

Two-Step and Three-Step Nilpotent Lie Algebras Constructed from Schreier Graphs

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Abstract. We associate a two-step nilpotent Lie algebra to an arbitrary Schreier graph. We then use properties of the Schreier graph to determine necessary and sufficient conditions for this Lie algebra to extend to a three-step nilpotent Lie algebra. As an application, if we start with pairs of non-isomorphic Schreier graphs coming from Gassmann-Sunada triples, we prove that the pair of associated two-step nilpotent Lie algebras are always isometric. In contrast, we use a well-known pair of Schreier graphs to show that the associated three-step nilpotent extensions need not be isometric.

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1. Introduction

The purpose of this paper is to introduce a new method for associating two-step nilpotent Lie algebras with Schreier graphs and then extend these Lie algebras (in certain cases) to three-step nilpotent Lie algebras. Also, we compare the pairs of resultant Lie algebras when associated with Gassmann-Sunada triples.

In 2004, S. G. Dani and M. G. Mainkar first presented a method for constructing two-step nilpotent Lie algebras from simple graphs [3]. They used the two-step nilpotent construction to find properties of a graph that would result in the constructed nilmanifold admitting Anosov automorphisms. J. Lauret and C. Will used this construction to find examples of nonisometric Einstein solvmanifolds [8], and H. Pouseele and P. Tirao used the construction to consider symplectic nilmanifolds [11]. Mainkar also proved that for simple graphs, the resulting Lie algebras are isomorphic if and only if the graphs are isomorphic [10], and in [9], she extended this construction to k -step nilpotent Lie algebras. Also, V. Grantcharov is currently working on extending the Dani-Mainkar construction on simple graphs to three-step solvable Lie algebras [6]. In the Dani-Mainkar construction, each vertex and each edge of the graph correspond to distinct elements in the Lie algebra;

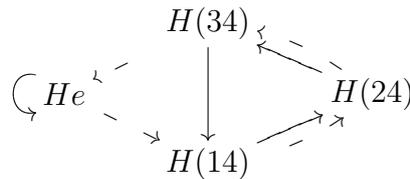
therefore for large graphs, the corresponding dimension of the Lie algebra is also large. For the higher-step construction, the dimension of the constructed Lie algebras grows more rapidly.

The focus of this paper is on Schreier graphs because of their inherent group structure. J. L. Gross proved that every connected regular graph of even degree is a Schreier graph, [7]. Schreier graphs; however, are often non-simple directed graphs, in which case the Dani-Mainkar construction is not defined. We therefore introduce a new method for associating Lie algebras with Schreier graphs, as suggested by C. S. Gordon. In §2, we discuss the definitions and notations that will be used in this paper. In §3, we detail this new construction of a two-step nilpotent Lie algebra associated with an arbitrary Schreier graph. We also provide necessary and sufficient conditions on the graph for this construction to extend to a three-step nilpotent Lie algebra. As an application, in §4 we prove that for any pair of Schreier graphs associated to a Gassmann-Sunada triple, the resultant two-step nilpotent Lie algebras are isometric, but we give an example where the pair of three-step nilpotent Lie algebras are non-isometric.

2. Background and Notation

Definition 2.1. [2] Let G be a finite group and H a subgroup of G . Let $C := \{z_1, \dots, z_c, z_1^{-1}, \dots, z_c^{-1}\}$ be a generating set of G not containing the identity that is closed under inverses, and let $C_{pos} := \{z_1, \dots, z_c\}$. The *Schreier graph* of G relative to H and C , written $\mathcal{G}(G, H, C)$ or simply \mathcal{G} if understood in context, is a directed edge-labeled graph defined by the following. The vertices of \mathcal{G} consist of the set of right cosets, $V(\mathcal{G}) = \{Hg : g \in G\}$. The edges consist of the set of ordered pairs $E(\mathcal{G}) = \{(Hg, Hgz_i^{-1}) : z_i \in C_{pos}\}$, and each edge (Hg, Hgz_i^{-1}) is given the label z_i .

Example 2.2. Let $G = S_4$, $H = S_3$, and $C_{pos} = \{(123), (1234)\}$. Then the Schreier graph \mathcal{G} with respect to H and C is



where the solid lines are with label (123) and dotted lines are with label (1234).

Remark 2.3. Note that while a Schreier graph is defined if C_{pos} contains an element of order 2, the edges associated to those elements will become trivial elements in the Lie algebras we construct in §3. Hence, in what follows, we assume our generating set C does not contain order 2 elements, i.e. $z \neq z^{-1}$ for all $z \in C$.

Remark 2.4. The structure of a Schreier graph implies that the group G acts on $V(\mathcal{G})$ by right inverse multiplication. To see this, we define $\alpha(z_i) : V(\mathcal{G}) \mapsto$

$V(\mathcal{G})$ for $z_i \in C$ by

$$\alpha(z_i)(Hg) = Hgz_i^{-1} \text{ for all } z_i \in C.$$

Then α extends from C to G because C generates G .

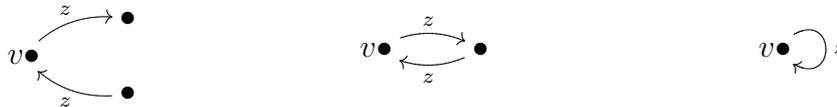
Definition 2.5. For a directed graph, a *walk* of length q from vertex v to w is a sequence of $q + 1$ vertices (and therefore q edges) where successive vertices in the sequence are connected by a directed edge in $E(\mathcal{G})$. If these vertices are all distinct, except possibly v and w , then this is called a *path* of length q , or a *q-path*. If $v = w$, then this path is called *closed*. We will denote q -paths by a $(q + 1)$ -tuple of vertices, $(v_1, v_2, \dots, v_{q+1})$ where $(v_i, v_{i+1}) \in E(\mathcal{G})$ for $i = 1, \dots, q$.

The following two facts about properties of a Schreier graph will be important in the proofs of the main theorems in §3 and §4.

Remark 2.6. Because the edges of a Schreier graph are associated with generators of a finite group, Schreier graphs will be the union of closed paths of a single label, where the length of a path with every edge labeled $z_i \in C_{pos}$ is less than or equal to the order of the generating element z_i .

Remark 2.7. If $|C_{pos}| = c$, then the Schreier graph \mathcal{G} will be $2c$ -regular where each vertex has a directed edge labeled z_i going out of the vertex and one going into the vertex, $i = 1, \dots, c$. This gives three different possibilities for each vertex v and each generator (and hence each label) z :

1. $\alpha(z)(v) \neq \alpha(z^{-1})(v)$
2. $\alpha(z)(v) = \alpha(z^{-1})(v) \neq v$,
3. $\alpha(z)(v) = \alpha(z^{-1})(v) = v$



The following two definitions for Lie algebras are standard, see e.g. P. Eberlein [4].

Definition 2.8. A Lie algebra \mathfrak{n} is k -step *nilpotent* if $\mathfrak{n}^{(k)} = 0$ for some $k \in \mathbb{Z}^+$ but $\mathfrak{n}^{(k-1)} \neq 0$, where $\mathfrak{n}^{(0)} = \mathfrak{n}$ and $\mathfrak{n}^{(k)} = [\mathfrak{n}^{(k-1)}, \mathfrak{n}]$, $k \in \mathbb{Z}^+$.

Definition 2.9. Given two Lie algebras, $(\mathfrak{n}_1, [,]_1)$ and $(\mathfrak{n}_2, [,]_2)$, a map $\phi : \mathfrak{n}_1 \rightarrow \mathfrak{n}_2$ is an *isomorphism* if ϕ is a linear bijection and $\phi([x, y]_1) = [\phi(x), \phi(y)]_2$ for all $x, y \in \mathfrak{n}_1$. If an inner product is specified on these Lie algebras we say that ϕ is an *isometry* if it is an isomorphism and in addition $\langle x, y \rangle_1 = \langle \phi(x), \phi(y) \rangle_2$ for all $x, y \in \mathfrak{n}_1$.

In §3, we develop a construction of two-step and three-step nilpotent Lie algebras associated with Schreier graphs, and in §4, we compare properties of pairs of Lie algebras arising from Gassmann-Sunada triples.

Definition 2.10. Let G be a finite group, with H_1 and H_2 subgroups of G

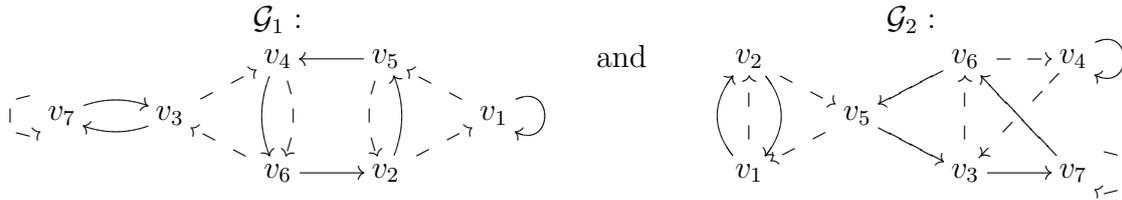
such that for every $g \in G$,

$$|[g] \cap H_1| = |[g] \cap H_2|, \tag{1}$$

where $[g]$ denotes the conjugacy class of g in G . In this case, H_1 and H_2 are called *almost conjugate* subgroups of G , and (G, H_1, H_2) is called a *Gassmann-Sunada triple*, [2, 12].

Example 2.11. Let G be the special linear group of degree 3 over the field \mathbb{F}_2 , $H_1 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$, and $H_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}$.

Let $C_{pos} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right\}$. In [1, 2], this is shown to be a Gassmann-Sunada triple. The Schreier graphs, $\mathcal{G}_1 = \mathcal{G}(G, H_1, C)$ and $\mathcal{G}_2 = \mathcal{G}(G, H_2, C)$, are



where the solid line corresponds to the first generator given in C_{pos} and the dotted line to the second generator.

3. Two-Step and Three-Step Nilpotent Lie Algebra Constructions

In this section, we construct a two-step nilpotent Lie algebra from an arbitrary Schreier graph and then give necessary and sufficient conditions under which this Lie algebra can extend to a three-step nilpotent Lie algebra.

Construction 3.1 (Two-Step Nilpotent Construction). From a Schreier graph $\mathcal{G} = \mathcal{G}(G, H, C)$ given by Definition 2.1, we let \mathfrak{v} be the space of formal linear combinations over \mathbb{R} of elements in $V(\mathcal{G})$ and \mathfrak{z} be the space of formal linear combinations over \mathbb{R} of elements in C_{pos} , where $|C_{pos}| = c$. We then define the Lie algebra $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{z}$ as the direct sum of vector spaces; we then require \mathfrak{z} to be contained in the center of \mathfrak{n} and define the Lie bracket by the following: $\forall v_i, v_j \in V(\mathcal{G}) \subseteq \mathfrak{v}$,

$$[v_i, v_j] = \sum_{p=1}^{|C_{pos}|} (\epsilon_p - \epsilon'_p) z_p, \tag{2}$$

where $\epsilon_p = \begin{cases} 1 & , \text{ if } v_j = \alpha(z_p)(v_i) \\ 0 & , \text{ otherwise,} \end{cases}$

and $\epsilon'_p = \begin{cases} 1 & , \text{ if } v_j = \alpha(z_p^{-1})(v_i) \\ 0 & , \text{ otherwise.} \end{cases}$

All other brackets not defined by linearity or skew-symmetry are set equal to zero.

To see that this does define a Lie algebra, consider the following. First, if there exists a $z_\ell = z_\ell^{-1} \in C_{pos}$, then $\forall v_i, v_j \in \mathfrak{v}$, the coefficient of z_ℓ in $[v_i, v_j]$ is 0.

This is why we exclude such elements from C_{pos} as we mentioned in Remark 2.3. Also note that $[v_i, v_i] = 0$ because for a fixed label z_p , either v_i has a loop with label z_p in which case $\epsilon_p = \epsilon'_p = 1$, or v_i does not have a loop with label z_p in which case $\epsilon_p = \epsilon'_p = 0$. In either case, $\epsilon_p - \epsilon'_p = 0$ for all p . Furthermore, this bracket will be skew-symmetric because $v_j = \alpha(z_p)(v_i)$ implies $v_i = \alpha(z_p^{-1})(v_j)$. Finally, note that because \mathfrak{z} is contained in the center of \mathfrak{n} , the Jacobi identity on the bracket given above is trivial, which makes $(\mathfrak{n}, [,])$ as defined above a two-step nilpotent Lie algebra.

Remark 3.2. In this paper when needed, we specify an inner product on $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ by requiring $\{V(\mathcal{G}), C_{pos}\}$ to be an orthonormal basis.

Remark 3.3. The two-step nilpotent Lie algebra defined in Construction 3.1 does not rely on the fact that the graph was a Schreier graph. A two-step nilpotent Lie algebra can be constructed similarly from any directed, edge-labeled graph by having a set of graph labels, $C_{pos} = \{z_1, \dots, z_c\}$, instead of having a set of generators of a group acting on the graph. The Lie bracket on $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{z}$ is then defined as in Construction 3.1, except now

$$\epsilon_p = \begin{cases} 1 & , \text{ if } (v_i, v_j) \text{ is an edge with label } z_p \\ 0 & , \text{ otherwise,} \end{cases}$$

$$\text{and } \epsilon'_p = \begin{cases} 1 & , \text{ if } (v_j, v_i) \text{ is an edge with label } z_p \\ 0 & , \text{ otherwise.} \end{cases}$$

In order to find necessary and sufficient conditions on a Schreier graph for the two-step nilpotent Lie algebra construction to extend to a three-step nilpotent Lie algebra, we must introduce the following definition.

Definition 3.4. For a Schreier graph $\mathcal{G} = \mathcal{G}(G, H, C)$, a label $z \in C_{pos}$ is called *admissible* if there exists a single closed path of length 3 or 4 with each edge labeled z , and all other closed paths with edges labeled z are of length 1 or 2. Otherwise, z is called *inadmissible*. We will denote the set of admissible labels by $\{z_{r_1}, \dots, z_{r_m}\}$ and the set of inadmissible labels by $\{z_{b_1}, \dots, z_{b_n}\}$. A path is called *admissible* if it is the single closed path of length 3 or 4 for an admissible label z_r .

Theorem 3.5. *Let G be a finite group, H a subgroup of G , C a generating set of G , and \mathcal{G} the Schreier graph of G with respect to H and C as in Definition 2.1. Let \mathfrak{n} be the two-step nilpotent Lie algebra associated with \mathcal{G} by Construction 3.1. Then \mathfrak{n} extends to a three-step nilpotent Lie algebra $\widehat{\mathfrak{n}}$ if and only if there exists at least one admissible label in C_{pos} . Moreover, up to the variations allowed in Construction 3.6 below, this is the only 3-step nilpotent extension of \mathfrak{n} .*

Construction 3.6 (Three-Step Nilpotent Construction). For each admissible label z_{r_k} , we define new elements $t_{r_{k,1}}$ and $t_{r_{k,2}}$ (at least one $t_{r_{k,\ell}} \neq 0$) such that the 3-step nilpotent extension of \mathfrak{n} is $\widehat{\mathfrak{n}} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{t}$, where \mathfrak{v} and \mathfrak{z} are defined as before and $\mathfrak{t} = \text{span}_{\mathbb{R}}\{t_{r_{k,1}}, t_{r_{k,2}} : z_{r_k} \text{ is admissible}\}$. The Lie bracket is then defined as in Construction 3.1 with the following additional nonzero brackets, and

then extend by linearity and skew-symmetry:

If the admissible path with label z_{r_k} is of length 4 and has successive vertices $(v_1, v_2, v_3, v_4, v_1)$, we set

$$\begin{aligned} [v_1, z_{r_k}] &= -[v_3, z_{r_k}] = t_{r_{k,1}}, \text{ and} \\ [v_2, z_{r_k}] &= -[v_4, z_{r_k}] = t_{r_{k,2}} \end{aligned} \tag{3}$$

If the admissible path with label z_{r_k} is of length 3 and has successive vertices (v_1, v_2, v_3, v_1) , we set

$$\begin{aligned} [v_1, z_{r_k}] &= t_{r_{k,1}}, \\ [v_2, z_{r_k}] &= t_{r_{k,2}}, \text{ and} \\ [v_3, z_{r_k}] &= -(t_{r_{k,1}} + t_{r_{k,2}}) \end{aligned} \tag{4}$$

For any other vertex v_i not in the admissible 3- or 4-path, we set

$$[v_i, z_{r_k}] = 0 \tag{5}$$

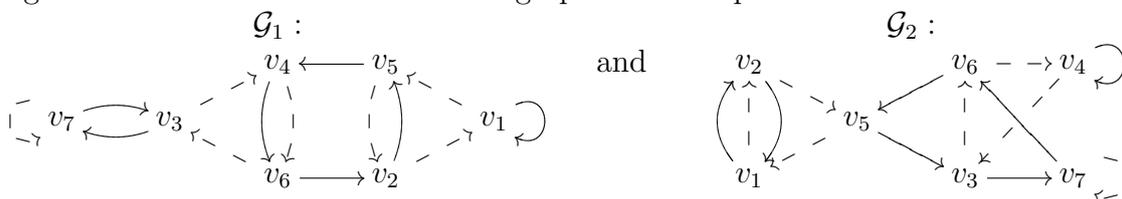
For any edge with inadmissible label z_b , we set

$$[v_j, z_b] = 0 \forall v_j \in \mathfrak{v}. \tag{6}$$

Remark 3.7. In order for $\hat{\mathfrak{n}}$ to be 3-step nilpotent, we must set at least one $t_{r_{k,\ell}} \neq 0$. The 3-step nilpotent extension of \mathfrak{n} is not unique. Distinct Lie algebra extensions can be obtained by defining relations between the various elements $t_{r_{k,\ell}}$, namely these elements may be linearly dependent. Because of these variations, we get $1 \leq \dim \mathfrak{t} \leq 2m$, where m is the number of admissible labels.

Remark 3.8. This paper does not address extensions where $[v_i, v_j] \in \mathfrak{t}$ since these do not seem to intuitively arise from graph properties, nor do they contribute to the extension being 3-step nilpotent.

Example 3.9. The following is a three-step nilpotent extension of the Lie algebras associated with the Schreier graphs in Example 2.11:



The solid lines correspond to the first generator in C_{pos} , denoted z_r because it is admissible, and the dotted lines correspond to the second generator, denoted z_b because it is inadmissible. If we delete the last column of bracket relations below, we have the two-step nilpotent Lie algebra as defined in Construction 3.1.

$$\begin{array}{l}
 \widehat{\mathbf{n}}_1 : \left[\begin{array}{ll|l}
 [v_1, v_2] = -z_b & [v_3, v_4] = z_b & [v_2, z_r] = t \\
 [v_1, v_5] = z_b & [v_3, v_6] = -z_b & [v_4, z_r] = -t \\
 [v_2, v_5] = z_r - z_b & [v_4, v_5] = -z_r & [v_5, z_r] = 0 \\
 [v_2, v_6] = -z_r & [v_4, v_6] = z_r + z_b & [v_6, z_r] = 0
 \end{array} \right. \\
 \\
 \widehat{\mathbf{n}}_2 : \left[\begin{array}{ll|l}
 [v_1, v_2] = z_b & [v_3, v_6] = z_b & [v_3, z_r] = t \\
 [v_1, v_5] = -z_b & [v_3, v_7] = z_r & [v_5, z_r] = 0 \\
 [v_2, v_5] = z_b & [v_4, v_6] = -z_b & [v_6, z_r] = -t \\
 [v_3, v_4] = -z_b & [v_5, v_6] = -z_r & [v_7, z_r] = 0 \\
 [v_3, v_5] = -z_r & [v_6, v_7] = -z_r &
 \end{array} \right.
 \end{array}$$

All other brackets not defined by skew-symmetry or linearity are equal to zero.

Proof of Theorem 3.5 (sufficient).

Define $\epsilon_{i,j}^{r_k} = \begin{cases} 1 & , \text{ if there is a } z_{r_k}\text{-edge connecting } v_i \text{ to } v_j \\ -1 & , \text{ if there is a } z_{r_k}\text{-edge connecting } v_j \text{ to } v_i \\ 0 & , \text{ otherwise} \end{cases}$

and similarly define $\epsilon_{i,j}^{b_\ell}$. We proceed by induction on the number of admissible labels. Assume that the Schreier graph has only one admissible label z_r , and the inadmissible labels, if any exist, are denoted z_{b_ℓ} , $\ell = 1, \dots, n$. If we pick any three vertices from the graph, say v_1, v_2, v_3 , then the following possibilities occur for the Jacobi identity on those three vertices:

Case 1: There are no edges labeled z_r connecting v_1, v_2 , or v_3 , in which case the Jacobi identity will be satisfied because

$$\begin{aligned}
 & [v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] \\
 &= [v_1, \epsilon_{2,3}^r z_r] + \sum_{\ell=1}^n [v_1, \epsilon_{2,3}^{b_\ell} z_{b_\ell}] \\
 &+ [v_2, \epsilon_{3,1}^r z_r] + \sum_{\ell=1}^n [v_2, \epsilon_{3,1}^{b_\ell} z_{b_\ell}] \\
 &+ [v_3, \epsilon_{1,2}^r z_r] + \sum_{\ell=1}^n [v_3, \epsilon_{1,2}^{b_\ell} z_{b_\ell}] && \text{by linearity of the bracket} \\
 &= [v_1, 0] + 0 + [v_2, 0] + 0 + [v_3, 0] + 0 && \text{by Equation 6 and definition of } \epsilon_{i,j}^r \\
 &= 0.
 \end{aligned}$$

Note that by the linearity of the Lie bracket, we can always take the Jacobi identity and separate the brackets containing z_{b_ℓ} terms, which will equal zero by Equation 6, so we only need to consider the Jacobi identity in relation to brackets containing z_{r_k} terms.

Case 2: Without loss of generality, there is precisely one z_r -edge connecting v_1 to v_2 , which implies that v_3 is not contained in the admissible path with label z_r . In this case,

$$\begin{aligned}
 & [v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] \\
 &= [v_1, 0] + [v_2, 0] + [v_3, z_r] && \text{by Equation 6 and definition of } \epsilon_{i,j}^r \\
 &= 0 && \text{by Equation 5.}
 \end{aligned}$$

Case 3a: There are precisely two edges labeled z_r between the vertices v_1, v_2, v_3 . The first way that this may occur is, without loss of generality, one edge connects v_1 to v_2 and the other connects v_2 to v_1 . This means that $[v_1, v_2] = 0$ by the definition of the Lie bracket in Equation 2. Therefore, the Jacobi identity

is satisfied because

$$\begin{aligned} & [v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] \\ &= [v_1, 0] + [v_2, 0] + [v_3, 0] && \text{by Equation 6 and 2} \\ &= 0. \end{aligned}$$

Case 3b: The second way that two edges labeled z_r may appear is, without loss of generality, one edge connects v_1 to v_2 and the other from v_2 to v_3 . Since there is no z_r -edge connecting v_3 to v_1 , this implies that the path with labels z_r must be an admissible 4-path so $[v_1, z_r] = -[v_3, z_r]$ by Equation 3. Again the Jacobi identity is satisfied because

$$\begin{aligned} & [v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] \\ &= [v_1, z_r] + [v_2, 0] + [v_3, z_r] && \text{by Equation 6 and definition of } \epsilon_{i,j}^r \\ &= 0 && \text{by Equation 3.} \end{aligned}$$

Case 4: There are three edges labeled z_r connecting v_1 to v_2 to v_3 back to v_1 . So the path here is an admissible 3-path with label z_r . The Jacobi equation becomes

$$\begin{aligned} & [v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] \\ &= [v_1, z_r] + [v_2, z_r] + [v_3, z_r] && \text{by Equation 6 and definition of } \epsilon_{i,j}^r \\ &= 0 && \text{by Equation 4.} \end{aligned}$$

These four cases cover all possibilities because of the properties of a Schreier graph discussed in Remark 2.7. Therefore, no matter which three vertices we pick in the graph and by the linearity of the Lie bracket, the Jacobi identity is always satisfied, making $\hat{\mathfrak{n}}$ a Lie algebra.

Now using induction, assume that we have a Lie algebra associated with a graph with admissible labels, z_{r_1}, \dots, z_{r_m} , and inadmissible labels, z_{b_1}, \dots, z_{b_n} . If we add an additional admissible label $z_{r_{m+1}}$ in C_{pos} , then the Jacobi identity for any three vertices becomes

$$\begin{aligned} & [v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] \\ &= [v_1, \epsilon_{2,3}^{r_{m+1}} z_{r_{m+1}}] + \sum_{k=0}^m [v_1, \epsilon_{2,3}^{r_k} z_{r_k}] \\ &+ [v_2, \epsilon_{3,1}^{r_{m+1}} z_{r_{m+1}}] + \sum_{k=0}^m [v_2, \epsilon_{3,1}^{r_k} z_{r_k}] \\ &+ [v_3, \epsilon_{1,2}^{r_{m+1}} z_{r_{m+1}}] + \sum_{k=0}^m [v_3, \epsilon_{1,2}^{r_k} z_{r_k}] && \text{by Equation 6 and linearity of the bracket} \\ &= ([v_1, \epsilon_{2,3}^{r_{m+1}} z_{r_{m+1}}] + [v_2, \epsilon_{3,1}^{r_{m+1}} z_{r_{m+1}}] + [v_3, \epsilon_{1,2}^{r_{m+1}} z_{r_{m+1}}]) \\ &+ \sum_{k=0}^m ([v_1, \epsilon_{2,3}^{r_k} z_{r_k}] + [v_2, \epsilon_{3,1}^{r_k} z_{r_k}] + [v_3, \epsilon_{1,2}^{r_k} z_{r_k}]) \\ &= [v_1, \epsilon_{2,3}^{r_{m+1}} z_{r_{m+1}}] + [v_2, \epsilon_{3,1}^{r_{m+1}} z_{r_{m+1}}] + [v_3, \epsilon_{1,2}^{r_{m+1}} z_{r_{m+1}}] + 0 && \text{by induction hypothesis.} \\ &= 0 && \text{because the proof of the base case of the induction proof showed that the} \\ & \text{Jacobi identity is satisfied for any single admissible label.} && \blacksquare \end{aligned}$$

Proof of Theorem 3.5 (necessary). Assume now that the Schreier graph \mathcal{G} has no admissible labels in C_{pos} . This means that for each label z_{b_ℓ} , at least one of the following occur:

1. Each closed path with edges labeled z_{b_ℓ} is of length 1 or 2.
2. There are at least two closed paths of length 3 or 4, with edges labeled z_{b_ℓ} .

3. There exists a closed path with label z_{b_ℓ} that is of length q , $q \geq 5$.

We continue by induction on the number of inadmissible labels z_{b_ℓ} in the Schreier graph. Assume that \mathcal{G} only has one inadmissible label z_b .

Case 1: If each closed path in \mathcal{G} with label z_b is of length 1 or 2, then $[v_i, v_j] = 0$ for all $v_i, v_j \in \mathfrak{v}$ by how the Lie bracket is defined in Equation 2. Therefore, $\dim \mathfrak{z} = 0 \implies \dim \mathfrak{t} = 0$ so there does not exist a three-step nilpotent extension of \mathfrak{n} .

Case 2: Assume that \mathcal{G} has at least two closed paths of length 3 or 4, with edges labeled z_b . Let v_i be a vertex in one of these paths and (v_j, v_k) be an edge in one of the other paths of length 3 or 4. Note that these two paths will not have any vertices in common by Remark 2.7. Because we are assuming that \mathfrak{n} is a Lie algebra and only considering when there is a 3-step nilpotent extension, we may assume that the Jacobi identity is satisfied for all $v \in \mathfrak{v}$. Therefore,

$$\begin{aligned} [v_i, [v_j, v_k]] + [v_j, [v_k, v_i]] + [v_k, [v_i, v_j]] &= 0 \\ \implies [v_i, z_b] + [v_j, 0] + [v_k, 0] &= 0 \text{ (by Equation 2)} \\ \implies [v_i, z_b] &= 0 \text{ for all } v_i \text{ in the closed path.} \end{aligned} \tag{7}$$

Since this was for an arbitrary v_i in a path of length 3 or 4, we can conclude that $[v_i, z_b] = 0$ for all v_i in any path of length 3 or 4. Now, let v_i be another vertex on this graph not contained in a closed path of length 3 or 4, and again let (v_j, v_k) be an edge in one of the closed paths of length 3 or 4. Then,

$$\begin{aligned} [v_i, [v_j, v_k]] + [v_j, [v_k, v_i]] + [v_k, [v_i, v_j]] &= 0 \\ \implies [v_i, z_b] + [v_j, 0] + [v_k, 0] &= 0 \text{ (by Equation 2)} \\ \implies [v_i, z_b] &= 0 \text{ for all } v_i \text{ not in a closed path} \end{aligned} \tag{8}$$

of length 3 or 4. Therefore, $[v_i, z_b] = 0$ for all $v_i \in \mathfrak{v}$ (by Equations 7 and 8), which implies that $\dim \mathfrak{t} = 0$ so a three-step extension of \mathfrak{n} of the type assumed does not exist.

Case 3: Assume that \mathcal{G} has a closed path of length q , $q \geq 5$, with edges labeled z_b . Let the successive vertices of this closed path be $(v_0, v_1, \dots, v_{q-1}, v_0)$. Because \mathfrak{n} is a Lie algebra, we assume that the Jacobi identity is satisfied for $v_i, v_{(i+2) \bmod q}$, and $v_{(i+3) \bmod q}$. This implies that $\forall i = 0, \dots, q-1$,

$$[v_i, [v_{(i+2) \bmod q}, v_{(i+3) \bmod q}]] + [v_{(i+2) \bmod q}, [v_{(i+3) \bmod q}, v_i]] + [v_{(i+3) \bmod q}, [v_i, v_{(i+2) \bmod q}]] = 0 \implies [v_i, z_b] + [v_{(i+2) \bmod q}, 0] + [v_{(i+3) \bmod q}, 0] = 0$$

because two nonconsecutive points in a closed path with labels z_b on a Schreier graph cannot have a z_b -edge connecting them.

$$\implies [v_i, z_b] = 0 \forall i = 0, \dots, q-1. \tag{9}$$

Now let v_j be a vertex not in this closed path of length q . Then

$$\begin{aligned}
 [v_j, [v_0, v_1]] + [v_0, [v_1, v_j]] + [v_1, [v_j, v_0]] &= 0 \\
 \implies [v_j, z_b] + [v_0, 0] + [v_1, 0] &= 0 \text{ (by Equation 2)} \\
 \implies [v_j, z_b] &= 0 \text{ for all } v_j \text{ not in the closed path of length } q.
 \end{aligned}
 \tag{10}$$

Therefore, $[v_j, z_b] = 0$ for all $v_j \in \mathfrak{v}$ (by Equations 9 and 10), which implies that $\dim \mathfrak{t} = 0$ so a three-step extension of \mathfrak{n} does not exist.

Now, assume that \mathcal{G} has inadmissible labels z_{b_1}, \dots, z_{b_n} , and also assume that a three-step extension of \mathfrak{n} does not exist, i.e. $[v_i, z_{b_\ell}] = 0 \forall v_i \in \mathfrak{v}$ and $\forall \ell = 1, \dots, n$. Now if we add an inadmissible label $z_{b_{n+1}} \in C_{pos}$, we see that for any $v_1, v_2, v_3 \in \mathfrak{v}$, $[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$
 $\implies ([v_1, \epsilon_{2,3}^{b_{n+1}} z_{b_{n+1}}] + [v_2, \epsilon_{3,1}^{b_{n+1}} z_{b_{n+1}}] + [v_3, \epsilon_{1,2}^{b_{n+1}} z_{b_{n+1}}]) + \sum_{\ell=0}^n ([v_1, \epsilon_{2,3}^{b_\ell} z_{b_\ell}] + [v_2, \epsilon_{3,1}^{b_\ell} z_{b_\ell}] + [v_3, \epsilon_{1,2}^{b_\ell} z_{b_\ell}]) = 0$ by linearity of the bracket
 $\implies [v_1, \epsilon_{2,3}^{b_{n+1}} z_{b_{n+1}}] + [v_2, \epsilon_{3,1}^{b_{n+1}} z_{b_{n+1}}] + [v_3, \epsilon_{1,2}^{b_{n+1}} z_{b_{n+1}}] = 0$ by induction hypothesis
 $\implies [v_i, z_{b_{n+1}}] = 0 \forall v_i \in \mathfrak{v}$ because the proof of the base case of this induction hypothesis showed that this is the result if the Jacobi identity is satisfied for any inadmissible label. ■

Proof of Theorem 3.5 (nilpotency). $[\widehat{\mathfrak{n}}, \widehat{\mathfrak{n}}] = \mathfrak{z} \oplus \mathfrak{t}$ and $\widehat{\mathfrak{n}}^{(2)} = [\widehat{\mathfrak{n}}, [\widehat{\mathfrak{n}}, \widehat{\mathfrak{n}}]] = \mathfrak{t} \subseteq Z(\widehat{\mathfrak{n}})$. Therefore, $\widehat{\mathfrak{n}}$ is a 3-step nilpotent Lie algebra. ■

4. Lie Algebras Associated with a Gassmann-Sunada Triple

We continue the notation of §2 and §3. The above construction does not require us to begin with a Gassmann-Sunada triple, but some interesting results occur when we look at the Lie algebras associated with a pair of Schreier graphs of a Gassmann-Sunada triple. Recall from Remark 3.2 that in this paper, we will take the union of the set of vertices, the set of labels in C_{pos} , and the set

$$\{t_{r_{k,1}}, t_{r_{k,2}} : z_{r_k} \text{ is admissible and } t_{r_{k,\ell}} \neq 0\}$$

to be an orthonormal basis for $\widehat{\mathfrak{n}} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{t}$.

Proposition 4.1. [5, Lecture 4] *Let (G, H_1, H_2) be a Gassmann-Sunada triple and \mathcal{G}_1 and \mathcal{G}_2 the pair of Schreier graphs associated with this triple. Let α_i be the group action of G on \mathfrak{v} as in Remark 2.4, which will be unitary under the assumed metric given in Remark 3.2. Because H_1 and H_2 are almost conjugate subgroups of G , the representations α_1 and α_2 are unitarily equivalent, i.e. there exists a unitary operator $T : \mathfrak{v}_1 \rightarrow \mathfrak{v}_2$ such that $T(\alpha_1(x)(H_1g)) = \alpha_2(x)(T(H_1g))$ for all $x \in G$ and for all $H_i g \in \mathfrak{v}_i, i = 1, 2$. This operator T is referred to as the transplantation or intertwining operator. For more information, see [5].*

Given a two-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where \mathfrak{v} and \mathfrak{z} are inner product spaces, we can define an operator $j : \mathfrak{z} \rightarrow so(\mathfrak{v})$ given by $j(z)(v) = (ad v)^*z$ where $(ad v)(w) = [v, w]$ and $*$ denotes the adjoint operator with respect

to the given inner product. In other words $j(z)v$ is the unique element in \mathfrak{v} such that

$$\langle j(z)v, w \rangle = \langle z, [v, w] \rangle \text{ for all } w \text{ in } \mathfrak{v}$$

(See Eberlein [4]).

Theorem 4.2. *The j -operator on the 2-step nilpotent metric Lie algebra, $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, associated with a Schreier graph by Construction 3.1 is given by*

$$(\forall z \in \mathfrak{z}, v \in \mathfrak{v}) \quad j(z)v = \alpha(z)(v) - \alpha(z^{-1})(v).$$

Proof. Fix basis elements $v \in \mathfrak{v}$ and $z \in \mathfrak{z}$. Let w be a basis element in \mathfrak{v} . Then,

$$\begin{aligned} \langle j(z)v, w \rangle &= \langle z, [v, w] \rangle \\ &= \begin{cases} \langle z, z \rangle = 1 & , \text{ if } w = \alpha(z)(v) \text{ and } \alpha(z)(v) \neq \alpha(z^{-1})(v) \\ \langle z, -z \rangle = -1 & , \text{ if } w = \alpha(z^{-1})(v) \text{ and } \alpha(z)(v) \neq \alpha(z^{-1})(v) \\ \langle z, z - z \rangle = 0 & , \text{ if } w = \alpha(z)(v) = \alpha(z^{-1})(v) \\ 0 & , \text{ otherwise.} \end{cases} \end{aligned}$$

Recall from Remark 2.7 that this covers all cases that can occur on a Schreier graph. On the other hand,

$$\begin{aligned} \langle \alpha(z)(v) - \alpha(z^{-1})(v), w \rangle &= \langle \alpha(z)(v), w \rangle - \langle \alpha(z^{-1})(v), w \rangle \\ &= \begin{cases} 1 - 0 = 1 & , \text{ if } w = \alpha(z)(v) \text{ and } \alpha(z)(v) \neq \alpha(z^{-1})(v) \\ 0 - 1 = -1 & , \text{ if } w = \alpha(z^{-1})(v) \text{ and } \alpha(z)(v) \neq \alpha(z^{-1})(v) \\ \langle 0, w \rangle = 0 & , \text{ if } w = \alpha(z)(v) = \alpha(z^{-1})(v) \\ 0 & , \text{ otherwise.} \end{cases} \end{aligned}$$

Since this is true for any basis elements w in \mathfrak{v} and by the uniqueness and linearity of the inner product, this implies that $j(z)v = \alpha(z)(v) - \alpha(z^{-1})(v)$. ■

Theorem 4.3. *Starting with a pair of Schreier graphs coming from a Gassmann-Sunada triple, let (\mathfrak{n}_1, j_1) and (\mathfrak{n}_2, j_2) be the associated pair of two-step nilpotent Lie algebras determined by Construction 3.1, and let T be the unitary intertwining operator guaranteed by the Gassmann-Sunada condition. Then,*

$$T(j_1(z)v) = j_2(z)(Tv) \quad \forall z \in \mathfrak{z}_1 \text{ and } \forall v \in \mathfrak{v}_1.$$

Proof. The following calculation will yield the proof:

$$\begin{aligned} T(j_1(z)v) &= T(\alpha_1(z)(v) - \alpha_1(z^{-1})(v)) \\ &= T(\alpha_1(z)(v)) - T(\alpha_1(z^{-1})(v)) \\ &= \alpha_2(z)(Tv) - \alpha_2(z^{-1})(Tv) \\ &= j_2(z)(Tv) \end{aligned}$$

Corollary 4.4. *Starting with a pair of Schreier graphs coming from a Gassmann-Sunada triple, let (\mathfrak{n}_1, j_1) and (\mathfrak{n}_2, j_2) be the associated pair of two-step nilpotent metric Lie algebras determined by Construction 3.1 with the metric defined in Remark 3.2. Then, (\mathfrak{n}_1, j_1) is isometric to (\mathfrak{n}_2, j_2) .*

Proof. Using [5, Lect. 8, Prop. 4.6], we get (\mathfrak{n}_1, j_1) is isomorphic to (\mathfrak{n}_2, j_2) by $\tilde{T} := T \oplus Id$. Also $\forall v, v' \in \mathfrak{v}_1$ and $\forall z, z' \in \mathfrak{z}_1$,
 $\langle (v, z), (v', z') \rangle_1 = \langle v, v' \rangle_1 + \langle z, z' \rangle_1$
 $= \langle T(v), T(v') \rangle_2 + \langle Id(z), Id(z') \rangle_2 = \langle \tilde{T}(v, z), \tilde{T}(v', z') \rangle_2$. ■

While the pair of two-step nilpotent Lie algebras associated with a Gassmann-Sunada triple are always isometric, the three-step nilpotent Lie algebra extensions determined by Construction 3.6 need not be.

Theorem 4.5. *The pair of three-step nilpotent Lie algebras given in Example 3.9 from §3 are non-isometric.*

Proof. For the full proof, see the appendix below. The idea of the proof is that we assume that there exists ϕ that is an isometry between $\widehat{\mathfrak{n}}_1$ and $\widehat{\mathfrak{n}}_2$. Then using the properties of Lie algebra isometries listed below, we obtain a contradiction. Therefore, the two Lie algebras are non-isometric.

1. $\phi : \mathfrak{v} \rightarrow \mathfrak{v}, \mathfrak{z} \rightarrow \mathfrak{z}$, and $\mathfrak{t} \rightarrow \mathfrak{t}$.
2. ϕ has to preserve the ascending central series.
3. The columns (and rows) of the matrix ϕ must be orthonormal to each other.
4. $\phi([x, y]_1) = [\phi(x), \phi(y)]_2$ for all $x, y \in \widehat{\mathfrak{n}}_1$. ■

Note: Because there is a choice in constructing the 3-step nilpotent Lie algebra, a similar argument shows that the following variations on $\widehat{\mathfrak{n}}_2$ are also non-isometric to $\widehat{\mathfrak{n}}_1$:

1. Interchanging t and $-t$, i.e. $[v_3, z_r] = -t$ and $[v_6, z_r] = t$.
2. Switching the t and 0 components, i.e. $[v_3, z_r] = 0, [v_5, z_r] = t, [v_6, z_r] = 0$, and $[v_7, z_r] = -t$.
3. Switching the t and 0 components and then interchanging t and $-t$.

5. Appendix

Let $\widehat{\mathfrak{n}}_1$ and $\widehat{\mathfrak{n}}_2$ be the three step nilpotent Lie algebras given in Example 3.9. Assume that $\phi : \widehat{\mathfrak{n}}_1 \rightarrow \widehat{\mathfrak{n}}_2$ is an isometry, where the entries of the matrix ϕ with respect to the orthonormal basis $\{v_1, \dots, v_7, z_r, z_b, t\}$ for both $\widehat{\mathfrak{n}}_1$ and $\widehat{\mathfrak{n}}_2$ are $(\gamma^{i,j})_{i,j=1}^{10}$.

We begin by computing the ascending central series of the two Lie algebras,

obtaining the following:

$$Z(\widehat{\mathbf{n}}_1) := \{w \in \widehat{\mathbf{n}}_1 : [w, \widehat{\mathbf{n}}_1] = 0\} = \text{span}_{\mathbb{R}}\{v_1 + v_2 + \cdots + v_6, v_7, z_b, t\} \quad (11)$$

$$Z(\widehat{\mathbf{n}}_2) := \{w \in \widehat{\mathbf{n}}_2 : [w, \widehat{\mathbf{n}}_2] = 0\} = \text{span}_{\mathbb{R}}\{v_1 + v_2 + v_5 + v_7, v_3 + v_4 + v_6, z_b, t\} \quad (12)$$

$$Z_2(\widehat{\mathbf{n}}_1) := \{w \in \widehat{\mathbf{n}}_1 : [w, \widehat{\mathbf{n}}_1] \subseteq Z(\widehat{\mathbf{n}}_1)\} = \text{span}_{\mathbb{R}}\{v_1, v_2 + v_4, v_3, v_5 + v_6, v_7, z_r, z_b, t\} \quad (13)$$

$$Z_2(\widehat{\mathbf{n}}_2) := \{w \in \widehat{\mathbf{n}}_2 : [w, \widehat{\mathbf{n}}_2] \subseteq Z(\widehat{\mathbf{n}}_2)\} = \text{span}_{\mathbb{R}}\{v_1, v_2, v_3 + v_6, v_4, v_5 + v_7, z_r, z_b, t\} \quad (14)$$

Next, we use the assumption that ϕ is an isometry to obtain the following properties about the matrix ϕ :

$$\phi \text{ an isometry} \implies \text{the matrix } \phi \text{ is orthonormal} \quad (15)$$

$$\phi : \mathfrak{t} \rightarrow \mathfrak{t} \implies \gamma_{10,j} = \gamma_{j,10} = 0 \text{ for } j = 1, \dots, 9 \quad (16)$$

$$\phi : \mathfrak{z} \rightarrow \mathfrak{z} \implies \gamma_{8,j} = \gamma_{j,8} = \gamma_{9,j} = \gamma_{j,9} = 0 \text{ for } j = 1, \dots, 7 \quad (17)$$

so that ϕ now is of the form

$$\left(\begin{array}{c|c|c} \mathbf{A} & 0 & 0 \\ \hline 0 & \mathbf{B} & 0 \\ \hline 0 & 0 & \mathbf{C} \end{array} \right)$$

where \mathbf{A} is of size 7×7 , \mathbf{B} is 2×2 , and \mathbf{C} is 1×1 .

Finally, we use a combination of the above results along with the property that $\phi([x, y]_1) = [\phi(x), \phi(y)]_2$ for all $x, y \in \widehat{\mathbf{n}}_1$ to obtain relations about the various entries in ϕ :

$$\gamma_{10,10} = \pm 1 \text{ (by 15 and 16)} \quad (18)$$

$$\phi(z_b) \in Z(\widehat{\mathbf{n}}_2) \implies \gamma_{8,9} = 0 \text{ (by 11 and 12)} \quad (19)$$

$$\implies \gamma_{9,9}, \gamma_{8,8} \in \{\pm 1\} \text{ (by 15)} \quad (20)$$

$$\text{and } \gamma_{9,8} = 0 \text{ (by 15)} \quad (21)$$

$$\phi(v_7) \in Z(\widehat{\mathbf{n}}_2) \implies \gamma_{1,7} = \gamma_{2,7} = \gamma_{5,7} = \gamma_{7,7}$$

$$\text{and } \gamma_{3,7} = \gamma_{4,7} = \gamma_{6,7} \text{ (by 11 and 12)} \quad (22)$$

$$\phi(v_1) \in Z_2(\widehat{\mathbf{n}}_2) \implies \gamma_{3,1} = \gamma_{6,1} \text{ and } \gamma_{5,1} = \gamma_{7,1} \text{ (by 13 and 14)} \quad (23)$$

$$\phi(v_3) \in Z_2(\widehat{\mathbf{n}}_2) \implies \gamma_{3,3} = \gamma_{6,3} \text{ and } \gamma_{5,3} = \gamma_{7,3} \text{ (by 13 and 14)} \quad (24)$$

$$\phi(v_2 + v_4) \in Z_2(\widehat{\mathbf{n}}_2) \implies \gamma_{7,4} = \gamma_{5,2} + \gamma_{5,4} - \gamma_{7,2} \text{ (by 13 and 14)} \quad (25)$$

$$\phi(v_5 + v_6) \in Z_2(\widehat{\mathbf{n}}_2) \implies \gamma_{7,6} = \gamma_{5,5} + \gamma_{5,6} - \gamma_{7,5} \text{ (by 13 and 14)} \quad (26)$$

$$[\phi(v_2), \phi(z_r)] = \phi(t) = \gamma_{10,10}t \implies \gamma_{6,2} = \gamma_{3,2} - \gamma_{8,8}\gamma_{10,10} \quad (27)$$

$$[\phi(v_4), \phi(z_r)] = \phi(-t) = -\gamma_{10,10}t \implies \gamma_{6,4} = \gamma_{3,4} + \gamma_{8,8}\gamma_{10,10} \quad (28)$$

$$[\phi(v_5), \phi(z_r)] = \phi(0) = 0 \implies \gamma_{3,5} = \gamma_{6,5} \quad (29)$$

$$[\phi(v_6), \phi(z_r)] = \phi(0) = 0 \implies \gamma_{3,6} = \gamma_{6,6} \quad (30)$$

$$\begin{aligned} (\text{row } 3) \cdot (\text{row } 6) &= 0 \text{ (by 15)} \\ \implies \gamma_{3,4} &= \gamma_{3,2} - \gamma_{8,8}\gamma_{10,10} \end{aligned} \quad (31)$$

$$\implies \gamma_{6,4} = \gamma_{3,2} \text{ (by 22, 23, 24, 27, 28, 29, 30)} \quad (32)$$

$$\begin{aligned} (\text{row } k) \cdot (\text{row } 3) &= (\text{row } k) \cdot (\text{row } 6) \text{ for } k = 1, 2, 4, 5, 7 \text{ (by 15)} \\ \implies \gamma_{k,4} &= \gamma_{k,2} \text{ for } k = 1, 2, 4, 5, 7 \text{ (by 22, 23, 24, 27, 28, 29, 30)} \end{aligned} \quad (33)$$

$$\begin{aligned} \implies \gamma_{5,2} &= \gamma_{5,4} = \gamma_{7,2} = \gamma_{7,4} \\ \implies \gamma_{7,4} &= \gamma_{7,2} = 2\gamma_{5,2} - \gamma_{7,2} \text{ (by 25)} \end{aligned} \quad (34)$$

$$[\phi(v_2), \phi(v_6)] = \phi(-z_r) = -\gamma_{8,8}z_r, \text{ just looking at the } z_r\text{-coefficient}$$

$$\begin{aligned} \implies \sum_{i < j} (\gamma_{i,2}\gamma_{j,6} - \gamma_{j,2}\gamma_{i,6})[v_i, v_j] &= -\gamma_{8,8}z_r \\ \implies -(\gamma_{3,2}\gamma_{5,6} - \gamma_{5,2}\gamma_{3,6}) + (\gamma_{3,2}\gamma_{7,6} - \gamma_{7,2}\gamma_{3,6}) \\ &\quad - (\gamma_{5,2}\gamma_{6,6} - \gamma_{6,2}\gamma_{5,6}) - (\gamma_{6,2}\gamma_{7,6} - \gamma_{7,2}\gamma_{6,6}) = -\gamma_{8,8} \\ \implies \gamma_{7,6} &= \gamma_{5,6} - \gamma_{10,10} \text{ (by 27, 30, 34)} \end{aligned} \quad (35)$$

$$\implies \gamma_{7,5} = \gamma_{5,5} + \gamma_{10,10} \text{ (by 26)} \quad (36)$$

$$[\phi(v_2), \phi(v_4)] = \phi(0) = 0, \text{ just looking at the } z_b\text{-coefficient}$$

$$\begin{aligned} \implies \sum_{i < j} (\gamma_{i,2}\gamma_{j,4} - \gamma_{j,2}\gamma_{i,4})[v_i, v_j] &= 0 \\ \implies (\gamma_{1,2}\gamma_{2,4} - \gamma_{2,2}\gamma_{1,4}) - (\gamma_{1,2}\gamma_{5,4} - \gamma_{5,2}\gamma_{1,4}) \\ &\quad + (\gamma_{2,2}\gamma_{5,4} - \gamma_{5,2}\gamma_{2,4}) - (\gamma_{3,2}\gamma_{4,4} - \gamma_{4,2}\gamma_{3,4}) \\ &\quad + (\gamma_{3,2}\gamma_{6,4} - \gamma_{6,2}\gamma_{3,4}) - (\gamma_{4,2}\gamma_{6,4} - \gamma_{6,2}\gamma_{4,4}) = 0 \\ \implies \gamma_{4,2} &= \gamma_{3,2} - 1/2\gamma_{8,8}\gamma_{10,10} \text{ (by 27, 31, 32, 33)} \end{aligned} \quad (37)$$

$$(\text{row } k) \cdot (\text{row } 5) = (\text{row } k) \cdot (\text{row } 7) \text{ for } k = 1, 2, 3, 4, 6 \text{ (by 15)}$$

$$\implies \gamma_{k,5} = \gamma_{k,6} \text{ for } k = 1, 2, 3, 4, 6 \text{ (by 22, 23, 24, 34, 35, 36)} \quad (38)$$

$$\|\text{row } 5\| - 1 = (\text{row } 5) \cdot (\text{row } 7) \text{ (by 15)}$$

$$\implies \gamma_{5,6} = \gamma_{5,5} + \gamma_{10,10} \text{ (by 22, 23, 24, 34, 35, 36)} \quad (39)$$

$$\implies \gamma_{7,6} = \gamma_{5,5} \text{ (by 35)} \quad (40)$$

$$\begin{aligned}
 [\phi(v_5), \phi(v_6)] &= \phi(0) = 0, \text{ just looking at the } z_b\text{-coefficient} \\
 &\implies \sum_{i < j} (\gamma_{i,5}\gamma_{j,6} - \gamma_{j,5}\gamma_{i,6})[v_i, v_j] = 0 \\
 &\implies (\gamma_{1,5}\gamma_{2,6} - \gamma_{2,5}\gamma_{1,6}) - (\gamma_{1,5}\gamma_{5,6} - \gamma_{5,5}\gamma_{1,6}) \\
 &\quad + (\gamma_{2,5}\gamma_{5,6} - \gamma_{5,5}\gamma_{2,6}) - (\gamma_{3,5}\gamma_{4,6} - \gamma_{4,5}\gamma_{3,6}) \\
 &\quad + (\gamma_{3,5}\gamma_{6,6} - \gamma_{6,5}\gamma_{3,6}) - (\gamma_{4,5}\gamma_{6,6} - \gamma_{6,5}\gamma_{4,6}) = 0 \\
 &\implies \gamma_{1,5} = \gamma_{2,5} \text{ (by 29, 30, 38, 39)} \tag{41}
 \end{aligned}$$

$$\begin{aligned}
 [\phi(v_4), \phi(v_5)] &= \phi(-z_r) = -\gamma_{8,8}z_r, \text{ just looking at the } z_b\text{-coefficient} \\
 &\implies \sum_{i < j} (\gamma_{i,4}\gamma_{j,5} - \gamma_{j,4}\gamma_{i,5})[v_i, v_j] = -\gamma_{8,8}z_r \\
 &\implies (\gamma_{1,4}\gamma_{2,5} - \gamma_{2,4}\gamma_{1,5}) - (\gamma_{1,4}\gamma_{5,5} - \gamma_{5,4}\gamma_{1,5}) \\
 &\quad + (\gamma_{2,4}\gamma_{5,5} - \gamma_{5,4}\gamma_{2,5}) - (\gamma_{3,4}\gamma_{4,5} - \gamma_{4,4}\gamma_{3,5}) \\
 &\quad + (\gamma_{3,4}\gamma_{6,5} - \gamma_{6,4}\gamma_{3,5}) - (\gamma_{4,4}\gamma_{6,5} - \gamma_{6,4}\gamma_{4,5}) = 0 \\
 &\implies \gamma_{1,2}\gamma_{1,5} - \gamma_{2,2}\gamma_{1,5} - \gamma_{1,2}\gamma_{5,5} + \gamma_{2,2}\gamma_{5,5} \\
 &\quad = -\gamma_{4,5}\gamma_{8,8}\gamma_{10,10} + \gamma_{3,5}\gamma_{8,8}\gamma_{10,10} \text{ (by 28, 29, 33, 41)} \tag{42}
 \end{aligned}$$

$$\begin{aligned}
 [\phi(v_4), \phi(v_6)] &= \phi(z_r + z_b) = \gamma_{8,8}z_r + \gamma_{9,9}z_b, \text{ just looking at the } z_b\text{-coefficient} \\
 &\implies \sum_{i < j} (\gamma_{i,4}\gamma_{j,6} - \gamma_{j,4}\gamma_{i,6})[v_i, v_j] = -\gamma_{8,8}z_r + \gamma_{9,9}z_b \\
 &\implies (\gamma_{1,4}\gamma_{2,6} - \gamma_{2,4}\gamma_{1,6}) - (\gamma_{1,4}\gamma_{5,6} - \gamma_{5,4}\gamma_{1,6}) \\
 &\quad + (\gamma_{2,4}\gamma_{5,6} - \gamma_{5,4}\gamma_{2,6}) - (\gamma_{3,4}\gamma_{4,6} - \gamma_{4,4}\gamma_{3,6}) \\
 &\quad + (\gamma_{3,4}\gamma_{6,6} - \gamma_{6,4}\gamma_{3,6}) - (\gamma_{4,4}\gamma_{6,6} - \gamma_{6,4}\gamma_{4,6}) = \gamma_{9,9} \\
 &\implies \gamma_{1,2}\gamma_{1,5} - \gamma_{2,2}\gamma_{1,5} - \gamma_{1,2}\gamma_{5,5} + \gamma_{2,2}\gamma_{5,5} \\
 &\quad = \gamma_{1,2}\gamma_{10,10} - \gamma_{2,2}\gamma_{10,10} - \gamma_{4,5}\gamma_{8,8}\gamma_{10,10} + \gamma_{3,5}\gamma_{8,8}\gamma_{10,10} + \gamma_{9,9} \\
 &\quad \text{(by 27, 29, 31, 33, 38, 39)} \tag{43}
 \end{aligned}$$

$$\begin{aligned}
 [\phi(v_2), \phi(v_5)] &= \phi(z_r - z_b) = \gamma_{8,8}z_r - \gamma_{9,9}z_b, \text{ just looking at the } z_b\text{-coefficient} \\
 &\implies \sum_{i < j} (\gamma_{i,2}\gamma_{j,5} - \gamma_{j,2}\gamma_{i,5})[v_i, v_j] = -\gamma_{8,8}z_r - \gamma_{9,9}z_b \\
 &\implies (\gamma_{1,2}\gamma_{2,5} - \gamma_{2,2}\gamma_{1,5}) - (\gamma_{1,2}\gamma_{5,5} - \gamma_{5,2}\gamma_{1,5}) \\
 &\quad + (\gamma_{2,2}\gamma_{5,5} - \gamma_{5,2}\gamma_{2,5}) - (\gamma_{3,2}\gamma_{4,5} - \gamma_{4,2}\gamma_{3,5}) \\
 &\quad + (\gamma_{3,2}\gamma_{6,5} - \gamma_{6,2}\gamma_{3,5}) - (\gamma_{4,2}\gamma_{6,5} - \gamma_{6,2}\gamma_{4,5}) = -\gamma_{9,9} \\
 &\implies \gamma_{1,2}\gamma_{1,5} - \gamma_{2,2}\gamma_{1,5} - \gamma_{1,2}\gamma_{5,5} + \gamma_{2,2}\gamma_{5,5} \\
 &\quad = -\gamma_{3,5}\gamma_{8,8}\gamma_{10,10} + \gamma_{4,5}\gamma_{8,8}\gamma_{10,10} - \gamma_{9,9} \text{ (by 27, 29, 41)} \tag{44}
 \end{aligned}$$

$$\text{So, } \gamma_{2,2} = \gamma_{1,2} + \gamma_{9,9}\gamma_{10,10} \text{ (by 42, 43)} \quad (45)$$

$$\text{and } \gamma_{4,5} = \gamma_{3,5} + 1/2\gamma_{8,8}\gamma_{9,9}\gamma_{10,10} \text{ (by 42, 44)} \quad (46)$$

$[\phi(v_1), \phi(v_5)] = \phi(z_b) = \gamma_{9,9}z_b$, just looking at the z_b -coefficient

$$\begin{aligned} &\implies \sum_{i<j} (\gamma_{i,1}\gamma_{j,5} - \gamma_{j,1}\gamma_{i,5})[v_i, v_j] = \gamma_{9,9}z_b \\ &\implies (\gamma_{1,1}\gamma_{2,5} - \gamma_{2,1}\gamma_{1,5}) - (\gamma_{1,1}\gamma_{5,5} - \gamma_{5,1}\gamma_{1,5}) \\ &\quad + (\gamma_{2,1}\gamma_{5,5} - \gamma_{5,1}\gamma_{2,5}) - (\gamma_{3,1}\gamma_{4,5} - \gamma_{4,1}\gamma_{3,5}) \\ &\quad + (\gamma_{3,1}\gamma_{6,5} - \gamma_{6,1}\gamma_{3,5}) - (\gamma_{4,1}\gamma_{6,5} - \gamma_{6,1}\gamma_{4,5}) = \gamma_{9,9} \\ &\implies \gamma_{1,1}\gamma_{1,5} - \gamma_{2,1}\gamma_{1,5} - \gamma_{1,1}\gamma_{5,5} + \gamma_{2,1}\gamma_{5,5} \\ &\quad = \gamma_{9,9} \text{ (by 23, 29, 41)} \end{aligned} \quad (47)$$

$[\phi(v_1), \phi(v_6)] = \phi(0) = 0$, just looking at the z_b -coefficient

$$\begin{aligned} &\implies \sum_{i<j} (\gamma_{i,1}\gamma_{j,6} - \gamma_{j,1}\gamma_{i,6})[v_i, v_j] = 0 \\ &\implies (\gamma_{1,1}\gamma_{2,6} - \gamma_{2,1}\gamma_{1,6}) - (\gamma_{1,1}\gamma_{5,6} - \gamma_{5,1}\gamma_{1,6}) \\ &\quad + (\gamma_{2,1}\gamma_{5,6} - \gamma_{5,1}\gamma_{2,6}) - (\gamma_{3,1}\gamma_{4,6} - \gamma_{4,1}\gamma_{3,6}) \\ &\quad + (\gamma_{3,1}\gamma_{6,6} - \gamma_{6,1}\gamma_{3,6}) - (\gamma_{4,1}\gamma_{6,6} - \gamma_{6,1}\gamma_{4,6}) = 0 \\ &\implies \gamma_{1,1}\gamma_{1,5} - \gamma_{2,1}\gamma_{1,5} - \gamma_{1,1}\gamma_{5,5} + \gamma_{2,1}\gamma_{5,5} \\ &\quad = \gamma_{1,1}\gamma_{10,10} - \gamma_{2,1}\gamma_{10,10} \text{ (by 23, 30, 38, 39, 41)} \end{aligned} \quad (48)$$

$$\text{So, } \gamma_{2,1} = \gamma_{1,1} - \gamma_{9,9}\gamma_{10,10} \text{ (by 47, 48)} \quad (49)$$

$$\text{and } \gamma_{5,5} = \gamma_{1,5} - \gamma_{10,10} \text{ (by 47, 48, 49)} \quad (50)$$

$[\phi(v_2), \phi(v_5)] = \phi(z_r - z_b) = \gamma_{8,8}z_r - \gamma_{9,9}z_b$, just looking at the z_b -coefficient

$$\begin{aligned} &\implies \sum_{i<j} (\gamma_{i,2}\gamma_{j,5} - \gamma_{j,2}\gamma_{i,5})[v_i, v_j] = -\gamma_{8,8}z_r - \gamma_{9,9}z_b \\ &\implies (\gamma_{1,2}\gamma_{2,5} - \gamma_{2,2}\gamma_{1,5}) - (\gamma_{1,2}\gamma_{5,5} - \gamma_{5,2}\gamma_{1,5}) \\ &\quad + (\gamma_{2,2}\gamma_{5,5} - \gamma_{5,2}\gamma_{2,5}) - (\gamma_{3,2}\gamma_{4,5} - \gamma_{4,2}\gamma_{3,5}) \\ &\quad + (\gamma_{3,2}\gamma_{6,5} - \gamma_{6,2}\gamma_{3,5}) - (\gamma_{4,2}\gamma_{6,5} - \gamma_{6,2}\gamma_{4,5}) = -\gamma_{9,9} \\ &\implies -3/2\gamma_{9,9} = -\gamma_{9,9} \text{ (by 27, 29, 41, 45, 46, 50)} \\ &\implies \gamma_{9,9} = 0 \text{ which contradicts equation 20} \end{aligned}$$

Therefore, $\widehat{\mathbf{n}}_1$ is not isometric to $\widehat{\mathbf{n}}_2$.

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