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Nested Punctual Hilbert Schemes and Commuting Varieties of Parabolic Subalgebras*

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Abstract. It is known that the variety parametrizing pairs of commuting nilpotent matrices is irreducible and that this provides a proof of the irreducibility of the punctual Hilbert scheme in the plane. We extend this link to the nilpotent commuting variety of some parabolic subalgebras of $M_n(\Bbbk)$ and to the punctual nested Hilbert scheme. By this method, we obtain a lower bound on the dimension of these moduli spaces. We characterize the cases where they are irreducible. In some reducible cases, we describe the irreducible components and their dimensions.
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Let $S^{[n]}$ denote the Hilbert scheme parametrizing the zero dimensional schemes z_n in the affine plane $S = \mathbb{A}^2 = Spec\ \Bbbk[x,y]$ with $length(z_n) = n$. Several variations from this original Hilbert scheme have been considered. For instance, Briançon studied the punctual Hilbert scheme $S_0^{[n]}$ which parametrizes the subschemes z_n with length n and support on the origin [Br], and Cheah has considered the nested Hilbert schemes parametrizing tuples of zero dimensional schemes $z_{k_1} \subset z_{k_2} \subset \cdots \subset z_{k_r}$ organised in a tower of successive inclusions [Ch98a, Ch98b].

Let $\mathcal{C}(M_n)$ be the commuting variety of M_n, *i.e.* the variety parametrizing the pairs of square matrices (X,Y) with $X \in M_n(\Bbbk), Y \in M_n(\Bbbk)$, $XY = YX$. Gerstenhaber [Ge] proved the irreducibility of $\mathcal{C}(M_n)$. Many variations in the same circle of ideas have been considered. For instance, one can consider $\mathcal{C}(\mathfrak{a})$, where $\mathfrak{a} \subset M_n$ is a subspace (often a Lie subalgebra), or $\mathcal{N}(\mathfrak{a}) \subset \mathcal{C}(\mathfrak{a})$ defined by the condition that X, Y be nilpotent (cf. *e.g.* [Pa, Bar, Pr, Bu, GR]).

There is a well known connection between Hilbert schemes and commuting varieties. If $z_n \in S^{[n]}$ is a zero dimensional subscheme, and if $b_1, \dots, b_n$ is a base of the structural sheaf $\mathcal{O}_{z_n} = \Bbbk[x,y]/I_{z_n}$, the multiplications by x and y on $\mathcal{O}_{z_n}$ are represented by a pair of commuting matrices X, Y. The scheme z_n is characterized by the pair of commuting matrices (X,Y) up to simultaneous conjugation. This

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link has been intensively used by Nakajima [Na]. Obviously, variations on the Hilbert scheme correspond to variations on the commuting varieties.

The goal of this paper is to study the punctual nested Hilbert schemes $S_0^{[k,n]}$ and $S_0^{[\![k,n]\!]}$ and their matrix counterparts $\mathcal{N}(\mathfrak{p}_{k,n})$ and $\mathcal{N}(\mathfrak{q}_{k,n})$. Here $S_0^{[k,n]} \subset S_0^{[k]} \times S_0^{[n]}$ parametrizes the pairs of punctual schemes z_k, z_n with $z_k \subset z_n$ and $S_0^{[\![k,n]\!]} \subset S_0^{[k]} \times S_0^{[k+1]} \times \cdots \times S_0^{[n]}$ parametrizes the tuples $z_k \subset z_{k+1} \cdots \subset z_n$, $\mathfrak{p}_{k,n} \subset M_n$ is a parabolic subalgebra defined by a flag $F_0 \subset F_k \subset F_n$ with $\dim F_i = i$ and $\mathfrak{q}_{k,n}$ is associated with a flag $F_0 \subset F_1 \cdots \subset F_k \subset F_n$.

Our interest in the nested punctual Hilbert schemes stems from the the creation and annihilation operators on the cohomology of the Hilbert scheme introduced by Nakajima and Grojnowski [Na, Groj]. The geometry of the nested Hilbert schemes controls these operators. A typical application is the vanishing of a cohomology class which is the push-down of the class of a variety under a contracting morphism. It is often necessary to describe the components of the nested Hilbert schemes and/or their dimension to simplify the computations [Na, Le, CE]. On the Lie algebra side, the subalgebras $\mathfrak{p}_{k,n} \subset M_n$ are the maximal parabolics, hence are prototypes for the study of general parabolics. On the other hand, the algebras $\mathfrak{q}_{k,n}$ are used as a tool to study some other cases and are well behaved for our computations.. Closely linked to this setting, note also that $\mathfrak{q}_{n,n}$ is a Borel subalgebra of M_n. Some properties of $\mathcal{N}(\mathfrak{q}_{n,n})$ can be found in [GR].

Let $P_{k,n}$, resp. $Q_{k,n}$, be the groups of invertible matrices in $\mathfrak{p}_{k,n}$, resp. $\mathfrak{q}_{k,n}$. It acts on $\mathfrak{p}_{k,n}$, resp. $\mathfrak{q}_{k,n}$, by conjugation. In the Lie algebra setting, $P_{k,n}$, resp. $Q_{k,n}$, is nothing but the parabolic subgroup of $\mathrm{GL}_n(\mathbb{k})$ with Lie algebra $\mathfrak{p}_{k,n}$, resp. $\mathfrak{q}_{k,n}$.

It is possible in our context to make precise the connection between Hilbert schemes and commuting varieties. Since zero dimensional schemes are characterized by pairs of commuting matrices up to the choice of the base, the expectation is that Hilbert schemes should be quotients of commuting varieties. This is correct in essence, provided that one takes care of the existence of cyclic vectors. Moreover, the acting groups $P_{k,n}$ and $Q_{k,n}$ are not reductive. Nevertheless, we will construct a geometric quotient in the sense of Mumford [MFK], as follows.

Let $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ and $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})$ be the open loci in $\mathcal{N}(\mathfrak{p}_{k,n})\times\mathbb{k}^n$ and $\mathcal{N}(\mathfrak{q}_{k,n})\times\mathbb{k}^n$ defined by the existence of a cyclic vector, *i.e.* these open loci parametrize the tuples $((X,Y),v)$ with $\mathbb{k}[X,Y](v) = \mathbb{k}^n$. They are stable under the respective action of $P_{k,n}$ and $Q_{k,n}$.

Theorem 2.2. *The following statements hold:*

1. *There exist geometric quotients*
   $q\colon \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n}) \to \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})/P_{k,n}$ *and* $q'\colon \widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n}) \to \widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})/Q_{k,n}$
   and they are principal bundles locally trivial for the Zariski topology.
2. *There exist surjective morphisms*
   $\widetilde{\pi}_{k,n}\colon rcl\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n}) \to S_0^{[n-k,n]}$ *and* $\widetilde{\pi}'_{k,n}\colon rcl\widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n}) \to S_0^{[\![n-k,n]\!]}$.
3. *There exist isomorphisms*
   $i : \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})/P_{k,n} \to S_0^{[n-k,n]}$ *and* $i' : \widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})/Q_{k,n} \to S_0^{[\![n-k,n]\!]}$.

These isomorphisms identify the projections to the Hilbert schemes with the geometric quotients, i.e. $i \circ q = \widetilde{\pi}_{k,n}$ *and* $i' \circ q' = \widetilde{\pi}'_{k,n}$.

This is directly inspired from the general construction of Nakajima's quiver varieties (see *e.g.* [Gi]), the cyclicity being a stability condition in the sense of [MFK]. It can straightforwardly be generalized to any parabolic subalgebra of M_n.

We then investigate the number of components of $\mathcal{N}(\mathfrak{p}_{k,n})$, $\mathcal{N}(\mathfrak{q}_{k,n})$, $S_0^{[k,n]}$, $S_0^{[\![k,n]\!]}$ and their dimension. Many of our proofs consider the problem for $\mathcal{N}(\mathfrak{p}_{k,n})$, $\mathcal{N}(\mathfrak{q}_{k,n})$ firstly and then use the above theorem and some geometric arguments to push down the information to the Hilbert schemes. Conversely, sometimes, we pull back the information from the Hilbert scheme to the commuting variety. The general philosophy is that the problems on the commuting varieties are in some sense "linear" versions of the corresponding problems on the Hilbert scheme which are "polynomial" problems. This explains why the most frequent direction of propagation of the information is from commuting varieties to Hilbert schemes.

Theorem 4.11. $S_0^{[k,n]}$ *is irreducible if and only if* $k \in \{0, 1, n-1, n\}$. *The variety* $\mathcal{N}(\mathfrak{p}_{k,n})$ *is irreducible if and only if* $k \in \{0, 1, n-1, n\}$.

Theorem 4.12. $S_0^{[\![k,n]\!]}$ *is irreducible if and only if* $k \in \{n-1, n\}$ *or* $n \leqslant 3$. $\mathcal{N}(\mathfrak{q}_{k,n})$ *is irreducible if and only if* $k \in \{0, 1\}$ *or* $n \leqslant 3$.

When $k = 2$ or $k = n-2$, we have precise results on the number of components and their dimensions.

Theorem 6.3. *Let* $\mathfrak{w} = \mathfrak{q}_{2,n}$ *or* $\mathfrak{p}_{2,n}$. *Then* $\mathcal{N}(\mathfrak{w})$ *is equidimensional of dimension* $\dim \mathfrak{w} - 1$. *It has* $\left\lfloor \frac{n}{2} \right\rfloor$ *components.*

Theorem 6.5. $S_0^{[2,n]}$, $S_0^{[n-2,n]}$, $S_0^{[\![n-2,n]\!]}$ *are equidimensional of dimension* $n-1$. *They have* $\left\lfloor \frac{n}{2} \right\rfloor$ *components.*

The similarity between $S_0^{[k,n]}$ and $S_0^{[n-k,n]}$ follows from a transposition isomorphism between $\mathcal{N}(\mathfrak{p}_{k,n})$ and $\mathcal{N}(\mathfrak{p}_{n-k,n})$. Note however that there might be profound differences between the Hilbert schemes and the corresponding commuting varieties because of the cyclicity condition, see remark 2.15.

Without any assumption on $k \in [\![0, n]\!]$, we have an estimate for the dimension of the components.

Proposition (Section 5). *Each irreducible component of* $S_0^{[\![k,n]\!]}$ *has dimension at least* $n-1$ *which is the dimension of the curvilinear component. Each irreducible component of* $S_0^{[k,n]}$ *has dimension at least* $n-2$, *which is the dimension of the curvilinear component minus one. Each irreducible component of* $\mathcal{N}(\mathfrak{q}_{k,n})$ *has dimension at least* $\dim \mathfrak{q}_{k,n} - 1$. *Each irreducible component of* $\mathcal{N}(\mathfrak{p}_{k,n})$ *has dimension at least* $\dim \mathfrak{p}_{k,n} - 2$.

Note that the result is not optimal for $\mathfrak{p}_{k,n}$ and $S_0^{[k,n]}$ as Theorems 6.3 and 6.5 show.

Our approach does not depend on the characteristic of $\Bbbk$. One reason that makes this possible is that we often rely on the key work of Premet in [Pr] made in arbitrary characteristic.

Several statements in the paper allow generalisations or abstract reformulations. To keep the paper readable by a large audience, we have chosen a presentation which minimizes the prerequisites. Hopefully, the paper is readable by a non specialist in at least one of the domains Hilbert schemes/commuting varieties.

1. Reducible nested Hilbert schemes

Throughout the paper, we work over an algebraically closed field $\Bbbk$ of arbitrary characteristic.

In this section, we produce examples of reducible nested Hilbert schemes, and we identify some of their components via direct computations.

Let $S = \mathbb{A}^2 = Spec\ \Bbbk[x,y]$ be the affine plane. We denote by $S^{[n]}$ the Hilbert scheme parametrizing the zero dimensional subschemes $z_n \subset \mathbb{A}^2$ of length n. We denote by $S^{[k,n]} \subset S^{[k]} \times S^{[n]}$ the Hilbert scheme parametrizing the pairs (z_k, z_n) with $z_k \subset z_n$. We denote by $S^{[\![k,n]\!]} \subset S^{[k]} \times S^{[k+1]} \times \cdots \times S^{[n]}$ the Hilbert scheme that parametrizes the tuples of subschemes $(z_k, z_{k+1}, \ldots, z_n)$ with $z_k \subset z_{k+1} \cdots \subset z_n$. An index 0 indicates that the schemes considered are supported on the origin. For instance, $S_0^{[k,n]} \subset S_0^{[k]} \times S_0^{[n]}$ is the Hilbert scheme parametrizing the pairs (z_k, z_n) with $z_k \subset z_n$ and $supp(z_k) = supp(z_n) = O$.

All these Hilbert schemes have a functorial description. For the original Hilbert scheme, see [Gr60a] or [HM] for a modern treatment. For the nested Hilbert schemes see [Ke]. For the versions supported on the origin, a good reference is [Be]. In section 2, we will recall the main technical descriptions that we need.

Proposition 1.1. *For $k \neq 0, 1, n-1, n$, $S_0^{[k,n]}$ is reducible.*

Proof. Recall that a curvilinear scheme of length n is a punctual scheme which can be defined by the ideal (x, y^n) in some system of coordinates *i.e.* this is a punctual scheme included in a smooth curve. The curvilinear schemes form an irreducible subvariety of $S_0^{[n]}$ of dimension $n-1$ [Br]. We prove that $S_0^{[k,n]}$ admits at least two components: the curvilinear component where z_k and z_n are both curvilinear (of dimension $n-1$ since $z_k = (x, y^k)$ is determined by $z_n = (x, y^n)$) and an other component of dimension greater or equal than $n-1$. The families that we exhibit below are special cases of more general constructions which give charts on the Hilbert schemes [Ev].

Consider the families of subschemes z_k, z_n, with equation I_k and I_n where $I_n = (x^{n-1}, yx + \sum_{i=2}^{n-2} a_i x^i, y^2 + \sum_{i=2}^{n-2} a_i y x^{i-1} + b x^{n-2})$. Let φ be the change of coordinates defined by $x \mapsto x$, $y \mapsto y - \sum_{i=2}^{n-2} a_i x^{i-1}$. Then

$$\varphi(I_n) = (x^{n-1}, yx, y^2 + bx^{n-2}).$$

In particular, for each choice of the parameters a_i, b, the scheme z_n has length n.

We may suppose $n \geq 4$, otherwise there are no integers k to consider in the proposition. Then all the generators of I_n have valuation at least two and it follows that z_n is not curvilinear.

For each z_n, there is a one dimensional family of subschemes $z_k \subset z_n$. We check this claim in the coordinate system where $I_n = (x^{n-1}, yx, y^2 + bx^{n-2})$. Consider $I_k = (x^k, y - cx^{k-1})$. Modulo I_k we have $x^{n-1} = 0$ and $yx = cx^k = 0$. Since $k \leq n-2$ and $k \geq 2$, $y^2 + bx^{n-2} = y^2 = (cx^{k-1})^2 = 0$. Thus $I_n \subset I_k$, as expected.

All the ideals I_n and I_k are pairwise distinct since their generators form a reduced Gröbner basis for the order $y >> x$ and a reduced Gröbner basis is unique ([Ei], Exercise 15.14). We thus have two families of dimension $n-1$, namely the curvilinear component and the family we constructed with the parameters (a_i, b, c). It remains to prove that they cannot be both included in a same component V of dimension $\geq n$. For this, we prove that the closure of the curvilinear locus is an irreducible component.

Let p be the projection $S_0^{[k,n]} \to S_0^{[n]}$. Let $C^n \subset S_0^{[n]}$ be the curvilinear locus and $C^{k,n} = (p^{-1}(C^n))_{red}$ be the reduced inverse image. Note that p restricts to a bijection between $C^{k,n}$ and C^n. Let V be an irreducible variety containing the curvilinear locus $C^{k,n}$. Since C^n is open in $p(V) \subset S_0^{[n]}$ by [Br] and since p restricts to a bijection between $C^{k,n}$ and C^n, we have $\dim V = \dim C^n = n-1$. ■

In general $S_0^{[k,n]}$ has more than the two components exhibited in Proposition 1.1. For instance, corollary 6.5 shows that $S_0^{[2,n]}$ is equidimensional with $\lfloor \frac{n}{2} \rfloor$ components. As a first step towards this goal, we count the number of components of dimension $n-1$.

Proposition 1.2. *$S_0^{[2,n]}$ contains exactly $\lfloor n/2 \rfloor$ components of dimension $n-1$.*

Proof. Consider the action of the torus $t.x = t^k x$ $(k >> 0)$, $t.y = ty$ on $\mathbb{k}[x,y]$ and the induced action on $S_0^{[n]}$. There is a Bialynicki-Birula decomposition of $S_0^{[n]}$ with respect to this action. According to [ES, Proof of Proposition 4.2], any cell is characterized by a partition of n, and the dimension of the cell with partition $\boldsymbol{\lambda} = (\lambda_1 \geqslant \dots \geqslant \lambda_{d_{\boldsymbol{\lambda}}})$ is $n - \lambda_1$.

There is a unique cell of dimension $n-1$ of $S_0^{[n]}$ and it is associated with the unique partition $\boldsymbol{\lambda} = (1, 1, \dots, 1)$ of n with $\lambda_1 = 1$. Geometrically, this cell parametrizes the curvilinear subschemes which intersect the vertical line $y = 0$ with multiplicity one. We call it the curvilinear cell and we denote it by F_{curv}. There are $\lfloor n/2 \rfloor$ cells $F_{\boldsymbol{\lambda}} \subset S_0^{[n]}$ of dimension $n-2$ corresponding to the partitions $\boldsymbol{\lambda}$ with n boxes and $\lambda_1 = 2$: one has to take $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{a,b} := (2^a, 1^{b-a})$, with $b \geq a \geq 1$ and $a + b = n$.

Following [Ev], we may be more explicit and describe the charts corresponding to the Bialynicki-Birula strata. Since $S_0^{[2,2]} \cong S_0^{[2]}$ is homeomorphic to $\mathbb{P}^1$, where $(c : d) \in \mathbb{P}^1$ corresponds to the subscheme $z_2 \in S_0^{[2]}$ with ideal $(cx + dy, x^2, y^2)$, the proposition is true for $n = 2$ and we may suppose $n \geq 3$. If $b = a$, the Bialynicki-Birula stratum $F_{\boldsymbol{\lambda}_{a,b}}$ is isomorphic to $Spec\ \mathbb{k}[c_{ij}]$ with universal ideal $(x^a, y^2 + \sum_{j\in\{0,1\}, i\in\{1,\dots,a-1\}} c_{ij} x^i y^j)$. If $b > a$, the stratum is $Spec\ \mathbb{k}[c_i, d_i, e_i]$ with universal ideal $(x^b, yx^a + \sum_{i\in\{1,\dots,b-a-1\}} c_i x^{a+i}, y^2 + \sum_{i\in\{1,\dots,b-a-1\}} c_i y x^i + \sum_{i\in\{1,\dots,a-1\}} d_i (yx^i + \sum_{j\in\{1,\dots,b-a-1\}} c_j x^{i+j}) + \sum_{i\in\{b-a,\dots b-1\}} e_i x^i)$

There is at most one term of degree one in the generators of the ideal, which appears when $(b-a=0, c_{10}\neq 0)$ or $(b-a=1, e_1\neq 0)$. In these cases, the corresponding point of the Bialynicki-Birula cell parametrizes a curvilinear scheme and it parametrizes a noncurvilinear scheme if $b-a\geq 2$ or $e_1=0$ or $c_{10}=0$. There are $\lfloor n/2\rfloor -1$ partitions $\boldsymbol{\lambda}_{a,b}$ with $b-a\geq 2$.

Consider the projection $p: S_0^{[2,n]}\to S_0^{[n]}$ and $z_n\in S_0^{[n]}$. The fiber $p^{-1}(z_n)$ is set-theoretically a point if z_n is curvilinear. If z_n is not curvilinear, the fiber is $S_0^{[2]}$ which is homeomorphic to $\mathbb{P}^1$.

It follows that $p^{-1}(F_{curv})$ and $p^{-1}(F_{\boldsymbol{\lambda}_{a,b}})$ with $b-a\geq 2$ are irreducible varieties of dimension $n-1$. There are $\lfloor \frac{n}{2}\rfloor$ such irreducible varieties. To prove that their closures are irreducible components, note that $S_0^{[2,n]}$ is a proper subscheme of the n dimensional irreducible variety $S_0^{[n]}\times S_0^{[2]}$. In particular, any irreducible closed subvariety of dimension $n-1$ in $S_0^{[2,n]}$ is an irreducible component.

It remains to prove that there are no other components. Let L be a component with dimension $n-1$. Since $S_0^{[2]}$ is one-dimensional, the generic fiber of the projection $L\to S_0^{[n]}$ has dimension 0 or 1 thus the projection has dimension at least $n-2$. If the projection has dimension $n-1$, then the generic point of L maps to the generic point of the curvilinear component for dimension reasons, and L is the curvilinear component $\overline{p^{-1}(F_{curv})}$. If the projection has dimension $n-2$, then the generic point of L maps to the generic point of a Bialynicki-Birula cell of dimension $n-2$, $F_{\boldsymbol{\lambda}_{a,b}}$, or to a non closed point of F_{curv}. Since the generic fiber has dimension 1, the generic point of L does not map to F_{curv} nor to the generic point of $F_{\boldsymbol{\lambda}_{a,b}}$, $b-a\leq 1$. Hence L is included in one of the components $\overline{p^{-1}(F_{\boldsymbol{\lambda}_{a,b}})}$ constructed above with $b-a\geq 2$, and the equality follows from the equality of dimensions. ∎

Remark 1.3. It is possible to prove along the same lines that $S_0^{[n-2,n]}$ has exactly $\lfloor n/2\rfloor$ components of dimension $n-1$. More precisely, the universal ideal $(P_0=x^b, P_1=yx^a+\sum_{i\in\{1,\ldots,b-a-1\}}c_ix^{a+i}, P_2=y^2+\sum_{i\in\{1,\ldots,b-a-1\}}c_iyx^i+\sum_{i\in\{1,\ldots,a-1\}}d_i(yx^i+\sum_{j\in\{1,\ldots,b-a-1\}}c_jx^{i+j})+\sum_{i\in\{b-a,\ldots b-1\}}e_ix^i)$ over $F_{\boldsymbol{\lambda}_{a,b}}$ with $b-a\geq 2$ as above defines a $n-2$ dimensional family of subschemes z_n of length n. For each such subscheme z_n, there is a one dimensional family of subschemes $z_{n-2}(t)$ parametrized by t with $z_{n-2}(t)\subset z_n$. In coordinates $z_{n-2}(t)$ is defined by the ideal $(P_0/x, P_1/x+tx^{b-1}, P_2)$ which is well defined since x divides both P_0 and P_1. By the above, the component containing the couples (z_{n-2},z_n) has dimension dimension $(n-2)+1=n-1$. Adding the curvilinear component, we obtain in this way the $\lfloor n/2\rfloor$ components of dimension $n-1$.

2. Hilbert schemes and commuting varieties

The goal of this section is to make precise the link between Hilbert schemes and commuting varieties in our context. More explicitly, we realize the Hilbert schemes $S_0^{[n-k,n]}$ and $S_0^{[\![n-k,n]\!]}$ as geometric quotients of the commuting schemes $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ and $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})$ by the groups $P_{k,n}$ and $Q_{k,n}$ (Theorem 2.2). As a consequence, we point out a precise connection between irreducible components of $S_0^{[n-k,n]}$ (resp. $S_0^{[\![n-k,n]\!]}$) and those of $\mathcal{N}^{cyc}(\mathfrak{p}_{k,n})$ (resp. $\mathcal{N}^{cyc}(\mathfrak{q}_{k,n})$) in Proposition 2.13.

Commuting varieties and their quotient.

We first introduce the notation to handle the commuting varieties. Let $M_{n,k}$ be the space of $n \times k$ matrices with entries in $\Bbbk$ and let $M_n := M_{n,n}$. The associative algebra M_n will more often be considered as a Lie algebra $\mathfrak{g}$ via $[A, B] := AB - BA$ and we will be interested in the action by conjugation of $G = \mathrm{GL}_n$ on it ($g \cdot X = gXg^{-1}$). If $\mathfrak{w}$ is a Lie subalgebra of M_n and $X \in \mathfrak{w}$, we denote the centralizer (also called commutant) of X in $\mathfrak{w}$ by

$$\mathfrak{w}^X := \{Y \in \mathfrak{w} \mid [Y, X] = 0\}.$$

The set of elements of $\mathfrak{w}$ which are nilpotent in M_n is denoted by $\mathfrak{w}^{nil}$. We define the nilpotent commuting variety of $\mathfrak{w}$:

$$\mathcal{N}(\mathfrak{w}) = \{(X, Y) \in (\mathfrak{w}^{nil})^2 \mid [X, Y] = 0\} \subset \mathfrak{w} \times \mathfrak{w}.$$

If a subgroup $Q \subset G$ normalizes $\mathfrak{w}$ then Q^X is the stabilizer of $X \in \mathfrak{w}$ in Q. The group Q acts on $\mathcal{N}(\mathfrak{w})$ diagonally ($q \cdot (X, Y) = (q \cdot X, q \cdot Y)$).

Theorem 2.1. *If X^0 denotes a regular nilpotent element of M_n, we have*

$$\mathcal{N}(M_n) = \overline{G \cdot (X^0, (M_n^{X^0})^{nil})}$$

In particular, the variety $\mathcal{N}(M_n)$ is irreducible of dimension $n^2 - 1$

Recall that an element $X \in M_n$ is said to be regular if it has a cyclic vector, *i.e.* an element $v \in \Bbbk^n$ such that $\langle X^k(v) | k \in \mathbb{N}\rangle = \Bbbk^n$. This easily implies, and is in fact equivalent to, $\dim G^X (= \dim M_n^X) = n$. There is only one regular nilpotent orbit. This is the orbit of nilpotent elements having only one Jordan block.

This theorem was first stated in [Bar] using the correspondence with Hilbert schemes (with a small correction in the proof of lemma 3, see MathReviews 1825165). We can find other proofs of this theorem in [Bas03] and [Pr]. In [Pr], the result is proved whithout any assumption on char $\Bbbk$.

Let $V = \Bbbk^n$ and $(e_1, \ldots, e_n)$ be its canonical basis. We will identify M_n with $\mathfrak{gl}(V)$, the set of endomorphisms of V, thanks to this basis. For $1 \leqslant i \leqslant n$, let $V_i = \langle e_1, \ldots e_i\rangle$. We define $\mathfrak{p}_{k,n}$ (resp. $\mathfrak{q}_{k,n}$) as the set of matrices $X \in \mathfrak{gl}(V)$ such that $X(V_k) \subseteq V_k$ (resp. $X(V_i) \subseteq V_i$ for all $1 \leqslant i \leqslant k$). Given $X \in \mathfrak{p}_{k,n}$, we denote by $X^{(k)}$ the linear map induced by X on V/V_k. Let $P_{k,n} \subset \mathrm{GL}_n$ (resp. $Q_{k,n} \subset \mathrm{GL}_n$) be the algebraic group of invertible matrices of $\mathfrak{p}_{k,n}$ (resp. $\mathfrak{q}_{k,n}$). In the Lie algebra vocabulary, $P_{k,n}$ and $Q_{k,n}$ (resp. $\mathfrak{p}_{k,n}$ and $\mathfrak{q}_{k,n}$) are parabolic subgroups of GL_n (resp. parabolic subalgebras of $\mathfrak{gl}(V)$) and $\mathrm{Lie}(P_{k,n}) = \mathfrak{p}_{k,n}$, $\mathrm{Lie}(Q_{k,n}) = \mathfrak{q}_{k,n}$. In fact, all the content of this section can easily be generalized to any parabolic subalgebra of $\mathfrak{gl}(V)$ and a corresponding nested Hilbert scheme. Namely, the parabolic subalgebra stabilizing a partial flag $F_0 \subset F_{k_1} \subset \cdots \subset F_{k_\ell} \subset F_n$ ($\dim F_j = j$) is in correspondence with the nested Hilbert scheme with length $n - k_\ell \leqslant \cdots \leqslant n - k_1 \leqslant n$.

In Definition 2.9 and Proposition 2.10, we define a scheme $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{w})$, whose $\Bbbk$-points are the triples (X, Y, v) with $(X, Y) \in \mathcal{N}(\mathfrak{w})$ and $v \in V$ is a cyclic vector for the pair of endomorphisms X, Y. Later, we also describe an action of the group

$P_{k,n}$ (resp. $Q_{k,n}$) on the scheme $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ (resp. $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})$). Set-theoretically, this action is given by $g\cdot(X,Y,v) = (gXg^{-1}, gYg^{-1}, gv)$.

The following theorem asserts that a GIT quotient in the sense of Mumford [MFK] exists, and that the quotients are nested punctual Hilbert schemes.

Theorem 2.2. *1. The geometric quotients* $q : \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n}) \twoheadrightarrow \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})/P_{k,n}$ *and* $q' : \widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n}) \twoheadrightarrow \widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})/Q_{k,n}$ *exist and they are principal bundles locally trivial for the Zariski topology.*

2. There exist surjective morphisms

$$\widetilde{\pi}_{k,n} : \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n}) \twoheadrightarrow S_0^{[n-k,n]},$$

$$\widetilde{\pi}'_{k,n} : \widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n}) \twoheadrightarrow S_0^{[\![n-k,n]\!]}.$$

3. There exist an isomorphism $i : \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})/P_{k,n} \xrightarrow{\sim} S_0^{[n-k,n]}$ *and an isomorphism* $i' : \widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})/Q_{k,n} \xrightarrow{\sim} S_0^{[\![n-k,n]\!]}$. *These isomorphisms identify the projections to the Hilbert schemes with the geometric quotients,* i.e. $i \circ q = \widetilde{\pi}_{k,n}$ *and* $i' \circ q' = \widetilde{\pi}'_{k,n}$.

Relative representability.

Hilbert schemes are often defined through their functor of points (see [EH] or [St] for an introduction). We will use this setting to prove Theorem 2.2. A useful example for us is the functor of points of the $\Bbbk$-vector space V. This is the functor which associates

- to any $\Bbbk$-algebra A, the set $V(A) := V \otimes_{\Bbbk} A \cong A^n$.
- to any morphism $A \to B$, the natural map $\begin{array}{ccc} V(A) & \to & V(B) = V(A) \otimes_A B \\ v & \mapsto & v \otimes 1 \end{array}$. (1)

In particular, the functor represented by M_n (resp. V_k, $\mathfrak{p}_{k,n}$) associates to any $\Bbbk$-algebra A, the set $M_n(A)$ of $n \times n$-matrices with coefficients in A (resp. $V_k(A) := V_k \otimes A \subset V(A)$, $\mathfrak{p}_{k,n}(A) := \{X \in M_n(A) \,|\, X(V_k(A)) \subset V_k(A)\}$), see [St, Example 2.1]. In the following, we will usually only make explicit the value of the functors on objects, their value on morphisms then being standard.

For more involved examples, the notion of relative representability of functors turns out to be useful. We recall this notion from [Gr60b], with some obvious adjustments to fit our context. We will use this language to prove the representability of our functors.

Let F, G be functors from the category of $\Bbbk$-algebras to sets. Suppose that F is a subfunctor of G, ie. for every $\Bbbk$-algebra A, $F(A)$ is a subset of $G(A)$. The inclusion $F \subset G$ is relatively representable if, for every $\Bbbk$-algebra A and every $g \in G(A)$, there exists a subscheme $Z \subset Spec(A)$ satisfying the following property: for every $\varphi : A \to B$, the morphism $Spec(B) \to Spec(A)$ factorizes through Z if and only if the element $f \in G(B)$ defined by $f = \varphi_*(g)$ satisfies $f \in F(B)$. Grothendieck, [Gr60b, Lemme 3.6] proves that if G is representable and if $F \subset G$ is relatively representable, then F is representable.

In intuitive words, a relatively representable subfunctor $F \subset G$ is a subfunctor of G defined by subscheme conditions on the base. We illustrate this through the following elementary lemma.

Lemma 2.3. *The functor which maps a $\mathbb{k}$-algebra A to the set $P_{k,n}(A) := \{X \in \mathfrak{p}_{k,n}(A) \,|\, \det X$ is invertible$\}$ is representable. The corresponding scheme is $P_{k,n}$.*

Proof. In the previous setting, we let $G(A) := \mathfrak{p}_{k,n}(A)$ and $F(A) := P_{k,n}(A)$. Given A and $g \in G(A)$, we set $Z := \{p \in Spec(A) | \det g \notin p\}$. Obviously, Z is an open subscheme of $Spec(A)$. For every $\varphi : A \to B$, we consider the element $f := \varphi_*(g) \in G(B)$. We have $f \in F(B) \Leftrightarrow \det \varphi_*(g) = \varphi(\det g)$ is invertible $\Leftrightarrow$ $\forall p \in Spec(B)\ \det g \notin \varphi^{-1}(p)$, that is, the comorphism $Spec(B) \to Spec(A)$ factorizes through Z. In particular, $F \subset G$ is relatively representable, hence F is representable by a subscheme of $\mathfrak{p}_{k,n}$. The $\mathbb{k}$-points of F are those of the open subscheme $P_{k,n} \subset \mathfrak{p}_{k,n}$. Hence $P_{k,n}$ with the open subscheme structure represents F. ∎

This also applies when F is a subfunctor of G defined by the inclusion of two families according to the following lemma, proved in [Ke, Lemma 1.1].

Lemma 2.4. *Let $X \subset Spec(A) \times W$, $Y \subset Spec(A) \times W$ be two families of subschemes of a scheme W with X finite and flat over $Spec(A)$. There exists a subscheme $Z \subset Spec(A)$ such that, for every morphism $f : Spec(B) \to Spec(A)$, the following two conditions are equivalent:*

- *f factorizes through Z*
- *$X \times_{Spec(A)} Spec(B) \subset Y \times_{Spec(A)} Spec(B)$*

Proposition 2.5. *Let $n_1 \geq n_2 \cdots \geq n_j > 0$ be integers. Let $F^{n_1,\dots,n_j}$ be the functor from $\mathbb{k}$-algebras to sets defined by $F^{n_1,\dots,n_j}(A) = \{(I_1, \dots, I_j)\}$ where*

- *for every i, $I_i \subset A[x_1, \dots, x_d]$ is an ideal,*
- *$A[x_1, \dots, x_d]/I_i$ is locally free on A of rank n_i,*
- *$(x_1, \dots, x_d)^{n_i} \subset I_i$,*
- *$I_1 \subset I_2 \cdots \subset I_j$.*

Then $F^{n_1,\dots,n_j}$ is representable.

Proof. For $j = 1$, the functor F^{n_1} parametrizes families of punctual subschemes of length n_1 in the closed subscheme W defined by the ideal $(x_1, \dots, x_d)^{n_1}$ in the affine space $Spec\ \mathbb{k}[x_1, \dots, x_d]$. It follows that this functor is representable by the Hilbert scheme $W^{[n_1]} = W_0^{[n_1]}$ We then proceed by induction. Let $G^{n_1,\dots,n_j}$ be the functor defined similarly to $F^{n_1,\dots,n_j}$, except that we replace the condition $I_1 \subset I_2 \cdots \subset I_j$ with the condition $I_1 \subset I_2 \cdots \subset I_{j-1}$. The functor $G^{n_1,\dots,n_j}$ is representable by $X_1 \times X_2$, where X_1 represents $F^{n_1,\dots,n_{j-1}}$, well defined by induction, and X_2 represents F^{n_j}. The inclusion of functors $F^{n_1,\dots,n_j} \subset G^{n_1,\dots,n_j}$ is defined by the extra condition $I_{j-1} \subset I_j$. According to the last lemma 2.4, this corresponds to a subscheme condition on the base of the families, ie. $F^{n_1,\dots,n_j} \subset G^{n_1,\dots,n_j}$ is relatively representable. It follows that $F^{n_1,\dots,n_j}$ is representable. ■

Functorial Definitions.

The functorial description of the Hilbert scheme $S^{[n]}$ is classical, but we need to precise the functorial description of $\widetilde{\mathcal{N}}^{cyc}$ and of the variants $S_0^{[n]}, S_0^{[k,n]}$ of the Hilbert scheme that we use.

Consider the Hilbert-Chow morphism $S^{[n]} \to Sym^n(\mathbb{A}^2)$, and compose it with the natural map $Sym^n(\mathbb{A}^2) \to Sym^n(\mathbb{A}^1) \times Sym^n(\mathbb{A}^1)$. We obtain a morphism $\rho : S^{[n]} \to Sym^n(\mathbb{A}^1) \times Sym^n(\mathbb{A}^1)$ which set-theoretically sends a subscheme z_n to the tuples of coordinates $(\{x_1,\dots,x_n\},\{y_1,\dots,y_n\})$ where (x_i,y_i) are the points of z_n counted with multiplicities. A morphism $Spec\ R \to S^{[n]}$ factorizes through $\rho^{-1}(0,0)$ if the corresponding ideal $I(Z) \subset R[x,y]$ satisfies $(x^n,y^n) \in I(Z)$. However, this property gives a special status to the lines $x=0$ and $y=0$ as shown by the following example, whose verification is straightforward.

Example 2.6. Let $R = \Bbbk[a,b]/(ab,b^2)$ and $I = (y+ax+b, x^2) \subset R[x,y]$. Then $x^2 \in I$, $y^2 \in I$, but for any $t \in \Bbbk^*$, $(x+ty)^2 \notin I$.

Consequently, we do not define $S_0^{[n]}$ as being $\rho^{-1}(0,0)$ and we ask for a coordinate-free definition. The dimension of the ambient space S plays no role in the definition. We shall give a general definition for the Hilbert scheme $Z_0^{[n]}$ parametrizing subschemes z_n of length n in a scheme Z of any dimension d supported on a smooth point $o \in Z$.

For this, we recall the well-known remark that a subscheme z_n of length n in a scheme Z is supported on a smooth point $o \in Z$ if and only if $I(o)^n \subset I(z_n)$, ie. if z_n is a subscheme of $Spec\,\Bbbk[x_1,\dots,x_d]/(x_1,\dots,x_d)^n$ where d is the dimension of Z at o. This leads to the following definitions for the localized Hilbert scheme $Z_0^{[n]}$ and the localized nested Hilbert scheme $Z_0^{[n_j,n_{j-1},\dots,n_1]}$. Of course, they include our two main objects of study $S_0^{[k,n]}$ and $S_0^{[\![k,n]\!]}$ with $Z = S = \mathbb{A}^2$ and $o = (0,0)$.

Definition 2.7. Let Z be a scheme over $\Bbbk$, $o \in Z$ a smooth point such that the local dimension of Z at o is d. The Hilbert scheme $Z_0^{[n]}$ is the scheme that represents the functor F^n of Proposition 2.5. Let $n_1 \geq n_2 \cdots \geq n_j > 0$ be integers. The Hilbert scheme $Z_0^{[n_j,n_{j-1},\dots,n_1]}$ is the scheme which represents the functor $F^{n_1,n_2,\dots,n_j}$.

As long as we consider topological properties, a superscript 1 plays no role since the schemes $S_0^{[1,n]}$ and $S_0^{[n]}$ are homeomorphic. In fact, the following proposition shows they are even isomorphic as varieties.

Proposition 2.8. *Let $(S_0^{[1,n]})_{red}$ and $(S_0^{[n]})_{red}$ be the varieties obtained from $S_0^{[1,n]}$ and $S_0^{[n]}$ with the reduced induced closed subscheme structure. Then $(S_0^{[1,n]})_{red} \cong (S_0^{[n]})_{red}$*

Proof. The functor $F^{n,1}$ associated with $S_0^{[1,n]}$ is defined by $F^{n,1}(A) = \{(I_1, I_2) \subset A[x,y]$ with $(x,y)^n \subset I_1 \subset I_2$, $(x,y) \subset I_2$, $A[x,y]/I_1$ locally free of rank

n, $A[x,y]/I_2$ locally free of rank $1\}$. In particular, $I_2 = (x,y)$ is the only possibility. In other words, if F^n denotes the functor associated with $S_0^{[n]}$, then $F^{n,1}$ can be seen as a subfunctor of F^n defined by the condition $I_1 \subset (x,y)$. By Keel's Lemma 2.4, this inclusion is relatively representable and $S_0^{[1,n]}$ is a closed subscheme of $S_0^{[n]}$. When $A = \Bbbk$, $\Bbbk[x,y]/(x,y)^n$ is a local ring with maximal ideal (x,y). It follows that the inclusion $I_2 \subset (x,y)$ is always satisfied or equivalently, that the embedding $S_0^{[1,n]} \subset S_0^{[n]}$ identifies the $\Bbbk$-points on both sides. This proves the proposition. ∎

Definition 2.9. Let A be a $\Bbbk$-algebra. Let $V(A)$, $V_k(A)$ and $\mathfrak{p}_{k,n}(A)$ be as (1) and below. Consider the functor m from $\Bbbk$-algrebras to sets where $m(A)$ is

$$\left\{ \begin{array}{c|c} (X,Y,v) \in & [X,Y]=0,\ X^n = X^{n-1}Y = \ldots = Y^n = 0, \\ \mathfrak{p}_{k,n}(A)\times\mathfrak{p}_{k,n}(A)\times V(A) & (X^{(k)})^{n-k} = \cdots = (Y^{(k)})^{n-k} = 0 \text{ on } V/V_k(A) \\ & ev_n \text{ and } ev_{n-k} \text{ are surjective} \end{array} \right\}$$

where

$$ev_n : \left\{ \begin{array}{rcl} A[x,y] & \to & V(A) \simeq A^n \\ P(x,y) & \mapsto & P(X,Y)(v) \end{array} \right. , ev_{n-k} : \left\{ \begin{array}{rcl} A[x,y] & \to & V(A)/V_k(A) \simeq A^{n-k} \\ P(x,y) & \mapsto & P(X,Y)(v) + V_k(A) \end{array} \right.$$

are the natural evaluation morphisms.

Proposition 2.10. *(Functorial definition of $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n}) \subset \mathfrak{p}_{k,n}\times\mathfrak{p}_{k,n}\times V$). The functor m is representable by a scheme $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$.*

Proof. We give a sketch of the proof. Let m' be the functor given by the same conditions as m except the surjectivity of ev_n and ev_{n-k}. In view of [St, Example 2.1], m' is representable by a closed affine subscheme of $\mathfrak{p}_{k,n}\times\mathfrak{p}_{k,n}\times V$. Then, the inclusion $m \subset m'$ is defined by surjectivity conditions, or equivalently by the invertibility of some determinant. It follows that this inclusion of functors is relatively representable, using the same argument as in the proof of Lemma 2.3. ∎

The first point of the following lemma shows that the closed points of $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ are the expected triples (X,Y,v). Since, on $\Bbbk$-points, we require X and Y to be nilpotent, it could seem natural in the above definition of the functor $A \mapsto m(A)$ to replace the condition $X^n = X^{n-1}Y = \ldots = Y^n = 0$ with the simpler condition $X^n = Y^n = 0$. The second point of the lemma shows that this would add extra embedded components to $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ and we are not interested in these components.

Lemma 2.11. *(i) Let $X, Y \in M_n(\Bbbk)$ be a pair of nilpotent commuting matrices. Then $X^iY^{n-i} = 0$ for all $i \in [\![0,n]\!]$.*
(ii) The above conclusion may fail when replacing $\Bbbk$ by an arbitrary (even noetherian) $\Bbbk$-algebra R.

Proof. (i) From reduction theory, it is an elementary fact that X and Y are simultaneously strictly upper trigonalisable. Hence the equalities.

(ii) Take $R = \Bbbk[a,b]/(ab,b^2)$, $X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} b & 0 \\ a & b \end{pmatrix}$. Then $X^2 = Y^2 = 0$ and $XY = YX = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$. ∎

Finally, we can define the action of $P_{k,n}$ on $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$, *i.e.* the morphism $\gamma : P_{k,n} \times \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n}) \to \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ at the functorial level. Let $g \in P_{k,n}(A)$, $t = (X,Y,v) \in m(A)$ so $(g,t) \in Hom(Spec\ A, P_{k,n} \times \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n}))$. Then the element $t' = (X',Y',v') \in m(A)$, image of (g,t) by the action morphism γ, is $X' = gXg^{-1}, Y' = gYg^{-1}, v' = gv$.

Proof of Theorem 2.2. The cases of $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ and $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})$ are similar and we consider only the first case. The strategy is the following. We first construct a categorical quotient. Using the functorial properties of both the categorical quotient and the Hilbert scheme, we construct the isomorphism between $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})/P_{k,n}$ and $S_0^{[n-k,n]}$. Finally, using the description of the quotient via the Hilbert scheme, we show that the categorical quotient turns out to be a geometric quotient.

Let $\Delta_{n-k} \subset \Delta_n$ be two sets of monomials $\{\delta_i = x^{\alpha_i}y^{\beta_i}\}$ of respective cardinality $n-k$ and n. Let $\Delta = \{\Delta_{n-k}, \Delta_n\}$. For each such Δ, there is an open subscheme $\widetilde{\mathcal{N}}^{cyc}_{\Delta} \subset \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ whose support is the locus where the evaluation morphisms ev_{n-k} and ev_n are surjective using only the images of the monomials in Δ. More precisely, let $A[\Delta_i]$ be the free A-module with basis Δ_i. The open subscheme $\widetilde{\mathcal{N}}^{cyc}_{\Delta}$ corresponds to the subfunctor $m_\Delta(A) \subset m(A)$ containing the triples $(X,Y,v) \in m(A)$ such that $ev_{\Delta_n} : A[\Delta_n] \to A^n$, $\delta_i \mapsto (\delta_i(X,Y)(v))$ and $ev_{\Delta_{n-k}} : A[\Delta_{n-k}] \to A^{n-k}$, $\delta_i \mapsto (\delta_i(X,Y)(v)) mod\ V_k(A)$ are surjective.

Recall that the surjectivity of the A-linear maps $ev_{\Delta_{n-k}}$ and ev_{Δ_n} is equivalent to their being an isomorphism ([AM], Exercice 3.15), thus to their determinant being invertible in A. In particular, $\widetilde{\mathcal{N}}^{cyc}_{\Delta}$ is defined by the nonvanishing of a determinant in $\mathcal{N}(\mathfrak{p}_{k,n}) \times \Bbbk^n$, hence it is affine.

Since we have a covering of $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ with open affine $P_{k,n}$-stable subschemes $\widetilde{\mathcal{N}}^{cyc}_{\Delta} \simeq Spec\ B_\Delta$, it is possible to construct a categorical quotient on each open subscheme as $\widetilde{\mathcal{N}}^{cyc}_{\Delta}/P_{k,n} := Spec\ B_\Delta^{P_{k,n}}$ with the invariant functions. Since the group is not reductive, $B_\Delta^{P_{k,n}}$ is not a priori finitely generated (and we cannot apply [MFK, Thm 1.1]). We have to show without the general theory that the local quotients are algebraic (*i.e.* of finite type over $\Bbbk$) and that the local constructions glue to produce a global categorical quotient.

Recall the functor h which defines the Hilbert scheme $S_0^{[k,n]}$. If Δ is as above, there is a subfunctor h_Δ of h. By definition, $h_\Delta(A)$ contains the pairs $(I,J) \in h(A)$ such that $A[x,y]/I$ (resp. $A[x,y]/J$) is free on A of rank $n-k$ (resp. of rank n) and such that the monomials δ_i in Δ_{n-k} (resp. in Δ_n) form a basis of $A[x,y]/I$ (resp. $A[x,y]/J$). This is a relatively representable subfunctor, which is representable by an open subscheme $S_\Delta \subset S_0^{[n-k,n]}$.

There is a morphism of functors $m \to h$ defined by

$$(X,Y,v) \in m(A) \mapsto (I = Ker(ev_{n-k}), J = Ker(ev_n)) \in h(A)$$

and a morphism of schemes $\widetilde{\pi}_{k,n} : \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n}) \to S_0^{[n-k,n]}$ associated with the morphism of functors. By construction, this map is invariant under the action of $P_{k,n}$. From the universal property of the categorical quotient, we obtain a factorisation $\widetilde{\mathcal{N}}_\Delta^{cyc}/P_{k,n} \to S_0^{[n-k,n]}$ whose image is in S_Δ, hence the factorisation $i_\Delta : \widetilde{\mathcal{N}}_\Delta^{cyc}/P_{k,n} \to S_\Delta$.

To prove that i_Δ is an isomorphism, we will construct an inverse ρ_Δ. Let $(I, J) \in h_\Delta(A)$. We choose a basis $b_1, \dots, b_n$ of $A[x, y]/J$ such that $b_{k+1}, \dots, b_n$ is a basis of $A[x, y]/I$. Such a basis exists since we can take b_i to be the monomials in Δ. If we replace each element $b_i, i \le k$ by a suitable combination $b_i + \sum_{j \ge k+1} a_{ij} b_j$, we may suppose that the kernel I/J of the map $A[x, y]/J \to A[x, y]/I$ is generated by $b_1, \dots, b_k$. This choice of our basis yields an effective isomorphism $A[x, y]/J \simeq A^n$. The multiplication by x and y on $A[x, y]/J$ then correspond to matrices $X, Y \in \mathfrak{p}_{k,n}(A)$. Choose $v = 1 \in A[x, y]/J$. Then $(X, Y, v) \in m(A)$ and corresponds to a morphism $\nu : Spec\ A \to \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$. This morphism is not canonically defined because of the arbitrary choice of the basis $b_1, \dots, b_n$. However, if ν_1 and ν_2 are two possible choices for the morphism ν, and if $\varphi \in P_{k,n}(A) = Hom(Spec\ A, P_{k,n})$ is the decomposition matrix of the basis defining ν_1 on the basis defining ν_2, then $\nu_2 = \gamma \circ (\varphi, \nu_1)$, where γ is the action morphism. Since ν_1 and ν_2 differ by the action of $P_{k,n}(A)$, it follows that the morphism $\eta = q \circ \nu_1 = q \circ \nu_2$ is well defined. The map which sends (I, J) to η is a morphism of functors. This is the functorial description of a scheme morphism $\rho_\Delta : S_\Delta \to \widetilde{\mathcal{N}}_\Delta^{cyc}/P_{k,n}$. By construction, ρ_Δ and i_Δ are mutually inverse.

Since we proved that our local quotients $\widetilde{\mathcal{N}}_\Delta^{cyc}/P_{k,n}$ are isomorphic to an open subscheme S_Δ of the Hilbert scheme $S_0^{[n-k,k]}$, these local quotients are algebraic. Gluing these local quotients to form a global quotient $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})/P_{k,n}$ is straightforward: this corresponds to the gluing of the open subschemes S_Δ in the Hilbert scheme $S_0^{[n-k,k]}$.

So far, we have proved that the Hilbert scheme $S_0^{[n-k,k]}$ is a categorical quotient of $\widetilde{\mathcal{N}}^{cyc}$. There remains to prove that this quotient is locally trivial in the Zariski topology. This will imply the remaining statements of the theorem, namely that the quotient is geometrical and the surjectivity of the quotient morphism. We shall prove the local triviality over S_Δ. More precisely, we shall exhibit a pair of inverse isomorphisms φ_1, φ_2 to prove that $S_\Delta \times P_{k,n}$ and $\widetilde{\mathcal{N}}_\Delta^{cyc}$ are isomorphic as schemes over S_Δ.

Remark that we have constructed a (non-canonical) map $h_\Delta(A) \mapsto m(A)$ sending (I, J) to ν. Since this map depends functorially on A, this functor corresponds to a section $s_\Delta : S_\Delta \to \widetilde{\mathcal{N}}_\Delta^{cyc}$ of the map $\widetilde{\pi}_{k,n} : \widetilde{\mathcal{N}}_\Delta^{cyc} \to S_\Delta$. We define φ_1 to be the composition

$$S_\Delta \times P_{k,n} \overset{(s_\Delta, Id)}{\to} \widetilde{\mathcal{N}}_\Delta^{cyc} \times P_{k,n} \to \widetilde{\mathcal{N}}_\Delta^{cyc}$$

where the second arrow is given by the group action.

The identity map $id_{\widetilde{\mathcal{N}}_\Delta^{cyc}}$ on $\widetilde{\mathcal{N}}_\Delta^{cyc} \simeq Spec(B_\Delta)$, is an element of $m_\Delta(B_\Delta)$. It yields an evaluation map $(ev_n)_1$ and the following diagram, where J is the kernel of $(ev_n)_1$ and I is the kernel of $\psi \circ (ev_n)_1$.

$$\begin{array}{ccccc} & & I & & I/J \\ & & \downarrow & & \downarrow \\ J & \hookrightarrow & B_\Delta[x,y] & \overset{(ev_n)_1}{\to} & V(B_\Delta) \\ & & & & \downarrow \psi \\ & & & & V(B_\Delta)/V_k(B_\Delta) \end{array} .$$

Using the map $s_\Delta \circ \widetilde{\pi}_{k,n} : \mathcal{N}^{cyc}_\Delta \to \widetilde{\mathcal{N}}^{cyc}_\Delta$ instead of the identity map, we get a similar diagram with $(ev_n)_2$ instead of $(ev_n)_1$ and I, J unchanged. The morphism $g = (ev_n)_1 \circ ((ev_n)_2)^{-1} \in \mathrm{GL}(V(B_\Delta))$ is then well defined. Since $((ev_n)_2)^{-1}(Ker(\psi)) = I$, the morphism g sends $I/J = Ker(\psi) = V_k(B_\Delta)$ to itself and $g \in P_{k,n}(B_\Delta) = Hom(Spec(B_\Delta), P_{k,n})$. We define $\varphi_2 : \widetilde{\mathcal{N}}^{cyc}_\Delta \to S_\Delta \times P_{k,n}$ by $\varphi_2 = (\widetilde{\pi}_{k,n}, g)$. By construction, the morphisms φ_1 and φ_2 are inverse. ∎

From $\widetilde{\mathcal{N}}$ to $\mathcal{N}$.

In the previous section, the Hilbert schemes $S_0^{[k,n]}$ ans $S_0^{[\![k,n]\!]}$ have been constructed as quotients of the schemes $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ and $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})$ which parametrize triples (X, Y, v). In this section, we show how to throw off the data v. From this point and until the end of the article, we only need to work with the underlying variety structure on our schemes. In particular, we will consider the following variety for $\mathfrak{w} = \mathfrak{p}_{k,n}$ or $\mathfrak{q}_{k,n}$:

$$\mathcal{N}^{cyc}(\mathfrak{w}) := \{(X,Y) \in \mathcal{N}(\mathfrak{w}) \mid \exists v \in V \text{ s.t. } (X,Y,v) \in \widetilde{\mathcal{N}}^{cyc}(\mathfrak{w})\}.$$

Lemma 2.12. *(i) The action of $P_{k,n}$ (resp. $Q_{k,n}$) on $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ (resp. $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})$) is free.*

(ii) Let $v_1, v_2 \in V$ such that $(X, Y, v_i) \in \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})$ (resp. $\widetilde{\mathcal{N}}^{cyc}(\mathfrak{q}_{k,n})$). Then (X, Y, v_1) and (X, Y, v_2) belong to the same $P_{k,n}$ (resp. $Q_{k,n}$)-orbit.

Proof. (i) Let $(X, Y, v) \in \widetilde{\mathcal{N}}^{cyc}(\mathfrak{w})$ and $g \in \mathrm{GL}(V)$ stabilizing (X, Y, v). Then g stabilizes each $X^iY^j(v)$ and, since these elements generates V, we have $g = Id$.
(ii) Let $g : \left\{ \begin{array}{ccc} V & \to & V \\ P(X,Y).v_1 & \mapsto & P(X,Y).v_2 \end{array} \right.$. It is well defined since $\{P \in \mathbb{k}[x,y] | P(X,Y).v_i = 0\} = \{P \in \mathbb{k}[x,y] | P(X,Y) = 0\}$ by the cyclicity condition. Moreover g is linear and $g.v_1 = v_2$.
For any $S \in \mathbb{k}[x,y]$, we have the equality $gXg^{-1}(S(X,Y).v_2) = gXS(X,Y)(v_1) = g(S'(X,Y)(v_1)) = S'(X,Y).v_2 = X(S(X,Y)(v_2))$ where $S' = xS \in \mathbb{k}[x,y]$. In particular, g stabilizes X by cyclicity of v_2 and the same holds for Y.
A similar argument shows that any subspace $V_i \subset V$ stable under X and Y is stabilized by g. The cyclicity property implies that $g.v_1 = S(X,Y)(v_1)$ and that V_i is generated by $(R_l(X,Y)(v_1))_l$ for some polynomials $S, (R_l)_l$ of $\mathbb{k}[x,y]$. Then $g.V_i$ is generated by $(g.R_l(X,Y)(v_1))_l = (R_l(X,Y)(g.v_1))_l = ((R_l(X,Y) \times S(X,Y))(v_1))_l = (S(X,Y)(R_l(X,Y)(v_1)))_l \subset V_i$. Hence g stabilizes each such subspace V_i and the result follows from the definitions of $P_{k,n}$ and $Q_{k,n}$. ∎

It follows from Lemma 2.12(ii) that the following set-theoretical quotient map

$$\pi_{k,n} : \begin{cases} \mathcal{N}^{cyc}(\mathfrak{p}_{k,n}) & \to & S_0^{[n-k,n]} \\ (X,Y) & \mapsto & (Ker(ev_{n-k}), Ker(ev_n)) \\ & & (= \widetilde{\pi}_{k,n}(X,Y,v)\ \forall v \in V \text{ s.t. } (X,Y,v) \in \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n})) \end{cases}$$

is well defined where

$$ev_{n-k} : \begin{cases} \Bbbk[x,y] & \to & \mathfrak{gl}(V/V_k) \\ P & \mapsto & P(X^{(k)}, Y^{(k)}) \end{cases} \text{ and } ev_n : \begin{cases} \Bbbk[x,y] & \to & \mathfrak{gl}(V) \\ P & \mapsto & P(X,Y). \end{cases}$$

This also allows to define $\pi'_{k,n} : \mathcal{N}^{cyc}(\mathfrak{q}_{k,n}) \to S_0^{[\![n-k,n]\!]}$.

Proposition 2.13. *$\pi_{k,n}$ induces a bijection between irreducible components of $S_0^{[n-k,n]}$ of dimension m and irreducible components of $\mathcal{N}^{cyc}(\mathfrak{p}_{k,n})$ of dimension $m + (\dim \mathfrak{p}_{k,n} - n)$. The same holds for $\pi'_{k,n}$, $S_0^{[\![n-k,n]\!]}$ and $\mathcal{N}^{cyc}(\mathfrak{q}_{k,n})$.*

Proof. As usual, we give a proof only for $\mathfrak{p}_{k,n}$.

Let Z_1, Z_2 be varieties and $f : Z_1 \to Z_2$ be an open surjective morphism with irreducible fibers. Then, the pre-image by f of any irreducible component of Z_2 is irreducible (*e.g.* see [TY, Proposition 1.1.7]). On the other hand, the image of any irreducible component of Z_1 by f is irreducible. Hence f induces a bijection between irreducible components of Z_1 and Z_2.

Then, since a geometric quotient by a connected group satisfies the above assumptions on f, we can apply the previous argument to $\widetilde{\pi}_{k,n}$. It also works for $pr : \begin{cases} \widetilde{\mathcal{N}}^{cyc}(\mathfrak{p}_{k,n}) & \to & \mathcal{N}^{cyc}(\mathfrak{p}_{k,n}) \\ (X,Y,v) & \mapsto & (X,Y) \end{cases}$. The dimension statement follows since fibers of $\widetilde{\pi}_{k,n}$ are of dimension $\dim \mathfrak{p}$ (Lemma 2.12 (i)) and those of pr are of dimension n (given (X,Y), the set $\{v |\, (X,Y,v) \in \widetilde{\mathcal{N}}(\mathfrak{p}_{k,n})\}$ is open in V). ■

The correspondence with commuting varieties allows us to see in an elementary way some non-trivial facts on the Hilbert scheme. We give an example.

Proposition 2.14. *Given a pair $(z_{n-k}, z_n) \in S_0^{[n-k,n]}$, there exists a chain of intermediate subschemes $z_{n-k} \subset z_{n-k+1} \subset \cdots \subset z_n$. In other words, the projection map $S_0^{[\![n-k,n]\!]} \to S_0^{[n-k,n]}$ is surjective. The same holds for the projection map $S_0^{[\![n-k,n]\!]} \to S_0^{[\![n-k',n]\!]}$ with $k \geqslant k'$.*

Proof. The first assertion follows from the fact that any commuting pair $(X_{|V_k}, Y_{|V_k}) \in \mathfrak{gl}(V_k)$ is simultaneously trigonalizable by an element of $\mathrm{GL}_{V_k} \subset P_{k,n}$. Hence, in the new basis, it stabilizes the flag $V_1 \subset V_2, \cdots \subset V_k$. The second one is the same argument applied to the pair $(X^{(k)}, Y^{(k)}) \in \mathfrak{gl}(V/V_k)$. ■

Remark 2.15. Note that there is a Lie algebra isomorphism between $\mathfrak{p}_{k,n}$ and $\mathfrak{p}_{n-k,n}$ (namely, minus the transposition with respect to the anti-diagonal). Hence the two varieties $\mathcal{N}(\mathfrak{p}_{k,n})$ and $\mathcal{N}(\mathfrak{p}_{n-k,n})$ are isomorphic.

$$\begin{array}{ccccc} \mathcal{N}^{cyc}(\mathfrak{p}_{n-k,n}) & \underset{\text{open}}{\hookrightarrow} & \mathcal{N}(\mathfrak{p}_{k,n}) & \underset{\text{open}}{\hookleftarrow} & \mathcal{N}^{cyc}(\mathfrak{p}_{k,n}) \; . \\ {\scriptstyle \pi_{n-k,n}}\downarrow & & & & \downarrow{\scriptstyle \pi_{k,n}} \\ S_0^{[k,n]} & & & & S_0^{[n-k,n]} \end{array}$$

We use this duality in Lemma 4.7 where we pull back informations related to irreducibility from $S_0^{[1,n]}$ to $\mathcal{N}(\mathfrak{p}_{n-1,n}) \cong \mathcal{N}(\mathfrak{p}_{1,n})$. Eventually, this turns out to be a key part of our proof of the irreducibility of $S_0^{[n-1,n]}$ (cf. Corollary 4.9).

However, the cyclicity condition breaks the symmetry and there might be profound differences between $\mathcal{N}^{cyc}(\mathfrak{p}_{k,n})$ and $\mathcal{N}^{cyc}(\mathfrak{p}_{n-k,n})$, hence between $S_0^{[n-k,n]}$ and $S_0^{[k,n]}$. For instance, $S_0^{[1,3]}$ and $S_0^{[2,3]}$ both contain a curvilinear locus as an open subvariety, and these curvilinear loci are isomorphic. On the boundary of this curvilinear locus, the two Hilbert schemes are quite different: when the scheme z_3 has equation (x^2, xy, y^2) there is set theoretically only one length 1 point z_1 in z_3, but there is a $\mathbb{P}^1$ of z_2 with length 2 satisfying $z_2 \subset z_3$.

3. Technical lemmas on matrices

In this section, we collect technical results that will be used later on. Most of these results aim to describe $\mathfrak{a}^{nil} \subset \mathfrak{a}$, where $\mathfrak{a}$ is a space of matrices commuting with a Jordan matrix of type $\boldsymbol{\lambda} \in \mathcal{P}(n)$ and $\mathfrak{a}^{nil}$ is the set of nilpotent matrices of $\mathfrak{a}$. In particular, we will make frequent use of Lemmas 3.3 and Proposition 3.5. Parts of the results shown are well known in the more general framework of Lie algebras. Our goal is to translate this in the matrix setting and to provide a low-level understanding of the involved phenomena.

Lemma 3.1.
(i) $(M_n)^{nil}$ *is an irreducible subvariety of codimension* n *in* M_n.
(ii) Assume that $\mathfrak{p}$ *is the parabolic subalgebra defined by* $\mathfrak{p} = \{X \in M_n |\ \forall j,\ X(V_{i_j}) \subset V_{i_j}\}$ *where the* i_j *are* $k+1$ *indices satisfying* $0 = i_0 \leqslant i_1 \leqslant \dots i_k = n$. *Then* $X \in \mathfrak{p}$ *is nilpotent if and only if the* k *extracted matrices*

$$X_j = \begin{pmatrix} X_{i_{j-1}+1, i_{j-1}+1} & \cdots & X_{i_{j-1}+1, i_j} \\ \vdots & & \vdots \\ X_{i_j, i_{j-1}+1} & \cdots & X_{i_j, i_j} \end{pmatrix} \in M_{i_j - i_{j-1}}, \qquad 1 \leqslant j \leqslant k,$$

are nilpotent.
(iii) If $\mathfrak{p}$ *is a parabolic subalgebra of* M_n *then* $\mathfrak{p}^{nil}$ *is an irreducible subvariety of* $\mathfrak{p}$ *of codimension* n.

Proof. (i) See [Bas03, Proposition 2.1] for an elementary proof of this classical fact.
(ii) First, note that X_j can be viewed as the matrix of the endomorphism induced by X on $V_{i_j}/V_{i_{j-1}}$. Then, as vector spaces,

$$\mathfrak{p} \overset{v.s.}{\cong} \mathfrak{l} \oplus \mathfrak{n} \qquad \text{where} \left\{ \begin{array}{l} \mathfrak{l} := \prod_{j=1}^{k} (\mathrm{End}(V_{i_j}/V_{i_j - 1})) \\ \mathfrak{n} := \{X \in \mathfrak{p} \,|\, X(V_{i_j}) \subset V_{i_{j-1}}\} \end{array} \right.$$

and $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{p}$. Hence $X = X_{\mathfrak{l}} + X_{\mathfrak{n}} \in \mathfrak{p}$ is nilpotent if and only if $X_{\mathfrak{l}}$ is nilpotent. This is equivalent to the nilpotency of each X_j.
(iii) Up to base change, one can assume that $\mathfrak{p}$ satisfies the hypothesis of (ii). Thus $\mathfrak{p}^{nil}$ is isomorphic to $\prod_{j=1}^{k}(\mathrm{End}(V_{i_j}/V_{i_j-1}))^{nil} \times \mathfrak{n}$. It then follows from (i) that $\mathfrak{p}^{nil}$ is an irreducible subvariety of $\mathfrak{p}$ of codimension $\sum_{j=1}^{k}(i_j - i_{j-1}) = n$. ∎

Let us explain (ii) in a more visual way.

Example 3.2. A matrix of the form

$$X = \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \\ 0 & 0 & k & l & m \\ 0 & 0 & n & o & p \\ 0 & 0 & q & r & s \end{pmatrix}$$

is nilpotent if and only if the two following submatrices are nilpotent

$$X_1 = \begin{pmatrix} a & b \\ f & g \end{pmatrix}, \qquad X_2 = \begin{pmatrix} k & l & m \\ n & o & p \\ q & r & s \end{pmatrix}$$

Fix an element $\boldsymbol{\lambda} = (\lambda_1 \geqslant \cdots \geqslant \lambda_{d_{\boldsymbol{\lambda}}})$ in $\mathcal{P}(n)$, the set of partitions of n. We define $X_{\boldsymbol{\lambda}} \in M_n$ as the nilpotent element in Jordan canonical form associated to $\boldsymbol{\lambda}$. In other words, in the basis $(f_j^i := e_{\sum_{\ell=1}^{i-1}\lambda_\ell + j})_{\substack{1 \leqslant i \leqslant d_{\boldsymbol{\lambda}} \\ 1 \leqslant j \leqslant \lambda_i}}$, we have

$$X_{\boldsymbol{\lambda}}(f_j^i) = \begin{cases} f_{j-1}^i & \text{if } j \neq 1, \\ 0 & \text{else.} \end{cases} \tag{2}$$

For $Y \in M_n$, we denote the entries of Y via $Y.f_{j'}^{i'} = \sum_{(i,j)} Y_{j,j'}^{i,i'} f_j^i$ and use the following notation

$$Y = \left(Y_{j,j'}^{i,i'}\right)_{(i,j),(i',j')}.$$

An explicit characterization of $M_n^{X_{\boldsymbol{\lambda}}} := \{Y \in M_n |\ [X_{\boldsymbol{\lambda}}, Y] = 0\}$ is given by the following classical lemma.

Lemma 3.3. *$Y \in M_n^{X_{\boldsymbol{\lambda}}}$ if and only if the following relations are satisfied:*

$$\begin{cases} Y_{j,j'}^{i,i'} = 0 & \text{if } j > j' \text{ or } \lambda_i - j < \lambda_{i'} - j', \\ Y_{j,j'}^{i,i'} = Y_{j-1,j'-1}^{i,i'} & \text{if } 2 \leqslant j \leqslant j' \text{ and } \lambda_i - j \geqslant \lambda_{i'} - j'. \end{cases}$$

Picturally, this means that Y can be decomposed into blocks $Y^{i,i'} \in M_{\lambda_i,\lambda_{i'}}$ where

$$Y^{i,i'} = \begin{pmatrix} Y^{i,i'}_{1,1} & Y^{i,i'}_{1,2} & \dots & Y^{i,i'}_{1,\lambda_{i'}} \\ 0 & Y^{i,i'}_{1,1} & \ddots & \vdots \\ \vdots & 0 & \ddots & Y^{i,i'}_{1,2} \\ \vdots & \vdots & \ddots & Y^{i,i'}_{1,1} \\ \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{if } \lambda_i \geqslant \lambda_{i'},$$

$$Y^{i,i'} = \begin{pmatrix} 0 & \dots & 0 & Y^{i,i'}_{\lambda_i,\lambda_{i'}} & \dots & Y^{i,i'}_{2,\lambda_{i'}} & Y^{i,i'}_{1,\lambda_{i'}} \\ 0 & \dots & \dots & 0 & \ddots & \ddots & Y^{i,i'}_{2,\lambda_{i'}} \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & Y^{i,i'}_{\lambda_i,\lambda_{i'}} \end{pmatrix} \quad \text{if } \lambda_i \leqslant \lambda_{i'}.$$

Proof. See [TA] or [Bas00, Lemma 3.2] for a more recent account. ■

Fix $\boldsymbol{\lambda} \in \mathcal{P}(n)$. For each length $\ell \in \mathbb{N}^*$ appearing in $\boldsymbol{\lambda}$ (i.e. $\exists i \in [\![1, d_{\boldsymbol{\lambda}}]\!], \lambda_i = \ell$), we define $\tau_\ell = \sharp\{i|\lambda_i = \ell\}$. Let $W_\ell := \langle f_1^i | \lambda_i \geqslant \ell\rangle$. This is a filtration of $W := W_1 = \langle f_1^i | i \in [\![1, d_{\boldsymbol{\lambda}}]\!]\rangle$ whose associated grading is given by the subspaces $W'_\ell := \langle f_1^i|\lambda_i \geqslant \ell\rangle / \langle f_1^i|\lambda_i > \ell\rangle$ of dimension τ_ℓ.

It follows from Lemma 3.3 that each W_ℓ is stable under $M_n^{X_{\boldsymbol{\lambda}}}$. Hence we have a Lie algebra morphism $M_n^{X_{\boldsymbol{\lambda}}} \stackrel{pr_{ext}}{\longrightarrow} M_{d_{\boldsymbol{\lambda}}}$ where the extracted matrix $pr_{ext}(Y) = Y_{ext} := (Y^{i,i'}_{1,1})_{i,i'}$ can be seen as the element induced by Y on $W = \operatorname{Ker} X_{\boldsymbol{\lambda}}$.

Lemma 3.4. *The image $(M_n^{X_{\boldsymbol{\lambda}}})_{ext}$ of the morphism pr_{ext} is the parabolic subalgebra*

$$\{Z \in M_{d_{\boldsymbol{\lambda}}} \,|\, Z(W_\ell) \subset W_\ell, \forall \ell \in \mathbb{N}^*\}.$$

Proof. It is an immediate consequence of Lemma 3.3. ■

Similarily we define the surjective (cf. Lemma 3.3) maps $M_n^{X_{\boldsymbol{\lambda}}} \stackrel{pr_\ell}{\to} M_{\tau_\ell} =: M_n^{X_{\boldsymbol{\lambda}}}(\ell) \cong \mathfrak{gl}(W'_\ell)$ where

$$pr_\ell(Y) = Y(\ell) := (Y^{i,i'}_{1,1})_{((i,i')|\lambda_i = \lambda_{i'} = \ell)} \tag{3}$$

can be seen as the element induced by Y on W'_ℓ. We also define $(M_n^{X_{\boldsymbol{\lambda}}})_{gr} := \prod_\ell M_n^{X_{\boldsymbol{\lambda}}}(\ell)$ and pr_{gr} as the surjective map: $\left\{\begin{array}{ccc} M_n^{X_{\boldsymbol{\lambda}}} & \to & (M_n^{X_{\boldsymbol{\lambda}}})_{gr} \\ Y & \mapsto & Y_{gr} = \prod_\ell Y(\ell) \end{array}\right.$. We have a natural section $\varphi : (M_n^{X_{\boldsymbol{\lambda}}})_{gr} \to M_n^{X_{\boldsymbol{\lambda}}}$ of the Lie algebra morphism pr_{gr} by setting $Z^{i,i'}_{j,j'} := \left\{\begin{array}{ll} Y^{i,i'} & \text{if } j = j',\ \lambda_i = \lambda_{i'} \\ 0 & \text{else} \end{array}\right.$ and $\varphi((Y^{i,i'})_{i,i'}) := (Z^{i,i'}_{j,j'})_{(i,j),(i',j')}$. Hence, we can view $(M_n^{X_{\boldsymbol{\lambda}}})_{gr}$ as a subalgebra of $M_n^{X_{\boldsymbol{\lambda}}}$ and

$$M_n^{X_{\boldsymbol{\lambda}}} \stackrel{v.s.}{\cong} (M_n^{X_{\boldsymbol{\lambda}}})_{gr} \oplus \mathfrak{n}, \tag{4}$$

where $\mathfrak{n} := \mathrm{Ker}(pr_{gr})$. A similar decomposition $pr_{ext}\colon M_n^{X_{\boldsymbol\lambda}} \overset{v.s.}{\cong} (M_n^{X_{\boldsymbol\lambda}})_{ext} \oplus \mathfrak{n}_1$ holds, where $\mathfrak{n}_1 := \mathrm{Ker}(pr_{ext})$.

Proposition 3.5.
(i) $Y \in M_n^{X_{\boldsymbol\lambda}}$ is nilpotent if and only if Y_{gr} is. In other words $(M_n^{X_{\boldsymbol\lambda}})^{nil} \cong (M_n^{X_{\boldsymbol\lambda}})^{nil}_{gr} \times \mathfrak{n}_1$
(ii) $Y \in M_n^{X_{\boldsymbol\lambda}}$ is nilpotent if and only if each $Y(\ell) \in M_{\tau_\ell}$ is.
(iii) $(M_n^{X_{\boldsymbol\lambda}})^{nil}$ is an irreducible subvariety of $M_n^{X_{\boldsymbol\lambda}}$ of codimension $d_{\boldsymbol\lambda}$.

Proof. We associate to each basis element f_j^i the weight $w(f_j^i) := (j-\lambda_i, j)$. We order the weights lexicographically. Lemma 3.3 asserts that $Y \in M_n^{X_{\boldsymbol\lambda}}$ is parabolic with respect to these weights, *i.e.* $Y(f_b^a) = \sum_{w(f_{b'}^{a'}) \leq w(f_b^a)} c_{b'}^{a'} f_{b'}^{a'}$. Remark that two elements f_j^i and $f_{j'}^{i'}$ have the same weight if and only if $\lambda_i = \lambda_{i'}$ and $j = j'$. We order the basis f_j^i with respect to their weight. The base change from the f_j^i lexicographically ordered by their index (i,j) to the f_j^i ordered by their weight transforms the matrix Y into a matrix Z.

Let w be a weight and $f_j^{i_1}, f_j^{i_2}, \dots f_j^{i_k}$ be the elements with weight w and $\ell := \lambda_{i_m}$ (for any $m \in [\![1,k]\!]$). The diagonal block of Z corresponding to the weight w is $Y(\ell)$ (Lemma 3.3). In other words, Z and Y_{gr} have the same diagonal blocks $Y(\ell)$, the difference being that the same block is repeated ℓ times in Z. In conclusion, Y_{gr} is nilpotent iff its diagonal blocks $Y(\ell)$ are nilpotent, iff Z and Y are nilpotent. This proves i) and ii). Since $(M_n^{X_{\boldsymbol\lambda}})_{gr} \cong \prod_{\ell \in \mathbb{N}^*} M_{\tau_\ell}$ and $\sum_{\ell \in \mathbb{N}^*} \tau_\ell = d_{\boldsymbol\lambda}$, Lemma 3.1 (i) allows us to conclude. ∎

In the Lie algebra vocabulary, $(M_n^{X_{\boldsymbol\lambda}})_{gr}$ is a reductive part (in M_n) of the centraliser of $X_{\boldsymbol\lambda}$ in M_n and $\mathfrak{n}$ is its nilpotent radical so (4) can be written as a Lie algebra isomorphism $M_n^{X_{\boldsymbol\lambda}} \overset{v.s.}{\cong} (M_n^{X_{\boldsymbol\lambda}})_{gr} \ltimes \mathfrak{n}$. See [Pr] for an analogue of Proposition 3.5 (ii) valid for a general reductive Lie algebra.

Example 3.6. Let $n = 12$, $\boldsymbol\lambda = (4,2,2,2,1,1)$ hence

$$X_{\boldsymbol\lambda} = \left(\begin{array}{cccc|cc|cc|cc|c|c}
0 & 1 & & & & & & & & & & \\
 & 0 & 1 & & & & & & & & & \\
 & & 0 & 1 & & & & & & & & \\
 & & & 0 & & & & & & & & \\
\hline
 & & & & 0 & 1 & & & & & & \\
 & & & & & 0 & & & & & & \\
\hline
 & & & & & & 0 & 1 & & & & \\
 & & & & & & & 0 & & & & \\
\hline
 & & & & & & & & 0 & 1 & & \\
 & & & & & & & & & 0 & & \\
\hline
 & & & & & & & & & & 0 & \\
\hline
 & & & & & & & & & & & 0
\end{array}\right),$$

$$M_n^{X_{\boldsymbol\lambda}} \ni Y = \left(\begin{array}{cccc|cc|cc|cc|c|c} a & b & c & d & h_1 & i_1 & h_2 & i_2 & h_3 & i_3 & j_1 & j_2 \\ & a & b & c & & h_1 & & h_2 & & h_3 & & \\ & & a & b & & & & & & & & \\ & & & a & & & & & & & & \\ \hline & & h_4 & i_4 & e_1 & f_1 & k_1 & l_1 & k_2 & l_2 & m_1 & m_2 \\ & & & h_4 & & e_1 & & k_1 & & k_2 & & \\ \hline & & h_5 & i_5 & k_3 & l_3 & e_2 & f_2 & k_4 & l_4 & m_3 & m_4 \\ & & & h_5 & & k_3 & & e_2 & & k_4 & & \\ \hline & & h_6 & i_6 & k_5 & l_5 & k_6 & l_6 & e_3 & f_3 & m_5 & m_6 \\ & & & h_6 & & k_5 & & k_6 & & e_3 & & \\ \hline & & & j_3 & & m_7 & & m_8 & & m_9 & g_1 & n_1 \\ \hline & & & j_4 & & m_{10} & & m_{11} & & m_{12} & n_2 & g_2 \end{array}\right).$$

Here $d_{\boldsymbol\lambda} = 6,\ \tau_4 = 1,\ \tau_2 = 3,\ \tau_1 = 2$ and

$$(M_n^{X_{\boldsymbol\lambda}})_{ext} \ni Y_{ext} = \left(\begin{array}{c|ccc|cc} a & h_1 & h_2 & h_3 & j_1 & j_2 \\ \hline & e_1 & k_1 & k_2 & m_1 & m_2 \\ & k_3 & e_2 & k_4 & m_3 & m_4 \\ & k_5 & k_6 & e_3 & m_5 & m_6 \\ \hline & & & & g_1 & n_1 \\ & & & & n_2 & g_2 \end{array}\right), \quad (M_n^{X_{\boldsymbol\lambda}})_{gr} \ni Y_{gr} = \left(\begin{array}{c|ccc|cc} a & & & & & \\ \hline & e_1 & k_1 & k_2 & & \\ & k_3 & e_2 & k_4 & & \\ & k_5 & k_6 & e_3 & & \\ \hline & & & & g_1 & n_1 \\ & & & & n_2 & g_2 \end{array}\right).$$

$$M_1 \cong (M_n^{X_{\boldsymbol\lambda}})(4) \ni Y(4) = \left(\ a\ \right),$$
$$M_3 \cong (M_n^{X_{\boldsymbol\lambda}})(2) \ni Y(2) = \begin{pmatrix} e_1 & k_1 & k_2 \\ k_3 & e_2 & k_4 \\ k_5 & k_6 & e_3 \end{pmatrix}, \quad M_2 \cong (M_n^{X_{\boldsymbol\lambda}})(1) \ni Y(1) = \begin{pmatrix} g_1 & n_1 \\ n_2 & g_2 \end{pmatrix}.$$

Y is nilpotent if and only if Y_{ext} is nilpotent if and only if the three matrices $Y(\ell)$ are nilpotent. This corresponds to the six$(= d_{\boldsymbol\lambda})$ independent conditions

$$\begin{cases} Tr(Y(4)) = a = 0, \quad \begin{array}{l} Tr(Y(2)) = 0 \\ e_1e_2 + e_2e_3 + e_3e_1 - k_1k_3 - k_4k_6 - k_2k_5 = 0 \\ \det(Y(2)) = 0 \end{array}, \quad \begin{array}{l} Tr(Y(1)) = 0 \\ \det(Y(1)) = 0 \end{array} \end{cases}.$$

Definition 3.7. Let $\boldsymbol\lambda \in \mathcal{P}(n)$ and $\mathfrak{w}$ be a subspace of M_n (*e.g.* a Lie subalgebra of M_n containing $X_{\boldsymbol\lambda}$). We define the following vector spaces

$$\mathfrak{w}^{X_{\boldsymbol\lambda}} := \mathfrak{w} \cap M_n^{X_{\boldsymbol\lambda}}, \qquad (\mathfrak{w}^{X_{\boldsymbol\lambda}})_{gr} := \{Y_{gr} | Y \in \mathfrak{w}^{X_{\boldsymbol\lambda}}\}.$$

The following lemmas relate the geometry of $(\mathfrak{w}^{X_{\boldsymbol\lambda}})^{nil}$ to the one of $(\mathfrak{w}^{X_{\boldsymbol\lambda}})^{nil}_{gr}$ or $(\mathfrak{w}^{X_{\boldsymbol\lambda}}(\ell))^{nil}$

Lemma 3.8.
(i) There exists a vector space $\mathfrak{n}_2$ such that the following isomorphisms of algebraic varieties holds

$$\mathfrak{w}^{X_{\boldsymbol\lambda}} \cong (\mathfrak{w}^{X_{\boldsymbol\lambda}})_{gr} \times \mathfrak{n}_2, \qquad (\mathfrak{w}^{X_{\boldsymbol\lambda}})^{nil} \cong (\mathfrak{w}^{X_{\boldsymbol\lambda}})^{nil}_{gr} \times \mathfrak{n}_2.$$

(ii) $(\mathfrak{w}^{X_{\boldsymbol\lambda}})^{nil}$ is irreducible if and only if $(\mathfrak{w}^{X_{\boldsymbol\lambda}})^{nil}_{gr}$ is and

$$codim_{\ \mathfrak{w}^{X_{\boldsymbol\lambda}}}(\mathfrak{w}^{X_{\boldsymbol\lambda}})^{nil} = codim_{\ (\mathfrak{w}^{X_{\boldsymbol\lambda}})_{gr}}(\mathfrak{w}^{X_{\boldsymbol\lambda}})^{nil}_{gr}$$

Proof. (i) Let $\mathfrak{n}_2 = \mathrm{Ker}((pr_{gr})_{|\mathfrak{w}^{X_\lambda}})$. The first equation follows and the statement about nilpotent elements is a consequence of Proposition 3.5 (i).
(ii) is a consequence of (i). ∎

Let $\mathfrak{w}^{X_\lambda}(\ell) := pr_\ell(\mathfrak{w}^{X_\lambda}) = \{Y(\ell) \,|\, Y \in \mathfrak{w}^{X_\lambda}\} \subseteq M_{\tau_\ell}$. We have a natural analogue of Proposition 3.5 (iii) in this case under some necessary restrictions.

Lemma 3.9. *Let $\mathfrak{w}$ be a subspace of M_n such that the decomposition $\mathfrak{w}_{gr}^{X_\lambda} = \prod_\ell (\mathfrak{w}^{X_\lambda})(\ell)$ holds.*
(i) The variety $(\mathfrak{w}^{X_\lambda})^{nil}$ is irreducible if and only if each $(\mathfrak{w}^{X_\lambda}(\ell))^{nil}$ is and

$$codim_{\,\mathfrak{w}^{X_\lambda}}(\mathfrak{w}^{X_\lambda})^{nil} = \sum_\ell codim_{\,\mathfrak{w}^{X_\lambda}(\ell)}(\mathfrak{w}^{X_\lambda}(\ell))^{nil}.$$

(ii) In particular if, for each ℓ, $\mathfrak{w}^{X_\lambda}(\ell)$ is isomorphic to M_{τ_ℓ}, $\mathfrak{p}_{k',\tau_\ell}$ or $\mathfrak{q}_{k',\tau_\ell}$ $(1 \leqslant k' \leqslant \tau_\ell)$ then $(\mathfrak{w}^{X_\lambda})^{nil}$ is irreducible and $codim_{\,\mathfrak{w}^{X_\lambda}}(\mathfrak{w}^{X_\lambda})^{nil} = d_\lambda$.

Proof. (i) follows from Lemma 3.8.
(ii) is then a consequence of Lemma 3.1. ∎

Remark 3.10. The previous lemma is in general sufficient for our applications. But, in some cases, we have $(\mathfrak{w}^{X_\lambda})_{gr} \subsetneq \prod_\ell \mathfrak{w}^{X_\lambda}(\ell)$. A slightly less precise decomposition may remain valid in these cases. Define $\mathfrak{w}^{X_\lambda}(\ell_1, \ell_2) := pr_{\ell_1,\ell_2}(\mathfrak{w}^{X_\lambda}) = \{(Y(\ell_1), Y(\ell_2)) | Y \in \mathfrak{w}^{X_\lambda}\} \subseteq \mathfrak{w}^{X_\lambda}(\ell_1) \times \mathfrak{w}^{X_\lambda}(\ell_2)$. Assume that there exists a decomposition of the form $\mathfrak{w}_{gr}^{X_\lambda} = (\mathfrak{w}^{X_\lambda})(\ell_1, \ell_2) \times \prod_{\ell \notin \{\ell_1,\ell_2\}} (\mathfrak{w}^{X_\lambda})(\ell)$.
Then $(\mathfrak{w}^{X_\lambda})^{nil}$ is irreducible if and only if $(\mathfrak{w}^{X_\lambda}(\ell_1, \ell_2))^{nil}$ and each $(\mathfrak{w}^{X_\lambda}(\ell))^{nil}$ are. Then

$$\begin{aligned} \mathrm{codim}_{\,\mathfrak{w}^{X_\lambda}}(\mathfrak{w}^{X_\lambda})^{nil} &= \mathrm{codim}_{\,\mathfrak{w}^{X_\lambda}(\ell_1,\ell_2)}(\mathfrak{w}^{X_\lambda}(\ell_1,\ell_2))^{nil} \\ &\quad + \sum_{\ell \notin \{\ell_1,\ell_2\}} \mathrm{codim}_{\,\mathfrak{w}^{X_\lambda}(\ell)}(\mathfrak{w}^{X_\lambda}(\ell))^{nil}. \end{aligned} \tag{5}$$

4. Irreducibility of $\mathcal{N}(\mathfrak{p}_{1,n})$ and $S_0^{[n-1,n]}$

The aim of this section is to prove that $\mathcal{N}(\mathfrak{p}_{1,n})$ is irreducible (Theorem 4.8). We obtain as a corollary that a necessary and sufficient condition for the irreducibility of $\mathcal{N}(\mathfrak{p}_{k,n})$ and $S_0^{[k,n]}$ is $k \in \{0, 1, n-1, n\}$ (Theorem 4.11).

In this section, we will use the simplifying notation $\mathfrak{p} := \mathfrak{p}_{1,n}$. The strategy is the following. We introduce a variety $\mathcal{M}(\mathfrak{p})$ of almost commuting matrices. Since $\mathcal{M}(\mathfrak{p})$ is easily described as a graph, we get its irreducibility and its dimension. The dimensions of the components of $\mathcal{N}(\mathfrak{p})$ are controlled through the equations defining $\mathcal{N}(\mathfrak{p})$ in $\mathcal{M}(\mathfrak{p})$. From this dimension estimate, we have a small list of candidates to be an irreducible component. We finally show that only one element in this list defines an irreducible component.

In this section we assume $n \geqslant 2$. Recall that $(e_1, \dots, e_n)$ is the canonical basis of $V = \Bbbk^n$, $V_i = \langle e_1, \dots, e_i \rangle$. Also, we note $U_i := \langle e_{i+1}, \dots, e_n \rangle$. We will mostly be interested in this section by $V_1 = \Bbbk e_1$ and $U_1 = \langle e_2, \dots, e_n \rangle$. Recall also that $\mathfrak{p} = \mathfrak{p}_{1,n} = \{X \in \mathfrak{gl}(V) \mid X(V_1) \subset V_1\}$. By virtue of Proposition 2.13, we can study $\mathcal{N}(\mathfrak{p})$ in order to get informations on $S_0^{[n-1,n]}$.

We have

$$\mathfrak{p} \overset{v.s.}{=} \mathfrak{gl}(V_1) \oplus \mathrm{Hom}(U_1, V_1) \oplus \mathfrak{gl}(U_1) \cong \Bbbk \oplus M_{1,n-1} \oplus M_{n-1} \tag{6}$$

With respect to this decomposition, for any $X \in \mathfrak{p}$, we set $X = X_1 + X_2 + X_3$ where $X_1 := X_{|V_1} \in \mathfrak{gl}(V_1) \cong \Bbbk$, $X_2 \in \mathrm{Hom}(U_1, V_1) \cong M_{1,n-1}$ and $X_3 \in \mathfrak{gl}(U_1) \cong M_{n-1}$. That is

$$X = \left(\begin{array}{c|c} X_1 & X_2 \\ \hline 0 & \\ \vdots & X_3 \\ 0 & \end{array}\right) \tag{7}$$

We will often identify $\mathrm{Hom}(U_1, V_1)$ with $E := \langle {}^te_2, \dots, {}^te_n \rangle$. Define

$$\mathcal{M}(\mathfrak{p}) := \left\{ (X,Y,j) \;\middle|\; \begin{array}{c} (X,Y) \in \mathfrak{p}^2, j \in \mathrm{Hom}(U_1,V_1) \\ X, Y \text{ nilpotent} \end{array}, \; [X,Y] - \left(\begin{array}{c|c} 0 & j \\ \hline 0 & \\ \vdots & (0) \\ 0 & \end{array}\right) = 0 \right\}$$

The following Proposition and Corollary are prototypes for several similar results of Section 5. The main ideas for this approach are taken from [Zo].

Proposition 4.1. *If $n \geqslant 2$, then $\mathcal{M}(\mathfrak{p})$ is irreducible of dimension $n^2 - 2$*

Proof.

Let us compute

$$[X,Y] = \left(\begin{array}{c|c} 0 & X_2Y_3 - Y_2X_3 \\ \hline 0 & \\ \vdots & [X_3, Y_3] \\ 0 & \end{array}\right). \tag{8}$$

Hence, we have an alternative definition of $\mathcal{M}(\mathfrak{p})$:

$$(X,Y,j) \in \mathcal{M}(\mathfrak{p}) \Leftrightarrow \left\{ \begin{array}{c} (X_3, Y_3) \in \mathcal{N}(\mathfrak{gl}(U_1)), \\ X_1 = Y_1 = 0, \\ j = X_2Y_3 - Y_2X_3. \end{array} \right. \tag{9}$$

In other words, $\mathcal{M}(\mathfrak{p})$ is isomorphic to the graph of the morphism

$$\begin{array}{rcl} \mathcal{N}(M_{n-1}) \times (M_{1,n-1})^2 & \to & M_{1,n-1} \\ ((X_3, Y_3), (X_2, Y_2)) & \mapsto & X_2Y_3 - Y_2X_3. \end{array}$$

and the result follows from Theorem 2.1. ∎

Corollary 4.2. *The dimension of each irreducible component of $\mathcal{N}(\mathfrak{p})$ is greater or equal than $n^2 - n - 1$.*

Proof. If $n = 1$, the result is obvious.

Else, we embed $\begin{array}{ccc} \mathcal{N}(\mathfrak{p}) & \hookrightarrow & \mathcal{M}(\mathfrak{p}) \\ (X,Y) & \mapsto & (X,Y,0) \end{array}$. Hence, $\mathcal{N}(\mathfrak{p})$ is defined in $\mathcal{M}(\mathfrak{p})$ by the $n-1$ equations $0 = j \in M_{1,n-1}$ (cf. (9)). Then, we conclude with Proposition 4.1. ∎

Let us consider the set of 1-*marked partitions* of n

$$\mathcal{P}'(n) := \{(\lambda_1, (\lambda_2 \geqslant \cdots \geqslant \lambda_{d_{\boldsymbol{\lambda}}})) \mid \sum_{i=1}^{d_{\boldsymbol{\lambda}}} \lambda_i = n, \lambda_1 \geqslant 1\}.$$

Given $\boldsymbol{\lambda} \in \mathcal{P}'(n)$, we let $g_j^i := e_{(\sum_{\ell=1}^{i-1} \lambda_\ell)+j}$ for $\left\{ \begin{array}{l} 1 \leqslant i \leqslant d_{\boldsymbol{\lambda}}, \\ 1 \leqslant j \leqslant \lambda_i \end{array}\right.$ and we define $X_{\boldsymbol{\lambda}} \in \mathfrak{p}$ via

$$X_{\boldsymbol{\lambda}}(g_j^i) = \left\{ \begin{array}{ll} g_{j-1}^i & \text{if } j > 1, \\ 0 & \text{if } j = 1. \end{array}\right. \tag{10}$$

Note that these $X_{\boldsymbol{\lambda}}$ with $\boldsymbol{\lambda} \in \mathcal{P}'(n)$ are a priori different from the $X_{\boldsymbol{\lambda}}$ with $\boldsymbol{\lambda} \in \mathcal{P}(n)$ in spite of the similar notation used.

Lemma 4.3 (Classification Lemma)**.** *Let $P := \{x \in \mathfrak{p} \mid \det x \neq 0\}$ be the connected subgroup of G with Lie algebra $\mathfrak{p}$ and let X be a nilpotent element of $\mathfrak{p}$. There exists a unique $\boldsymbol{\lambda} \in \mathcal{P}'(n)$ such that $P \cdot X = P \cdot X_{\boldsymbol{\lambda}}$.*

Proof. Let us describe the P-action on $\mathfrak{p}^{nil}$.

Let $X = \left(\begin{array}{c|c} 0 & X_2 \\ \hline 0 & \\ \vdots & X_3 \\ 0 & \end{array}\right) \in \mathfrak{p}^{nil}$ and $p = \left(\begin{array}{c|c} p_1 & p_2 \\ \hline 0 & \\ \vdots & p_3 \\ 0 & \end{array}\right) \in P$ (hence, $p_1 \in \Bbbk^*$, $p_3 \in \mathrm{GL}(U_1) \cong \mathrm{GL}_{n-1}$ and $p^{-1} = \left(\begin{array}{c|c} p_1^{-1} & -p_1^{-1}p_2p_3^{-1} \\ \hline 0 & \\ \vdots & p_3^{-1} \\ 0 & \end{array}\right)$). Then

$$p \cdot X = pXp^{-1} = \left(\begin{array}{c|c} 0 & p_2X_3p_3^{-1} + p_1X_2p_3^{-1} \\ \hline 0 & \\ \vdots & p_3X_3p_3^{-1} \\ 0 & \end{array}\right). \tag{11}$$

Hence, in order to classify P-orbits of $\mathfrak{p}^{nil}$, we can restrict ourselves to the case where X_3 is in Jordan normal form and study $P' \cdot X$ where $P' = \{p \in P \mid p_3 \in \mathrm{GL}_{n-1}^{X_3}\}$. More precisely, we fix $\boldsymbol{\mu} \in \mathcal{P}(n-1)$ and $f_j^i := e_{(\sum_{\ell=1}^{i-1} \mu_\ell)+j+1}$ $(1 \leqslant i \leqslant d_{\boldsymbol{\mu}}, 1 \leqslant j \leqslant \mu_i)$ and assume that

$$X_3(f_j^i) = \left\{ \begin{array}{ll} f_{j-1}^i & \text{if } j > 1, \\ 0 & \text{if } j = 1. \end{array}\right.$$

Recall that we identify $\mathrm{Hom}(U_1, V_1)$ with $E = \langle {}^t f_j^i \rangle_{i,j} \cong \Bbbk^{n-1}$. The action of GL_{n-1} on this vector space that we consider is the natural right action. For any $p_3 \in \mathrm{GL}_{n-1}^{X_3}$, we have $p_2 X_3 p_3^{-1} = p_2 p_3^{-1} X_3$ and $\{p_2 p_3^{-1} X_3 \mid p_2 \in E\} = \mathrm{Im}({}^t X_3) = \langle {}^t f_j^i \mid j \neq 1 \rangle$ for any $p_3 \in \mathrm{GL}_{n-1}^{X_3}$. On the other hand, set $i_0 = \min\{i \mid X_2(f_1^{i'}) \neq 0 \text{ for some } i' \text{ such that } \mu_i = \mu_{i'}\}$ (If $X_2 = 0$, set $i_0 := d_{\boldsymbol{\mu}} + 1$, $\mu_{i_0} = 0$ and ${}^t f_1^{i_0} = 0$). We have

$$\begin{aligned}\left\{ p_1 X_2 p_3^{-1} \,\middle|\, \begin{array}{c} p_1 \in \Bbbk^*, \\ p_3 \in \mathrm{GL}_{n-1}^{X_3} \end{array} \right\} + \mathrm{Im}({}^t X_3) &\overset{(p_1 Id_{n-1} \subset \mathrm{GL}_{n-1}^{X_3})}{=} \left\{ X_2 p_3^{-1} \mid p_3 \in \mathrm{GL}_{n-1}^{X_3} \right\} + \mathrm{Im}({}^t X_3) \\ &\overset{(\text{Lemma 3.4})}{=} \langle {}^t f_1^i \mid \mu_i = \mu_{i_0} \rangle \setminus \{0\} \\ &\quad + \langle {}^t f_j^i \mid j \neq 1 \text{ or } \mu_{i_0} > \mu_i \rangle \end{aligned} \tag{12}$$

As a consequence, the P-orbit of X is uniquely determined by $\boldsymbol{\mu}$ and i_0. A representative of $P \cdot X$ is $Y = \begin{pmatrix} 0 & {}^t f_1^{i_0} \\ \hline 0 & \\ \vdots & X_3 \\ 0 & \end{pmatrix}$. Finally, an elementary base change in P obtained by a re-ordering of the $(f_j^i)_{i,j}$ sends Y on $X_{\boldsymbol{\lambda}}$ where $\boldsymbol{\lambda} := (\mu_{i_0} + 1, (\mu_2 \geqslant \cdots \widehat{\mu_{i_0}} \cdots \geqslant \mu_{d_\mu}))$. ∎

Remark 4.4. In the special case $X^0 := X_{\boldsymbol{\lambda}^0}$ where $\boldsymbol{\lambda}^0 := (n, \emptyset) \in \mathcal{P}'(n)$, we also get

$$\overline{P' \cdot X^0} = X_3^0 + \mathrm{Hom}(U_1, V_1)$$

as a consequence of (12), where P' is the subgroup of P defined in the previous proof.

When $\boldsymbol{\lambda} \in \mathcal{P}'(n)$, we say that $X_{\boldsymbol{\lambda}}$ is in canonical form in $\mathfrak{p}$. Let

$$\mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p}) := P \cdot (X_{\boldsymbol{\lambda}}, (\mathfrak{p}^{X_{\boldsymbol{\lambda}}})^{nil}). \tag{13}$$

Then

$$\begin{aligned} \dim \mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p}) &= \dim P \cdot X_{\boldsymbol{\lambda}} + \dim (\mathfrak{p}^{X_{\boldsymbol{\lambda}}})^{nil} \\ &= \dim \mathfrak{p} - \dim \mathfrak{p}^{X_{\boldsymbol{\lambda}}} + \dim (\mathfrak{p}^{X_{\boldsymbol{\lambda}}})^{nil} \\ &= \dim \mathfrak{p} - \mathrm{codim}_{\,\mathfrak{p}^{X_{\boldsymbol{\lambda}}}} (\mathfrak{p}^{X_{\boldsymbol{\lambda}}})^{nil}. \end{aligned} \tag{14}$$

Lemma 4.5.

$$\mathcal{N}(\mathfrak{p}) = \bigsqcup_{\boldsymbol{\lambda} \in \mathcal{P}'(n)} \mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p})$$

Moreover, $(\mathfrak{p}^{X_{\boldsymbol{\lambda}}})^{nil}$ *and* $\mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p})$ *are irreducible and* $\dim \mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p}) = n^2 - n + 1 - d_{\boldsymbol{\lambda}}$.

Proof. The decomposition into a disjoint union follows from Lemma 4.3.

Let $\boldsymbol{\lambda} \in \mathcal{P}'(n)$ and use notation of (10). In order to apply results of section 3, we have to define a new basis (f_j^i) in which $X := X_{\boldsymbol{\lambda}}$ is in canonical form for

M_n as in (2). Set $i_0 := \max(\{i|\lambda_i > \lambda_1\} \cup \{1\})$ and

$$f_j^i := \begin{cases} g_j^1 & \text{if } i = i_0 \\ g_j^{i+1} & \text{if } i < i_0 \\ g_j^i & \text{if } i > i_0 \end{cases}, \qquad \mu_i := \begin{cases} \lambda_1 & \text{if } i = i_0 \\ \lambda_{i+1} & \text{if } i < i_0 \\ \lambda_i & \text{if } i > i_0 \end{cases}.$$

In this basis, X becomes $X_{\boldsymbol{\mu}}$ with $\boldsymbol{\mu} = (\mu_1 \geqslant \cdots \geqslant \mu_{d_{\boldsymbol{\lambda}}}) \in \mathcal{P}(n)$ and $\mathfrak{p}$ is defined in M_n by the single property $Y \in \mathfrak{p} \Leftrightarrow Y(f_1^{i_0}) \subset \Bbbk f_1^{i_0}$. Hence, the subspace $(\mathfrak{p}^X)_{gr}$ (cf. Definition 3.7) is also characterized in $(M_n^X)_{gr}$ by the single property

$$Y_{gr} \in (\mathfrak{p}^X)_{gr} \Leftrightarrow Y_{gr}(f_1^{i_0}) \subset \Bbbk f_1^{i_0}.$$

In particular, letting $\tau_\ell := \sharp\{i \,|\, \lambda_i = \ell\} = \sharp\{i \,|\, \mu_i = \ell\}$, we have

$$\mathfrak{p}^X(\ell) \cong \begin{cases} M_{\tau_\ell} & \text{if } \ell \neq \lambda_1 \\ \mathfrak{p}_{1,\tau_\ell} & \text{if } \ell = \lambda_1 \end{cases}, \text{ and } (\mathfrak{p}^X)_{gr} = \prod_\ell \mathfrak{p}^X(\ell).$$

Then, Lemma 3.9 (ii) provides the irreducibility statement for $(\mathfrak{p}^X)^{nil}$ and hence for $\mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p})$. Together with (14), it also provides the dimension of $\mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p})$. ■

Combining this with corollary 4.2, we get that the irreducible components of $\mathcal{N}(\mathfrak{p})$ are of the form $\overline{\mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p})}$ where $\boldsymbol{\lambda} \in \mathcal{P}'(n)$ has at most two parts ($d_{\boldsymbol{\lambda}} \leqslant 2$). The unique irreducible component of maximal dimension is associated with $\boldsymbol{\lambda}^0 = (n, \emptyset) \in \mathcal{P}'(n)$.

There remains to show that

$$\mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p}) \subset \overline{\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p})} \tag{15}$$

when $\boldsymbol{\lambda}$ has two parts. In order to prove this, we distinguish two cases.

Lemma 4.6. *If $\boldsymbol{\lambda} = (\lambda_1, (\lambda_2)) \in \mathcal{P}'(n)$ with $\lambda_1 \leqslant \lambda_2 + 1$, property (15) is satisfied.*

Proof. For $(X_3, Y_3) \in \mathcal{N}(\mathfrak{gl}(U_1))$, we look at the fiber over (X_3, Y_3) in $\mathcal{N}(\mathfrak{p})$ and $\overline{\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p})}$:

$$F_{X_3,Y_3} := \{(X_2, Y_2) \in (\mathrm{Hom}(U_1, V_1))^2 \mid (X_2 + X_3, Y_2 + Y_3) \in \mathcal{N}(\mathfrak{p})\},$$

$$F'_{X_3,Y_3} := \{(X_2, Y_2) \in (\mathrm{Hom}(U_1, V_1))^2 \mid (X_2 + X_3, Y_2 + Y_3) \in \overline{\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p})}\}.$$

Since $F_{X_3,Y_3} = \{(X_2, Y_2) \mid {}^tX_3{}^tY_2 = {}^tY_3{}^tX_2\}$ (cf. (8)) is a vector space, it is irreducible. On the other hand, the two varieties F_{X_3,Y_3} and F'_{X_3,Y_3} are closed and satisfy $F'_{X_3,Y_3} \subset F_{X_3,Y_3}$. So

$$F_{X_3,Y_3} = F'_{X_3,Y_3} \Leftrightarrow \dim F_{X_3,Y_3} = \dim F'_{X_3,Y_3}. \tag{16}$$

We can compute the dimension of F_{X_3,Y_3} in the following way:

$$\begin{aligned} \dim F_{X_3,Y_3} &= \dim(\mathrm{Im}({}^tX_3) \cap \mathrm{Im}({}^tY_3)) + \dim \mathrm{Ker}({}^tX_3) + \dim \mathrm{Ker}({}^tY_3) \\ &= \dim \mathrm{Im}({}^tX_3) + \dim \mathrm{Im}({}^tY_3) - \dim(\mathrm{Im}({}^tX_3) + \mathrm{Im}({}^tY_3)) \\ &\quad + \dim \mathrm{Ker}({}^tX_3) + \dim \mathrm{Ker}({}^tY_3) \\ &= 2(n-1) - \dim(\mathrm{Im}({}^tX_3) + \mathrm{Im}({}^tY_3)). \end{aligned}$$

Set $X^0 := X_{\boldsymbol{\lambda}^0}$. Then, identifying $\mathrm{Hom}(U_1, V_1)$ with $\langle {}^te_2, \dots, {}^te_n\rangle$ and using notation of (7), we have $\mathrm{Im}({}^tX_3^0) = \langle {}^te_3, \dots {}^te_n\rangle$ and for any $Y_3 \in (\mathfrak{gl}(U_1)^{X_3^0})^{nil}$, the inclusion $\mathrm{Im}({}^tY_3) \subset \mathrm{Im}({}^tX_3^0)$ holds. Since $\dim \mathrm{Im}({}^tX_3^0) = n-2$, we get $\dim F_{X_3^0,Y_3} = n$. An other consequence of the inclusion $\mathrm{Im}({}^tY_3) \subset \mathrm{Im}({}^tX_3^0)$ is the following: for any $X_2 \in \mathrm{Hom}(U_1, V_1)$, there exists $Y_2 \in \mathrm{Hom}(U_1, V_1)$ such that $(X_2, Y_2) \in F_{X_3^0,Y_3}$. Combining this with Remark 4.4, we get that $X_3^0 + X_2 \in P.X_3^0$ for a general element $(X_2, Y_2) \in F_{X_3^0,Y_3}$ and

$$\overline{\mathcal{N}_{\lambda^0}(\mathfrak{p})} = \overline{\mathrm{GL}(U_1)\cdot\left\{(X_3^0 + X_2, Y_3 + Y_2) \;\middle|\; \begin{array}{c} Y_3 \in (\mathfrak{gl}(U_1)^{X_3^0})^{nil} \\ (X_2, Y_2) \in F_{X_3^0,Y_3} \end{array}\right\}}.$$

In particular, a general element (X, Y) of the irreducible variety $\overline{\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p})}$ satisfies $\dim F'_{X_3,Y_3} = n$. Moreover, since $\mathcal{N}(\mathfrak{gl}(U_1)) = \overline{\mathrm{GL}(U_1).(X_3^0, (\mathfrak{gl}(U_1)^{X_3^0})^{nil})}$ (Theorem 2.1), we see that any $(X_3, Y_3) \in \mathcal{N}(\mathfrak{gl}(U_1))$ lies in fact in $\overline{\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p})}$ by considering the inclusion $\mathcal{N}(\mathfrak{gl}(U_1)) \subset \mathcal{N}(\mathfrak{p})$ given by $X_2 = Y_2 = 0$. Hence $F'_{X_3,Y_3} \neq \emptyset$ and

$$\forall (X_3, Y_3) \in \mathcal{N}(\mathfrak{gl}(U_1)), \quad \dim F'_{X_3,Y_3} \geqslant n. \tag{17}$$

From now on, we fix $X := X_{\boldsymbol{\lambda}}$ and want to show that a general element Y of $(\mathfrak{p}^X)^{nil}$ satisfies $(X, Y) \in \overline{\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p})}$. This will prove the Lemma since $(\mathfrak{p}^X)^{nil}$ is irreducible (Lemma 4.5) and we will then have $(X, (\mathfrak{p}^X)^{nil}) \subset \overline{\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p})}$. Define $Z \in \mathfrak{p}$ by

$$Z(g_j^i) = \begin{cases} g_{j-1}^2 & \text{if } i = 1,\ j > 1, \\ 0 & \text{else.} \end{cases}$$

We have $Z \in (\mathfrak{p}^X)^{nil}$ under the hypothesis made on $\boldsymbol{\lambda}$ (Lemma 3.3) and $\mathrm{Im}({}^tZ_3) + \mathrm{Im}({}^tX_3) = \langle g_2^1, \dots, g_{\lambda_1}^1, g_2^2, \dots, g_{\lambda_2}^2\rangle$ so $\dim F_{X_3,Z_3} = n$. Since the application $\begin{cases} (\mathfrak{p}^X)^{nil} & \to & \mathbb{N} \\ Y & \mapsto & \dim F_{X_3,Y_3} \end{cases}$ is upper semi-continuous, it follows from (17) that $W := \left\{Y \in (\mathfrak{p}^X)^{nil} \mid \dim F_{X_3,Y_3} = n = \dim F'_{X_3,Y_3}\right\}$ is a non-empty open subset of $(\mathfrak{p}^X)^{nil}$. For $Y \in W$, we have $(X, Y) \in (X_3, Y_3) + F_{X_3,Y_3} \subset \overline{\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p})}$ by (16). ■

The following Lemma can be proved with purely matricial arguments. However, we find the given proof more interesting. It uses the isomorphism $\mathfrak{p}_{1,n} \cong \mathfrak{p}_{n-1,n}$ and enlighten a bit the correspondence between $S_0^{[1,n]}$ and $S_0^{[n-1,n]}$ mentioned in remark 2.15.

Lemma 4.7. *If $\boldsymbol{\lambda} = (\lambda_1, (\lambda_2)) \in \mathcal{P}'(n)$ with $\lambda_1 \geqslant \lambda_2$, then Property* (15) *is satisfied.*

Proof. Seen as varieties, we have $S_0^{[1,n]} \overset{var}{\cong} S_0^{[n]}$ (Proposition 2.8). In particular, the irreducibility of $S_0^{[1,n]}$ follows from that of $S_0^{[n]}$ [Br, Pr] and $\mathcal{N}^{cyc}(\mathfrak{p}_{n-1,n})$ is then also irreducible (Proposition 2.13).

We have a Lie algebra isomorphism given by

$$\psi' : \begin{cases} \mathfrak{p}_{1,n} & \to & \mathfrak{p}_{n-1,n} \\ X & \mapsto & -s({}^tX)s^{-1} \end{cases} \tag{18}$$

where s is defined on the basis $(e_i)_{i\in[\![1,n]\!]}$ by $s(e_i) := e_{n-i}$. In particular, the restriction $\psi : \mathcal{N}(\mathfrak{p}_{1,n}) \to \mathcal{N}(\mathfrak{p}_{n-1,n})$ is an isomorphism of varieties. Moreover, $\psi(X,Y)$ has a cyclic vector if and only if $({}^tX, {}^tY)$ does.

Note that $\psi(\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p}_{1,n})) = \mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p}_{n-1,n})$ and that $\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p}_{1,n})$ is open in $\mathcal{N}(\mathfrak{p}_{1,n})$. It is then straightforward to check that $\psi(\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p}_{1,n})) \subset \mathcal{N}^{cyc}(\mathfrak{p}_{n-1,n})$. Hence, it follows from Lemma 4.5 and the irreducibility of the open subvariety $\mathcal{N}^{cyc}(\mathfrak{p}_{n-1,n}) \subset \mathcal{N}(\mathfrak{p}_{n-1,n})$ that $\overline{\psi(\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p}_{1,n}))} = \overline{\mathcal{N}^{cyc}(\mathfrak{p}_{n-1,n})}$.

Consider now $X_{\boldsymbol{\lambda}}$ given by (10). We can define $Y \in (\mathfrak{p}_{1,n})^{nil}$ via

$$Y(g^i_j) := \begin{cases} g^1_j & \text{if i=2,} \\ 0 & \text{if i=1.} \end{cases}$$

Under our hypothesis on $\boldsymbol{\lambda}$, we have $Y \in \mathfrak{p}_{1,n}^{X_{\boldsymbol{\lambda}}}$ (Lemma 3.3) and ${}^tg^1_1$ is a cyclic vector for $({}^tX_{\boldsymbol{\lambda}}, {}^tY)$. In particular, $\psi(\mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p}_{1,n})) \cap \mathcal{N}^{cyc}(\mathfrak{p}_{n-1,n}) \neq \emptyset$ so the irreducible subset $\psi(\mathcal{N}_{\boldsymbol{\lambda}}(\mathfrak{p}_{1,n}))$ is contained in $\overline{\mathcal{N}^{cyc}(\mathfrak{p}_{n-1,n})} = \overline{\psi(\mathcal{N}_{\boldsymbol{\lambda}^0}(\mathfrak{p}_{1,n}))}$. Since ψ is an isomorphism, (15) is proved in our case. ■

Finally, it follows from discussion above (15) that the following theorem holds.

Theorem 4.8. *The variety $\mathcal{N}(\mathfrak{p}_{1,n})$ is irreducible of dimension $n^2 - n = \dim \mathfrak{p}_{1,n} - 1$.*

Hence, by Proposition 2.13:

Corollary 4.9. *$S_0^{[n-1,n]}$ is an irreducible variety of dimension $n-1$.*

Remark 4.10. The above corollary was already proved in [CE] with other techniques (Bialynicki-Birula stratifications and Gröbner basis computations).

Theorem 4.11. *$S_0^{[k,n]}$ is irreducible if and only if $k \in \{0, 1, n-1, n\}$. $\mathcal{N}(\mathfrak{p}_{k,n})$ is irreducible if and only if $k \in \{0, 1, n-1, n\}$.*

Proof. Since $S_0^{[n]}$ is irreducible [Br, Pr], since $S_0^{[1,n]}$ is homeomorphic to $S_0^{[n]}$ and $S_0^{[n,n]}$ is isomorphic to $S_0^{[n]}$, this proves together with Proposition 1.1 the assertion of the Theorem for $S_0^{[k,n]}$. The variety $\mathcal{N}(\mathfrak{p}_{k,n})$ is irreducible for $k = 1$ by Theorem 4.8. The transposition isomorphism of (18) implies that $\mathcal{N}(\mathfrak{p}_{n-1,n}) \simeq \mathcal{N}(\mathfrak{p}_{1,n})$ is irreducible too. The varieties $\mathcal{N}(\mathfrak{p}_{0,n}) = \mathcal{N}(\mathfrak{p}_{n,n}) = \mathcal{N}(M_n)$ are irreducible by Theorem 2.1. Finally, the number of components in $\mathcal{N}(\mathfrak{p}_{k,n})$ is greater or equal than the number of components of $\mathcal{N}^{cyc}(\mathfrak{p}_{k,n})$ which is, in turn, equal to the number of components in $S_0^{[n-k,n]}$ (Proposition 2.13). It follows that $\mathcal{N}(\mathfrak{p}_{k,n})$ is reducible for $k \in \{2, \dots, n-2\}$ ■

Corollary 4.12. *$S_0^{[\![k,n]\!]}$ is irreducible if and only if $k \in \{n-1, n\}$ or $n \leqslant 3$. $\mathcal{N}(\mathfrak{q}_{k,n})$ is irreducible if and only if $k \in \{0, 1\}$ or $n \leqslant 3$.*

Proof. Note that $S_0^{[\![k,n]\!]} \simeq S_0^{[k,n]}$ for $k = n-1, n$ and $\mathcal{N}(\mathfrak{q}_{k,n}) = \mathcal{N}(\mathfrak{p}_{k,n})$ for $k = 0, 1$. The "if" part then follows from Theorem 4.11 and easy computations when $n \leqslant 3$. For $k \geqslant 2$, we have a sequence of surjective morphisms $\mathcal{N}^{cyc}(\mathfrak{q}_{k,n}) \to S_0^{[\![n-k,n]\!]} \to S_0^{[\![n-2,n]\!]} \to S_0^{[n-2,n]}$ (Propositions 2.13 and 2.14). Since $S_0^{[n-2,n]}$ is reducible when $n \geqslant 4$, the corollary follows. ■

5. General lower bounds for the dimension of the components

The goal of this section is to give lower bounds for the dimension of the components of $\mathcal{N}(\mathfrak{p}_{k,n})$, $\mathcal{N}(\mathfrak{q}_{k,n})$, $S_0^{[k,n]}$, $S_0^{[\![k,n]\!]}$ which are valid for all k, n.

Let $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$.

Proposition 5.1. *Each irreducible component of $\mathcal{N}(\mathfrak{q}_{k,n})$ has dimension at least* $\dim \mathfrak{q}_{k,n} - 1$.

Proof. We proceed by induction on k, the case $k = 1$ being proved in Theorem 4.8 since $\mathfrak{q}_{1,n} = \mathfrak{p}_{1,n}$. Assume now $k \geqslant 2$. The proof mainly follows those of Proposition 4.1 and Corollary 4.2.

For any $X \in \mathfrak{q}_{k,n}$, we decompose $X = X_1 + X_2 + X_3$ as in (7), with $X_3 \in (\mathfrak{gl}(U_1) \cap \mathfrak{q}_{k,n}) \cong \mathfrak{q}_{k-1,n-1}$. We let

$$\mathcal{M}(\mathfrak{q}_{k,n}) := \left\{ (X,Y,j) \in (\mathfrak{q}_{k,n}^{nil})^2 \times \operatorname{Hom}(U_1, V_1) \;\middle|\; [X,Y] - \left(\begin{array}{c|c} 0 & j \\ \hline 0 & \\ \vdots & 0 \\ 0 & \end{array} \right) = 0 \right\}$$

Then

$$(X,Y,j) \in \mathcal{M}(\mathfrak{q}_{k,n}) \Leftrightarrow \begin{cases} (X_3, Y_3) \in \mathcal{N}(\mathfrak{q}_{k-1,n-1}), \\ X_1 = Y_1 = 0, \\ j = X_2 Y_3 - Y_2 X_3 \end{cases}$$

Thus $\mathcal{M}(\mathfrak{q}_{k,n})$ is isomorphic to the graph of

$$\varphi : \begin{array}{ccc} \mathcal{N}(\mathfrak{q}_{k-1,n-1}) \times (M_{1,n-1})^2 & \to & M_{1,n-1} \\ ((X_3,Y_3),(X_2,Y_2)) & \mapsto & X_2Y_3 - Y_2X_3 \end{array}.$$

Hence, by induction, each irreducible component of $\mathcal{M}(\mathfrak{q}_{k,n})$ has a dimension greater or equal than $\dim \mathfrak{q}_{k-1,n-1} - 1 + 2(n-1)$.

Since $k - 1 \geqslant 1$, and $X_3, Y_3 \in \mathfrak{gl}(U_1)$ are nilpotent, we get $X_3(e_2) = Y_3(e_2) = 0$. So the image of φ lies in $\langle {}^t e_3, \dots {}^t e_n \rangle$ and $\mathcal{N}(\mathfrak{q}_{k,n})$ is defined in $\mathcal{M}(\mathfrak{q}_{k,n})$ by $n-2$ equations. Then the dimension of each of its irreducible component is greater or equal than $\dim \mathfrak{q}_{k-1,n-1} - 1 + 2(n-1) - (n-2) = \dim \mathfrak{q}_{k-1,n-1} + n - 1 = \dim \mathfrak{q}_{k,n} - 1$. ■

Hence, by Proposition 2.13:

Corollary 5.2. *Each irreducible component of $S_0^{[\![n-k,n]\!]}$ has dimension at least $n-1$ which is the dimension of the curvilinear component.*

Remark 5.3. When $k = n$, Proposition 5.1 provides a lower bound for the dimension of the nilpotent commuting variety of the Borel subalgebra $\mathfrak{q}_{n,n}$. In this case, a simpler proof is given by considering the bracket map $\mathfrak{n} \times \mathfrak{n} \to [\mathfrak{n}, \mathfrak{n}]$ where $\mathfrak{n}$ is the nilradical of a Borel $\mathfrak{b}$. Its fibers, in particular its null one which is equal to $\mathcal{N}(\mathfrak{b})$, are of dimension greater or equal than $2 \dim \mathfrak{n} - \dim[\mathfrak{n}, \mathfrak{n}] = \dim \mathfrak{b}$ in an arbitrary semisimple Lie algebra. When $\mathfrak{b}$ acts on $\mathfrak{n}$ with finitely many orbits, a computation similar to (14) then shows that $\mathcal{N}(\mathfrak{b})$ is an equidimensional variety. This simplifies some of the arguments of [GR], where this result was first proved, since it allows to avoid Strategy 2.10 (2-3) in this case.

Unfortunately, concerning $\mathfrak{p}_{k,n}$ we are only able to give the following less effective bound.

Proposition 5.4. *Each irreducible component of* $\mathcal{N}(\mathfrak{p}_{k,n})$ *has dimension at least* $\dim \mathfrak{p}_{k,n} - 2$.

Proof. Let

$$\mathcal{M}(\mathfrak{p}_{k,n}) := \left\{ (X,Y,B) \in \mathfrak{p}_{k,n}^2 \times \mathrm{Hom}(U_k, V_k) \;\middle|\; [X,Y] - \left(\begin{array}{c|c} 0 & B \\ \hline 0 & \\ \vdots & 0 \\ 0 & \end{array}\right) = 0 \right\}.$$

Once again, we proceed in a similar way to Proposition 4.1.

Hence, $\mathcal{M}(\mathfrak{p}_{k,n})$ is isomorphic to the graph of a morphism with an irreducible domain of dimension $(k^2-1) + ((n-k)^2 - 1) + 2k(n-k)$ and $\mathcal{N}(\mathfrak{p}_{k,n})$ is defined in $\mathcal{M}(\mathfrak{p}_{k,n})$ by $k(n-k)$ equations. Hence, the dimension of each irreducible components of $\mathcal{N}(\mathfrak{p}_{k,n})$ is greater or equal than $k^2 + (n-k)^2 + k(n-k) - 2 = \dim \mathfrak{p}_{k,n} - 2$. ■

Finally, we have the following consequence about nested Hilbert schemes (cf. Proposition 2.13).

Corollary 5.5. *Each irreducible component of* $S_0^{[n-k,n]}$ *has dimension at least* $n-2$*, which is the dimension of the curvilinear component minus one.*

Applying naively the same argument to a general parabolic subalgebra $\mathfrak{p}$ whose Levi part has ℓ blocks, one can show that the dimension of any irreducible component of $\mathcal{N}(\mathfrak{p})$ (resp. of the corresponding Hilbert scheme) has dimension at least $D-(\ell-1)$ with $D = \dim \mathfrak{p} - 1$ (resp. $D = n-1$). We think that the correct dimension should be D but were only able to prove this in special cases such as $\mathfrak{q}_{k,n}$. In fact, in this case as in some others, the extra codimension yielded by the Levi-blocks of size 1 can be discarded easily, hence the optimal result.

6. Detailed study of $S_0^{[2,n]}$

In the special cases $k = 2$ and $k = n-2$, we have a more precise estimate for the dimension of the components. The goal of this section is to describe the number and the dimension of the components for $\mathcal{N}(\mathfrak{p}_{2,n}) \simeq \mathcal{N}(\mathfrak{p}_{n-2,n})$, $\mathcal{N}(\mathfrak{q}_{2,n})$, $S_0^{[2,n]}$, $S_0^{[n-2,n]}$, $S_0^{[\![n-2,n]\!]}$. The general strategy is the same as in Section 4.

Our first aim is a classification of orbits.

We identify $\mathfrak{q}_{2,n}$ with

$$\mathfrak{gl}(V_1) \oplus \mathrm{Hom}(U_1, V_1) \oplus \{X \in \mathfrak{gl}(U_1) | X(e_2) \in \mathbb{C}e_2\} \overset{v.s.}{\cong} \Bbbk \oplus M_{1,n-1} \oplus \mathfrak{q}_{1,n-1}.$$

Again, we decompose each $X \in \mathfrak{q}_{2,n}$ with respect to this direct sum

$$X = \left(\begin{array}{c|c} X_1 & X_2 \\ \hline 0 & \\ \vdots & X_3 \\ 0 & \end{array}\right). \tag{19}$$

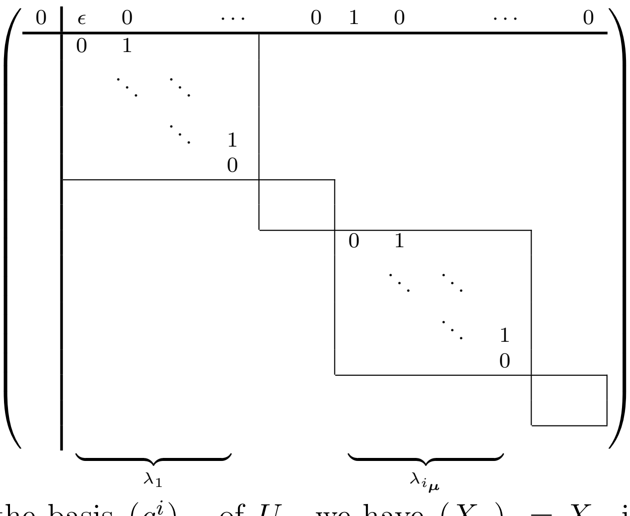
For any $\boldsymbol{\lambda} = (\lambda_1, (\lambda_2 \geqslant \cdots \geqslant \lambda_{d_{\boldsymbol\lambda}})) \in \mathcal{P}'(n)$, we set $\lambda_{d_{\boldsymbol\lambda}+1} = 0$. Let

$$\mathcal{P}''(n) := \left\{ (\boldsymbol\lambda, l, \epsilon) \in \mathcal{P}'(n-1) \times \mathbb{N} \times \{0,1\} \;\middle|\; \begin{array}{l} l = \lambda_i \text{ for some } i \in [\![2, d_{\boldsymbol\lambda}+1]\!] \\ \epsilon = 1 \Rightarrow (l > \lambda_1 \text{ or } l = 0) \end{array} \right\}. \tag{20}$$

For $\boldsymbol\mu = (\boldsymbol\lambda, l, \epsilon) \in \mathcal{P}''(n)$, we define $g_j^i := e_{\sum_{\ell=1}^{i-1} \lambda_\ell + j + 1}$ and $i_{\boldsymbol\mu} := \min\{i' > 1 \mid l = \lambda_{i'}\} \in [\![2, d_{\boldsymbol\lambda}+1]\!]$. In the basis $\left(e_1, (g_j^i)_{\left(\substack{1 \leqslant i \leqslant d_{\boldsymbol\lambda} \\ 1 \leqslant j \leqslant \lambda_i}\right)}\right)$, we define $X_{\boldsymbol\mu} \in \mathfrak{q}_{2,n}$ via

$$X_{\boldsymbol\mu}(e_1) := 0, \qquad X_{\boldsymbol\mu}(g_j^i) := \begin{cases} g_{j-1}^i & \text{if } j > 1 \\ \epsilon\, e_1 & \text{if } i = 1,\, j = 1 \\ e_1 & \text{if } i = i_{\boldsymbol\mu} \text{ and } j = 1 \\ 0 & \text{else} \end{cases}.$$

$X_{\boldsymbol\mu} =$



Note that in the basis $(g_j^i)_{i,j}$ of U_1, we have $(X_{\boldsymbol\mu})_3 = X_{\boldsymbol\lambda}$ in the notation of (10). We claim that $(X_{\boldsymbol\mu})_{\boldsymbol\mu \in \mathcal{P}''(n)}$ is a set of representatives of nilpotent orbits of $\mathfrak{q}_{2,n}$.

Lemma 6.1. *Each nilpotent element of $\mathfrak{q}_{2,n}$ (resp. $\mathfrak{p}_{2,n}$) is $Q_{2,n}$ (resp. $P_{2,n}$)-conjugated to $X_{\boldsymbol\mu}$ for some $\boldsymbol\mu \in \mathcal{P}''(n)$.*
Moreover $Q_{2,n} \cdot X_{\boldsymbol\mu} = Q_{2,n} \cdot X_{\boldsymbol\mu'}$ if and only if $\boldsymbol\mu = \boldsymbol\mu'$.

Proof. Thanks to the inclusion $(\mathrm{GL}(V_2)\times Id_{U_2})\subset P_{2,n}$, we can trigonalize the $\mathfrak{gl}(V_2)$-part of any element of $\mathfrak{p}_{2,n}$, hence each element of $\mathfrak{p}_{2,n}$ is $P_{2,n}$-conjugated to an element of $\mathfrak{q}_{2,n}$. Since $Q_{2,n}\subset P_{2,n}$, it is therefore sufficient to prove the result for $\mathfrak{q}_{2,n}$.

Let $X=X_1+X_2+X_3\in\mathfrak{q}_{2,n}$ be a nilpotent element. We have $X_1=0$. The element X_3 is nilpotent so, up to conjugacy by an element of $(Id_{V_1}\times Q_{1,n-1})\subset Q_{2,n}$, we may assume that $X_3=X_{\boldsymbol{\lambda}}$ for some fixed $\boldsymbol{\lambda}\in\mathcal{P}'(n-1)$ (Lemma 4.3). Let $Q'\subset Q_{2,n}$ be the subgroup of elements stabilizing this part $X_3=X_{\boldsymbol{\lambda}}$, that is $Q'=\left\{q=\left(\begin{array}{c|c} q_1 & q_2\\ \hline 0 & \\ \vdots & q_3\\ 0 & \end{array}\right)\;\middle|\; q_3\in Q_{1,n}^{X_{\boldsymbol{\lambda}}}\right\}$. For $q\in Q'$ we get (cf. (11)):

$$q\cdot X=\left(\begin{array}{c|c} 0 & q_1X_2q_3^{-1}+q_2X_{\boldsymbol{\lambda}}q_3^{-1}\\ \hline 0 & \\ \vdots & X_{\boldsymbol{\lambda}}\\ 0 & \end{array}\right)=\left(\begin{array}{c|c} 0 & X_2q_1q_3^{-1}+q_2q_3^{-1}X_{\boldsymbol{\lambda}}\\ \hline 0 & \\ \vdots & X_{\boldsymbol{\lambda}}\\ 0 & \end{array}\right).$$

Hence, we are reduced to classify the different Q'-orbits in $\mathrm{Hom}(U_1,V_1)\overset{v.s.}{\cong}\langle {}^tg_j^i\rangle_{i,j}\cong\mathbb{k}^{n-1}$ with respect to the action of Q' given by

$$q\cdot X_2=X_2q_1q_3^{-1}+q_2q_3^{-1}X_{\boldsymbol{\lambda}}.$$

In particular, $Q'\cdot X_2=X_2\mathbb{k}^*Q_{1,n-1}^{X_{\boldsymbol{\lambda}}}+(\mathbb{k}^{n-1})X_{\boldsymbol{\lambda}}=X_2Q_{1,n-1}^{X_{\boldsymbol{\lambda}}}+\mathrm{Im}({}^tX_{\boldsymbol{\lambda}})$. We have $\mathrm{Im}({}^tX_{\boldsymbol{\lambda}})=\langle {}^tg_j^i\mid j\geqslant 2\rangle$ and this subspace is stable under the right action of $Q_{1,n-1}^{X_{\boldsymbol{\lambda}}}$. There remains to understand the $Q_{1,n-1}^{X_{\boldsymbol{\lambda}}}$-action on the quotient space $\mathbb{k}^{n-1}/\mathrm{Im}({}^tX_{\boldsymbol{\lambda}})\cong\langle {}^tg_1^i\mid i\in[\![1,d_{\boldsymbol{\lambda}}]\!]\rangle$. Under notation of section 3, this corresponds to the right action of $(Q_{1,n-1}^{X_{\boldsymbol{\lambda}}})_{ext}$ on $\mathcal{W}:=\langle {}^tg_1^i\mid i\in[\![1,d_{\boldsymbol{\lambda}}]\!]\rangle$. In the left action setting on $\langle g_1^i\mid i\in[\![1,d_{\boldsymbol{\lambda}}]\!]\rangle$, $(Q_{1,n-1}^{X_{\boldsymbol{\lambda}}})_{ext}$ can be described as the subgroup stabilizing $\langle g_1^1\rangle$ in the parabolic subgroup stabilizing each $W_\ell=\langle g_1^i|\lambda_i\geqslant\ell\rangle$ (Lemma 3.4).

Picturally, this corresponds to a group of the following form:

$$(Q_{1,n-1}^{X_{\boldsymbol{\lambda}}})_{ext}=\left(\begin{array}{cccccccccc}
* & 0 & \dots & \dots & 0 & * & * & * & * & *\\
0 & * & * & * & * & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & * & * & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & 0 & 0 & * & * & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & * & * & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & 0 & 0 & * & * & * & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & * & * & * & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & * & * & * & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & * & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right).$$
$$\underbrace{\quad}_{i=1}\qquad\underbrace{\qquad\qquad}_{\{i|\lambda_i>\lambda_1\}}\qquad\underbrace{\qquad\qquad}_{\{i|\lambda_i\leqslant\lambda_1\}}$$

In the right action setting, let $\mathcal{W}_\ell:=\langle {}^tg_1^i\,|\,\lambda_i\leqslant\ell, i\neq 1\rangle$. We see that $(Q_{1,n-1}^{X_{\boldsymbol{\lambda}}})_{ext}$ is the subgroup of $M_{d_{\boldsymbol{\lambda}}}$ stabilizing $\mathbb{k}\,{}^tg_1^1\oplus\mathcal{W}_{\lambda_1}$ and each $\mathcal{W}_\ell$ ($\ell\in\mathbb{N}^*$).

Let $i_0 := \min(\{i > 1 \mid X_2(g_1^{i'}) \neq 0 \text{ for some } i' > 1 \text{ such that } \lambda_i = \lambda_{i'}\} \cup \{d_{\boldsymbol\lambda}+1\})$ and, if $i_0 = d_{\boldsymbol\lambda}+1$, we let ${}^t g_1^{i_0} := 0$. We get

$$X_2 \cdot (Q_{1,n-1}^{X_{\boldsymbol\lambda}})_{ext} \supseteq \left\langle {}^t g_1^i \,\middle|\, \begin{array}{c} i>1 \\ \lambda_i = \lambda_{i_0} \end{array} \right\rangle \setminus \{0\} + \left\langle {}^t g_1^i \,\middle|\, \begin{array}{c} i>1 \\ \lambda_i < \lambda_{i_0} \end{array} \right\rangle =: A.$$

On the other hand, if $X_2(g_1^1) \neq 0$, we set $\epsilon = 1$; otherwise we set $\epsilon = 0$. Then:

$$X_2 \cdot (Q_{1,n-1}^{X_{\boldsymbol\lambda}})_{ext} = \begin{cases} A + \Bbbk^*({}^t g_1^1) + \left\langle {}^t g_1^i \,\middle|\, \begin{array}{c} i>1, \\ \lambda_i \leqslant \lambda_1 \end{array} \right\rangle & \text{if } \epsilon = 1, \\ A & \text{if } \epsilon = 0. \end{cases} \tag{21}$$

Hence, if $\epsilon = 1$ and $\lambda_{i_0} \leqslant \lambda_1$, we have $X \in Q_{2,n} \cdot X_{\boldsymbol\mu}$ with $\boldsymbol\mu := (\boldsymbol\lambda, 0, 1)$. Else, we have $X \in Q_{2,n} \cdot X_{\boldsymbol\mu}$ with $\boldsymbol\mu := (\boldsymbol\lambda, \lambda_{i_0}, \epsilon)$.

Moreover, given $\boldsymbol\lambda \in \mathcal{P}'(n)$, different elements $(\boldsymbol\lambda, l, \epsilon), (\boldsymbol\lambda, l', \epsilon') \in \mathcal{P}''(n)$ give rise to different $(Q_{1,n-1}^{X_{\boldsymbol\lambda}})_{ext}$-orbits thanks to (21). So if $\boldsymbol\mu \neq \boldsymbol\mu'$, we have $Q_{2,n} \cdot X_{\boldsymbol\mu} \neq Q_{2,n} \cdot X_{\boldsymbol\mu'}$. ■

Note that we may have $P_{2,n} \cdot X_{\boldsymbol\mu} = P_{2,n} \cdot X_{\boldsymbol\mu'}$ with $\boldsymbol\mu \neq \boldsymbol\mu'$. A full classification of nilpotent orbits should throw away those cases. However, the description of Lemma 6.1 will be sufficient for our purpose.

If $\boldsymbol\mu = (\boldsymbol\lambda, \epsilon, l) \in \mathcal{P}''(n)$, we denote by $d_{\boldsymbol\mu}$ the number of parts in the partition of n associated to $\mathrm{GL}_n \cdot X_{\boldsymbol\mu}$. That is

$$d_{\boldsymbol\mu} = \begin{cases} d_{\boldsymbol\lambda} + 1 & \text{if } \epsilon = 0 \text{ and } l = 0, \\ d_{\boldsymbol\lambda} & \text{else.} \end{cases}$$

It follows from Lemma 6.1 that

$$\mathcal{N}(\mathfrak{p}_{2,n}) := \bigcup_{\boldsymbol\mu \in \mathcal{P}''(n)} \mathcal{N}_{\boldsymbol\mu}(\mathfrak{p}_{2,n}), \quad \text{where} \quad \mathcal{N}_{\boldsymbol\mu}(\mathfrak{p}_{2,n}) = P_{2,n} \cdot (X_{\boldsymbol\mu}, (\mathfrak{p}_{2,n}^{X_{\boldsymbol\mu}})^{nil}),$$

$$\mathcal{N}(\mathfrak{q}_{2,n}) := \bigsqcup_{\boldsymbol\mu \in \mathcal{P}''(n)} \mathcal{N}_{\boldsymbol\mu}(\mathfrak{q}_{2,n}), \quad \text{with} \quad \mathcal{N}_{\boldsymbol\mu}(\mathfrak{q}_{2,n}) = Q_{2,n} \cdot (X_{\boldsymbol\mu}, (\mathfrak{q}_{2,n}^{X_{\boldsymbol\mu}})^{nil}).$$

Lemma 6.2. *Let $\mathfrak{w} = \mathfrak{q}_{2,n}$ or $\mathfrak{p}_{2,n}$ and $\boldsymbol\mu = (\boldsymbol\lambda, \epsilon, l) \in \mathcal{P}''(n)$.*

1. *$(\mathfrak{w}^{X_\mu})^{nil}$ is an irreducible subvariety of $\mathfrak{w}^{X_\mu}$ of codimension*

$$c_{\boldsymbol\mu} = \begin{cases} d_{\boldsymbol\mu} - 1 & \text{if } \varepsilon = 1 \text{ and } l > 0, \\ d_{\boldsymbol\mu} & \text{else.} \end{cases}$$

2. *$\overline{\mathcal{N}_{\boldsymbol\mu}(\mathfrak{w})}$ is a closed irreducible subvariety of $\mathcal{N}(\mathfrak{w})$ of dimension* $\dim \mathfrak{w} - c_{\boldsymbol\mu}$

Proof. The computation (14) remains valid when one replaces $\mathfrak{p}(= \mathfrak{p}_{1,n})$ by $\mathfrak{p}_{2,n}$ or $\mathfrak{q}_{2,n}$. Hence, the second assertion is a consequence of the first one.

The proof is based on case by case considerations on $(\mathfrak{w}^{X_\mu})^{nil}_{gr}$ and the use of Lemma 3.9 (or Remark 3.10) in a similar manner as in Lemma 4.5.

Firstly, assume that $\varepsilon = 0$ or $l = 0$. The proof of Lemma 4.5 can easily be translated here. An elementary base change $(f_j^i)_{i,j}$ based on a reordering of $(e_1, (g_j^i)_{i,j})$ transforms $X_{\boldsymbol{\mu}}$ into an element in Jordan canonical form in M_n with partition $\boldsymbol{\mu}' \in \mathcal{P}(n)$. In these cases, $(\mathfrak{w}^{X_{\boldsymbol{\mu}}})_{gr}$ is defined in $(M_{d_{\boldsymbol{\mu}}}^{X_{\boldsymbol{\mu}}})_{gr}$ by a condition of one of the types given in the RHS below, for some i_0 and possibly i_1.

$$Y \in (\mathfrak{w}^{X_{\boldsymbol{\mu}}})_{gr} \Leftrightarrow \text{ (or)} \begin{cases} Y_{gr}(f_1^{i_0}) \in \Bbbk f_1^{i_0} & (\epsilon = 1, l = 0) \\ Y_{gr}(f_1^{i_0}) \in \Bbbk f_1^{i_0} \text{ and } Y_{gr}(f_1^{i_1}) \in \Bbbk f_1^{i_1} & \left(\begin{smallmatrix} \epsilon = 0, l \neq \lambda_1 \\ l+1 = \mu'_{i_0} \neq \mu'_{i_1} = \lambda_1 + 1 \end{smallmatrix} \right) \\ Y_{gr}(f_1^{i_0}), Y_{gr}(f_1^{i_1}) \in \langle f_1^{i_0}, f_1^{i_1} \rangle & \left(\begin{smallmatrix} \mathfrak{w} = \mathfrak{p}_{2,n}, \epsilon = 0, l = \lambda_1, \\ \mu'_{i_0} = \mu'_{i_1} = l+1 \end{smallmatrix} \right) \\ Y_{gr}(f_1^{i_0}) \in \Bbbk f_1^{i_0} \text{ and } Y_{gr}(f_1^{i_1}) \in \langle f_1^{i_0}, f_1^{i_1} \rangle & \left(\begin{smallmatrix} \mathfrak{w} = \mathfrak{q}_{2,n}, \epsilon = 0, l = \lambda_1, \\ \mu'_{i_0} = \mu'_{i_1} = l+1 \end{smallmatrix} \right) \end{cases}$$

In particular, $(\mathfrak{w}^{X_{\boldsymbol{\mu}}})_{gr} = \prod_\ell \mathfrak{w}^{X_{\boldsymbol{\mu}}}(\ell)$ and each $\mathfrak{w}^{X_{\boldsymbol{\mu}}}(\ell)$ is isomorphic to M_{τ_ℓ}, $\mathfrak{p}_{1,\tau_\ell}$, $\mathfrak{p}_{2,\tau_\ell}$ or $\mathfrak{q}_{2,\tau_\ell}$. We then finish as in Lemma 4.5.

If $\varepsilon = 1$ and $l > 0$, we have a more subtle base change to operate. Let $i_0 = \max\{i | \lambda_i > \lambda_1\}$. Recall that the condition $\epsilon = 1$ implies the inequality $1 < i_{\boldsymbol{\mu}} \leqslant i_0$ (cf. (20)). Let

$$f_j^i := \begin{cases} g_j^{i+1} & \text{if } i < i_0,\ i+1 \neq i_{\boldsymbol{\mu}} \text{ and } 1 \leqslant j \leqslant \lambda_{i+1}, \\ g_{j-1}^{i+1} & \text{if } i+1 = i_{\boldsymbol{\mu}} \text{ and } 1 < j \leqslant \lambda_{i_{\boldsymbol{\mu}}} + 1, \\ e_1 & \text{if } i+1 = i_{\boldsymbol{\mu}} \text{ and } j = 1, \\ g_j^1 - g_j^{i_{\boldsymbol{\mu}}} & \text{if } i = i_0 \text{ and } 1 \leqslant j \leqslant \lambda_1. \\ g_j^i & \text{if } i > i_0 \text{ and } 1 \leqslant j \leqslant \lambda_i, \end{cases} \tag{22}$$

In this new basis, $X_{\boldsymbol{\mu}}$ is in Jordan canonical form associated to a partition $\boldsymbol{\mu}' = (\mu'_1 \geqslant \cdots \geqslant \mu'_{d_{\boldsymbol{\lambda}}}) \in \mathcal{P}(n)$ and $\mathfrak{q}_{2,n}$ (resp. $\mathfrak{p}_{2,n}$) is characterized by the two conditions

$$Y \in \mathfrak{q}_{2,n} \text{ (resp. } \mathfrak{p}_{2,n}) \Leftrightarrow \begin{cases} Y(f_1^{i_{\boldsymbol{\mu}}-1}) \in \Bbbk f_1^{i_{\boldsymbol{\mu}}-1} \quad (\text{resp. } Y(f_1^{i_{\boldsymbol{\mu}}-1}) \in \langle f_1^{i_{\boldsymbol{\mu}}-1}, f_1^{i_0} + f_2^{i_{\boldsymbol{\mu}}-1} \rangle), \\ Y(f_1^{i_0} + f_2^{i_{\boldsymbol{\mu}}-1}) \in \langle f_1^{i_{\boldsymbol{\mu}}-1}, f_1^{i_0} + f_2^{i_{\boldsymbol{\mu}}-1} \rangle. \end{cases} \tag{23}$$

Define $\ell_1 := \mu'_{i_{\boldsymbol{\mu}}-1} = \lambda_{i_{\boldsymbol{\mu}}} + 1 = l+1$ and $\ell_2 := \mu'_{i_0} = \lambda_1$. From now on, we assume that $Y \in M_n^{X_{\boldsymbol{\mu}}}$. Then $Y(f_1^{i_{\boldsymbol{\mu}}-1})$ has no component in $f_2^{i_{\boldsymbol{\mu}}-1}$ (Lemma 3.3). Hence, for such Y, the two conditions on the first line of (23) are both equivalent to the existence of some $\alpha \in \Bbbk$ such that $Y(f_1^{i_{\boldsymbol{\mu}}-1}) = \alpha f_1^{i_{\boldsymbol{\mu}}-1}$.

Now, write $Y(f_1^{i_0}) = \sum_i \beta_i f_1^i$ and $Y(f_2^{i_{\boldsymbol{\mu}}-1}) = \sum_i \gamma_i f_1^i + \gamma'_i f_2^i$ (Lemma 3.3). We note that $\gamma'_{i_{\boldsymbol{\mu}}-1} = \alpha$ and, since $\mu'_{i_{\boldsymbol{\mu}}-1} = \lambda_{i_{\boldsymbol{\mu}}} + 1 \geqslant \lambda_1 + 2 = \mu'_{i_0} + 2$, we have $\gamma_i = 0$ for all i such that $\mu'_i = \mu'_{i_0}$ (Lemma 3.3). Hence the second condition of (23), $Y(f_1^{i_0} + f_2^{i_{\boldsymbol{\mu}}-1}) = \xi f_1^{i_{\boldsymbol{\mu}}-1} + \delta(f_1^{i_0} + f_2^{i_{\boldsymbol{\mu}}-1})$, implies $\beta_{i_0} = \delta = \gamma'_{i_{\boldsymbol{\mu}}-1} = \alpha$ and $\beta_i = 0$ for all $i \neq i_0$ such that $\mu'_i = \mu'_{i_0}$. Thus, we have the following characterization of $(\mathfrak{w}^{X_{\boldsymbol{\mu}}})_{gr}$ in $M_n^{X_{\boldsymbol{\mu}}}$:

$$Y_{gr} \in (\mathfrak{w}^{X_{\boldsymbol{\mu}}})_{gr} \Leftrightarrow Y(\ell_1), Y(\ell_2) = \left(\begin{pmatrix} \alpha & A_1 \\ 0 & \\ \vdots & B_1 \\ 0 & \end{pmatrix}, \begin{pmatrix} \alpha & A_2 \\ 0 & \\ \vdots & B_2 \\ 0 & \end{pmatrix} \right), \quad \begin{matrix} \alpha \in \Bbbk, \\ A_j \in M_{1,\tau_{\ell_j}-1}, \\ B_j \in M_{\tau_{\ell_j}-1}. \end{matrix}$$

Hence $\mathfrak{w}_{gr}^{X_{\boldsymbol{\lambda}}} = \mathfrak{w}^{X_{\boldsymbol{\lambda}}}(\ell_1, \ell_2) \times \prod_{\ell \notin \{\ell_1, \ell_2\}} \mathfrak{w}^{X_{\boldsymbol{\lambda}}}(\ell)$; $\mathfrak{w}^{X_{\boldsymbol{\mu}}}(\ell) = M_{\tau_\ell}$ for $\ell \neq \ell_1, \ell_2$ and $(\mathfrak{w}^{X_{\boldsymbol{\mu}}}(\ell_1, \ell_2))^{nil}$ is characterized in $\mathfrak{w}^{X_{\boldsymbol{\mu}}}(\ell_1, \ell_2)$ by the conditions $\alpha = 0$,

B_1, B_2 nilpotent (Lemma 3.1). Thus $(\mathfrak{w}^{X_\mu}(\ell_1,\ell_2))^{nil}$ is an irreducible variety of codimension $\tau_{\ell_1}+\tau_{\ell_2}-1$ in $\mathfrak{w}^{X_\mu}(\ell_1,\ell_2)$ (Lemma 3.1); the variety $(\mathfrak{w}^{X_\mu})^{nil}$ is also irreducible and $\operatorname{codim}_{\mathfrak{w}^{X_\mu}}(\mathfrak{w}^{X_\mu})^{nil} = d_{\boldsymbol{\mu}}-1$ (Remark 3.10). Hence we have proved the first assertion follows in this last case. ■

Theorem 6.3. *Let* $\mathfrak{w} = \mathfrak{q}_{2,n}$ *or* $\mathfrak{p}_{2,n}$. *Then* $\mathcal{N}(\mathfrak{w})$ *is equidimensional of dimension* $\dim \mathfrak{w} - 1$. *It has* $\left\lfloor \frac{n}{2} \right\rfloor$ *components.*

Proof. We have $\min\{c_{\boldsymbol{\mu}} | \boldsymbol{\mu} \in \mathcal{P}''(n)\} = 1$. Hence, it follows from Lemma 6.2 and Proposition 5.1 that each irreducible component of $\mathcal{N}(\mathfrak{q}_{2,n})$ has dimension $\dim \mathfrak{q}_{2,n} - 1$. There are two types of $\boldsymbol{\mu} \in \mathcal{P}''(n)$ such that $c_{\boldsymbol{\mu}} = 1$.

- $\boldsymbol{\mu} = ((n-1,\emptyset),0,1)$ which is the only element whose associated partition of n has just one part.
- $\boldsymbol{\mu} = ((\lambda_1,\lambda_2),\lambda_2,1)$ with $\lambda_2 > \lambda_1$. Its associated partition of n has two parts: $(\lambda_2+1 \geqslant \lambda_1)$, cf. (22) for more details. Note that this covers (the transpose of) the partitions involved in the proof of Proposition 1.2 since $\lambda_2 > \lambda_1 \Leftrightarrow (\lambda_2+1)-\lambda_1 \geqslant 2$.

There are $\left\lfloor \frac{n}{2} \right\rfloor$ such elements, whence the statement for $\mathfrak{w} = \mathfrak{q}_{2,n}$.
It follows from the description above that the map $\{\boldsymbol{\mu} \in \mathcal{P}''(n) | c_{\boldsymbol{\mu}} = 1\} \to \mathcal{P}(n)$ which sends $\boldsymbol{\mu}$ to the partition associated to $\mathrm{GL}_n . X_{\boldsymbol{\mu}}$ is injective. In particular, the different such $X_{\boldsymbol{\mu}}$ belong to different $P_{2,n}$-orbits. So the associated varieties $\overline{\mathcal{N}_{\boldsymbol{\mu}}(\mathfrak{p}_{2,n})}$, which are the irreducible components of maximal dimension of $\mathcal{N}(\mathfrak{p}_{2,n})$, are all distinct.

There remains to prove that there is no other irreducible component in $\mathcal{N}(\mathfrak{p}_{2,n})$. Let $(X,Y) \in \mathcal{N}(\mathfrak{p}_{2,n})$. The pair $(X_{|V_2}, Y_{|V_2})$ is a commuting pair in $\mathfrak{gl}(V_2)$ hence, up to $\mathrm{GL}(V_2) \times Id_{U_2} (\subset P_{2,n})$-conjugacy, we can assume that $X(e_1) = Y(e_1) = 0$. That is $(X,Y) \in \mathcal{N}(\mathfrak{q}_{2,n})$. In particular, there exists $\boldsymbol{\mu} \in \mathcal{P}''(n)$ such that $(X,Y) \in \overline{\mathcal{N}_{\boldsymbol{\mu}}(\mathfrak{q}_{2,n})} \subset \overline{\mathcal{N}_{\boldsymbol{\mu}}(\mathfrak{p}_{2,n})}$ and $c_{\boldsymbol{\mu}} = 1$. We have therefore shown that

$$\mathcal{N}(\mathfrak{p}_{2,n}) \subset \bigcup_{c_{\boldsymbol{\mu}}=1} \overline{\mathcal{N}_{\boldsymbol{\mu}}(\mathfrak{p}_{2,n})},$$

and we are done. ■

Remark 6.4. (i) The key point of this last proof in the case $\mathfrak{w} = \mathfrak{p}_{2,n}$ is that $\dim \mathcal{N}_{\boldsymbol{\mu}}(\mathfrak{q}_{2,n})$ and $\dim \mathcal{N}_{\boldsymbol{\mu}}(\mathfrak{p}_{2,n})$ are both related to the same integer $c_{\boldsymbol{\mu}}$. This is what allows us to carry out the equidimensionality property from $\mathcal{N}(\mathfrak{q}_{2,n})$ to $\mathcal{N}(\mathfrak{p}_{2,n})$
(ii) The method used in this section is deeply based on the decomposition of $\mathcal{N}(\mathfrak{w})$ as a finite union of the irreducible subvarieties $\mathcal{N}_{\boldsymbol{\mu}}(\mathfrak{w})$. For this, the classification into finitely many orbits of Lemma 6.1 plays a key role. This situation breaks down in general for $\mathfrak{p}_{k,n}$. Using quiver theory and techniques similar to [Bo], M. Reineke communicated to us an example of an infinite family of $P_{6,12}$-orbits in $\mathfrak{p}_{6,12}$.
(iii) Similarly, in [GR], the authors show that some continuous families of $Q_{n,n}$-orbits exist in $\mathfrak{q}_{n,n}$ (Borel case) as soon as $n \geqslant 6$. From this, they deduce the

existence of irreducible components of $\mathcal{N}(\mathfrak{q}_{n,n})$ of dimension greater or equal than $\dim \mathfrak{q}_{n,n}$ showing that the variety is not equidimensional in these cases.

Corollary 6.5. *$S_0^{[2,n]}, S_0^{[n-2,n]}, S_0^{[\![n-2,n]\!]}$ are equidimensional of dimension $n-1$. They have $\lfloor \frac{n}{2} \rfloor$ components.*

Proof. The number of components in $S_0^{[2,n]}$ is (Proposition 2.13) the number of components in $\mathcal{N}^{cyc}(\mathfrak{p}_{n-2,n})$, thus at most the number $\lfloor \frac{n}{2} \rfloor$ of components in the variety $\mathcal{N}(\mathfrak{p}_{n-2,n})$ which may contain noncyclic components. On the other hand, we have exhibited $\lfloor \frac{n}{2} \rfloor$ components of dimension $n-1$ in $S_0^{[2,n]}$ in Proposition 1.2, hence the conclusion for $S_0^{[2,n]}$. The same argument applies to $S_0^{[n-2,n]}$, using Remark 1.3.

Finally, from the existence of a surjective morphism $S_0^{[\![n-2,n]\!]} \to S_0^{[n-2,n]}$ (Proposition 2.14), we see that $S_0^{[\![n-2,n]\!]}$ has at least $\lfloor \frac{n}{2} \rfloor$ components. But Theorem 6.3 implies that there are at most $\lfloor \frac{n}{2} \rfloor$ components, and that these components have dimension $n-1$. The result follows. ∎

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