d}{4}(r-1) \right).$$

Therefore,

$$\begin{aligned} |\gamma_{\mathbf{k}}(\mathbf{s})| &\leq \sum_{l_0, \ldots, l_{r-1}; \text{finite}} |a_{l_0, \ldots, l_{r-1}}| \gamma(D_{0\frac{d}{2}})(\mathbf{s})^{l_0} \cdots \gamma(D_{(r-1)\frac{d}{2}})(\mathbf{s})^{l_{r-1}} \\ &\leq C \prod_{l=1}^{r} \left(|s_l| + \frac{d}{4}(r-1) \right)^N . \end{aligned}$$

■

Lemma 2.9. *For all* $\mathbf{m}, \mathbf{k} \in \mathscr{P}$*, we have*

$$\gamma_{\mathbf{k}}(\mathbf{m} - \rho) \geq 0. \tag{2.31}$$

Proof. Since $\gamma_{\mathbf{k}}(\mathbf{m} - \rho) = \frac{1}{d_{\mathbf{k}}} \left(\frac{n}{r}\right)_{\mathbf{k}} \binom{\mathbf{m}}{\mathbf{k}}$ and $d_{\mathbf{k}}, \left(\frac{n}{r}\right)_{\mathbf{k}} > 0$, it suffices to show $\binom{\mathbf{m}}{\mathbf{k}} \geq 0$ for all $\mathbf{m}, \mathbf{k} \in \mathscr{P}$. From [18], the generalized binomial coefficients are written as

$$\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} = \frac{P^*_{\mathbf{k}}\left(\mathbf{m}; \frac{d}{2}\right)}{H_{\left(\frac{d}{2}\right)}(\mathbf{k})},$$

where $P^*_{\mathbf{k}}\left(\mathbf{m}; \frac{d}{2}\right)$ is the shifted Jack polynomial (see also [19], [16] and [18]) and $H_{\left(\frac{d}{2}\right)}(\mathbf{k}) > 0$ is a deformation of the hook length. Moreover, by using (5.2) in [18]

$$P^*_{\mathbf{k}}\left(\mathbf{m}; \frac{d}{2}\right) = \frac{\frac{d}{2}\text{-}\dim \mathbf{m}/\mathbf{k}}{\frac{d}{2}\text{-}\dim \mathbf{m}} |\mathbf{m}|(|\mathbf{m}|-1)\cdots(|\mathbf{m}|-|\mathbf{k}|+1).$$

Further, the positivity of the generalized dimensions of the skew Young diagram, $\frac{d}{2}\text{-}\dim \mathbf{m}/\mathbf{k}$, follows from (5.1) of [18] and Chapter VI. 6 of [17]. Therefore, we obtain the positivity of the shifted Jack polynomial and the conclusion. ■

Theorem 2.10. (1) *For* $w \in \mathcal{D}, \mathbf{k} \in \mathscr{P}, \alpha \in \mathbb{C}$*, we have*

$$(\alpha)_{\mathbf{k}} \Delta(e-w)^{-\alpha} \Phi_{\mathbf{k}}(w(e-w)^{-1}) = \sum_{\mathbf{x} \in \mathscr{P}} d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x} - \rho) \Phi_{\mathbf{x}}(w). \tag{2.32}$$

Here, we choose the branch of $\Delta(e-w)^{-\alpha}$ *which takes the value* 1 *at* $w=0$*.*
(2) *For* $w \in V^{\mathbb{C}}, \mathbf{k} \in \mathscr{P}$*, the* K*-invariant analytic function* $e^{\mathrm{tr}\, w}\Phi_{\mathbf{k}}(w)$ *has the following expansion*

$$e^{\mathrm{tr}\, w}\Phi_{\mathbf{k}}(w) = \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w). \tag{2.33}$$

Proof. (1) We take $w = u\sum_{j=1}^{r}\lambda_j c_j \in \mathcal{D}$ with $u \in U$ and $1 > \lambda_1 \geq \ldots \geq \lambda_r \geq 0$. By Lemmas 2.6 and 2.8, there exist some $C>0$ and $N \in \mathbb{Z}_{\geq 0}$ such that

$$\begin{aligned}\sum_{\mathbf{x}\in\mathscr{P}} \left| d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w)\right| &\leq \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{|(\alpha)_{\mathbf{x}}|}{\left(\frac{n}{r}\right)_{\mathbf{x}}} |\gamma_{\mathbf{k}}(\mathbf{x}-\rho)||\Phi_{\mathbf{x}}(w)| \\ &\leq C\prod_{l=1}^{r}\sum_{x_l\geq 0} \frac{(|\alpha|+d(r-1))_{x_l}}{x_l!}\left(x_l + \frac{d}{2}(r-1)\right)^N \lambda_l^{x_l} \\ &< \infty.\end{aligned}$$

Therefore, the right hand side of (2.32) converges absolutely. By analytic continuation, it is sufficient to show the assertion when $\mathrm{Re}\,\alpha > \frac{d}{2}(r-1)$ and $w \in \Omega \cap (e-\Omega) \subset \mathcal{D}$.

$$\begin{aligned}\Phi_{\mathbf{k}}(\partial_z)\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\Phi_{\mathbf{x}}(z)\Phi_{\mathbf{x}}(w)\Big|_{z=e} &= \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\Phi_{\mathbf{k}}(\partial_z)\Phi_{\mathbf{x}}(z)|_{z=e}\Phi_{\mathbf{x}}(w) \\ &= \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w).\end{aligned}$$

On the other hand,

$$\begin{aligned}\Phi_{\mathbf{k}}(\partial_z)\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\Phi_{\mathbf{x}}(z)\Phi_{\mathbf{x}}(w)\Big|_{z=e} &= \Phi_{\mathbf{k}}(\partial_z)\Delta(w)^{-\alpha}\int_K \Delta(kw^{-1}-z)^{-\alpha}\,dk\Big|_{z=e} \\ &= \Delta(w)^{-\alpha}\int_K \Phi_{\mathbf{k}}(\partial_z)\Delta(kw^{-1}-z)^{-\alpha}\big|_{z=e}\,dk.\end{aligned}$$

Here, from $kw^{-1}-z \in T_\Omega$ for all $k \in K$ and Lemma 2.4,

$$\begin{aligned}\Phi_{\mathbf{k}}(\partial_z)\Delta(kw^{-1}-z)^{-\alpha}\big|_{z=e} &= \Phi_{\mathbf{k}}(\partial_z)\frac{1}{\Gamma_\Omega(\alpha)}\int_\Omega e^{-(x|kw^{-1}-z)}\Delta(x)^{\alpha}\Delta(x)^{-\frac{n}{r}}\,dx\Big|_{z=e} \\ &= \frac{1}{\Gamma_\Omega(\alpha)}\int_\Omega \Phi_{\mathbf{k}}(\partial_z)e^{(x|z)}|_{z=e}e^{-(x|kw^{-1})}\Delta(x)^{\alpha}\Delta(x)^{-\frac{n}{r}}\,dx \\ &= \frac{1}{\Gamma_\Omega(\alpha)}\int_\Omega \Phi_{\mathbf{k}}(x)e^{-(kx|(w^{-1}-e))}\Delta(x)^{\alpha}\Delta(x)^{-\frac{n}{r}}\,dx \\ &= (\alpha)_{\mathbf{k}}\Delta(w^{-1}-e)^{-\alpha}\Phi_{\mathbf{k}}((w^{-1}-e)^{-1}).\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Phi_{\mathbf{k}}(\partial_z) \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(z)\Phi_{\mathbf{x}}(w)\Big|_{z=e} \\
&\quad = \Delta(w)^{-\alpha} \int_K (\alpha)_{\mathbf{k}} \Delta(w^{-1}-e)^{-\alpha} \Phi_{\mathbf{k}}((w^{-1}-e)^{-1})\, dk \\
&\quad = (\alpha)_{\mathbf{k}} \Delta(e-w)^{-\alpha} \Phi_{\mathbf{k}}(w(e-w)^{-1}).
\end{aligned}$$

(2) Since the right hand side of (2.33) converges absolutely due to a similar argument of (1), we have

$$\begin{aligned}
e^{\operatorname{tr} w}\Phi_{\mathbf{k}}(w) &= \lim_{\alpha\to\infty} (\alpha)_{\mathbf{k}} \Delta\left(e - \frac{w}{\alpha}\right)^{-\alpha} \Phi_{\mathbf{k}}\left(\frac{w}{\alpha}\left(e-\frac{w}{\alpha}\right)^{-1}\right) \\
&= \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho) \lim_{\alpha\to\infty} (\alpha)_{\mathbf{x}} \Phi_{\mathbf{x}}\left(\frac{w}{\alpha}\right) \\
&= \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w).
\end{aligned}$$

■

Next we consider the gradient for a $\mathbb{C}$-valued function f on a simple Euclidean Jordan algebra V. In this part we refer to [4]. For a scalar or vector valued differentiable function f we define the gradient $\nabla f(x)$ by

$$(\nabla f(x)|u) = D_u f(x) = \frac{d}{dt} f(x+tu)\Big|_{t=0}.$$

For a $\mathbb{C}$-valued function $f = f_1 + if_2$, we define $\nabla f = \nabla f_1 + i\nabla f_2$. For $z = x+iy \in V^{\mathbb{C}}$, we define $D_z = D_x + iD_y$. Moreover, if $\{e_1,\dots,e_n\}$ is an orthonormal basis of V and $x = \sum_{j=1}^n x_j e_j \in V^{\mathbb{C}}$, then

$$\nabla f(x) = \sum_{j=1}^n \frac{\partial f(x)}{\partial x_j} e_j.$$

We remark that this expression is independent of the choice of an orthonormal basis of V.

For a V-valued function $f : V \to V$ expressed by $f(x) = \sum_{j=1}^r f_j(x)e_j$, we define ∇f by

$$\nabla f(x) = \sum_{j,l=1}^n \frac{\partial f_j(x)}{\partial x_l} e_j e_l.$$

That is also well defined. Let us present some derivation formulas.

Lemma 2.11. (1) *The product rule of differentiation: For V-valued function f, h, we have*

$$\operatorname{tr}\,(\nabla(f(x)h(x))) = \operatorname{tr}\,(\nabla f(x))h(x) + f(x)\operatorname{tr}\,(\nabla h(x)). \tag{2.34}$$

For $\mathbb{C}$-valued functions f, h,

$$\nabla(f(x)h(x)) = (\nabla f(x))h(x) + f(x)(\nabla h(x)). \tag{2.35}$$

(2)

$$\nabla x = \frac{n}{r} e. \tag{2.36}$$

(3) *For any invertible element* $x \in V^{\mathbb{C}}$,

$$\mathrm{tr}\ (x\nabla)x^{-1} := \mathrm{tr}\ (x(\nabla x^{-1})) = -\frac{n}{r}\,\mathrm{tr}\ x^{-1}. \tag{2.37}$$

(4) *For* $\beta \in \mathbb{C}$ *and an invertible element* $x \in V^{\mathbb{C}}$,

$$\nabla(\Delta(x)^{\beta}) = \beta\Delta(x)^{\beta}x^{-1}. \tag{2.38}$$

(1), (2), and (4) are well known (see [6], [4], and [7]). (3) follows from (1), (2), and $\nabla(xx^{-1}) = \nabla(e) = 0$.

The following recurrence formulas for the spherical functions, some of which involve the gradient, are also well known (see [4] and [7]).

Lemma 2.12. *Let* $\mathbf{s} \in \mathbb{C}^r$ *and* $x \in V^{\mathbb{C}}$. *Put*

$$a_j(\mathbf{s}) := \frac{c(\mathbf{s})}{c(\mathbf{s}+\epsilon_j)} = \prod_{k\neq j} \frac{s_j - s_k + \frac{d}{2}}{s_j - s_k}, \tag{2.39}$$

where $\epsilon_j := (0,\ldots,0,\overset{\overset{j}{\vee}}{1},0,\ldots,0)$. *Then,*

$$(\mathrm{tr}\ x)\varphi_{\mathbf{s}}(x) = \sum_{j=1}^{r} a_j(\mathbf{s})\varphi_{\mathbf{s}+\epsilon_j}(x), \tag{2.40}$$

$$(\mathrm{tr}\ \nabla)\varphi_{\mathbf{s}}(x) = \sum_{j=1}^{r} \left(s_j + \frac{d}{4}(r-1)\right) a_j(-\mathbf{s})\varphi_{\mathbf{s}-\epsilon_j}(x), \tag{2.41}$$

$$(\mathrm{tr}\ (x^2\nabla))\varphi_{\mathbf{s}}(x) = \sum_{j=1}^{r} \left(s_j - \frac{d}{4}(r-1)\right) a_j(\mathbf{s})\varphi_{\mathbf{s}+\epsilon_j}(x). \tag{2.42}$$

2.2. Multivariate Laguerre polynomials and their unitary picture. In this subsection, we promote a unitary picture associated with the multivariate Laguerre polynomials and provide some fundamental lemmas based on [1], [6] and [7].

First, we recall some function spaces and their complete orthogonal basis as in the case of one variable. Let $\alpha > 2\frac{n}{r} - 1$, $\mathbf{m} \in \mathscr{P}$.

(1) $\psi_{\mathbf{m}}^{(\alpha)}$; Multivariate Laguerre polynomials (up to an exponential factor)

$$L_{\alpha}^{2}(\Omega)^{K} := \{\psi : \Omega \longrightarrow \mathbb{C} \mid \psi \text{ is } K\text{-invariant and } \|\psi\|_{\alpha,\Omega}^{2} < \infty\},$$

$$\|\psi\|_{\alpha,\Omega}^{2} := \frac{2^{r\alpha}}{\Gamma_{\Omega}(\alpha)} \int_{\Omega} |\psi(u)|^2 \Delta(u)^{\alpha-\frac{n}{r}}\, du,$$

$$\psi_{\mathbf{m}}^{(\alpha)}(u) := e^{-\mathrm{tr}\ u} L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(2u).$$

Here, $L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(u)$ is the multivariate Laguerre polynomial defined by

$$L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(u) := d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}}(-1)^{|\mathbf{k}|}\binom{\mathbf{m}}{\mathbf{k}}\frac{1}{(\alpha)_{\mathbf{k}}}\Phi_{\mathbf{k}}(u)$$
$$= d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}}(-1)^{|\mathbf{k}|}d_{\mathbf{k}}\frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)}{\left(\frac{n}{r}\right)_{\mathbf{k}}(\alpha)_{\mathbf{k}}}\Phi_{\mathbf{k}}(u).$$

(2) $F_{\mathbf{m}}^{(\alpha)}$; Cayley transform of the spherical polynomials

$$\mathcal{H}_\alpha^2(T_\Omega)^K := \{F : T_\Omega \longrightarrow \mathbb{C} \mid F \text{ is } K\text{-invariant and analytic in } T_\Omega, \text{ and } \|F\|_{\alpha,T_\Omega}^2 < \infty\},$$

$$\|F\|_{\alpha,T_\Omega}^2 := \frac{1}{(4\pi)^n}\frac{\Gamma_\Omega(\alpha)}{\Gamma_\Omega\left(\alpha-\frac{n}{r}\right)}\int_{T_\Omega}|F(z)|^2\Delta(x)^{\alpha-\frac{2n}{r}}\,m(dz),$$

$$F_{\mathbf{m}}^{(\alpha)}(z) := d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Delta\left(\frac{e+z}{2}\right)^{-\alpha}\Phi_{\mathbf{m}}((z-e)(z+e)^{-1}).$$

(3) $f_{\mathbf{m}}^{(\alpha)}$; spherical polynomials

$$\mathcal{H}_\alpha^2(\mathcal{D})^K := \{f : \mathcal{D} \longrightarrow \mathbb{C} \mid f \text{ is } K\text{-invariant and analytic in } \mathcal{D}, \text{ and } \|f\|_{\alpha,\mathcal{D}}^2 < \infty\},$$

$$\|f\|_{\alpha,\mathcal{D}}^2 := \frac{1}{\pi^n}\frac{\Gamma_\Omega(\alpha)}{\Gamma_\Omega\left(\alpha-\frac{n}{r}\right)}\int_{\mathcal{D}}|f(w)|^2 h(w)^{\alpha-\frac{2n}{r}}\,m(dw),$$

$$h(w) := \mathrm{Det}\left(I_{V^{\mathbb{C}}} - 2w\Box\overline{w} + P(w)P(\overline{w})\right)^{\frac{r}{2n}},$$

$$f_{\mathbf{m}}^{(\alpha)}(w) := d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(u).$$

Here, Det stands for the usual determinant of a complex linear operator on $V^{\mathbb{C}}$.

We remark that

$$\|\psi_{\mathbf{m}}^{(\alpha)}\|_{\alpha,\Omega}^2 = \|F_{\mathbf{m}}^{(\alpha)}\|_{\alpha,T_\Omega}^2 = \|f_{\mathbf{m}}^{(\alpha)}\|_{\alpha,\mathcal{D}}^2 = d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}$$

and the orthogonality relations of $\psi_{\mathbf{m}}^{(\alpha)}$ also hold for $\alpha > \frac{n}{r}-1$.

Next, similar to the one variable case, we will consider some unitary isomorphisms.

Modified Laplace transform

$$\mathcal{L}_\alpha : L_\alpha^2(\Omega)^K \xrightarrow{\simeq} \mathcal{H}_\alpha^2(T_\Omega)^K,\ \ (\mathcal{L}_\alpha\psi)(z) := \frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)}\int_\Omega e^{-(z|u)}\Delta(u)^{\alpha-\frac{n}{r}}\psi(u)\,du.$$

Modified Cayley transform

$$C_\alpha^{-1} : \mathcal{H}_\alpha^2(T_\Omega)^K \xrightarrow{\simeq} \mathcal{H}_\alpha^2(\mathcal{D})^K,\ \ (C_\alpha^{-1}F)(w) := \Delta(e-w)^{-\alpha}F\left((e+w)(e-w)^{-1}\right).$$

To summarize the above, we obtain the following picture.

$$\begin{array}{ccccc} L_\alpha^2(\Omega)^K & \xrightarrow[\mathcal{L}_\alpha]{\simeq} & \mathcal{H}_\alpha^2(T_\Omega)^K & \xrightarrow[C_\alpha^{-1}]{\simeq} & \mathcal{H}_\alpha^2(\mathcal{D})^K. \\ \in & & \in & & \in \\ \psi_m^{(\alpha)} & \longmapsto & F_m^{(\alpha)} & \longmapsto & f_m^{(\alpha)} \\ (\mathbf{1}) & & (\mathbf{2}) & & (\mathbf{3}) \end{array} \tag{2.43}$$

Lemma 2.13. *For any $\alpha \in \mathbb{C}, u \in \Omega$ and $z \in \mathcal{D}$, we have*

$$\sum_{\mathbf{m}\in\mathscr{P}} L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(u)\Phi_{\mathbf{m}}(z) = \Delta(e-z)^{-\alpha}\int_K e^{-(ku|z(e-z)^{-1})}\,dk. \tag{2.44}$$

Proof. By referring to [3] (see Proposition 2. 8), (2.44) holds for $\alpha > \frac{n}{r} - 1 = d(r-1)$. Moreover, the right hand side of (2.44) is well defined for any $\alpha \in \mathbb{C}$. Hence, by analytic continuation, it is sufficient to show the absolute convergence of the left hand side under the assumption. By Lemmas 2.1, 2.6 and 2.7,

$$\begin{aligned}\sum_{\mathbf{m}\in\mathscr{P}} |L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(u)\Phi_{\mathbf{m}}(z)| &\le \sum_{\mathbf{m}\in\mathscr{P}}\sum_{\mathbf{k}\subset\mathbf{m}}\left| d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\binom{\mathbf{m}}{\mathbf{k}}\frac{(-1)^{\mathbf{k}}}{(\alpha)_{\mathbf{k}}}\Phi_{\mathbf{k}}(u)\right|\Phi_{\mathbf{m}}(a_1)\\
&\le \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}}\frac{1}{(|\alpha|+d(r-1))_{\mathbf{k}}}\Phi_{\mathbf{k}}(u)\\
&\quad \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(|\alpha|+d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1)\\
&= (1-a_1)^{-r|\alpha|-dr(r-1)}\sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}}\Phi_{\mathbf{k}}\left(\frac{a_1}{1-a_1}u\right)\\
&= (1-a_1)^{-r|\alpha|-dr(r-1)}e^{\frac{a_1}{1-a_1}\operatorname{tr} u} < \infty.\end{aligned} \tag{2.45}$$

■

Let us consider the operators $D_\alpha^{(j)}$ for $j = 1, 2, 3$. The operator $D_\alpha^{(3)}$ is a first order differential operator on the domain $\mathcal{D}$:

$$D_\alpha^{(3)} := 2\operatorname{tr}(w\nabla_w). \tag{2.46}$$

Since this is the Euler operator,

$$D_\alpha^{(3)} f_{\mathbf{m}}^{(\alpha)}(w) = 2|\mathbf{m}| f_{\mathbf{m}}^{(\alpha)}(w).$$

The operators $D_\alpha^{(2)}$ and $D_\alpha^{(1)}$ are respectively defined by $C_\alpha^{-1}D_\alpha^{(2)} = D_\alpha^{(3)}C_\alpha^{-1}$ and $\mathcal{L}_\alpha D_\alpha^{(1)} = D_\alpha^{(2)}\mathcal{L}_\alpha$. Hence, $D_\alpha^{(2)}F_{\mathbf{m}}^{(\alpha)}(w) = 2|\mathbf{m}|F_{\mathbf{m}}^{(\alpha)}(w)$ and

$$D_\alpha^{(1)}\psi_{\mathbf{m}}^{(\alpha)}(u) = 2|\mathbf{m}|\psi_{\mathbf{m}}^{(\alpha)}(u). \tag{2.47}$$

Moreover, they have the following expressions.

$$D_\alpha^{(2)} = \operatorname{tr}((z^2-e)\nabla_z + \alpha(z-e)), \tag{2.48}$$

$$D_\alpha^{(1)} = \operatorname{tr}(-u\nabla_u^2 - \alpha\nabla_u + u - \alpha e). \tag{2.49}$$

Lemma 2.14. (1)

$$\begin{aligned}D_\alpha^{(1)}\varphi_{\mathbf{s}}(u) &= \sum_{j=1}^r a_j(\mathbf{s})\varphi_{\mathbf{s}+\epsilon_j}(u) - r\alpha\varphi_{\mathbf{s}}(u)\\
&\quad -\sum_{j=1}^r\left(s_j + \frac{d}{4}(r-1)\right)\left(s_j+\alpha-\frac{d}{4}(r-1)-1\right)a_j(-\mathbf{s})\varphi_{\mathbf{s}-\epsilon_j}(u).\end{aligned} \tag{2.50}$$

(2)

$$D_\alpha^{(1)}\Phi_{\mathbf{x}}(u) = \sum_{j=1}^{r} \tilde{a_j}(\mathbf{x})\Phi_{\mathbf{x}+\epsilon_j}(u) - r\alpha\Phi_{\mathbf{x}}(u) - \sum_{j=1}^{r}\left(x_j + \frac{d}{2}(r-j)\right)\left(x_j + \alpha - 1 - \frac{d}{2}(j-1)\right)\tilde{a_j}(-\mathbf{x})\Phi_{\mathbf{x}-\epsilon_j}(u). \tag{2.51}$$

Here,

$$\tilde{a_j}(\mathbf{x}) := a_j(\mathbf{x}-\rho) = \prod_{k\neq j}\frac{x_j - x_k - \frac{d}{2}(j-k-1)}{x_j - x_k - \frac{d}{2}(j-k)}. \tag{2.52}$$

(3) *For any* $C \in \mathbb{C}$*,*

$$e^{C\,\mathrm{tr}\,u}D_\alpha^{(1)}e^{-C\,\mathrm{tr}\,u}\Phi_{\mathbf{x}}(u) = (1-C^2)\sum_{j=1}^{r}\tilde{a_j}(\mathbf{x})\Phi_{\mathbf{x}+\epsilon_j}(u) + \sum_{j=1}^{r}(C(2x_j+\alpha)-\alpha)\Phi_{\mathbf{x}}(u) - \sum_{j=1}^{r}\left(x_j + \frac{d}{2}(r-j)\right)\left(x_j+\alpha-1-\frac{d}{2}(j-1)\right) \cdot \tilde{a_j}(-\mathbf{x})\Phi_{\mathbf{x}-\epsilon_j}(u). \tag{2.53}$$

Remark 2.15. Though (2.50) is a corollary of Lemma 5.8 in [7] (version. 1) essentially, Faraut and Wakayama's lemma is deleted in their revised version. Hence, we re-prove it according to their proof.

Proof. (1) The modified Laplace transform of $\varphi_{\mathbf{s}}$ is given by

$$(\mathcal{L}_\alpha\varphi_{\mathbf{s}})(z) = \frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)}\int_\Omega e^{-(z|u)}\varphi_{\mathbf{s}}(u)\Delta(u)^{\alpha-\frac{n}{r}}\,du = 2^{r\alpha}\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\alpha)}\varphi_{-\mathbf{s}-\alpha}(z).$$

Thus, from the definition of $D_\alpha^{(1)}$ and Lemma 2.12,

$$\begin{aligned}\mathcal{L}_\alpha(D_\alpha^{(1)}\varphi_{\mathbf{s}})(z) &= D_\alpha^{(2)}(\mathcal{L}_\alpha\varphi_{\mathbf{s}})(z)\\ &= 2^{r\alpha}\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\alpha)}D_\alpha^{(2)}\varphi_{-\mathbf{s}-\alpha}(z)\\ &= \sum_{j=1}^{r}\left(s_j+\alpha-\frac{d}{4}(r-1)\right)a_j(\mathbf{s}+\alpha)2^{r\alpha}\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\alpha)}\varphi_{-\mathbf{s}-\alpha-\epsilon_j}(z)\\ &\quad - r\alpha 2^{r\alpha}\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\alpha)}\varphi_{-\mathbf{s}-\alpha}(z)\\ &\quad - \sum_{j=1}^{r}\left(s_j+\frac{d}{4}(r-1)\right)a_j(-\mathbf{s}-\alpha)2^{r\alpha}\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\alpha)}\varphi_{-\mathbf{s}-\alpha+\epsilon_j}(z).\end{aligned}$$

Since

$$\varphi_{-\mathbf{s}-\alpha\pm\epsilon_j}(z) = \frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\mathbf{s}+\alpha+\rho\pm\epsilon_j)}\mathcal{L}_\alpha(\varphi_{\mathbf{s}\pm\epsilon_j})(z),$$

we have

$$\begin{aligned}
&D_\alpha^{(2)}(\mathcal{L}_\alpha\varphi_{\mathbf{s}})(z)\\
&\quad=\sum_{j=1}^{r}\left(s_j+\alpha-\frac{d}{4}(r-1)\right)a_j(\mathbf{s})\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\mathbf{s}+\alpha+\rho+\epsilon_j)}\mathcal{L}_\alpha(\varphi_{\mathbf{s}+\epsilon_j})(z)\\
&\quad\quad-r\alpha(\mathcal{L}_\alpha\varphi_{\mathbf{s}})(z)\\
&\quad\quad-\sum_{j=1}^{r}\left(s_j+\frac{d}{4}(r-1)\right)a_j(-\mathbf{s})\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\mathbf{s}+\alpha+\rho-\epsilon_j)}\mathcal{L}_\alpha(\varphi_{\mathbf{s}-\epsilon_j})(z)\\
&\quad=\mathcal{L}_\alpha\left(\sum_{j=1}^{r}a_j(\mathbf{s})\varphi_{\mathbf{s}+\epsilon_j}(u)-r\alpha\varphi_{\mathbf{s}}(u)\right.\\
&\quad\quad\left.-\sum_{j=1}^{r}\left(s_j+\frac{d}{4}(r-1)\right)\left(s_j+\alpha-\frac{d}{4}(r-1)-1\right)a_j(-\mathbf{s})\varphi_{\mathbf{s}-\epsilon_j}(u)\right)(z).
\end{aligned}$$

(2) Put $\mathbf{s}=\mathbf{m}-\rho$ in (2.50).

(3) By

$$e^{C\,\mathrm{tr}\,u}\nabla_u e^{-C\,\mathrm{tr}\,u}=-Ce,\quad e^{C\,\mathrm{tr}\,u}\,\mathrm{tr}\,(u\nabla_u^2)e^{-C\,\mathrm{tr}\,u}=C^2\,\mathrm{tr}\,u$$

(e in $-Ce$ is the unit element of V) and the product rule of differentiation, we remark that

$$\begin{aligned}
e^{C\,\mathrm{tr}\,u}\,\mathrm{tr}\,(u\nabla_u^2)e^{-C\,\mathrm{tr}\,u}\Phi_{\mathbf{x}}(u)&=\mathrm{tr}\,(u\nabla_u^2)\Phi_{\mathbf{x}}(u)\\
&\quad+2\,\mathrm{tr}\,(ue^{C\,\mathrm{tr}\,u}\nabla_u(e^{-C\,\mathrm{tr}\,u})\nabla_u(\Phi_{\mathbf{x}}(u)))\\
&\quad+e^{C\,\mathrm{tr}\,u}\Phi_{\mathbf{x}}(u)\,\mathrm{tr}\,(u\nabla_u^2)e^{-C\,\mathrm{tr}\,u}\\
&=\left\{\mathrm{tr}\,(u\nabla_u^2)+C^2\,\mathrm{tr}\,u-2C|\mathbf{x}|\right\}\Phi_{\mathbf{x}}(u)
\end{aligned}$$

and

$$\begin{aligned}
e^{C\,\mathrm{tr}\,u}\,\mathrm{tr}\,(\nabla_u)e^{-C\,\mathrm{tr}\,u}\Phi_{\mathbf{x}}(u)&=\Phi_{\mathbf{x}}(u)\,\mathrm{tr}\,(e^{C\,\mathrm{tr}\,u}\nabla_u e^{-C\,\mathrm{tr}\,u})+\mathrm{tr}\,(\nabla_u)\Phi_{\mathbf{x}}(u)\\
&=-Cr\Phi_{\mathbf{x}}(u)+\mathrm{tr}\,(\nabla_u)\Phi_{\mathbf{x}}(u).
\end{aligned}$$

Hence,

$$e^{C\,\mathrm{tr}\,u}D_\alpha^{(1)}e^{-C\,\mathrm{tr}\,u}\Phi_{\mathbf{x}}(u)=D_\alpha^{(1)}\Phi_{\mathbf{x}}(u)-C^2\,\mathrm{tr}\,u\Phi_{\mathbf{x}}(u)+C(2|\mathbf{x}|+r\alpha)\Phi_{\mathbf{x}}(u).$$

Therefore, from (2.51) and (2.40),

$$
\begin{aligned}
&e^{C\,\mathrm{tr}\,u} D_\alpha^{(1)} e^{-C\,\mathrm{tr}\,u} \Phi_{\mathbf{x}}(u) \\
&\quad = \sum_{j=1}^{r} \tilde{a_j}(\mathbf{x}) \Phi_{\mathbf{x}+\epsilon_j}(u) - r\alpha \Phi_{\mathbf{x}}(u) \\
&\qquad - \sum_{j=1}^{r} \left(x_j + \frac{d}{2}(r-j)\right) \left(x_j + \alpha - 1 - \frac{d}{2}(j-1)\right) \tilde{a_j}(-\mathbf{x}) \Phi_{\mathbf{x}-\epsilon_j}(u) \\
&\qquad - C^2 \sum_{j=1}^{r} \tilde{a_j}(\mathbf{x}) \Phi_{\mathbf{x}+\epsilon_j}(u) + C(2|\mathbf{x}| + r\alpha) \Phi_{\mathbf{x}}(u) \\
&\quad = (1 - C^2) \sum_{j=1}^{r} \tilde{a_j}(\mathbf{x}) \Phi_{\mathbf{x}+\epsilon_j}(u) + \sum_{j=1}^{r} (C(2x_j + \alpha) - \alpha) \Phi_{\mathbf{x}}(u) \\
&\qquad - \sum_{j=1}^{r} \left(x_j + \frac{d}{2}(r-j)\right) \left(x_j + \alpha - 1 - \frac{d}{2}(j-1)\right) \tilde{a_j}(-\mathbf{x}) \Phi_{\mathbf{x}-\epsilon_j}(u).
\end{aligned}
$$

■

3. Multivariate Meixner, Charlier and Krawtchouk polynomials

In this section, we assume that $\mathbf{m}, \mathbf{n}, \mathbf{x}, \mathbf{y} \in \mathscr{P}, \alpha \in \mathbb{C}, c, a, p \in \mathbb{C}^*, N \in \mathbb{Z}_{\geq 0}$ and

$$z = u_1 \sum_{j=1}^{r} a_j c_j, \quad w = u_2 \sum_{j=1}^{r} b_j c_j \in V^{\mathbb{C}},$$

with $u_1, u_2 \in U$, $a_1 \geq \cdots \geq a_r \geq 0$, $b_1 \geq \cdots \geq b_r \geq 0$ unless otherwise specified.

3.1. Definitions.

Definition 3.1. We define the multivariate Meixner, Charlier and Krawtchouk polynomials as follows.

$$M_{\mathbf{m}}(\mathbf{x};\alpha,c) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{(\alpha)_{\mathbf{k}}} \binom{\mathbf{m}}{\mathbf{k}} \binom{\mathbf{x}}{\mathbf{k}} \left(1-\frac{1}{c}\right)^{|\mathbf{k}|} \tag{3.1}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{x}-\rho)}{(\alpha)_{\mathbf{k}}} \left(1-\frac{1}{c}\right)^{|\mathbf{k}|} \tag{3.2}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}(\mathbf{x}-\rho)}{\left(\frac{n}{r}\right)_{\mathbf{k}}(\alpha)_{\mathbf{k}}} \left(1-\frac{1}{c}\right)^{|\mathbf{k}|}, \tag{3.3}$$

$$C_{\mathbf{m}}(\mathbf{x};a) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \left(\frac{n}{r}\right)_{\mathbf{k}} \binom{\mathbf{m}}{\mathbf{k}} \binom{\mathbf{x}}{\mathbf{k}} \left(-\frac{1}{a}\right)^{|\mathbf{k}|} \tag{3.4}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho) \left(-\frac{1}{a}\right)^{|\mathbf{k}|} \tag{3.5}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}(\mathbf{x}-\rho)}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \left(-\frac{1}{a}\right)^{|\mathbf{k}|}, \tag{3.6}$$

$$K_{\mathbf{m}}(\mathbf{x};p,N) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{(-N)_{\mathbf{k}}} \binom{\mathbf{m}}{\mathbf{k}} \binom{\mathbf{x}}{\mathbf{k}} \left(\frac{1}{p}\right)^{|\mathbf{k}|} \quad (\mathbf{m}\subset N=(N,\dots,N)) \tag{3.7}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{x}-\rho)}{(-N)_{\mathbf{k}}} \left(\frac{1}{p}\right)^{|\mathbf{k}|} \tag{3.8}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}(\mathbf{x}-\rho)}{\left(\frac{n}{r}\right)_{\mathbf{k}}(-N)_{\mathbf{k}}} \left(\frac{1}{p}\right)^{|\mathbf{k}|}. \tag{3.9}$$

When $r=1$, these polynomials become the usual Meixner, Charlier and Krawtchouk polynomials. By the definition, we immediately obtain a duality property for these polynomials.

Proposition 3.2. (1) *For all* $\mathbf{m},\mathbf{x}\in\mathscr{P}$*, we have*

$$M_{\mathbf{m}}(\mathbf{x};\alpha,c) = M_{\mathbf{x}}(\mathbf{m};\alpha,c). \tag{3.10}$$

(2) *For all* $\mathbf{m},\mathbf{x}\in\mathscr{P}$*, we have*

$$C_{\mathbf{m}}(\mathbf{x};a) = C_{\mathbf{x}}(\mathbf{m};a). \tag{3.11}$$

(3) *For all* $\mathbf{m},\mathbf{x}\subset N$*, we have*

$$K_{\mathbf{m}}(\mathbf{x};p,N) = K_{\mathbf{x}}(\mathbf{m};p,N). \tag{3.12}$$

We only remark the proof of (3.12). Since $\mathbf{m},\mathbf{x}\subset N$,

$$K_{\mathbf{m}}(\mathbf{x};p,N) = \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{(-N)_{\mathbf{k}}} \binom{\mathbf{m}}{\mathbf{k}} \binom{\mathbf{x}}{\mathbf{k}} \left(\frac{1}{p}\right)^{|\mathbf{k}|}$$

$$= \sum_{\mathbf{k}\subset\mathbf{x}} \frac{1}{d_{\mathbf{k}}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{(-N)_{\mathbf{k}}} \binom{\mathbf{x}}{\mathbf{k}} \binom{\mathbf{m}}{\mathbf{k}} \left(\frac{1}{p}\right)^{|\mathbf{k}|} = K_{\mathbf{x}}(\mathbf{m};p,N).$$

We also obtain the following relations by the definitions.

Proposition 3.3. (1) (Relation of Meixne and Krawtchouk):

$$M_{\mathbf{m}}\left(\mathbf{x}; -N, \frac{p}{p-1}\right) = K_{\mathbf{m}}(\mathbf{x}; p, N). \tag{3.13}$$

(2)

$$\lim_{\alpha\to\infty} M_{\mathbf{m}}\left(\mathbf{x}; \alpha, \frac{a}{a+\alpha}\right) = C_{\mathbf{m}}(\mathbf{x}; a). \tag{3.14}$$

(3)

$$\lim_{N\to\infty} K_{\mathbf{m}}\left(\mathbf{x}; \frac{a}{N}, N\right) = C_{\mathbf{m}}(\mathbf{x}; a). \tag{3.15}$$

Actually, (1) follows from the definitions. For (2) and (3), we remark that

$$\lim_{\alpha\to\infty} \frac{\alpha^{|\mathbf{k}|}}{(\alpha)_{\mathbf{k}}} = \lim_{\alpha\to\infty} \prod_{j=1}^{r} \frac{\alpha^{k_j}}{\left(\alpha - \frac{d}{2}(j-1)\right)_{k_j}} = 1,$$

$$\lim_{N\to\infty} \frac{N^{|\mathbf{k}|}}{(-N)_{\mathbf{k}}} = \lim_{N\to\infty} \prod_{j=1}^{r} \frac{N^{k_j}}{\left(-N - \frac{d}{2}(j-1)\right)_{k_j}} = (-1)^{|\mathbf{k}|}.$$

3.2. Generating functions.

To present our key lemma which is a summation formula of the above polynomials, we need to prove their convergence.

Lemma 3.4. (1) *If* $1 > a_1 \geq \cdots \geq a_r \geq 0$, $b_1 \geq \cdots \geq b_r \geq 0$, *then*

$$\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x}; \alpha, c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \right| \leq e^{rb_1\left(1+\frac{a_1}{1-a_1}\left(\frac{1}{c}-1\right)\right)} (1-a_1)^{-r(|\alpha|+2n)}. \tag{3.16}$$

(2) *For any* $z, w \in V^{\mathbb{C}}$, *we have*

$$\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} C_{\mathbf{m}}(\mathbf{x}, a)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \right| \leq e^{r\left(a_1+b_1+\frac{a_1 b_1}{a}\right)}. \tag{3.17}$$

Proof. (1) By Lemma 2.1, Lemma 2.6 and Lemma 2.9,

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x}; \alpha, c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \right| \\
&\leq \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}} (|\alpha| + d(r-1))_{\mathbf{k}}} \left(\frac{1}{c} - 1\right)^{|\mathbf{k}|} \\
&\quad \cdot \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(|\alpha| + d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m} - \rho)\Phi_{\mathbf{m}}(a_1) \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x} - \rho)\Phi_{\mathbf{x}}(b_1).
\end{aligned}$$

Moreover, from (2.32) and (2.33) of Theorem 2.10,

$$\begin{aligned}
&\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(|\alpha|+d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1) \\
&\quad = (|\alpha|+d(r-1))_{\mathbf{k}}(1-a_1)^{-r|\alpha|-dr(r-1)-|\mathbf{k}|}a_1^{|\mathbf{k}|}, \\
&\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(b_1) = e^{rb_1}{b_1}^{|\mathbf{k}|}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \right| \\
&\le e^{rb_1}(1-a_1)^{-r(|\alpha|+d(r-1))} \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}\left(\left(\frac{1}{c}-1\right)\frac{a_1b_1}{1-a_1}\right) \\
&= e^{rb_1\left(1+\frac{a_1}{1-a_1}\left(\frac{1}{c}-1\right)\right)}(1-a_1)^{-r(|\alpha|+d(r-1))} < \infty.
\end{aligned}$$

(2) By a similar argument,

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} C_{\mathbf{m}}(\mathbf{x},a)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \right| \\
&\le \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} a^{-|\mathbf{k}|} \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1) \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(b_1) \\
&= e^{r(a_1+b_1)} \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}\left(\frac{a_1b_1}{a}\right) \\
&= e^{r\left(a_1+b_1+\frac{a_1b_1}{a}\right)} < \infty.
\end{aligned}$$

∎

The following theorem is the key result in our theory.

Theorem 3.5. (1) *For $z \in \mathcal{D}, w \in V^{\mathbb{C}}$, we obtain*

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}} e^{\operatorname{tr} w} L_{\mathbf{m}}^{\left(\alpha-\frac{n}{r}\right)}\left(\left(\frac{1}{c}-1\right)w\right)\Phi_{\mathbf{m}}(z) &= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) \\
&\quad \cdot d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \qquad (3.18) \\
&= \Delta(e-z)^{-\alpha} \int_K e^{(kw|(e-\frac{1}{c}z)(e-z)^{-1})}\, dk. \qquad (3.19)
\end{aligned}$$

(2) *For $w, z \in V^{\mathbb{C}}$, we obtain*

$$\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} e^{\mathrm{tr}\ w} \Phi_{\mathbf{m}}\left(e - \frac{1}{a}w\right)\Phi_{\mathbf{m}}(z) = \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} C_{\mathbf{m}}(\mathbf{x}; a)\Phi_{\mathbf{m}}(z) \cdot d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \tag{3.20}$$

$$= e^{\mathrm{tr}\ (w+z)} \int_K e^{-\frac{1}{a}(kw|z)}\, dk. \tag{3.21}$$

Remark 3.6. We remark for any $\mathbf{m}, \mathbf{x} \in \mathscr{P}$ and $\alpha \in \mathbb{C}$,

$$L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(0) = d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}$$

and $M_{\mathbf{m}}(\mathbf{x}; \alpha, 1) = 1$. Therefore, for $c = 1$, (3.18) is trivial and (3.19) degenerates to

$$\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(z) \cdot \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) = \Delta(e-z)^{-\alpha} e^{\mathrm{tr}\ w},$$

which is well known formula.

Proof. (1) By the above lemma, the series converges absolutely under the conditions. Therefore, we derive

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x}; \alpha, c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \\
&= \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(z) \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{1}{(\alpha)_{\mathbf{k}}} \left(1 - \frac{1}{c}\right)^{|\mathbf{k}|} \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w) \\
&= \sum_{\mathbf{m}\in\mathscr{P}} e^{\mathrm{tr}\ w} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{(-1)^{\mathbf{k}}}{(\alpha)_{\mathbf{k}}} \Phi_{\mathbf{k}}\left(\left(\frac{1}{c} - 1\right) w\right) \Phi_{\mathbf{m}}(z) \\
&= \sum_{\mathbf{m}\in\mathscr{P}} e^{\mathrm{tr}\ w} L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}\left(\left(\frac{1}{c} - 1\right) w\right) \Phi_{\mathbf{m}}(z).
\end{aligned}$$

(3.19) follows from (2.44).
(2) Put $c = \frac{a}{a+\alpha}, w \to \frac{w}{\alpha}, a, \alpha \in \mathbb{R}_{>0}$ in (1) of Theorem 3.5 and take the limit of $\alpha \to \infty$. ∎

The generating formulas of our polynomials are a corollary of the above theorem.

Theorem 3.7. (1) *For $z \in \mathcal{D}, \mathbf{x} \in \mathscr{P}$, we have*

$$\Delta(e-z)^{-\alpha} \Phi_{\mathbf{x}}\left(\left(e - \frac{1}{c}z\right)(e-z)^{-1}\right) = \sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}} \frac{(\alpha)_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}} M_{\mathbf{n}}(\mathbf{x}; \alpha, c)\Phi_{\mathbf{n}}(z). \tag{3.22}$$

(2) *For $z \in \mathcal{D}, \mathbf{x} \in \mathscr{P}$, we have*

$$e^{\operatorname{tr} z}\Phi_{\mathbf{x}}\left(e-\frac{1}{a}z\right)=\sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{n}}}C_{\mathbf{n}}(\mathbf{x};a)\Phi_{\mathbf{n}}(z). \tag{3.23}$$

(3) *For $z \in \mathcal{D}, \mathbf{x} \subset N$, we have*

$$\Delta(e+z)^N\Phi_{\mathbf{x}}\left(\left(e-\frac{1-p}{p}z\right)(e+z)^{-1}\right)=\sum_{\mathbf{n}\subset N}\binom{N}{\mathbf{n}}K_{\mathbf{n}}(\mathbf{x};p,N)\Phi_{\mathbf{n}}(z). \tag{3.24}$$

Proof. (1) We evaluate the spherical Taylor expansion of (3.19) with respect to w:

$$\begin{aligned}
&\Phi_{\mathbf{x}}(\partial_w)\Delta(e-z)^{-\alpha}\int_K e^{(kw|(e-\frac{1}{c}z)(e-z)^{-1})}\,dk\Big|_{w=0}\\
&=\Delta(e-z)^{-\alpha}\int_K \Phi_{\mathbf{x}}(\partial_w)e^{(w|k(e-\frac{1}{c}z)(e-z)^{-1})}|_{w=0}\,dk\\
&=\Delta(e-z)^{-\alpha}\int_K \Phi_{\mathbf{x}}\left(k\left(\left(e-\frac{1}{c}z\right)(e-z)^{-1}\right)\right)\,dk\\
&=\Delta(e-z)^{-\alpha}\Phi_{\mathbf{x}}\left(\left(e-\frac{1}{c}z\right)(e-z)^{-1}\right).
\end{aligned}$$

On the other hand, by (3.18),

$$\Phi_{\mathbf{x}}(\partial_w)\Delta(e-z)^{-\alpha}\int_K e^{(kw|(e-\frac{1}{c}z)(e-z)^{-1})}\,dk\Big|_{w=0}=\sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}}\frac{(\alpha)_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}}M_{\mathbf{n}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{n}}(z).$$

Therefore, we obtain the conclusion.
(2) The result is proved by a similar argument as in (1). That is, by (2) of Theorem 3.5, we have

$$\begin{aligned}
\sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{n}}}C_{\mathbf{n}}(\mathbf{x};a)\Phi_{\mathbf{n}}(z)&=\Phi_{\mathbf{x}}(\partial_w)e^{\operatorname{tr}(w+z)}\int_K e^{-\frac{1}{a}(kw|z)}\,dk\Big|_{w=0}\\
&=e^{\operatorname{tr} z}\int_K \Phi_{\mathbf{x}}(\partial_w)e^{(w|k(e-\frac{1}{a}z))}|_{w=0}\,dk\\
&=e^{\operatorname{tr} z}\int_K \Phi_{\mathbf{x}}\left(k\left(e-\frac{1}{a}z\right)\right)\,dk\\
&=e^{\operatorname{tr} z}\Phi_{\mathbf{x}}\left(e-\frac{1}{a}z\right).
\end{aligned}$$

(3) From the assumption $\mathbf{x} \subset N$ and (3.22), we have

$$\begin{aligned}
&\Delta(e-z)^N \Phi_{\mathbf{x}}\left(\left(e-\frac{1}{c}z\right)(e-z)^{-1}\right) \\
&= \lim_{\alpha\to -N} \Delta(e-z)^{-\alpha}\Phi_{\mathbf{x}}\left(\left(e-\frac{1}{c}z\right)(e-z)^{-1}\right) \\
&= \sum_{\mathbf{n}\in\mathscr{P}} \frac{d_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}} \sum_{\mathbf{k}\subset\mathbf{n}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{d_{\mathbf{k}}} \lim_{\alpha\to -N} \frac{(\alpha)_{\mathbf{n}}}{(\alpha)_{\mathbf{k}}} \binom{\mathbf{n}}{\mathbf{k}}\binom{\mathbf{x}}{\mathbf{k}}\left(1-\frac{1}{c}\right)^{|\mathbf{k}|} \\
&= \sum_{\mathbf{n}\subset N} d_{\mathbf{n}} \frac{(-N)_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}} \sum_{\mathbf{k}\subset\mathbf{x}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{d_{\mathbf{k}}} \frac{1}{(-N)_{\mathbf{k}}} \binom{\mathbf{n}}{\mathbf{k}}\binom{\mathbf{x}}{\mathbf{k}}\left(1-\frac{1}{c}\right)^{|\mathbf{k}|} \\
&= \sum_{\mathbf{n}\subset N} \binom{N}{\mathbf{n}} M_{\mathbf{n}}(\mathbf{x};-N,c)\Phi_{\mathbf{n}}(-z).
\end{aligned}$$

Since this series is a finite sum, we can take $c = \frac{p}{p-1}$ above. Therefore, we obtain

$$\begin{aligned}
\Delta(e-z)^N \Phi_{\mathbf{x}}\left(\left(e+\frac{1-p}{p}z\right)(e-z)^{-1}\right) &= \sum_{\mathbf{n}\subset N} \binom{N}{\mathbf{n}} M_{\mathbf{n}}\left(\mathbf{x};-N,\frac{p}{p-1}\right)\Phi_{\mathbf{n}}(-z) \\
&= \sum_{\mathbf{n}\subset N} \binom{N}{\mathbf{n}} K_{\mathbf{n}}(\mathbf{x};p,N)\,\Phi_{\mathbf{n}}(-z).
\end{aligned}$$

■

Remark 3.8. For $c=-1$, (3.22) was obtained by Davidson-Ólafsson-Zhang [1] (Lemma 4.1) as a generating function of multivariate Meixner-Pollaczeck polynomials which are called generalized Hermite polynomials in their paper.

Next we apply the unitary transformations in (2.43) to Theorem 3.5. Here, we also check convergence.

Lemma 3.9. *Fix $0<c<1$ and let $0<\varepsilon<1$ and $w,z\in\mathcal{D}$ satisfy that*

$$\begin{aligned}
&\left(c+(1-c)\frac{\varepsilon}{1-\varepsilon}\right)\left(1+(1-c)\frac{\varepsilon}{1-c\varepsilon}\right)<1, \\
&|\Phi_{\mathbf{m}}(w)|,|\Phi_{\mathbf{m}}(z)|<\Phi_{\mathbf{m}}(\varepsilon)=\varepsilon^{|\mathbf{m}|}.
\end{aligned} \tag{3.25}$$

Then,

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m},\mathbf{n}\in\mathscr{P}} \left| d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{n}} \frac{(\alpha)_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}} M_{\mathbf{n}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{n}}(cw)\right| \\
&\quad < ((1-c)(1-2(1+c)\varepsilon+(4c-1)\varepsilon^2))^{-r|\alpha|-dr(r-1)}.
\end{aligned} \tag{3.26}$$

Proof. By Lemma 2.1 and Lemma 2.9, we have

$$
\begin{aligned}
(\mathrm{LHS}) \le \sum_{\mathbf{x}\in\mathscr{P}} & d_{\mathbf{x}} \frac{(|\alpha|+d(r-1))_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|} \\
& \cdot \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} \frac{1}{(|\alpha|+d(r-1))_{\mathbf{k}}} \left(\frac{1}{c}-1\right)^{|\mathbf{k}|} \sum_{\mathbf{l}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{l}} \frac{1}{(|\alpha|+d(r-1))_{\mathbf{l}}} \left(\frac{1}{c}-1\right)^{|\mathbf{l}|} \\
& \cdot \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(|\alpha|+d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(\varepsilon) \\
& \cdot \sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}} \frac{(|\alpha|+d(r-1))_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}} \gamma_{\mathbf{l}}(\mathbf{n}-\rho)\Phi_{\mathbf{n}}(c\varepsilon).
\end{aligned}
$$

Furthermore, from Lemma 2.10 and the definition of the generalized binomial coefficients (2.21), we derive

$$
\begin{aligned}
(\mathrm{LHS}) &\le ((1-\varepsilon)(1-c\varepsilon))^{-r|\alpha|-dr(r-1)} \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(|\alpha|+d(r-1))_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|} \\
&\quad \cdot \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} \left(\frac{1}{c}-1\right)^{|\mathbf{k}|} \Phi_{\mathbf{k}}\left(\frac{\varepsilon}{1-\varepsilon}\right) \sum_{\mathbf{l}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{l}} \left(\frac{1}{c}-1\right)^{|\mathbf{l}|} \Phi_{\mathbf{l}}\left(\frac{c\varepsilon}{1-c\varepsilon}\right) \\
&= ((1-\varepsilon)(1-c\varepsilon))^{-r|\alpha|-dr(r-1)} \\
&\quad \cdot \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(|\alpha|+d(r-1))_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}\left(\left(c+(1-c)\frac{\varepsilon}{1-\varepsilon}\right)\left(1+(1-c)\frac{\varepsilon}{1-c\varepsilon}\right)\right).
\end{aligned}
$$

Finally, by using the assumption and Lemma 2.10, we obtain

$$
\begin{aligned}
(\mathrm{LHS}) &\le \left((1-\varepsilon)(1-c\varepsilon)\left(1-\left(c+(1-c)\frac{\varepsilon}{1-\varepsilon}\right)\left(1+(1-c)\frac{\varepsilon}{1-c\varepsilon}\right)\right)\right)^{-r|\alpha|-dr(r-1)} \\
&= ((1-c)(1-2(1+c)\varepsilon+(4c-1)\varepsilon^2))^{-r|\alpha|-dr(r-1)}.
\end{aligned}
$$

■

From this lemma, we can consider the following generating functions.

Theorem 3.10. (1) *For* $z \in \mathcal{D}, u \in V^{\mathbb{C}}$*, we obtain*

$$
\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}} \psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z) &= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) \\
&\qquad \cdot d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \left(\frac{2c}{1-c}\right)^{|\mathbf{x}|} e^{-\frac{1+c}{1-c}\operatorname{tr} u}\Phi_{\mathbf{x}}(u) \qquad (3.27) \\
&= \Delta(e-z)^{-\alpha} \int_K e^{-(ku|(e+z)(e-z)^{-1})}\, dk.
\end{aligned}
$$

(2) *Fix* $0 < c < 1$ *and assume that* $w, z \in \mathcal{D}$ *satisfy the condition in* (1) *of*

Lemma 3.9. We obtain

$$\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) = (1-c)^{r\alpha}\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z)$$
$$\cdot d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}c^{|\mathbf{x}|}\Delta(e-cw)^{-\alpha}\Phi_{\mathbf{x}}((e-w)(e-cw)^{-1}) \tag{3.28}$$
$$= \Delta(z)^{-\alpha}\int_K \Delta(kz^{-1}-w)^{-\alpha}\,dk.$$

(3) *For $w, z \in V^{\mathbb{C}}$, we obtain*

$$\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) = e^{-ra}\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{a^{|\mathbf{m}|}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}C_{\mathbf{m}}(\mathbf{x};a)\Phi_{\mathbf{m}}(z)$$
$$\cdot d_{\mathbf{x}}\frac{a^{|\mathbf{x}|}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}e^{\mathrm{tr}\, w}\Phi_{\mathbf{x}}\left(e-\frac{1}{a}w\right) \tag{3.29}$$
$$= e^{\mathrm{tr}\, w}\int_K e^{-a(kw|e-z)}\,dk.$$

Proof. As (1) and (3) follow immediately from Theorem 3.5, we only prove (2).

First, we remark that the right hand side of (3.28) converges absolutely under the conditions in Lemma 3.9. Moreover, we also remark that since (2.45)

$$\sum_{\mathbf{m}\in\mathscr{P}} |\psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)| \le (1-a_1)^{-r|\alpha|-dr(r-1)}e^{-\frac{1-3a_1}{1-a_1}},$$

the exchange of unitary transformations $\mathcal{L}_\alpha$, $\mathcal{M}_{\alpha,\theta}$ and $\mathcal{F}_{\alpha,\nu}^{-1}$, and the summation are justified under these restrictions. Therefore, to obtain the results, we apply the unitary transforms to both sides of (3.27). We will perform these calculations.

For (2), we apply transform $C_\alpha^{-1}\circ\mathcal{L}_\alpha$ to both sides of (3.27). From Lemma 2.4, we have

$$\mathcal{L}_\alpha(e^{-\frac{1+c}{1-c}\mathrm{tr}\, u}\Phi_{\mathbf{x}})(z) = \frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)}\int_\Omega e^{-\left(\frac{1+c}{1-c}e+z\middle|u\right)}\Phi_{\mathbf{x}}(u)\Delta(u)^{\alpha-\frac{n}{r}}\,du$$
$$= 2^{r\alpha}(\alpha)_{\mathbf{x}}\Delta\left(\frac{1+c}{1-c}e+z\right)^{-\alpha}\Phi_{\mathbf{x}}\left(\left(\frac{1+c}{1-c}e+z\right)^{-1}\right).$$

Furthermore,

$$\begin{aligned}
&C_\alpha^{-1} \circ \mathcal{L}_\alpha(e^{-\frac{1+c}{1-c}\operatorname{tr} u}\Phi_{\mathbf{x}})(w) \\
&= 2^{r\alpha}(\alpha)_{\mathbf{x}} C_\alpha^{-1}\left(\Delta\left(\frac{1+c}{1-c}e+z\right)^{-\alpha}\Phi_{\mathbf{x}}\left(\left(\frac{1+c}{1-c}e+z\right)^{-1}\right)\right)(w) \\
&= (\alpha)_{\mathbf{x}}\Delta(e-w)^{-\alpha}\Delta\left(\frac{1}{2}\left((e+w)(e-w)^{-1}+\frac{1+c}{1-c}e\right)\right)^{-\alpha} \\
&\quad \cdot \Phi_{\mathbf{x}}\left(\left((e+w)(e-w)^{-1}+\frac{1+c}{1-c}e\right)^{-1}\right) \\
&= (\alpha)_{\mathbf{x}}(1-c)^{r\alpha}\Delta(e-cw)^{-\alpha}\Phi_{\mathbf{x}}\left(\frac{1-c}{2}(e-w)(e-cw)^{-1}\right).
\end{aligned}$$

Hence, the right-hand side of (3.27) becomes the right-hand side of (3.28). Therefore, since $C_\alpha^{-1} \circ \mathcal{L}_\alpha(\psi_{\mathbf{m}}^{(\alpha)})(w) = d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(w)$, we obtain the conclusion. ∎

3.3. Orthogonality relations. We provide the orthogonality relations for our discrete orthogonal polynomials as a corollary of Theorem 3.10.

Theorem 3.11. (1) *For $\alpha > \frac{n}{r}-1, 0<c<1$, we obtain*

$$\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}c^{|\mathbf{x}|}M_{\mathbf{m}}(\mathbf{x};\alpha,c)M_{\mathbf{n}}(\mathbf{x};\alpha,c) = \frac{c^{-|\mathbf{m}|}}{(1-c)^{r\alpha}}\frac{1}{d_{\mathbf{m}}}\frac{\left(\frac{n}{r}\right)_{\mathbf{m}}}{(\alpha)_{\mathbf{m}}}\delta_{\mathbf{m},\mathbf{n}} \geq 0. \tag{3.30}$$

(2) *For $a>0$, we obtain*

$$\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{a^{|\mathbf{x}|}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}C_{\mathbf{m}}(\mathbf{x};a)C_{\mathbf{n}}(\mathbf{x};a) = a^{-|\mathbf{m}|}e^{ra}\frac{\left(\frac{n}{r}\right)_{\mathbf{m}}}{d_{\mathbf{m}}}\delta_{\mathbf{m},\mathbf{n}} \geq 0. \tag{3.31}$$

(3) *For $0<p<1$, we obtain*

$$\sum_{\mathbf{x}\subset N}\binom{N}{\mathbf{x}}p^{|\mathbf{x}|}(1-p)^{rN-|\mathbf{x}|}K_{\mathbf{m}}(\mathbf{x};p,N)K_{\mathbf{n}}(\mathbf{x};p,N) = \left(\frac{1-p}{p}\right)^{|\mathbf{m}|}\binom{N}{\mathbf{m}}^{-1}\delta_{\mathbf{m},\mathbf{n}} \geq 0. \tag{3.32}$$

Proof. (1) From (3.28) and (3.22), we have

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) &= (1-c)^{r\alpha}\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) \\
&\quad \cdot d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}c^{|\mathbf{x}|}\Delta(e-cw)^{-\alpha}\Phi_{\mathbf{x}}((e-w)(e-cw)^{-1}) \\
&= \sum_{\mathbf{m},\mathbf{n}\in\mathscr{P}}(1-c)^{r\alpha}d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}d_{\mathbf{n}}\frac{(\alpha)_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}}c^{|\mathbf{n}|}\Phi_{\mathbf{m}}(z)\Phi_{\mathbf{n}}(w) \\
&\quad \cdot \left\{\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}c^{|\mathbf{x}|}M_{\mathbf{m}}(\mathbf{x};\alpha,c)M_{\mathbf{n}}(\mathbf{x};\alpha,c)\right\}.
\end{aligned}$$

Therefore, by comparing the coefficients of $\Phi_{\mathbf{m}}(z)\Phi_{\mathbf{n}}(w)$ on both sides of this equation, we obtain (3.30).

(2) From (3.29) and (3.23), we derive

$$\begin{aligned}\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) &= e^{-ra}\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{a^{|\mathbf{m}|}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}C_{\mathbf{m}}(\mathbf{x};a)\Phi_{\mathbf{m}}(z)\\ &\quad\cdot d_{\mathbf{x}}\frac{a^{|\mathbf{x}|}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}e^{\operatorname{tr} w}\Phi_{\mathbf{x}}\left(e-\frac{1}{a}w\right)\\ &= \sum_{\mathbf{m},\mathbf{n}\in\mathscr{P}} e^{-ra}d_{\mathbf{m}}\frac{a^{|\mathbf{m}|}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}d_{\mathbf{n}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{n}}}\Phi_{\mathbf{m}}(z)\Phi_{\mathbf{n}}(w)\\ &\quad\cdot\left\{\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{a^{|\mathbf{x}|}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}C_{\mathbf{m}}(\mathbf{x};a)C_{\mathbf{n}}(\mathbf{x};a)\right\}.\end{aligned}$$

Then, by comparing the coefficients of $\Phi_{\mathbf{m}}(z)\Phi_{\mathbf{n}}(w)$, we have the conclusion.

(3) In (3.28), taking $\alpha=-N$, one has

$$\begin{aligned}\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(-N)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(-z) &= \sum_{\mathbf{m}\subset N}\binom{N}{\mathbf{m}}\Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z)\\ &=(1-c)^{-rN}\sum_{\mathbf{x},\mathbf{m}\subset N} d_{\mathbf{m}}\frac{(-N)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}M_{\mathbf{m}}(\mathbf{x};-N,c)\Phi_{\mathbf{m}}(-z)\\ &\quad\cdot d_{\mathbf{x}}\frac{(-N)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}c^{|\mathbf{x}|}\Delta(e-cw)^{N}\Phi_{\mathbf{x}}((e-w)(e-cw)^{-1}).\end{aligned}$$

The first equality follows from (2.28). Since the above sum is finite, we can put $c=\frac{p}{p-1}$, $(0<p<1)$. Hence,

$$\begin{aligned}\sum_{\mathbf{m}\subset N}\binom{N}{\mathbf{m}}\Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) &= (1-p)^{rN}\sum_{\mathbf{x},\mathbf{m}\subset N}\binom{N}{\mathbf{m}}K_{\mathbf{m}}(\mathbf{x};p,N)\Phi_{\mathbf{m}}(z)\binom{N}{\mathbf{x}}\left(\frac{p}{1-p}\right)^{|\mathbf{x}|}\\ &\quad\cdot\Delta\left(e+\frac{p}{1-p}w\right)^{N}\Phi_{\mathbf{x}}\left((e-w)\left(e+\frac{p}{1-p}w\right)^{-1}\right).\end{aligned}$$

From (3.24), we have

$$\begin{aligned}&\Delta\left(e+\frac{p}{1-p}w\right)^{N}\Phi_{\mathbf{x}}\left((e-w)\left(e+\frac{p}{1-p}w\right)^{-1}\right)\\ &=\sum_{\mathbf{n}\subset N}\binom{N}{\mathbf{n}}K_{\mathbf{n}}(\mathbf{x};p,N)\left(\frac{p}{1-p}\right)^{|\mathbf{n}|}\Phi_{\mathbf{n}}(w).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\mathbf{m}\subset N}\binom{N}{\mathbf{m}}\Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) &= \sum_{\mathbf{m},\mathbf{n}\subset N}\binom{N}{\mathbf{m}}\left(\frac{p}{1-p}\right)^{|\mathbf{n}|}\binom{N}{\mathbf{n}}\Phi_{\mathbf{m}}(z)\Phi_{\mathbf{n}}(w)\\ &\quad\cdot\left\{\sum_{\mathbf{x}\subset N}\binom{N}{\mathbf{x}}p^{|\mathbf{x}|}(1-p)^{rN-|\mathbf{x}|}K_{\mathbf{m}}(\mathbf{x};p,N)K_{\mathbf{n}}(\mathbf{x};p,N)\right\}.\end{aligned}$$

■

3.4. Difference equations and recurrence relations. In this subsection, we derive the difference equations and recurrence formulas for our polynomials from (2.47), Lemma 2.14 and (1) of Theorem 3.10.

Theorem 3.12. (1) *For* $\mathbf{x}, \mathbf{m} \in \mathscr{P}$, *we have*

$$\begin{aligned} & d_{\mathbf{x}}(c-1)|\mathbf{m}|M_{\mathbf{m}}(\mathbf{x};\alpha,c) \\ &\quad = \sum_{j=1}^{r} d_{\mathbf{x}+\epsilon_j}\tilde{a}_j(-\mathbf{x}-\epsilon_j)\left(x_j+\alpha-\frac{d}{2}(j-1)\right)cM_{\mathbf{m}}(\mathbf{x}+\epsilon_j;\alpha,c) \\ &\quad\quad - \sum_{j=1}^{r} d_{\mathbf{x}}(x_j+(x_j+\alpha)c)M_{\mathbf{m}}(\mathbf{x};\alpha,c) \\ &\quad\quad + \sum_{j=1}^{r} d_{\mathbf{x}-\epsilon_j}\tilde{a}_j(\mathbf{x}-\epsilon_j)\left(x_j+\frac{d}{2}(r-j)\right)M_{\mathbf{m}}(\mathbf{x}-\epsilon_j;\alpha,c). \end{aligned} \tag{3.33}$$

(2) *For* $\mathbf{x}, \mathbf{m} \in \mathscr{P}$, *we have*

$$\begin{aligned} -d_{\mathbf{x}}|\mathbf{m}|C_{\mathbf{m}}(\mathbf{x};a) &= \sum_{j=1}^{r} d_{\mathbf{x}+\epsilon_j}\tilde{a}_j(-\mathbf{x}-\epsilon_j)aC_{\mathbf{m}}(\mathbf{x}+\epsilon_j;a) \\ &\quad - \sum_{j=1}^{r} d_{\mathbf{x}}(x_j+a)C_{\mathbf{m}}(\mathbf{x};a) \\ &\quad + \sum_{j=1}^{r} d_{\mathbf{x}-\epsilon_j}\tilde{a}_j(\mathbf{x}-\epsilon_j)\left(x_j+\frac{d}{2}(r-j)\right)C_{\mathbf{m}}(\mathbf{x}-\epsilon_j;a). \end{aligned} \tag{3.34}$$

(3) *For* $\mathbf{x}, \mathbf{m} \subset N$, *we have*

$$\begin{aligned} & -d_{\mathbf{x}}|\mathbf{m}|K_{\mathbf{m}}(\mathbf{x};p,N) \\ &\quad = \sum_{j=1}^{r} d_{\mathbf{x}+\epsilon_j}\tilde{a}_j(-\mathbf{x}-\epsilon_j)\left(N-x_j+\frac{d}{2}(j-1)\right)pK_{\mathbf{m}}(\mathbf{x}+\epsilon_j;p,N) \\ &\quad\quad - \sum_{j=1}^{r} d_{\mathbf{x}}(p(N-x_j)+x_j(1-p))K_{\mathbf{m}}(\mathbf{x};p,N) \\ &\quad\quad + \sum_{j=1}^{r} d_{\mathbf{x}-\epsilon_j}\tilde{a}_j(\mathbf{x}-\epsilon_j)\left(x_j+\frac{d}{2}(r-j)\right)(1-p)K_{\mathbf{m}}(\mathbf{x}-\epsilon_j;p,N). \end{aligned} \tag{3.35}$$

Proof. (1) Let us apply operator $\frac{c-1}{2}e^{\frac{1+c}{1-c}\operatorname{tr} u}D_{\alpha}^{(1)}$ to both sides of (3.27). Since

$D_\alpha^{(1)}\psi_{\mathbf{m}}^{(\alpha)}(u) = 2|\mathbf{m}|\psi_{\mathbf{m}}^{(\alpha)}(u)$, we have

$$\begin{aligned}\frac{c-1}{2}e^{\frac{1+c}{1-c}\operatorname{tr} u}D_\alpha^{(1)}\left(\sum_{\mathbf{m}\in\mathscr{P}}\psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)\right) &= \sum_{\mathbf{m}\in\mathscr{P}}(c-1)e^{\frac{1+c}{1-c}\operatorname{tr} u}|\mathbf{m}|\psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)\\ &= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\left(\frac{2c}{1-c}\right)^{|\mathbf{x}|}\Phi_{\mathbf{x}}(u)\Phi_{\mathbf{m}}(z)\\ &\quad\cdot d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}(c-1)|\mathbf{m}|M_{\mathbf{m}}(\mathbf{x};\alpha,c).\end{aligned}$$

On the other hand, by (2.53), we have

$$\begin{aligned}&\frac{c-1}{2}e^{\frac{1+c}{1-c}\operatorname{tr} u}D_\alpha^{(1)}\left(\sum_{\mathbf{m}\in\mathscr{P}}\psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)\right)\\ &\quad= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z)d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\left(\frac{2c}{1-c}\right)^{|\mathbf{x}|}\frac{c-1}{2}\\ &\qquad\cdot e^{\frac{1+c}{1-c}\operatorname{tr} u}D_\alpha^{(1)}(e^{-\frac{1+c}{1-c}\operatorname{tr} u}\Phi_{\mathbf{x}}(u))\\ &\quad= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z)d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\left(\frac{2c}{1-c}\right)^{|\mathbf{x}|}\\ &\qquad\cdot\left\{\frac{2c}{1-c}\sum_{j=1}^r\tilde{a}_j(\mathbf{x})\Phi_{\mathbf{x}+\epsilon_j}(u) - \sum_{j=1}^r(x_j+(x_j+\alpha)c)\Phi_{\mathbf{x}}(u)\right.\\ &\qquad\quad\left.+\frac{1-c}{2}\sum_{j=1}^r\left(x_j+\frac{d}{2}(r-j)\right)\left(x_j+\alpha-1-\frac{d}{2}(j-1)\right)\tilde{a}_j(-\mathbf{x})\Phi_{\mathbf{x}-\epsilon_j}(u)\right\}\\ &\quad= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\left(\frac{2c}{1-c}\right)^{|\mathbf{x}|}\Phi_{\mathbf{x}}(u)\Phi_{\mathbf{m}}(z)\\ &\qquad\cdot\left\{\sum_{j=1}^r d_{\mathbf{x}+\epsilon_j}\tilde{a}_j(-\mathbf{x}-\epsilon_j)\frac{x_j+1+\frac{d}{2}(r-j)}{\left(\frac{n}{r}\right)_{\mathbf{x}+\epsilon_j}}\left(x_j+\alpha-\frac{d}{2}(j-1)\right)\right.\\ &\qquad\quad\cdot cM_{\mathbf{m}}(\mathbf{x}+\epsilon_j;\alpha,c)\\ &\qquad\quad-\sum_{j=1}^r d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}(x_j+(x_j+\alpha)c)M_{\mathbf{m}}(\mathbf{x};\alpha,c)\\ &\qquad\quad\left.+\sum_{j=1}^r d_{\mathbf{x}-\epsilon_j}\tilde{a}_j(\mathbf{x}-\epsilon_j)\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}-\epsilon_j}}M_{\mathbf{m}}(\mathbf{x}-\epsilon_j;\alpha,c)\right\}.\end{aligned}$$

Finally, the conclusion is obtained by

$$\left(\frac{n}{r}\right)_{\mathbf{x}+\epsilon_j} = \left(x_j+1+\frac{d}{2}(r-j)\right)\left(\frac{n}{r}\right)_{\mathbf{x}}$$

and comparing the coefficients in the above.

(2) Put $c = \frac{a}{a+\alpha}$ in (3.33) and take the limit as $\alpha \to \infty$. Then, by (3.14), we have the conclusion.
(3) Put $c = \frac{p}{p-1}, \alpha = -N$ and multiply $1-p$ in (3.33). Then, by (3.13), we have the conclusion. ■

The recurrence formulas follow immediately from Theorem 3.12 and Proposition 3.2.

Theorem 3.13. (1) *For* $\mathbf{x}, \mathbf{m} \in \mathscr{P}$, *we have*

$$\begin{aligned} &d_{\mathbf{m}}(c-1)|\mathbf{x}|M_{\mathbf{m}}(\mathbf{x};\alpha,c) \\ &\quad = \sum_{j=1}^{r} d_{\mathbf{m}+\epsilon_j}\tilde{a}_j(-\mathbf{m}-\epsilon_j)\left(m_j+\alpha-\frac{d}{2}(j-1)\right)cM_{\mathbf{m}+\epsilon_j}(\mathbf{x};\alpha,c) \\ &\qquad - \sum_{j=1}^{r} d_{\mathbf{m}}(m_j+(m_j+\alpha)c)M_{\mathbf{m}}(\mathbf{x};\alpha,c) \\ &\qquad + \sum_{j=1}^{r} d_{\mathbf{m}-\epsilon_j}\tilde{a}_j(\mathbf{m}-\epsilon_j)\left(m_j+\frac{d}{2}(r-j)\right)M_{\mathbf{m}-\epsilon_j}(\mathbf{x};\alpha,c). \end{aligned} \tag{3.36}$$

(2) *For* $\mathbf{x}, \mathbf{m} \in \mathscr{P}$, *we have*

$$\begin{aligned} -d_{\mathbf{m}}|\mathbf{x}|C_{\mathbf{m}}(\mathbf{x};a) &= \sum_{j=1}^{r} d_{\mathbf{m}+\epsilon_j}\tilde{a}_j(-\mathbf{m}-\epsilon_j)aC_{\mathbf{m}+\epsilon_j}(\mathbf{x};a) \\ &\quad - \sum_{j=1}^{r} d_{\mathbf{m}}(m_j+a)C_{\mathbf{m}}(\mathbf{x};a) \\ &\quad + \sum_{j=1}^{r} d_{\mathbf{m}-\epsilon_j}\tilde{a}_j(\mathbf{m}-\epsilon_j)\left(m_j+\frac{d}{2}(r-j)\right)C_{\mathbf{m}-\epsilon_j}(\mathbf{x};a). \end{aligned} \tag{3.37}$$

(3) *For* $\mathbf{x}, \mathbf{m} \subset N$, *we have*

$$\begin{aligned} &- d_{\mathbf{m}}|\mathbf{x}|K_{\mathbf{m}}(\mathbf{x};p,N) \\ &\quad = \sum_{j=1}^{r} d_{\mathbf{m}+\epsilon_j}\tilde{a}_j(-\mathbf{m}-\epsilon_j)\left(N-m_j+\frac{d}{2}(j-1)\right)pK_{\mathbf{m}+\epsilon_j}(\mathbf{x};p,N) \\ &\qquad - \sum_{j=1}^{r} d_{\mathbf{m}}(p(N-m_j)+m_j(1-p))K_{\mathbf{m}}(\mathbf{x};p,N) \\ &\qquad + \sum_{j=1}^{r} d_{\mathbf{m}-\epsilon_j}\tilde{a}_j(\mathbf{m}-\epsilon_j)\left(m_j+\frac{d}{2}(r-j)\right)(1-p)K_{\mathbf{m}-\epsilon_j}(\mathbf{x};p,N). \end{aligned} \tag{3.38}$$

3.5. Determinant formulas. In this subsection, we assume $d = 2$ (In particular, we remark that $\frac{n}{r} = r$). In this case the spherical polynomials $\Phi_{\mathbf{m}}$ are proportional to the Schur polynomials $s_{\mathbf{m}}$ (recall (2.19)). Further, there are some determinant formulas for the multivariate Meixner, Charlier and Krawtchouk polynomials. Before stating the main theorem, we provide the following lemma needed to prove the determinant formulas.

Lemma 3.14 ([12] Theorem 1.2.1). *Consider r power series of single variable $z \in \mathbb{C}$*

$$f_\mu(z) = \sum_{m \geq 0} A_m^{(\mu)} z^m \quad (\mu = 1, \cdots, r).$$

Then,

$$\frac{\det\,(f_\mu(z_\nu))}{V(z_1, \ldots, z_r)} = \sum_{\mathbf{m} \in \mathscr{P}} A_{\mathbf{m}} s_{\mathbf{m}}(z_1, \ldots, z_r), \tag{3.39}$$

where $V(z_1, \ldots, z_r)$ denote the Vandermonde determinant

$$V(z_1, \ldots, z_r) := \prod_{1 \leq \mu < \nu \leq r} (z_\mu - z_\nu),$$

and

$$A_{\mathbf{m}} := \det\,(A^{(\nu)}_{m_\mu + r - \mu}).$$

Theorem 3.15. (1) *For any $\mathbf{m}, \mathbf{x} \in \mathscr{P}$, we obtain*

$$M_{\mathbf{m}}(\mathbf{x}; \alpha, c) = \frac{1}{\delta!} \frac{\left(1 - \frac{1}{c}\right)^{-\frac{r(r-1)}{2}}}{s_{\mathbf{m}}(1, \ldots, 1) s_{\mathbf{x}}(1, \ldots, 1)} \prod_{j=1}^{r} (\alpha - r + 1)_{j-1} \cdot \det\,(M_{m_\mu + r - \mu}(x_\nu + r - \nu; \alpha - r + 1, c)). \tag{3.40}$$

Here, $M_{m_\mu + r - \mu}(x_\nu + r - \nu; \alpha - r + 1, c)$ is a one variable Meixner polynomial.
(2) *For any $\mathbf{m}, \mathbf{x} \in \mathscr{P}$, we obtain*

$$C_{\mathbf{m}}(\mathbf{x}; a) = \frac{1}{\delta!} \frac{(-a)^{\frac{r(r-1)}{2}}}{s_{\mathbf{m}}(1, \ldots, 1) s_{\mathbf{x}}(1, \ldots, 1)} \det\,(C_{m_\mu + r - \mu}(x_\nu + r - \nu; a)). \tag{3.41}$$

Here, $C_{m_\mu + r - \mu}(x_\nu + r - \nu; a)$ is a one variable Charlier polynomial.
(3) *For any $\mathbf{m}, \mathbf{x} \subset N = (N, \ldots, N)$, we obtain*

$$K_{\mathbf{m}}(\mathbf{x}; p, N) = \frac{1}{\delta!} \frac{p^{\frac{r(r-1)}{2}}}{s_{\mathbf{m}}(1, \ldots, 1) s_{\mathbf{x}}(1, \ldots, 1)} \prod_{j=1}^{r} (-N - r + 1)_{j-1} \cdot \det\,(K_{m_\mu + r - \mu}(x_\nu + r - \nu; p, N + r - 1)). \tag{3.42}$$

Here, $K_{m_\mu + r - \mu}(x_\nu + r - \nu; p, N + r - 1)$ is a one variable Krawtchouk polynomial.

Proof. (1) Let put $z = \sum_{j=1}^{r} z_j c_j, (0 < z_1, \ldots, z_r < 1)$. Since

$$s_{\mathbf{m}}(z_1, \ldots, z_r) = s_{\mathbf{m}}(1, \ldots, 1) \Phi_{\mathbf{m}}(z),$$

the generating function of the multivariate Meixner polynomials (3.22) express

$$\begin{aligned}\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{(r)_{\mathbf{m}}}M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) &= \Delta(e-z)^{-\alpha}\Phi_{\mathbf{x}}\left(\left(e-\frac{1}{c}z\right)(e-z)^{-1}\right)\\ &= \frac{1}{s_{\mathbf{x}}(1,\dots,1)}\det\left((1-z_\mu)^{-\alpha}\left(\frac{1-\frac{1}{c}z_\mu}{1-z_\mu}\right)^{x_\nu+r-\nu}\right)\\ &\quad\cdot\frac{1}{V\left(\frac{1-\frac{1}{c}z_1}{1-z_1},\dots,\frac{1-\frac{1}{c}z_r}{1-z_r}\right)}.\end{aligned}$$

Further, noticing that

$$\frac{1-\frac{1}{c}z_\mu}{1-z_\mu}-\frac{1-\frac{1}{c}z_\nu}{1-z_\nu}=\left(1-\frac{1}{c}\right)\frac{z_\mu-z_\nu}{(1-z_\mu)(1-z_\nu)},$$

we obtain

$$\begin{aligned}\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{(r)_{\mathbf{m}}}M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) &= \frac{\left(1-\frac{1}{c}\right)^{-\frac{r(r-1)}{2}}}{s_{\mathbf{x}}(1,\dots,1)}\\ &\quad\cdot\frac{\det\left((1-z_\mu)^{-(\alpha-r+1)}\left(\frac{1-\frac{1}{c}z_\mu}{1-z_\mu}\right)^{x_\nu+r-\nu}\right)}{V(z_1,\dots,z_r)}.\end{aligned}$$

Here, by (3.22), we remark

$$\begin{aligned}f_\nu(z_\mu) &= (1-z_\mu)^{-(\alpha-r+1)}\left(\frac{1-\frac{1}{c}z_\mu}{1-z_\mu}\right)^{x_\nu+r-\nu}\\ &= \sum_{m\geq 0}\frac{(\alpha-r+1)_m}{m!}M_m(x_\nu+r-\nu;\alpha-r+1,c)z_\mu.\end{aligned}$$

Therefore, we expand the above determinant expression in Schur function series by using Lemma 3.14.

$$\begin{aligned}\sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}}\frac{(\alpha)_{\mathbf{n}}}{(r)_{\mathbf{n}}}M_{\mathbf{n}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{n}}(z) &= \frac{1}{\delta!}\frac{\left(1-\frac{1}{c}\right)^{-\frac{r(r-1)}{2}}}{s_{\mathbf{x}}(1,\dots,1)}\prod_{j=1}^{r}(\alpha-r+1)_{j-1}\\ &\quad\cdot\sum_{\mathbf{m}\in\mathscr{P}}\frac{(\alpha)_{\mathbf{m}}}{(r)_{\mathbf{m}}}\det\big(M_{m_\mu+r-\mu}(x_\nu+r-\nu;\alpha-r+1,c)\big)\\ &\quad\cdot s_{\mathbf{m}}(1,\dots,1)\Phi_{\mathbf{m}}(z).\end{aligned}$$

Finally, by comparing of $\Phi_{\mathbf{m}}(z)$ on the above equation for $\mathbf{m}\in\mathscr{P}$, we obtain (3.40). Since both sides of (3.40) which are rational functions for α and c hold for $\alpha\in\mathbb{C}\setminus\mathbb{Z}_{\leq r-1}, c\neq 0$.

(2) From (3.14) and (3.40), we have

$$
\begin{aligned}
C_{\mathbf{m}}(\mathbf{x}; a) &= \lim_{\alpha \to \infty} M_{\mathbf{m}} \left(\mathbf{x}; \alpha, \frac{a}{a+\alpha} \right) \\
&= \frac{1}{\delta !} \frac{(-a)^{\frac{r(r-1)}{2}}}{s_{\mathbf{m}}(1, \dots, 1) s_{\mathbf{x}}(1, \dots, 1)} \\
&\quad \cdot \lim_{\alpha \to \infty} \prod_{j=1}^{r} \frac{(\alpha - r + 1)_{j-1}}{\alpha^{j-1}} \det \left(M_{m_\mu + r - \mu} \left(x_\nu + r - \nu; \alpha - r + 1, \frac{a}{a+\alpha} \right) \right) \\
&= \frac{1}{\delta !} \frac{(-a)^{\frac{r(r-1)}{2}}}{s_{\mathbf{m}}(1, \dots, 1) s_{\mathbf{x}}(1, \dots, 1)} \det \left(C_{m_\mu + r - \mu}(x_\nu + r - \nu; a) \right).
\end{aligned}
$$

(3) From (3.13) and (3.40), we have

$$
\begin{aligned}
K_{\mathbf{m}}(\mathbf{x}; p, N) &= M_{\mathbf{m}} \left(\mathbf{x}; -N, \frac{p}{p-1} \right) \\
&= \frac{1}{\delta !} \frac{p^{\frac{r(r-1)}{2}}}{s_{\mathbf{m}}(1, \dots, 1) s_{\mathbf{x}}(1, \dots, 1)} \prod_{j=1}^{r} (-N - r + 1)_{j-1} \\
&\quad \cdot \det \left(M_{m_\mu + r - \mu} \left(x_\nu + r - \nu; -N - r + 1, \frac{p}{p-1} \right) \right).
\end{aligned}
$$

Here, by using (3.13) again,

$$
M_{m_\mu + r - \mu} \left(x_\nu + r - \nu; -N - r + 1, \frac{p}{p-1} \right) = K_{m_\mu + r - \mu}(x_\nu + r - \nu; p, N + r - 1).
$$

Therefore, we obtain the conclusion. ∎

4. Concluding remarks

Interesting problems remain that are related to the multivariate Meixner, Charlier and Krawtchouk polynomials. First, we may consider a generalization of our discrete orthogonal polynomials for an arbitrary real value of multiplicity $d > 0$. Actually, we can consider the multivariate Meixner, Charlier and Krawtchouk polynomials and their orthogonality without using analysis on the symmetric cones as follows.

Let $n := r + \frac{d}{2} r(r-1)$, $d > 0$

$$
\begin{aligned}
d_{\mathbf{m}} &:= \prod_{j=1}^{r} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} j\right) \Gamma\left(\frac{d}{2}(j-1) + 1\right)} \\
&\quad \cdot \prod_{1 \le p < q \le r} \left(m_p - m_q + \frac{d}{2}(q - p) \right) \frac{\Gamma\left(m_p - m_q + \frac{d}{2}(q - p + 1)\right)}{\Gamma\left(m_p - m_q + \frac{d}{2}(q - p - 1) + 1\right)}.
\end{aligned}
$$

$$
\Gamma_\Omega(\mathbf{s}) := (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(s_j - \frac{d}{2}(j-1) \right),
$$

$$
(\mathbf{s})_{\mathbf{k}} := \prod_{j=1}^{r} \left(s_j - \frac{d}{2}(j-1) \right)_{k_j}.
$$

Further, $P_{\mathbf{k}}^{(\frac{2}{d})}(\lambda_1,\ldots,\lambda_r)$ is an r-variable Jack polynomial and

$$\Phi_{\mathbf{k}}^{(d)}(\lambda_1,\ldots,\lambda_r) := \frac{P_{\mathbf{k}}^{(\frac{2}{d})}(\lambda_1,\ldots,\lambda_r)}{P_{\mathbf{k}}^{(\frac{2}{d})}(1,\ldots,1)}. \tag{4.1}$$

Furthermore, we introduce the generalized (Jack) binomial coefficients based on [18] by

$$\Phi_{\mathbf{m}}^{(d)}(1+\lambda_1,\ldots,1+\lambda_r) = \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \Phi_{\mathbf{k}}^{(d)}(\lambda_1,\ldots,\lambda_r).$$

Definition 4.1. We define the generalized multivariate Meixner, Charlier and Krawtchouk polynomials by

$$M_{\mathbf{m}}^{(d)}(\mathbf{x};\alpha,c) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{(\alpha)_{\mathbf{k}}} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \binom{\mathbf{x}}{\mathbf{k}}_{\frac{d}{2}} \left(1-\frac{1}{c}\right)^{|\mathbf{k}|}, \tag{4.2}$$

$$C_{\mathbf{m}}^{(d)}(\mathbf{x};a) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \left(\frac{n}{r}\right)_{\mathbf{k}} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \binom{\mathbf{x}}{\mathbf{k}}_{\frac{d}{2}} \left(-\frac{1}{a}\right)^{|\mathbf{k}|}, \tag{4.3}$$

$$K_{\mathbf{m}}^{(d)}(\mathbf{x};p,N) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{(-N)_{\mathbf{k}}} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \binom{\mathbf{x}}{\mathbf{k}}_{\frac{d}{2}} \left(\frac{1}{p}\right)^{|\mathbf{k}|} \quad (\mathbf{m}\subset N=(N,\ldots,N)). \tag{4.4}$$

By the definitions, Proposition 3.2 and 3.3 also hold for the generalized multivariate Meixner, Charlier and Krawtchouk polynomials. Therefore, we think the following conjecture is natural.

Conjecture 4.2. Generating functions, orthogonality, difference equations and recurrence formulas also hold for the generalized multivariate Meixner, Charlier and Krawtchouk polynomials, as in Theorems 3.7, 3.11, 3.12 and 3.13 respectively. Here, we consider $\Delta(e-z)=(1-z_1)\cdots(1-z_r)$.

We remark that when $d=1,2,4$ or $r=2, d\in\mathbb{Z}_{>0}$ or $r=3, d=8$, this conjecture is proved by this paper and the classification of irreducible symmetric cones. However, it may be necessary to consider an algebraic treatment to prove the general case. In particular, since the difference equation for the multivariate Meixner polynomials is equivalent to the differential equation for the multivariate Laguerre polynomials which is explained by the degenerate double affine Hecke algebra [14], we expect the existence of a particular algebraic structure related to this algebra for our polynomials. Once we obtain such an interpretation, we may not only succeed in proving the above conjecture but also in providing further generalizations of our polynomials associated with root systems.

It is also valuable to give a group theoretic picture of our multivariate discrete orthogonal polynomials. In the one variable case, there are many geometric interpretations for these polynomials [22], [23]. Moreover, in the multivariate case for the Aomoto-Gelfand hypergeometric series, such group theoretic interpretations have recently been studied [8], [9]. On the other hand, since our multivariate discrete orthogonal polynomials have many rich properties which are generalizations of the one variable case, they are considered to be a good multivariate analogue of the Meixner, Charlier and Krawtchouk polynomials. Hence, for our multivariate discrete orthogonal polynomials, it seems that there are some group theoretic interpretations as some matrix elements or some spherical functions etc. We are also interested in a connection between our multivariate discrete orthogonal polynomials and the Aomoto-Gelfand type.

We are interested in whether we can apply our method to other discrete orthogonal polynomials, for example, the Hahn polynomials which are special orthogonal polynomials in the Askey scheme [15],

$$\begin{aligned} Q_m(x;\alpha,\beta,N) &= {}_3F_2\left(\begin{matrix}-m, m+\alpha+\beta+1, -x \\ \alpha+1, -N\end{matrix}; 1\right) \\ &= \sum_{k=0}^{m} \frac{k!}{(-N)_k} \frac{(m+\alpha+\beta+1)_k}{(\alpha+1)_k} \binom{m}{k}\binom{x}{k} \quad (m=0,1,\cdots,N). \end{aligned}$$

Namely, by considering “some generating functions of the generating functions” for these discrete orthogonal polynomials, we expect to obtain a correspondence between the Hahn polynomials and other orthogonal polynomials, for example, the Jacobi polynomials.

Finally, we would like to raise the issue of applications of our multivariate Meixner, Charlier and Krawtchouk polynomials. The standard Meixner, Charlier and Krawtchouk polynomials of single discrete variable have found numerous applications in combinatorics, stochastic processes, probability theory and mathematical physics (for their reference, see the introduction in [9]). Hence, we hope that our multivariate polynomials can be applied to various situations and we intend to investigate these in research tasks in the future.

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