

Pseudogroups via Pseudoactions: Unifying Local, Global, and Infinitesimal Symmetry

Anthony D. Blaom *

Communicated by K.-H. Neeb

Abstract. A multiplicatively closed, horizontal foliation on a Lie groupoid may be viewed as a ‘pseudoaction’ on the base manifold M . A pseudoaction generates a pseudogroup of transformations of M in the same way an ordinary Lie group action generates a transformation group. Infinitesimalizing a pseudoaction, one obtains the action of a Lie algebra on M , possibly twisted. A global converse to Lie’s third theorem proven here states that every twisted Lie algebra action is integrated by a pseudoaction. When the twisted Lie algebra action is complete it integrates to a twisted Lie group action, according to a generalization of Palais’ global integrability theorem.

Mathematics Subject Classification 2010: Primary 58H05; Secondary 58D19, 54H15.

Key Words and Phrases: Lie algebroid, pseudogroup, Cartan connection, Lie algebra, pseudoaction.

Dedicated to Richard W. Sharpe

1. Introduction

Pseudoactions. Unlike transformation groups, pseudogroups of transformations capture simultaneously the phenomena of global and *local* symmetry. On the other hand, a transformation group can be replaced by the action of an abstract group which may have nice properties — a Lie group in the best scenario. Here we show how to extend the abstraction of group actions to handle certain pseudogroups as well. These pseudogroups include: (i) all *Lie* pseudogroups of finite type, both transitive and intransitive, and hence the isometry pseudogroups of suitably regular geometric structures of finite type; (ii) all pseudogroups generated by the local flows of a smooth vector field; and, more generally (iii) any pseudogroup generated by the infinitesimal action of a finite-dimensional Lie algebra. Our generalization of a group action on M , here called a *pseudoaction*, consists of a multiplicatively

* Part of the research reported here was carried out during the author’s visit in 2014 to Utrecht University, whose support is gratefully acknowledged.

closed, horizontal foliation on a Lie groupoid G over M . Ordinary group actions are recovered as action groupoids equipped with the canonical horizontal foliation. Details appear in Section 2 below.

Differentiating a pseudoaction, one obtains a flat Cartan connection on a Lie algebroid. Cartan connections on Lie algebroids, introduced in [1] and studied further in [2, 3], are related to the classical connections bearing the same name. By definition, a Cartan connection on a Lie algebroid is just an ordinary linear connection ∇ , suitably respecting the underlying Lie algebroid structure. Every Lie algebroid over M equipped with a flat Cartan connection is the quotient by covering transformations of an action algebroid over the universal cover of M (see Section 2 below and [3]). For this reason we call Lie algebroids equipped with such a connection *twisted Lie algebra actions*. However, we emphasise that it is the Cartan connection, and not the action, that is ordinarily manifest in practice. In particular, this applies to the infinitesimal isometries of finite type geometric structures [2] and, more generally, to PDE's whose infinitesimal symmetries are of finite type.

As far as we know, pseudoactions were first introduced by Tang [19], using the language of étale groupoids. We discovered the notion independently, sketching a relationship between pseudoactions and their infinitesimal analogues in [2, Appendix A].

Integrating infinitesimal symmetry. A very well-known observation of Sophus Lie is sometimes phrased as follows:

Theorem 1.1 (c.1880 [11]). *Differentiating the action of an r -dimensional group of transformations on M at the identity, one obtains an r -dimensional Lie subalgebra of vector fields on M .*

The naive converse statement is false: Not every Lie algebra of vector fields on M is the infinitesimalization of a transformation group on M . Several partial converse statements, and generalizations thereof, are well-known. For example, Lie himself showed that every r -dimensional subalgebra of vector fields is generated by a *local* Lie group of transformations on M ; see e.g., [10]. (Of course all Lie's groups were local.) On the other hand, *any* Lie algebra is isomorphic to the Lie algebra \mathfrak{g}_0 of right-invariant vector fields of some Lie group G_0 . This is the result nowadays referred to as 'Lie III' but is actually due to Élie Cartan [5, 6].¹ For a beautiful generalization of Lie III to Lie groupoids, see Crainic and Fernandes [7, 8].

With the abstraction of Lie groups and Lie algebras and their actions in hand, one has the following partial converse to Lie's theorem, due to Richard Palais:

Theorem 1.2 (Palais' local integrability theorem [17]). *Every action of an*

¹In this article 'Lie I for groups' refers to the theorem that any Lie algebra integrated by a Lie group is in fact integrated by a simply-connected Lie group, unique up to isomorphism; 'Lie II for groups' is the theorem that every morphism of Lie algebras $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is integrated by a unique Lie group morphism $G_1 \rightarrow G_2$, where G_1 is a simply-connected Lie group integrating \mathfrak{g}_1 , and G_2 any Lie group integrating \mathfrak{g}_2 .

abstract Lie algebra \mathfrak{g}_0 on M is integrated by a local action of the simply-connected Lie group G_0 having \mathfrak{g}_0 as its Lie algebra.

In Section 4 we offer a short and novel proof of Palais' theorem using Lie groupoids. This can be compared to Palais' original proof or a contemporary proof such as [15, Theorem 6.5]. An interesting question, addressed by Kamber and Michor [12] but not here, is whether the space M can be 'completed' to larger space on which the action of G_0 becomes global.

For the special case of complete Lie algebra actions, Palais has already provided the following global result:

Theorem 1.3 (Palais' global integrability theorem [17]). *If \mathfrak{g}_0 above acts by complete vector fields, then the action is integrated by a global action of G_0 on M .*

Now according to the constructions outlined in Section 1 above, we have the following generalization of Lie's theorem cited above: *Differentiating a pseudoaction one obtains a twisted Lie algebra action.* The main preoccupation of the present paper will be to prove the following *global* converse:

Theorem 1.4 (Lie III for pseudoactions). *Every twisted Lie algebra action is integrated by some pseudoaction.*

Here we mean 'integrated' in the strongest sense: the pseudogroup defined by the pseudoaction, and the pseudogroup generated by local flows of the infinitesimal generators of the Lie algebra action, coincide. The integrating pseudoaction is uniquely determined, up to isomorphism, if we insist the underlying groupoid has simply-connected source fibres. See the restatement Theorem 2.6 below for details.

Under an appropriate completeness hypothesis we also obtain a generalization of Palais' global integrability theorem:

Theorem 1.5. *Every complete twisted Lie algebra action is integrated by a twisted Lie group action.*

Twisted Lie group actions are defined in Section 2; completeness in Section 2.

To complete this brief survey of old and new integrability results, we note a recently published theorem which applies in the special case of *transitive* actions:

Theorem 1.6 ([3]). *A smooth manifold M admits a transitive, geometrically closed, twisted Lie algebra action if and only if it is locally homogeneous.*

Recall here that M is *locally homogeneous* if M admits an atlas of coordinate charts modelled on a homogeneous space G/H , where the transition functions are right translations by elements of G . The closedness condition (not defined here) is needed to ensure G/H is a Hausdorff manifold. Locally homogeneous manifolds are less general than transitive pseudogroups generated by a pseudoaction, but more general than transitive twisted Lie group actions (manifolds homogeneous

‘up to cover’, see op. cit.).

Paper outline. Pseudoactions, and the pseudogroups they define, are described in Section 2, which includes a detailed formulation of the two new results alluded to above, reformulated as Theorem 2.6 — what we refer to as ‘Lie III for pseudoactions’ — and Theorem 2.7, a generalization of Palais’ global integrability theorem to the twisted case. It is an immediate corollary of Theorems 2.6 and 2.8 that all the pseudogroups on the list in Section 1 above can be encoded by pseudoactions. The simplest non-trivial pseudoactions, twisted Lie group actions, are also described in Section 2. As mentioned above and reviewed in Section 2, their infinitesimal counterparts amount to flat Cartan connections on a Lie algebroid. We conclude Section 2 by revisiting Olver’s example of a non-associative local Lie group [16] from the pseudoaction point of view.

In Section 4 we prove Lie III for pseudoactions in the untwisted case. The proof is based on an explicit model for a Lie groupoid integrating a given action algebroid, whose existence is guaranteed by a theorem of Dazord [9], but which differs from Dazord’s model (which seems unsuitable for our purposes). However, to show our model is well-defined, we must appeal to Palais’ local integrability theorem, and our new proof of this theorem depends on Dazord’s existence result.

In order to generalize to the untwisted case, we first formulate and prove versions of Lie I and Lie II for pseudoactions, which are of independent interest; these are Theorems 5.6 and 5.4 respectively. Theorem 5.4 gives sufficient conditions for integrating morphisms of twisted Lie algebra actions to morphisms of pseudoactions, but is restricted to the case of morphisms covering a local diffeomorphism of base manifolds (sufficient for present purposes). We conjecture that Theorems 5.6 and 5.4 hold for general morphisms and suggest the use of Salazar’s cocycle techniques [18] to extend our proofs.

With Lie I and Lie II in hand, the untwisted version of Lie III implies the general case, as we show in Section 6. Finally, the case of complete twisted Lie algebra actions is treated in Section 7.

Notation. Throughout this paper G denotes a Lie groupoid, G_0 a Lie group, \mathfrak{g} a Lie algebroid, and \mathfrak{g}_0 a Lie algebra.

2. Detailed formulation of main results

The reader is assumed to have some familiarity with Lie groupoids and Lie algebroids — say what is contained in [4]. For more detail, see [8, 14].

Pseudotransformations and pseudoactions. Let G be a Lie groupoid over a smooth, connected, paracompact manifold M , and call an immersed submanifold $\Sigma \subset G$ a *pseudotransformation* if the restrictions to Σ of the groupoid’s source and target maps are local diffeomorphisms. For example, if $\dim M = n$, then any smooth n -dimensional submanifold of $M \times M$ that is locally a graph over *both* projections $M \times M \rightarrow M$ is a pseudotransformation of the pair groupoid $M \times M$. Thus every local diffeomorphism of M may be regarded as a pseudotransformation, but so also may its ‘inverse’.

A *pseudoaction* on G is any smooth foliation \mathcal{F} on G such that:

- (1) The leaves of \mathcal{F} are pseudotransformations.
- (2) \mathcal{F} is *multiplicatively closed*.

To define what is meant in (2), regard local bisections of G as immersed submanifolds and let $\widehat{\mathcal{F}}$ denote the collection of all local bisections that intersect each leaf of \mathcal{F} in an *open* subset. Let \widehat{G} denote the collection of *all* local bisections of G , this being a groupoid over the collection of all open subsets of M . Then condition (2) is the requirement that $\widehat{\mathcal{F}} \subset \widehat{G}$ be a subgroupoid.

Given a pseudoaction \mathcal{F} of G on M , each element $b \in \widehat{\mathcal{F}}$ defines a local transformation ϕ_b of M . We let $\text{pseud}(\mathcal{F})$ denote the set of all local transformations $\psi: U \rightarrow V$ of M that are locally of this form; that is, for every $m \in U$, there exists a neighbourhood $U' \subset U$ of m such that $\psi|_{U'} = \phi_b$ for some $b \in \widehat{\mathcal{F}}$. Then, by (2), $\text{pseud}(\mathcal{F})$ is a pseudogroup.

The basic prototype of a pseudoaction is any smooth action of a Lie group G_0 on M : take as Lie groupoid the action groupoid $G := G_0 \times M$ and as foliation \mathcal{F} the one with leaves $\{g\} \times M$, $g \in G_0$. In this case $\text{pseud}(\mathcal{F})$ consists of all local transformations $\phi: U \rightarrow V$ of M such that restrictions of ϕ to connected components of U are of the form $m \mapsto g \cdot m$, for some $g \in G_0$.

Pseudoactions as flat Cartan connections. Let G be a Lie groupoid and J^1G the corresponding Lie groupoid of one-jets of local bisections of G . Then a right-inverse $S: G \rightarrow J^1G$ for the canonical projection $J^1G \rightarrow G$ determines a rank- n distribution $D \subset TG$ on G , where $n = \dim M$. If $S: G \rightarrow J^1G$ is additionally a morphism of Lie groupoids, then we call D (or S) a *Cartan connection* on G . We prove the following elementary observation in Section 3:

Proposition 2.1. *An arbitrary foliation \mathcal{F} on G is a pseudoaction if and only if its tangent distribution D is a Cartan connection on G .*

Now let \mathcal{F} be a pseudoaction of G and $S: G \rightarrow J^1G$ the corresponding Cartan connection. Infinitesimalizing S , we obtain a splitting $s: \mathfrak{g} \rightarrow J^1\mathfrak{g}$ of the exact sequence

$$0 \longrightarrow T^*M \otimes \mathfrak{g} \longrightarrow J^1\mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Here $J^1\mathfrak{g}$ denotes the Lie algebroid of one-jets of sections of the Lie algebroid \mathfrak{g} of G . (A canonical identification of the Lie algebra of J^1G with $J^1\mathfrak{g}$ is recalled in Section 3 below.) In the category of vector bundles, splittings s of the above sequence are in one-to-one correspondence with linear connections ∇ on \mathfrak{g} ; this correspondence² is given by

$$(3) \quad sX = J^1X + \nabla X; \quad X \in \Gamma(\mathfrak{g}).$$

When $s: \mathfrak{g} \rightarrow J^1\mathfrak{g}$ is a morphism of Lie algebroids, as is the case here, then ∇ is called an (*infinitesimal*) *Cartan connection* on \mathfrak{g} .

Certainly not all Cartan connections on \mathfrak{g} arise from smooth pseudoactions. Rather, one has the following fundamental result:

²According to our sign conventions, the inclusion $T^*M \otimes \mathfrak{g} \rightarrow J^1\mathfrak{g}$ induces the following map on sections: $df \otimes X \mapsto fJ^1X - J^1(fX)$.

Theorem 2.2. *Assume G is source-connected. Then a Cartan connection D is tangent to a pseudoaction if and only if the corresponding infinitesimal Cartan connection ∇ is flat.*

This theorem follows from a result proven recently by Salazar [18, Theorem 6.4.1] for arbitrary ‘multiplicatively closed’ distributions on a Lie groupoid, of which the distribution D defined by a Cartan connection is a special case. An independently obtained proof along different lines will be given elsewhere. Both proofs involve techniques not needed in the rest of the paper.

In general, an arbitrary Cartan connection on a Lie groupoid G defines a rank- n distribution on G that is not integrated by any foliation \mathcal{F} .

Twisted Lie group actions. To describe the simplest non-trivial examples of pseudoactions, let G_0 be a Lie group acting on the universal cover \widetilde{M} of M . Assume the action respects the group of covering transformations $\Lambda \cong \pi_1(M)$ in the following sense: If G_0 acts effectively and is identified with a subgroup of $\text{Diff}(M)$, then require that $\Lambda \subset \text{Diff}(M)$ be contained in the normaliser of G_0 . More generally, we require the existence of a group homomorphism

$$\lambda \mapsto \nu_\lambda: \Lambda \rightarrow \text{Aut}(G_0),$$

such that

$$(4) \quad \lambda(g \cdot \widetilde{m}) = \nu_\lambda(g) \cdot \lambda(\widetilde{m}); \quad \lambda \in \Lambda, g \in G_0, \widetilde{m} \in \widetilde{M}.$$

(We say that $\lambda: \widetilde{M} \rightarrow \widetilde{M}$ is G_0 -equivariant with twist ν_λ .) This requirement can always be satisfied when G_0 contains Λ as a subgroup, provided the action of G_0 on \widetilde{M} extends the tautological action of Λ ; in that case take $\nu_\lambda(g) = \lambda g \lambda^{-1}$. In any case, assuming (4) holds, Λ acts on the action groupoid $G_0 \times \widetilde{M}$ by Lie groupoid automorphisms and the quotient $G_0 \times_\nu M := (G_0 \times \widetilde{M})/\Lambda$ becomes a Lie groupoid over M . Moreover, the canonical foliation of $G_0 \times M$ by copies of M drops to a pseudoaction \mathcal{F} on $G_0 \times_\nu M$. Such pseudoactions will be called *twisted Lie group actions*.

It is easy to see $\text{pseud}(\mathcal{F})$ is the pseudogroup generated by all local transformations $\varphi: U \rightarrow V$ of M covered by a local transformation of \widetilde{M} of the form $\widetilde{m} \rightarrow g \cdot \widetilde{m}$, $g \in G_0$.

Example 2.3. Take G_0 to be the three-dimensional group of isometries of the Euclidean plane \mathbb{R}^2 to obtain a twisted Lie group action on the cylinder $S^1 \times \mathbb{R} \cong \mathbb{R}^2/\mathbb{Z}$. The associated pseudogroup of transformations is the pseudogroup of local and global isometries of $S^1 \times \mathbb{R}$. This pseudogroup includes the isometry *group* of $S^1 \times \mathbb{R}$ (i.e., the global isometries) which is only two-dimensional.

Twisted Lie algebra actions. According to Theorem 2.2, the infinitesimalization of a pseudoaction is a Lie algebroid \mathfrak{g} equipped with a flat Cartan connection ∇ . We call such a pair (\mathfrak{g}, ∇) a *twisted Lie algebra action*. To justify this terminology, let \mathfrak{g}_0 be a finite-dimensional Lie algebra acting smoothly from the left

on the universal cover \widetilde{M} of M , and denote the corresponding Lie algebra homomorphism $\mathfrak{g}_0 \rightarrow \Gamma(T\widetilde{M})$ by $\xi \mapsto \xi^\dagger$. We assume this action respects the group of covering transformations Λ in the following sense: there exists a representation $\mu \mapsto \mu_\lambda: \Lambda \rightarrow \text{Aut}(\mathfrak{g}_0)$ of Λ by Lie algebra automorphisms such that

$$\lambda_* \xi^\dagger = (\mu_\lambda \xi)^\dagger; \quad \lambda \in \Lambda, \xi \in \mathfrak{g}_0.$$

Here λ_* denotes pushforward. (This requirement is just the infinitesimal form of (4); we say $\lambda: \widetilde{M} \rightarrow \widetilde{M}$ is \mathfrak{g}_0 -equivariant with twist μ_λ .) Then the action of Λ on the action algebroid $\mathfrak{g}_0 \times \widetilde{M}$, defined by $\lambda \cdot (\xi, \widetilde{m}) = (\mu_\lambda \xi, \lambda(\widetilde{m}))$, is by Lie algebroid automorphisms, and the quotient $\mathfrak{g}_0 \times_\mu M := (\mathfrak{g}_0 \times \widetilde{M})/\Lambda$ becomes a Lie algebroid over M . Moreover, the canonical flat connection on $\mathfrak{g}_0 \times \widetilde{M}$, which is Cartan, drops to a flat Cartan connection ∇ on $\mathfrak{g}_0 \times_\mu M$.

Conversely, we have the following trivial observation, recorded for later on:

Remark 2.4. The formal pullback of $\mathfrak{g}_0 \times_\mu M$ to a Lie algebroid $\widetilde{\mathfrak{g}}$ over \widetilde{M} is canonically isomorphic to $\mathfrak{g}_0 \times \widetilde{M}$, with the canonical action of Λ on $\widetilde{\mathfrak{g}}$ being represented by the action of Λ on $\mathfrak{g}_0 \times \widetilde{M}$ described above.

The main point, however, is:

Proposition 2.5 ([3]). *Every Lie algebroid \mathfrak{g} over M , equipped with a flat Cartan connection ∇ — i.e., every twisted Lie algebra action — is naturally isomorphic to $\mathfrak{g}_0 \times_\mu M$, for some \mathfrak{g}_0 and μ as above.*

In detail: Let $\widetilde{\mathfrak{g}}$ denote the pullback of \mathfrak{g} to a Lie algebroid over \widetilde{M} , $\widetilde{\nabla}$ the pullback connection, and \mathfrak{g}_0 the finite-dimensional vector space of $\widetilde{\nabla}$ -parallel sections of $\widetilde{\mathfrak{g}}$. Then $\mathfrak{g}_0 \subset \Gamma(\widetilde{\mathfrak{g}})$ is a Lie subalgebra acting on \widetilde{M} according to $\xi^\dagger(\widetilde{m}) = \# \xi(\widetilde{m})$ where $\#$ denotes anchor; the monodromy μ of the flat connection ∇ is a representation of $\pi_1(M) \cong \Lambda$ on \mathfrak{g}_0 by Lie algebra automorphisms; and an isomorphism $\mathfrak{g}_0 \times_\mu M \cong \mathfrak{g}$ is given by

$$(\xi, \widetilde{m}) \bmod \Lambda \mapsto \pi(\xi(\widetilde{m})),$$

where $\pi: \widetilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ is the canonical projection. For further details see [3].

Pseudogroups generated by infinitesimal data. Let (\mathfrak{g}, ∇) be a twisted Lie algebra action on M and, for each local ∇ -parallel section X of \mathfrak{g} , let $\Phi_{\#X}^t$ denote the corresponding time- t flow map³ of the vector field $\#X$. Here $\#: \mathfrak{g} \rightarrow TM$ denotes the anchor of \mathfrak{g} . Then the collection all local transformations of M of the form $\Phi_{\#X}^t$, for some such X and $t \in \mathbb{R}$, generates a pseudogroup denoted $\text{pseud}(\nabla)$. Locally, each element of $\text{pseud}(\nabla)$ is of the form

$$\Phi_{\#X_1}^{t_1} \circ \Phi_{\#X_2}^{t_2} \circ \cdots \circ \Phi_{\#X_k}^{t_k},$$

³We do not restrict t but allow $\Phi_{\#X}^t$ to have empty domain, i.e., to be the ‘empty transformation’, whose composition with any other is itself.

for some locally defined sections X_1, X_2, \dots, X_k of \mathfrak{g} and $t_1, t_2, \dots, t_k \in \mathbb{R}$. In the special case that (\mathfrak{g}, ∇) is an action algebroid, this is the usual pseudogroup of transformations generated by flows of the infinitesimal generators of the action.

Integrability and formal integrability. A pseudoaction \mathcal{F} on a Lie groupoid G over M will be said to *integrate* a twisted Lie algebra action (\mathfrak{g}, ∇) on M if $\text{pseud}(\mathcal{F}) = \text{pseud}(\nabla)$. We will say that \mathcal{F} *formally integrates* (\mathfrak{g}, ∇) if G integrates \mathfrak{g} — i.e., \mathfrak{g} is the Lie algebroid of G — and if ∇ is the infinitesimalization of the Cartan connection D tangent to \mathcal{F} , as we have described in Section 2. Note that an integration \mathcal{F} need not be a formal integration.

Here is our main result, proven in Section 6:

Theorem 2.6 (Lie III for pseudoactions). *For every twisted Lie algebra action (\mathfrak{g}, ∇) there exists a pseudoaction \mathcal{F} that simultaneously integrates and formally integrates (\mathfrak{g}, ∇) . Moreover, any pseudoaction \mathcal{F} on a Lie groupoid G formally integrating (\mathfrak{g}, ∇) is a bona fide integration if G has connected source-fibres.*

Furthermore, by Lie I for pseudoactions (Theorem 5.6), \mathcal{F} is *uniquely* determined up to isomorphism, if we insist that G have simply-connected source fibres.

Complete twisted Lie algebra actions. In Section 7 we prove the following generalization of Palais' global integrability theorem quoted in Section 1, which strengthens the conclusion of the preceding theorem under an additional hypothesis:

Theorem 2.7. *Every complete twisted Lie algebra action (\mathfrak{g}, ∇) on M is formally integrated (and hence integrated) by a twisted Lie group action.*

In detail: Let $(\tilde{\mathfrak{g}}, \tilde{\nabla})$ denote the pullback of (\mathfrak{g}, ∇) to the universal cover \tilde{M} , and $\mathfrak{g}_0 \subset \Lambda(\tilde{\mathfrak{g}})$ the Lie subalgebra of $\tilde{\nabla}$ -parallel sections. Let $\mu: \Lambda \rightarrow \text{Aut}(\mathfrak{g}_0)$ denote the monodromy representation of the flat connection ∇ . Then this representation is by Lie algebra automorphisms and, by Lie II for Lie groups, lifts to a Lie group homomorphism $\nu: \Lambda \rightarrow \text{Aut}(G_0)$, where G_0 is the simply-connected Lie group integrating \mathfrak{g}_0 . The twisted Lie algebra action (\mathfrak{g}, ∇) is integrated by the twisted Lie group action $G_0 \times_\nu M$.

By *complete* we mean that ∇ should have complete geodesics. A *geodesic* of ∇ is a smooth \mathfrak{g} -path $t \mapsto X_t \in \mathfrak{g}$ such that $\nabla_{\#X_t} X_t = 0$, where $\#: \mathfrak{g} \rightarrow TM$ denotes the anchor. A *\mathfrak{g} -path* is a path $t \mapsto X_t \in \mathfrak{g}$ such that $\#X_t = \dot{m}_t$, where $m_t \in M$ is the base point of X_t and a dot denotes derivative.

Compactness of M is not sufficient for completeness of ∇ unless M is simply-connected. A counterexample, and some sufficient conditions for completeness, are offered in [3].

Finite type Lie pseudogroups. Let $J^k(M \times M)$ denote the Lie groupoid of k -jets of diffeomorphisms $\varphi: U \rightarrow V$ from an open set $U \subset M$ to an open set $V \subset M$. Let Γ denote a pseudogroup of infinitely differentiable transformations on M . Prolonging, one obtains a tower of surjective maps between groupoids,

$$\Gamma^0 \leftarrow \Gamma^1 \leftarrow \Gamma^2 \leftarrow \dots$$

Here $\Gamma^i := \{J_m^i \varphi \mid \varphi \in \Gamma, m \in \text{dom}(\varphi)\}$ and Γ^0 is the foliation by orbits of the pseudogroup, understood as a subgroupoid of $M \times M$. We will call Γ a *Lie pseudogroup of finite type k* if the subgroupoids $\Gamma^k \subset J^k(M \times M)$ and $\Gamma^{k+1} \subset J^{k+1}(M \times M)$ are *Lie* subgroupoids, and if the surjection $\Gamma^{k+1} \rightarrow \Gamma^k$ is a diffeomorphism.

Finite type Lie pseudogroups commonly arise as the symmetries of sufficiently regular geometric structures and nonlinear PDE's.

Theorem 2.8. *For every Lie pseudogroup Γ of finite type k , there exists a canonical pseudoaction \mathcal{F} on Γ^k such that $\Gamma = \text{pseud}(\mathcal{F})$.*

Applying Theorem 2.6 we obtain:

Corollary 2.9. *If the Lie groupoid Γ^k is source-connected, then the Lie pseudogroup Γ defining it is generated by infinitesimal data — i.e., by the flows determined by a flat Cartan connection ∇ on the Lie algebroid of Γ^k , as described above.*

The construction of \mathcal{F} is as follows: Regarding $\Gamma^{k+1} \subset J^{k+1}(M \times M)$ as a subgroupoid of $J^1(J^k(M \times M))$, we in fact have $\Gamma^{k+1} \subset J^1\Gamma^k$. The smooth inverse $S: \Gamma^k \rightarrow \Gamma^{k+1} \subset J^1\Gamma^k$ of $\Gamma^{k+1} \rightarrow \Gamma^k$ (well-defined by $S(J_m^k \varphi) = J_m^{k+1} \varphi$) becomes a right-inverse for the natural projection $J^1\Gamma^k \rightarrow \Gamma^k$ and is necessarily a groupoid morphism. In other words, S is a Cartan connection on Γ^k .

The corresponding distribution D on Γ^k is integrable. Indeed, each point in Γ^k is of the form $g = J_{m_0}^k \varphi$ for some $\varphi \in \Gamma$ and $m_0 \in M$; a local bisection of Γ^k whose image integrates D and contains g is given by $m \mapsto J_m^k \varphi$. The integrating foliation \mathcal{F} is a pseudoaction, by Proposition 2.2.

The proof that $\Gamma = \text{pseud}(\mathcal{F})$ is postponed to Section 3

While every Lie pseudogroup of finite type is generated by a pseudoaction, not all pseudogroups generated by a pseudoaction are Lie pseudogroups. For example, the pseudogroup of local flows of a vector field on \mathbb{R} , vanishing at zero to all orders of differentiability, but vanishing at no other point, is not a Lie pseudogroup of finite type. However, Theorem 2.6 guarantees that every pseudogroup generated by the flow of a vector field is generated by a pseudoaction: take \mathfrak{g} to be the action algebroid $\mathbb{R} \times M$.

A finite-dimensional pseudoaction cannot generate a Lie pseudogroup of infinite type.

Non-associative local Lie groups. We now describe what is perhaps the simplest non-trivial example of a pseudoaction on a simply-connected domain (where there can be no monodromy twist). The example is interesting because the underlying Lie algebra is commutative but the corresponding pseudogroup is not.

Let $\mathbb{R}^2 \setminus 0$ denote the punctured plane and $\pi: M \rightarrow \mathbb{R}^2 \setminus 0$ its universal covering. The coordinate vector fields $\partial/\partial x$ and $\partial/\partial y$ on \mathbb{R}^2 pull back to commuting vector fields v_1, v_2 on M which are consequently the infinitesimal generators of

an action of the Lie algebra \mathbb{R}^2 on M . This action is evidently not complete. If $\pi(m) = (-1/\sqrt{2}, -1/\sqrt{2})$ then $\exp(v_1)(\exp(v_2)m) \neq \exp(v_2)(\exp(v_1)m)$, which shows the pseudogroup generated by this Lie algebra action is not commutative. Consequently, while M is locally isomorphic to the the Abelian group \mathbb{R}^2 , there can be no way to extend a local isomorphism to a global one. Olver [16] shows that one can give M the structure of a local Lie group, but multiplication in this structure is not associative.

The action algebroid in this example is isomorphic to TM which is the Lie algebroid of the pair groupoid $M \times M$. Now

$$\Omega(m_1, m_2) = \pi(m_2) - \pi(m_1)$$

defines a Lie groupoid morphism $\Omega: M \times M \rightarrow \mathbb{R}^2$ into the Abelian Lie group \mathbb{R}^2 and the derivative of Ω is evidently the Lie algebroid morphism $\omega: TM \rightarrow \mathbb{R}^2$ satisfying $\omega(v_1(m)) = (1, 0)$ and $\omega(v_2(m)) = (0, 1)$. It is easy to see that the connected components of fibres of $\Omega: M \times M \rightarrow \mathbb{R}^2$ must be the leaves of a pseudoaction \mathcal{F} on $M \times M$ and that this pseudoaction formally integrates — and hence integrates — the Lie algebra action.

The lack of globalizability noted above is reflected in interesting topology in the corresponding pseudotransformations (the leaves of \mathcal{F}). To describe this topology, first note that the target map $\alpha(m_1, m_2) = m_1$ maps $\Omega^{-1}(u)$ onto $M_u := M \setminus \pi^{-1}(-u)$, $u \in \mathbb{R}^2$. If $u = 0$ then $M_u = M \cong \mathbb{R}^2$; if $u \neq 0$ then M_u is a plane with a countably infinite number of discrete punctures.

Now identify the group of covering transformations of $\pi: M \rightarrow \mathbb{R}^2 \setminus 0$ with \mathbb{Z} and let this group act on $M \times M$ by acting on the second factor. Then \mathbb{Z} acts freely and transitively on the fibres of the restriction $\alpha: \Omega^{-1}(u) \rightarrow M_u$, so that $\Omega^{-1}(u)$ is a principal \mathbb{Z} -bundle over M_u . If $u = 0$ this bundle is trivial, $\Omega^{-1}(0) \cong \mathbb{Z} \times M$, and the connected components of $\Omega^{-1}(0)$ may be viewed as bona fide global transformations of M , namely the covering transformations of $\pi: M \rightarrow \mathbb{R}^2 \setminus 0$. However, for $u \neq 0$ the principal \mathbb{Z} -bundle $\Omega^{-1}(u)$ is non-trivial (one can show the associated Čech cohomology class $c \in H^1(M_u, \mathbb{Z})$ is non-trivial). It is, however, connected, and hence a leaf of \mathcal{F} . From these observations it follows that the leaf space of \mathcal{F} is the plane with a countable infinity of origins.

The example above was pointed out to us by a referee who queries the extent to which our results depend upon unwitting assumptions of associativity. The only associativity used in this paper is associativity of multiplication in a (global) Lie group and associativity of map composition (pseudogroups are always associative).

3. Elementary considerations

Elements of J^1G and its Lie algebroid. Let J^1G denote the Lie groupoid of one-jets of local bisections of a Lie groupoid G . Given an arrow $g \in G$ from m to m' , we will view an element of J_g^1G as a linear map $\mu: T_mM \rightarrow T_gG$ such that $T_m\alpha \circ \mu = \text{id}_{T_mM}$ and $T_m\beta \circ \mu: T_mM \rightarrow T_{m'}M$ is invertible. Here α and β denote the source and target maps respectively.

If \mathfrak{g} is the Lie algebroid of G then by convention elements of \mathfrak{g} will be regarded as vectors tangent at some $m \in M \subset G$ to a fibre of the source projection $\alpha: G \rightarrow M$. Let $J^1\mathfrak{g}$ denote the vector bundle of one-jets of sections of \mathfrak{g} . Implicit in Section 2 is an identification of $J^1\mathfrak{g}$ with the (abstract) Lie algebroid of J^1G . This identification is given by

$$J_m^1X \mapsto \left. \frac{d}{dt} T_m(\Phi_{X^R}^t \circ \iota_M) \right|_{t=0}; \quad X \in \Gamma(\mathfrak{g}), m \in M.$$

Here X^R denotes the right-invariant vector field on G corresponding to X , $\Phi_{X^R}^t$ its time- t flow map, and $\iota_M: M \rightarrow G$ the inclusion.

Local bisections integrating a foliation. It will be convenient henceforth to view a local bisection of a Lie groupoid G as a right inverse $b: U \rightarrow G$ for the target projection $\beta: G \rightarrow M$. We say b integrates a foliation \mathcal{F} on G if the image of any connected subset of U lies within a single leaf of \mathcal{F} . Assuming the leaves of \mathcal{F} are pseudotransformations, the set $\widehat{\mathcal{F}}$ defined in Section 2 then consists of all local bisections of G integrating \mathcal{F} , as it is not hard to show. We record the following elementary observations, whose proofs are left to the reader:

Proposition 3.1. *Let \mathcal{F} be a foliation on G whose leaves are pseudotransformations. Then:*

- (5) *Any local bisection $b: U \rightarrow G$ integrates \mathcal{F} if and only if the image of the tangent map $Tb: TU \rightarrow TG$ is contained in the distribution $D \subset TG$ tangent to \mathcal{F} .*
- (6) *Any two local bisections integrating \mathcal{F} , and agreeing at a single point $m \in M$ in their common domain U , agree on all of U^m , where U^m is the connected component of U containing m .*
- (7) *If \mathcal{F} is a pseudoaction, then every element of $\text{pseud}(\mathcal{F})$ is, locally, of the form $U \xrightarrow{b} G \xrightarrow{\beta} M$, for some local bisection b integrating \mathcal{F} .*

The proof of Proposition 2.1. Let \mathcal{F} be an arbitrary foliation on a Lie groupoid G and $D \subset TG$ its tangent distribution. We omit the straightforward proof that D being Cartan is necessary for \mathcal{F} to be a pseudoaction, and prove only sufficiency.

Let $S: G \rightarrow J^1G$ denote the Lie groupoid morphism corresponding to the distribution D tangent to \mathcal{F} , which we suppose is a Cartan connection. So, if $g \in G$ is an arrow in G beginning at $m \in M$, then $S(g)v \in D(g)$ for all $v \in T_mM$. From the definition of a Cartan connection, it is not hard to see the leaves of \mathcal{F} must be pseudotransformations. To show \mathcal{F} is multiplicatively closed, let $b_1: U_1 \rightarrow G$ and $b_2: U_2 \rightarrow G$ be two local bisections integrating \mathcal{F} , and suppose they are composable, i.e., $U_1 = \beta(b_2(U_2))$; we need to show that the product $b_1b_2: U_2 \rightarrow G$ also integrates \mathcal{F} .

Let $m \in U_2$ be arbitrary. Then, using the definition of products in J^1G , it is not hard to show that

$$T(b_1b_2) \cdot v = J_m^1(b_1b_2)v = (J_{m'}^1b_1 J_m^1b_2)v; \quad v \in T_mM, m' := \beta(b_2(m)).$$

On the other hand, since b_1 and b_2 integrate \mathcal{F} , it follows from (5) above that $J_{m'}^1 b_1 = S(b_1(m'))$ and $J_m^1 b_2 = S(b_1(m))$. Substituting into the above, and using the fact that S is a Lie groupoid morphism, we conclude

$$T(b_1 b_2) \cdot v = S(b_1(m') b_2(m)) v \in D.$$

Since m and v are arbitrary, the image of $T(b_1 b_2): TU_2 \rightarrow TG$ is contained in D . By (5) above, $b_1 b_2$ integrates \mathcal{F} , so that $\widehat{\mathcal{F}} \subset \widehat{G}$ is closed under multiplication.

The proof that $\widehat{\mathcal{F}} \subset \widehat{G}$ is closed under inversion is similar and omitted. ■

Proof of Theorem 2.8. The construction of a pseudoaction \mathcal{F} on Γ^k , for any Lie pseudogroup Γ of finite type k , was given in Section 2; it remains to show that $\Gamma = \text{pseud}(\mathcal{F})$. It is easy to see that $\Gamma \subset \text{pseud}(\mathcal{F})$. For the reverse inclusion, suppose $\phi \in \text{pseud}(\mathcal{F})$. To show $\phi \in \Gamma$ it suffices, by the collating property of pseudogroups, to construct, for any m_0 in the domain U of ϕ , an open neighbourhood V of m_0 such that $\phi|_V \in \Gamma$.

Since $\phi \in \text{pseud}(\mathcal{F})$ we have, shrinking $U \ni m_0$ if necessary, $\phi = \beta \circ b$, for some local bisection $b: U \rightarrow \Gamma^k$ integrating \mathcal{F} . Here $\beta: \Gamma^k \rightarrow M$ denotes the target map of Γ^k . Furthermore, we have $b(m_0) = J_{m_0}^k \varphi$ for some $\varphi \in \Gamma$, with domain $U' \ni m_0$, say. But in that case we obtain a second local bisection $b': U' \rightarrow \Gamma^k$ integrating \mathcal{F} , defined by $b'(m) = J_m^k \varphi$, and satisfying $b'(m_0) = b(m_0)$. By the local uniqueness of integrating bisections (6) b and b' coincide on some open neighbourhood V of m_0 , giving us $\phi|_V = \beta \circ b'|_V = \varphi|_V \in \text{pseud}(\Gamma)$. ■

4. Integrating a Lie algebra action

In this central section of the paper we show that every Lie algebra action is integrated by a pseudoaction \mathcal{F} which is also a formal integration. We will construct our integration with the help of Palais' local integrability theorem, of which we offer a novel proof. While in principle the reader may take Palais' result as given — and read the remainder of the section independently of our proof of it — we feel our proof puts the later constructions into better context.

Let \mathfrak{g}_0 be a finite-dimensional Lie algebra acting smoothly from the left on M , and denote the corresponding Lie algebra homomorphism $\mathfrak{g}_0 \rightarrow \Gamma(TM)$ by $\xi \mapsto \xi^\dagger$. The simply-connected Lie group with Lie algebra \mathfrak{g}_0 will be denoted by G_0 .

Local actions. By a *local action* of G_0 on M we shall mean an open neighbourhood $W \subset G_0 \times M$ of the identity section $\text{id} \times M$, together with a smooth map $(g, m) \mapsto \phi_g(m): W \rightarrow M$ such that:

(8) $\phi_{\text{id}}(m) = m$ for all $m \in M$.

(9) $\phi_h(\phi_g(m)) = \phi_{hg}(m)$ whenever both sides are well-defined.

Without loss of generality, we can suppose any local action additionally satisfies:

(10) $(g, m) \in W \Leftrightarrow (g^{-1}, \phi_g(m)) \in W$.

For if not, we can replace W with $W \cap W^{-1}$, where

$$W^{-1} = \{(g^{-1}, \phi_g(m) \mid (g, m) \in W\}.$$

Theorem 4.1 (Palais’ local integrability theorem [17]). *There exists a local action $(m, g) \mapsto \phi_g$ of G_0 on M such that ϕ_g is the restriction to some open set of the time-one flow map of some infinitesimal generator ξ^\dagger , $\xi \in \mathfrak{g}_0$.*

Before turning to the proof, we record a simple lemma that will be needed both in the proof and later on.

Lemma 4.2. *Let G be a Lie groupoid whose Lie algebroid is the action algebroid $\mathfrak{g}_0 \times M$. For each $\xi \in \mathfrak{g}_0$ let ξ_c denote the constant section of $\mathfrak{g}_0 \times M$ and ξ_c^R the corresponding right-invariant vector field on G . Then the flow of the vector fields ξ^\dagger and ξ_c^R are related by*

$$(11) \quad \beta(\Phi_{\xi_c^R}^t(p)) = \Phi_{\xi^\dagger}^t(\beta(p)); \quad p \in G,$$

whenever both sides are defined. Here β denotes the target projection of G . Also, if $\Phi_{\xi^\dagger}^t(\beta(p))$ is defined for some $t \in \mathbb{R}$, then so is $\Phi_{\xi_c^R}^t(p)$, i.e.,

$$(12) \quad \text{dom } \Phi_{\xi_c^R}^t = \beta^{-1}(\text{dom } \Phi_{\xi^\dagger}^t).$$

Finally, if $\Omega: G \rightarrow G_0$ is a Lie groupoid morphism whose derivative is the canonical projection $\mathfrak{g}_0 \times M \rightarrow \mathfrak{g}_0$, then

$$(13) \quad \Omega(\Phi_{\xi_c^R}^t(p)) = \exp(t\xi)\Omega(p); \quad \xi \in \mathfrak{g}_0, \quad p \in G.$$

Proof. Equation (11) is an immediate consequence of the fact that ξ_c^R and ξ^\dagger are β -related. To see that (12) holds note that integral paths $t \mapsto m(t)$ of $\xi^\dagger = \#\xi_c$ lift to \mathfrak{g} -paths $t \mapsto (\xi, m(t))$ ($\mathfrak{g} = \mathfrak{g}_0 \times M$), which in turn integrate to families of G -paths (paths lying in source-fibres).⁴ These G -paths are necessarily integral paths of ξ_c^R mapped by β to integral paths of ξ^\dagger .

To prove (13), let ξ^R denote the right-invariant vector field on the Lie group G_0 corresponding to $\xi \in \mathfrak{g}_0$. Then ξ_c^R and ξ^R will be Ω -related. It follows that Ω maps integral paths of ξ_c^R to integral paths of ξ^R , and so (13) holds. ■

The proof of Theorem 4.1. According to Dazord, all action algebroids are integrable. That is to say, there exists a Lie groupoid G whose Lie algebroid is $\mathfrak{g}_0 \times M$. We refer to Dazord’s original article [9] for the elegant proof, but will make no use of the explicit model for G constructed there. (An alternative model appears below.) By Lie I for Lie groupoids — see e.g., [8] — we may take G to be source simply-connected; then using Lie II for Lie groupoids, we obtain a Lie groupoid morphism $\Omega: G \rightarrow G_0$ whose derivative is the canonical projection $\omega: \mathfrak{g}_0 \times M \rightarrow \mathfrak{g}_0$.

⁴To see that \mathfrak{g} -paths lift to G -paths the essential points are: (i) The base-paths of \mathfrak{g} -paths always lie on orbits of G ; and (ii) The restriction of $\beta: G \rightarrow M$ to any source-fibre of G is a principal bundle over the corresponding orbit. See, e.g., [8].

We next construct ‘tubular neighbourhood’ of M in G modelled on a neighbourhood of $\text{id} \times M$ in $G_0 \times M$. Let $B \subset G_0$ be any open neighbourhood of the identity that is the diffeomorphic image of some convex open neighbourhood of zero in \mathfrak{g}_0 , under the exponential map $\exp: \mathfrak{g}_0 \rightarrow G_0$. Then, by local existence and uniqueness theorems for ODE’s, there exists an open neighbourhood $W_{\text{big}} \subset B \times M$ of $\text{id} \times M$ such that $E: W_{\text{big}} \rightarrow G$, given by $E(\exp(\xi), m) = \Phi_{\xi^{\mathbb{R}}}^1(m)$, is well-defined (notation is as in Lemma 4.2).

Evidently, the tangent map $TE: TW_{\text{big}} \rightarrow TG$ has full rank at all points of $\text{id} \times M$, implying that E is a local diffeomorphism in some neighbourhood of $\text{id} \times M$. By a topological argument familiar from proofs of the tubular neighbourhood theorem (see, e.g., [13, IV, §5, p. 109]) the reader will agree that E is in fact a global diffeomorphism, when restricted to a possibly smaller neighbourhood of $\text{id} \times M$ in $B \times M$, which we nevertheless continue to denote by W_{big} .

Notice that by construction $\alpha(E(g, m)) = m$, where α denotes the source map, and that

$$(14) \quad \Omega(E(g, m)) = g,$$

by Equation (11) of the preceding lemma. Consequently, $\Omega \times \alpha: E(W_{\text{big}}) \rightarrow W_{\text{big}}$ is the inverse of E ; in particular $\Omega \times \alpha$ is injective on $Z_{\text{big}} := E(W_{\text{big}})$. Note that Z_{big} is paracompact and Hausdorff, because $W_{\text{big}} \subset B \times M$ is evidently paracompact and Hausdorff.

We now apply the following fact proven in Appendix A, which generalizes a well-known observation about Lie groups:

Proposition 4.3. *Let $Z_{\text{big}} \subset G$ be a paracompact, Hausdorff, open neighbourhood of M in a Lie groupoid G over M . Then there exists a neighbourhood $Z \subset Z_{\text{big}}$ of M such that for all $h, g \in Z$ we have $hg \in Z_{\text{big}}$, whenever h and g are multipliable.*

Setting $W = E^{-1}(Z)$, we seek to show that $\phi_g(m) := \beta(E(g, m))$ defines a local action $(g, m) \mapsto \phi_g(m): W \rightarrow M$ of G_0 on M ; recall β denotes the target map.

The requirement (8) follows immediately. To establish (9), let us suppose $(m, g) \in W$, $(h, \phi_g(m)) \in W$ and $(hg, m) \in W$. Then $E(h, \phi_g(m))$ and $E(g, m)$ are multipliable elements of $Z \subset G$. Moreover, we claim

$$(15) \quad E(h, \phi_g(m))E(g, m) = E(hg, m).$$

As $ZZ \subset Z_{\text{big}}$ (in the sense of the proposition) both sides of the equation are elements of Z_{big} , on which $\Omega \times \alpha$ is injective. To show equality holds in (15) it therefore suffices to show equality after applying $\Omega \times \alpha$ to both sides of the equation. But this follows easily from (14) and the fact that $\Omega: G \rightarrow G_0$ is a groupoid morphism.

We now compute

$$\begin{aligned} \phi_h(\phi_g(m)) &= \beta\left(E(h, \phi_g(m))\right) = \beta\left(E(h, \phi_g(m))E(g, m)\right) \\ &= \beta\left(E(hg, m)\right) = \phi_{hg}(m). \end{aligned}$$

The third equality follows from (15). This completes the proof that $(g, m) \mapsto \phi_g(m): W \rightarrow M$ is a local action.

Note that the definition of E and (11) imply that $\phi_{\exp(\xi)} = \Phi_{\xi t}^1$, so that the local transformations ϕ_g have the form asserted in the theorem. ■

An explicit Lie groupoid integrating $\mathfrak{g}_0 \times M$. Fix a local action

$$(m, g) \mapsto \phi_g(m): W \rightarrow M$$

of G_0 on M integrating the action of \mathfrak{g}_0 on M , in the sense of Theorem 4.1. In the sequel a statement such as ‘ $m \in \text{dom } \phi_g$ ’ should be interpreted as ‘ $(g, m) \in W$ ’. We assume (10) holds, i.e., $m \in \text{dom } \phi_g \Leftrightarrow \phi_g(m) \in \text{dom } \phi_{g^{-1}}$. Evidently, $\phi_{g^{-1}} = \phi_g^{-1}$.

Let a *chain* for the local action be a finite sequence $(g_1, g_2, \dots, g_k, m)$, where $g_1, g_2, \dots, g_k \in G_0$ and $m \in M$ are such that $m \in \text{dom } \phi_{g_1} \circ \phi_{g_2} \circ \dots \circ \phi_{g_k}$. Explicitly, this means:

$$(16) \quad m_k := m \in \text{dom } \phi_{g_k},$$

$$(17) \quad m_j := \phi_{g_{j+1}}(m_{j+1}) \in \text{dom } \phi_{g_j}, \text{ for } j = k - 1, k - 2, \dots, 1.$$

An equivalence relation \sim on chains is defined by declaring

$$(g_1, g_2, \dots, g_k, m) \sim (\bar{g}_1, \bar{g}_2, \dots, \bar{g}_k, \bar{m})$$

if:

$$(18) \quad m = \bar{m},$$

$$(19) \quad g_1 g_2 \dots g_k = \bar{g}_1 \bar{g}_2 \dots \bar{g}_k,$$

$$(20) \quad \text{germ}_m \phi_{g_1} \circ \phi_{g_2} \circ \dots \circ \phi_{g_k} = \text{germ}_{\bar{m}} \phi_{\bar{g}_1} \circ \phi_{\bar{g}_2} \circ \dots \circ \phi_{\bar{g}_k}.$$

We define G to be the set of all chains modulo the equivalence relation \sim . The class represented by a chain $(g_1, g_2, \dots, g_k, m)$ will be denoted $[g_1, g_2, \dots, g_k, m]$.

Theorem 4.4. *The set G is a Lie groupoid over M with Lie algebroid $\mathfrak{g}_0 \times M$. Furthermore, the map $\Omega: G \rightarrow G_0$ defined by*

$$\Omega([g_1, g_2, \dots, g_k, m]) = g_1 g_2 \dots g_k$$

is a Lie groupoid morphism whose derivative is $\omega: \mathfrak{g}_0 \times M \rightarrow \mathfrak{g}_0$.

That G is a set-theoretic groupoid over M is clear. The groupoid operations closely resemble those for action groupoids: The identity at $m \in M$ is $[\text{id}, m]$. The source and target maps $\alpha, \beta: G \rightarrow M$ are given by $\alpha([g_1, g_2, \dots, g_k, m]) = m$, $\beta([g_1, g_2, \dots, g_k, m]) = (\phi_{g_1} \circ \phi_{g_2} \circ \dots \circ \phi_{g_k})(m)$. Multiplication is defined by

$$[g_1, g_2, \dots, g_k, m][\bar{g}_1, \bar{g}_2, \dots, \bar{g}_k, \bar{m}] = [g_1, g_2, \dots, g_k, \bar{g}_1, \bar{g}_2, \dots, \bar{g}_k, \bar{m}].$$

The inverse of $[g_1, g_2, \dots, g_k, m]$ is $[g_k^{-1}, g_{k-1}^{-1}, \dots, g_1^{-1}, (\phi_{g_1} \circ \phi_{g_2} \circ \dots \circ \phi_{g_k})(m)]$, which is well-defined on account of the requirement (10).

After defining the smooth structure of G in Section 4 below, it is a straightforward exercise left to the reader to show this structure is compatible with the groupoid operations, and that the following map determines an isomorphism between the action algebroid $\mathfrak{g}_0 \times M$ and the abstract Lie algebroid of G :

$$(\xi, m) \mapsto \left. \frac{d}{dt} [\exp(t\xi), m] \right|_{t=0}.$$

The claim regarding Ω is immediately verified.

The smooth structure of G . Define $B \subset G_0$ as in the proof of Theorem 4.1 and define a ‘ball of radius r ’ by $B_r := \exp(r \exp^{-1}(B)) \subset B$; here $0 < r \leq 1$. To construct a smooth atlas on M will require the following preliminary observations:

Lemma 4.5. (21) *For every relatively compact open set $V \subset M$ there exists $r > 0$ such that $V \subset \text{dom } \phi_g$ for all $g \in B_r$.*

(22) *Given $r > 0$ and $g \in B_r$, there exists $\rho > 0$ such that $B_\rho g \subset B_r$.*

Proof. As V is relatively compact, one can readily construct a neighbourhood of V in W of the form $B_r \times V$, for some $r > 0$. The claim (21) immediately follows. Conclusion (22) follows from the continuity of multiplication in G_0 . ■

For each $[g_1, g_2, \dots, g_k, m_0] \in G$ we define a local parameterization φ of G as follows. First, let U be a relatively compact open neighbourhood of m_0 contained in the domain of $\phi_{g_1} \circ \phi_{g_2} \circ \dots \circ \phi_{g_k}$ and put

$$V := (\phi_{g_1} \circ \phi_{g_2} \circ \dots \circ \phi_{g_k})(U).$$

Applying (21), choose $r > 0$ small enough that $V \subset \text{dom } \phi_g$ for all $g \in B_r$. Then $\varphi: B_r \times U \rightarrow G$ is well-defined by

$$\varphi(g, m) = [g, g_1, g_2, \dots, g_k, m].$$

That φ is injective is a triviality.

For the purpose of examining the transition functions associated with the collection of all such local parameterizations, consider a second point

$$[\bar{g}_1, \bar{g}_2, \dots, \bar{g}_k, \bar{m}_0] \in G,$$

and an associated local parameterization $\bar{\varphi}: B_{\bar{r}} \times \bar{U} \rightarrow G$, defined analogously. Furthermore, suppose that the images of the parameterizations have non-trivial overlap

$$\mathcal{O} := \varphi(B_r \times U) \cap \bar{\varphi}(B_{\bar{r}} \times \bar{U}) \subset G.$$

Then, for some $m' \in U \cap \bar{U}$, there exists $g_0 \in B_r$ and $\bar{g}_0 \in B_{\bar{r}}$ such that $\varphi(g_0, m') = \bar{\varphi}(\bar{g}_0, m')$. In particular, we will have

$$(23) \quad g_0 g_1 g_2 \cdots g_k = \bar{g}_0 \bar{g}_1 \bar{g}_2 \cdots \bar{g}_k.$$

Given this, one readily shows that the transition function

$$\bar{\varphi}^{-1} \circ \varphi: \varphi^{-1}(\mathcal{O}) \rightarrow \bar{\varphi}^{-1}(\mathcal{O})$$

is given by $(g, m) \mapsto (gg_0^{-1}\bar{g}_0, m)$ which, as a map on $B_r \times U$, is smooth. To show the local parameterizations define a smooth structure on G , it remains only to establish the following fact, whose proof will rest crucially on the fundamental property (9) of the local action:

Lemma 4.6. *The set $\varphi^{-1}(\mathcal{O})$ is open in $B_r \times U$.*

Proof. An arbitrary point of $\varphi^{-1}(\mathcal{O})$ is of the form (g_0, m') for some $g_0 \in B_r$ and $m' \in U \cap \bar{U}$ satisfying $\varphi(g_0, m') = \bar{\varphi}(\bar{g}_0, m')$, for some $\bar{g}_0 \in B_{\bar{r}}$. This means (23) holds and

$$\text{germ}_{m'} \phi_{g_0} \circ \phi_{g_1} \circ \phi_{g_2} \circ \cdots \circ \phi_{g_k} = \text{germ}_{m'} \phi_{\bar{g}_0} \circ \phi_{\bar{g}_1} \circ \phi_{\bar{g}_2} \circ \cdots \circ \phi_{\bar{g}_k}.$$

In particular, there exists an open neighbourhood $U' \subset U \cap \bar{U}$ of m' such that

$$(24) \quad \phi_{g_0} \circ \phi_{g_1} \circ \phi_{g_2} \circ \cdots \circ \phi_{g_k} \Big|_{U'} = \phi_{\bar{g}_0} \circ \phi_{\bar{g}_1} \circ \phi_{\bar{g}_2} \circ \cdots \circ \phi_{\bar{g}_k} \Big|_{U'}.$$

To prove the lemma, we will show that the neighbourhood $B_\rho g_0 \times U'$ of (g_0, m') lies in $\varphi^{-1}(\mathcal{O})$, for some $\rho > 0$. To this end, apply Lemma 4.5 to find $\rho > 0$ small enough that all the following hold, for all $\delta \in B_\rho$:

$$(25) \quad B_\rho g_0 \subset B_r,$$

$$(26) \quad B_\rho \bar{g}_0 \subset B_{\bar{r}},$$

$$(27) \quad \phi_{g_0}(V) \subset \text{dom } \phi_\delta,$$

$$(28) \quad \phi_{\bar{g}_0}(\bar{V}) \subset \text{dom } \phi_\delta.$$

With $\delta \in B_\rho$ arbitrary henceforth, we have, by (25) and (26),

$$(29) \quad V \subset \text{dom } \phi_{\delta g_0},$$

$$(30) \quad \bar{V} \subset \text{dom } \phi_{\delta \bar{g}_0}.$$

By (9), we have

$$(31) \quad \phi_{\delta g_0} \Big|_V = \phi_\delta \circ \phi_{g_0} \Big|_V,$$

$$(32) \quad \phi_{\delta \bar{g}_0} \Big|_{\bar{V}} = \phi_\delta \circ \phi_{\bar{g}_0} \Big|_{\bar{V}},$$

where all maps appearing are well-defined, on account of equations (27)–(30).

By (25) and (26), the open set $B_\rho g_0 \times U'$ lies in $B_r \times U = \text{dom } \varphi$ and the open set $B_\rho \bar{g}_0 \times U'$ lies in $B_{\bar{r}} \times \bar{U} = \text{dom } \bar{\varphi}$. Our proof of the lemma is complete if we can show $\varphi(B_\rho g_0 \times U') \subset \bar{\varphi}(B_\rho \bar{g}_0 \times U')$. It suffices to show that $\varphi(\delta g_0, m) = \bar{\varphi}(\delta \bar{g}_0, m)$ for arbitrary $\delta \in B_\rho$ and $m \in U'$. This is equivalent to showing:

$$\delta g_0 g_1 g_2 \cdots g_k = \delta \bar{g}_0 \bar{g}_1 \bar{g}_2 \cdots \bar{g}_k$$

and $\text{germ}_m \phi_{\delta g_0} \circ \phi_{g_1} \circ \phi_{g_2} \circ \cdots \circ \phi_{g_k} = \text{germ}_m \phi_{\delta \bar{g}_0} \circ \phi_{\bar{g}_1} \circ \phi_{\bar{g}_2} \circ \cdots \circ \phi_{\bar{g}_k}.$

The first equation follows immediately from (23). Regarding the second, we have

$$\begin{aligned} \text{germ}_m \phi_{\delta_{g_0}} \circ \phi_{g_1} \circ \phi_{g_2} \circ \cdots \circ \phi_{g_k} &= \text{germ}_m \phi_{\delta} \circ \phi_{g_0} \circ \phi_{g_1} \circ \phi_{g_2} \circ \cdots \circ \phi_{g_k}, & \text{by (31)} \\ &= \text{germ}_m \phi_{\delta} \circ \phi_{\bar{g}_0} \circ \phi_{\bar{g}_1} \circ \phi_{\bar{g}_2} \circ \cdots \circ \phi_{\bar{g}_k}, & \text{by (24)} \\ &= \text{germ}_m \phi_{\delta_{\bar{g}_0}} \circ \phi_{\bar{g}_1} \circ \phi_{\bar{g}_2} \circ \cdots \circ \phi_{\bar{g}_k}, & \text{by (32).} \blacksquare \end{aligned}$$

Integrability. Let ∇ denote the canonical flat Cartan connection on $\mathfrak{g}_0 \times M$. The following result is a special instance of Lie III for pseudoactions (Theorem 2.6):

Proposition 4.7. *There exists a source-connected Lie groupoid G supporting a pseudoaction \mathcal{F} that both integrates and formally integrates $(\mathfrak{g}_0 \times M, \nabla)$.*

Specifically, let G denote the Lie groupoid integrating $\mathfrak{g}_0 \times M$ defined after the proof of Proposition 4.3, and let $\Omega: G \rightarrow G_0$ the Lie groupoid morphism whose derivative is the canonical projection $\omega: \mathfrak{g}_0 \times M \rightarrow \mathfrak{g}_0$; an explicit formula for Ω is given in Theorem 4.4. Replace G by its source-connected component⁵ and let \mathcal{F} be the pseudoaction whose leaves are the connected components of fibres of the submersion $\Omega: G \rightarrow G_0$.

Proof that \mathcal{F} integrates $\mathfrak{g}_0 \times M$. Our task is to prove $\text{pseud}(\nabla) = \text{pseud}(\mathcal{F})$. To prove $\text{pseud}(\nabla) \subset \text{pseud}(\mathcal{F})$, it suffices to show that for any $\xi \in \mathfrak{g}_0$ and $t \in \mathbb{R}$, we have $\Phi_{\xi^\dagger}^t \in \text{pseud}(\mathcal{F})$. Here and below we adopt the notation of Lemma 4.2. By (12), $U := \text{dom } \Phi_{\xi^\dagger}^t \subset M \subset G$ lies in $\text{dom } \Phi_{\xi_{\mathbb{R}}}^t$ and a local bisection $b: U \rightarrow G$ is therefore well-defined by $b(m) := \Phi_{\xi_{\mathbb{R}}}^t(m)$. Moreover, by (13), we have

$$(33) \quad \Omega(b(m)) = \exp(t\xi), \quad \text{for all } m \in U,$$

which shows that b integrates \mathcal{F} , because leaves of \mathcal{F} are connected components of fibres of Ω . It follows that $\beta \circ b \in \text{pseud}(\mathcal{F})$. But, by (11), $\beta \circ b = \Phi_{\xi^\dagger}^t$, implying $\Phi_{\xi^\dagger}^t \in \text{pseud}(\mathcal{F})$, as desired.

Suppose $\phi = \beta \circ b$ for some local bisection $b: U \rightarrow G$ integrating \mathcal{F} , $U = \text{dom } \phi$. Then as all elements of $\text{pseud}(\mathcal{F})$ are locally of this form, it suffices in establishing $\text{pseud}(\mathcal{F}) \subset \text{pseud}(\nabla)$ to show an arbitrary point $m \in U$ has an open neighbourhood \tilde{U} such that $\phi|_{\tilde{U}} \in \text{pseud}(\nabla)$. This follows from the collating property of pseudogroups.

We have $b(m) = [g_1, g_2, \dots, g_k, m]$ for some $g_1, g_2 \in B$. In that case $U' = \text{dom } \phi_{g_1} \circ \phi_{g_2} \circ \cdots \circ \phi_{g_k}$ is an open neighbourhood of m and the map $b^m: U' \rightarrow G$ given by $b^m(m') = [g_1, g_2, \dots, g_k, m']$ is a local bisection integrating \mathcal{F} such that $b^m(m) = b(m)$. By the local uniqueness (6) of local bisections integrating \mathcal{F} we conclude $b|_{\tilde{U}} = b^m|_{\tilde{U}}$, where \tilde{U} is the connected component of $U \cap U'$ containing m . Consequently

$$\phi|_{\tilde{U}} = \phi_{g_1} \circ \phi_{g_2} \circ \cdots \circ \phi_{g_k}|_{\tilde{U}}$$

and whence $\phi|_{\tilde{U}} \in \text{pseud}(\nabla)$ as claimed. ■

⁵We believe G is already source-connected — indeed source-simply-connected — but do not prove or need this fact here.

Proof that \mathcal{F} formally integrates $\mathfrak{g}_0 \times M$. Recalling the definitions of Section 2 and, in particular, the correspondence (3), we see that showing \mathcal{F} formally integrates $(\mathfrak{g}_0 \times M, \nabla)$ amounts to showing $dS(x) = J_m^1 X + (\nabla X)(m)$, for any $x \in \mathfrak{g}_0 \times M$, where X is an arbitrary extension of x to a local section of $\mathfrak{g}_0 \times M$. Here $S: G \rightarrow J^1 G$ is the Cartan connection on G which, as a distribution on G , is tangent to the leaves of \mathcal{F} ; indeed, from the explicit form of local bisections integrating \mathcal{F} mentioned in the preceding paragraph, S is given by

$$(34) \quad S([g_1, g_2, \dots, g_k, m])\dot{\gamma}(0) = \left. \frac{d}{ds}[g_1, g_2, \dots, g_k, \gamma(s)] \right|_{s=0},$$

for any path $s \mapsto \gamma(s) \in M$ with $\dot{\gamma}(0) \in T_m M$.

Adopting the notation of Lemma 4.2, we take $x = (\xi, m)$, and $X = \xi_c$, and need to prove

$$(35) \quad dS(\xi_c(m)) = J_m^1 \xi_c.$$

First, one readily sees that

$$\xi_c^R([g_1, g_2, \dots, g_k, m]) = \left. \frac{d}{dt}[\exp(t\xi)g_1, g_2, \dots, g_k, m] \right|_{t=0}$$

for any $[g_1, g_2, \dots, g_k, m] \in G$. This vector field can be explicitly integrated locally; in particular,

$$(36) \quad \Phi_{\xi_c^R}^t(m) = [\exp(t\xi), m].$$

Next, we claim

$$(37) \quad S(\Phi_{\xi_c^R}^t(m)) = T_m(\Phi_{\xi_c^R}^t \circ \iota_M),$$

where $\iota: M \rightarrow G$ is the inclusion. Indeed, for any path $s \mapsto \gamma(s) \in M$ with $\dot{\gamma}(0) \in T_m M$,

$$\begin{aligned} S\left(\Phi_{\xi_c^R}^t(m)\right)\dot{\gamma}(0) &= S([\exp(t\xi), m])\dot{\gamma}(0), && \text{by (36)} \\ &= \left. \frac{d}{ds}[\exp(t\xi), \gamma(s)] \right|_{s=0}, && \text{by (34)} \\ &= \left. \frac{d}{ds}\Phi_{\xi_c^R}^t(\gamma(s)) \right|_{s=0}, && \text{by (36)} \\ &= T_m(\Phi_{\xi_c^R}^t \circ \iota_M)\dot{\gamma}(0). \end{aligned}$$

Using (37) we compute

$$\begin{aligned} dS(\xi_c(m)) &= \left. \frac{d}{dt}S(\Phi_{\xi_c^R}^t(m)) \right|_{t=0} = \left. \frac{d}{dt}T_m(\Phi_{\xi_c^R}^t(m)) \right|_{t=0} \\ &= J_m^1 \xi_c^R, \end{aligned}$$

completing the proof of (35). Note that the last equality follows from our implicit identification of the Lie algebroid of $J^1 G$ with $J^1 \mathfrak{g}$, as discussed in Section 3. ■

5. Integrating morphisms

Morphisms of pseudoactions. By a *morphism* of pseudoactions $\Pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ let us mean a Lie groupoid morphism $\Pi: G_1 \rightarrow G_2$ of the underlying Lie groupoids that maps leaves of \mathcal{F}_1 into leaves of \mathcal{F}_2 . If, in addition, Π covers a local diffeomorphism $f: M_1 \rightarrow M_2$, then the restriction of Π to any leaf of \mathcal{F}_1 will be a local diffeomorphism into a leaf of \mathcal{F}_2 (because Π respects source and target maps).

Proposition 5.1. *Let $\Pi: G_1 \rightarrow G_2$ be a morphism of Lie groupoids covering a local diffeomorphism $f: M_1 \rightarrow M_2$ and assume G_1 is source-connected. Let \mathcal{F}_1 and \mathcal{F}_2 be pseudoactions on G_1 and G_2 respectively, and ∇^1 and ∇^2 the corresponding infinitesimal Cartan connections. Then Π is a morphism $\mathcal{F}_1 \rightarrow \mathcal{F}_2$ if and only if its derivative $\pi = d\Pi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ respects the connections ∇^1, ∇^2 .*

Remark 5.2. Since $\pi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ covers a local diffeomorphism $f: M_1 \rightarrow M_2$ by hypothesis, that π should respect connections simply means that $\pi_*(\nabla_V^1 X_1) = \nabla_{f_*V}^2 \pi_* X_1$ for all local sections X_1 of \mathfrak{g}_1 , and vector fields V on M_1 , whose domain of definition is small enough that the pushforward operations π_* and f_* make sense (see the proof below).

Proof. With the help of Proposition 3.1 it is not hard to establish the following:

Lemma 5.3. *Under the hypotheses of the above proposition, Π is a morphism $\mathcal{F}_1 \rightarrow \mathcal{F}_2$ if and only if the following diagram is commutative:*

$$\begin{array}{ccc} J^1G_1 & \xrightarrow{J^1\Pi} & J^1G_2 \\ s_1 \uparrow & & \uparrow s_2 \\ G_1 & \xrightarrow{\Pi} & G_2 \end{array} \cdot$$

Here S_1 and S_2 denote the corresponding (global) Cartan connections.

Since Π is not a base-preserving morphism, one must take care to interpret the map $J^1\Pi: J^1G_1 \rightarrow J^1G_2$ appropriately: Since we assume Π covers a local diffeomorphism, each local bisection $b: U \rightarrow G_1$ pushes forward to a local bisection $\Pi_*b := \Pi \circ b \circ f^{-1}: f(U) \rightarrow G_2$, provided U is sufficiently small that the local inverse $f^{-1}: f(U) \rightarrow U$ is defined. Then $J^1\Pi(J_m^1b) := J_{f(m)}^1(\Pi_*b)$.

We define a map $J^1\pi: J^1\mathfrak{g}_1 \rightarrow J^1\mathfrak{g}_2$ similarly: Every local section X of \mathfrak{g}_1 , defined on a sufficiently small domain, has a pushforward $\pi_*X = \pi \circ X \circ f^{-1}$, and we define $J^1\pi(J_m^1X) = J_{f(m)}^1\pi_*X$. We leave it to the reader to verify that $d(J^1\Pi) = J^1(d\Pi)$, i.e., $d(J^1\Pi) = J^1\pi$. Then, taking derivatives, commutativity of the diagram in the lemma guarantees commutativity of

$$(38) \quad \begin{array}{ccc} J^1\mathfrak{g}_1 & \xrightarrow{J^1\pi} & J^1\mathfrak{g}_2 \\ s_1 \uparrow & & \uparrow s_2 \\ \mathfrak{g}_1 & \xrightarrow{\pi} & \mathfrak{g}_2 \end{array} \cdot$$

Here $s_1 = dS_1$ and $s_2 = dS_2$. In fact, commutativity of each diagram is equivalent to that of the other, because two Lie groupoid morphisms $J^1G_1 \rightarrow G_2$ having the same derivative must coincide (because G_1 , and hence J^1G_1 , has connected source-fibres). So, to prove the proposition it suffices, by the lemma, to show that diagram (38) commutes if and only if π preserves the connections ∇^1, ∇^2 . Since the two connections ∇_1 and ∇_2 are flat, the latter is equivalent to the assertion that the pushforward π_*X of every ∇^1 -parallel local section X of \mathfrak{g}_1 is ∇^2 -parallel. That this is equivalent to commutativity of (38) follows easily from the equations $s_1X = J^1X + \nabla^1X$ and $s_2Y = J^1Y + \nabla^2Y$ defining the connections ∇^1 and ∇^2 . ■

Integrating morphisms. Using the preceding proposition, we prove the following result on integrating morphisms between twisted Lie algebra actions:

Theorem 5.4 (Lie II for pseudoactions). *Let $(\mathfrak{g}_1, \nabla^1)$ and $(\mathfrak{g}_2, \nabla^2)$ be twisted Lie algebra actions formally integrated by pseudoactions \mathcal{F}_1 on G_1 and \mathcal{F}_2 on G_2 , and assume G_1 is source-simply-connected. Suppose $\pi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebroid morphism respecting the connections ∇^1, ∇^2 , and that π covers a local diffeomorphism $f: M_1 \rightarrow M_2$. Then there exists a unique morphism $\Pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ whose derivative is π .*

Proof. By Lie II for Lie groupoid morphisms, there exists a unique Lie groupoid morphism $\Pi: G_1 \rightarrow G_2$ whose derivative is π . Proposition 5.1 implies Π is a morphism of pseudoactions $\mathcal{F}_1 \rightarrow \mathcal{F}_2$. ■

Coverings of pseudoactions. Let us say that a Lie groupoid \tilde{G} over M covers a Lie groupoid G over M if \tilde{G} and G are source-connected and have the same Lie algebroid \mathfrak{g} , and if there is a base-preserving Lie groupoid morphism $\Pi: \tilde{G} \rightarrow G$ whose derivative $\pi: \mathfrak{g} \rightarrow \mathfrak{g}$ is the identity. Any covering $\Pi: \tilde{G} \rightarrow G$ is necessarily a *surjective* local diffeomorphism.

A covering of a pseudoaction \mathcal{F} on \mathcal{G} is a covering $\Pi: \tilde{G} \rightarrow G$ of Lie groupoids, together with a pseudoaction $\tilde{\mathcal{F}}$ on \tilde{G} , such that Π is a morphism of pseudoactions $\tilde{\mathcal{F}} \rightarrow \mathcal{F}$. It follows in that case, from Proposition 5.1, that $\tilde{\mathcal{F}}$ and \mathcal{F} formally integrate the same twisted Lie algebra action on M .

Proposition 5.5. *For any covering $\tilde{\mathcal{F}} \rightarrow \mathcal{F}$ of pseudoactions one has*

$$\text{pseud}(\tilde{\mathcal{F}}) = \text{pseud}(\mathcal{F}).$$

Proof. Let $\Pi: \tilde{G} \rightarrow G$ denote the underlying morphism of Lie groupoids. As it is more-or-less immediate that $\text{pseud}(\tilde{\mathcal{F}}) \subset \text{pseud}(\mathcal{F})$, we prove only the reverse inclusion. Suppose $\phi \in \text{pseud}(\mathcal{F})$. Then it suffices to show that each m_0 in the domain U of ϕ has a neighbourhood V such that $\phi|_V \in \text{pseud}(\tilde{\mathcal{F}})$. Shrinking $U \ni m_0$ if necessary, we have $\phi = \beta \circ b$, for some local bisection $b: U \rightarrow G$ integrating \mathcal{F} ; we are denoting both target projections $\tilde{G} \rightarrow M$ and $G \rightarrow M$ by β . Put $g = b(m_0)$. Since $\Pi: \tilde{G} \rightarrow G$ is surjective (the crucial point) there exists $\tilde{g} \in \tilde{G}$ with $\Pi(\tilde{g}) = g$. Since Π is morphism of pseudoactions $\tilde{\mathcal{F}} \rightarrow \mathcal{F}$,

Π maps the leaf $\tilde{\mathcal{L}}$ of $\tilde{\mathcal{F}}$ through \tilde{g} into the leaf \mathcal{L} of \mathcal{F} through g . In fact, as Π is base-preserving, and $\tilde{\mathcal{L}}$ and \mathcal{L} are pseudotransformations, the restriction $\Pi: \tilde{\mathcal{L}} \rightarrow \mathcal{L}$ is a local diffeomorphism. There consequently exists some open sets $V \subset U$ and $W \subset \tilde{\mathcal{L}}$ such that W is mapped diffeomorphically onto $b(V) \subset \mathcal{L}$ by Π , and diffeomorphically onto V by the source projection. The local bisection $\tilde{b}: V \rightarrow \tilde{G}$ of $\tilde{\mathcal{F}}$ whose image is W integrates $\tilde{\mathcal{F}}$, so that $\beta \circ \tilde{b} \in \text{pseud}(\tilde{\mathcal{F}})$. But by construction, $\beta \circ \tilde{b}$ coincides with $\phi = \beta \circ b$ on V , which shows $\phi|_V \in \text{pseud}(\tilde{\mathcal{F}})$. ■

Theorem 5.6 (Lie I for pseudoactions). *Let (\mathfrak{g}, ∇) be a twisted Lie algebra action formally integrated by a pseudoaction \mathcal{F} on a source-connected Lie groupoid G . Then there exists a second pseudoaction $\tilde{\mathcal{F}}$ formally integrating (\mathfrak{g}, ∇) whose underlying Lie groupoid \tilde{G} is source-simply-connected. The pseudoaction $\tilde{\mathcal{F}}$ is uniquely defined, up to isomorphism.*

We will temporarily need the following addendum (ultimately rendered redundant by Theorem 2.6 proven in the next section):

Addendum 5.7. *If additionally \mathcal{F} integrates (\mathfrak{g}, ∇) , then we may suppose $\tilde{\mathcal{F}}$ does too.*

Proof. By Lie I and II for Lie groupoids (see, e.g., [8]) there exists a source-simply-connected Lie groupoid \tilde{G} covering G , unique up to isomorphism. We equip \tilde{G} with the pseudoaction whose leaves are the connected components of pre-images of leaves of \mathcal{F} under the covering map $\tilde{G} \rightarrow G$. Then $\tilde{\mathcal{F}}$ evidently covers \mathcal{F} and formally integrates (\mathfrak{g}, ∇) . The addendum is a consequence of the preceding proposition.

If $\tilde{\mathcal{F}}'$ is a second pseudoaction on \tilde{G} formally integrating (\mathfrak{g}, ∇) then the identity morphism on \tilde{G} is an isomorphism of pseudoactions $\tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}'$, by Proposition 5.1. ■

6. Integrating a twisted Lie algebra action

We now combine the results of Section 4 and Section 5 to prove Theorem 2.6 (Lie III for pseudoactions). Throughout (\mathfrak{g}, ∇) denotes a twisted Lie algebra action on M .

The integration of (\mathfrak{g}, ∇) . Let $(\tilde{\mathfrak{g}}, \tilde{\nabla})$ denote the pullback of (\mathfrak{g}, ∇) to the universal cover \tilde{M} under the covering map $f: \tilde{M} \rightarrow M$. Since \tilde{M} is simply-connected, $\tilde{\mathfrak{g}}$ is naturally isomorphic to an action algebroid (by Proposition 2.5, for example). According to Proposition 4.7, there exists a source-connected Lie groupoid \tilde{G} supporting a pseudoaction $\tilde{\mathcal{F}}$ that integrates and formally integrates $(\tilde{\mathfrak{g}}, \tilde{\nabla})$. The former property means

$$(39) \quad \text{pseud}(\tilde{\mathcal{F}}) = \text{pseud}(\tilde{\nabla}).$$

By Lie I for pseudoactions (Theorem 5.6) and its Addendum, we may take \tilde{G} to be source-simply-connected.

Next, let Λ denote the group of covering transformations of \widetilde{M} . Then Λ acts canonically on $\widetilde{\mathfrak{g}}$ by Lie algebroid isomorphisms, and each such isomorphism respects the connection $\widetilde{\nabla}$. By Lie II for pseudoactions (Theorem 5.4), this action lifts to an action of Λ on \widetilde{G} by isomorphisms of $\widetilde{\mathcal{F}}$ (Lie groupoid isomorphisms mapping leaves of $\widetilde{\mathcal{F}}$ to leaves of $\widetilde{\mathcal{F}}$). This action is totally discontinuous, since the action of Λ on \widetilde{M} is already totally discontinuous. The quotient $G := \widetilde{G}/\Lambda$ is a well-defined Lie groupoid over M whose Lie algebroid is isomorphic to \mathfrak{g} . Because Λ acts by isomorphisms of $\widetilde{\mathcal{F}}$, the pseudoaction $\widetilde{\mathcal{F}}$ drops to a foliation \mathcal{F} on G that is actually a pseudoaction, as is not too hard to see. The infinitesimalization of \mathcal{F} is evidently ∇ . The derivative of the covering $\Pi: \widetilde{G} \rightarrow G$ is the canonical projection $\pi: \widetilde{\mathfrak{g}} \rightarrow \mathfrak{g}$, which evidently respects the connections $\widetilde{\nabla}, \nabla$. By Proposition 5.1, Π maps leaves of $\widetilde{\mathcal{F}}$ into leaves of \mathcal{F} . In fact, since Π covers a local diffeomorphism of the base manifolds we have:

(40) *The restriction of Π to any leaf of $\widetilde{\mathcal{F}}$ is a local diffeomorphism into the corresponding leaf of \mathcal{F} .*

By construction, \mathcal{F} formally integrates (\mathfrak{g}, ∇) . We now use (39) and (40) to show that \mathcal{F} is a bona fide integration as well, i.e., $\text{pseud}(\mathcal{F}) = \text{pseud}(\nabla)$.

Proof that $\text{pseud}(\nabla) \subset \text{pseud}(\mathcal{F})$. Let X be a local ∇ -parallel section of \mathfrak{g} and $t \in \mathbb{R}$ such that $\Phi_{\#X}^t$ has non-empty domain U . As elements of $\text{pseud}(\nabla)$ are generated by transformations of this form, it will suffice to prove $\Phi_{\#X}^t \in \text{pseud}(\mathcal{F})$. Indeed, it will suffice to construct, for each $m_0 \in U$, an open neighbourhood $V \subset U$ of m_0 such that $\Phi_{\#X}^t|_V \in \text{pseud}(\mathcal{F})$.

Pull the section X back to a section \widetilde{X} of $\widetilde{\mathfrak{g}}$, and let $\widetilde{m}_0 \in \widetilde{M}$ be a point covering m_0 . Shrinking $U \ni m_0$ if necessary, there exists a connected neighbourhood \widetilde{U} of \widetilde{m}_0 mapped diffeomorphically onto U by the projection $f: \widetilde{M} \rightarrow M$, and we will have $\widetilde{U} \subset \text{dom } \Phi_{\#\widetilde{X}}^t$. Again supposing U to have been chosen sufficiently small, we have, by (39), $\Phi_{\#\widetilde{X}}^t = \beta \circ \widetilde{b}$, for some local bisection $\widetilde{b}: \widetilde{U} \rightarrow \widetilde{G}$ integrating $\widetilde{\mathcal{F}}$. Put $\widetilde{g} = \widetilde{b}(\widetilde{m}_0)$. As the restriction of Π to the leaf of $\widetilde{\mathcal{F}}$ through \widetilde{g} is a local diffeomorphism into the leaf of \mathcal{F} through $g = \Pi(\widetilde{g})$, there exists an open neighbourhood $\widetilde{V} \subset \widetilde{U}$ of \widetilde{m}_0 covering some neighbourhood $V \subset U$ of m_0 , and a local bisection $b: V \rightarrow G$ integrating \mathcal{F} , such that $f: \widetilde{M} \rightarrow M$ maps \widetilde{V} diffeomorphically onto V , and

$$(41) \quad b(f(\widetilde{m})) = \Pi(\widetilde{b}(\widetilde{m})); \quad \widetilde{m} \in \widetilde{V}.$$

But then, for any $\widetilde{m} \in \widetilde{V}$, we have

$$\beta(b(f(\widetilde{m}))) = \beta(\Pi(\widetilde{b}(\widetilde{m}))) = f(\beta(\widetilde{b}(\widetilde{m}))) = f(\Phi_{\#\widetilde{X}}^t(\widetilde{m})) = \Phi_{\#X}^t(f(\widetilde{m})),$$

the last equality following from the fact that $\#\widetilde{X}$ and $\#X$ are f -related. This shows that $\beta \circ b = \Phi_{\#X}^t|_V$. Since $\beta \circ b \in \text{pseud}(\mathcal{F})$, we have $\Phi_{\#X}^t|_V \in \text{pseud}(\mathcal{F})$, as required. \blacksquare

Proof that $\text{pseud}(\mathcal{F}) \subset \text{pseud}(\nabla)$. Let $\phi \in \text{pseud}(\mathcal{F})$. To show $\phi \in \text{pseud}(\nabla)$ is suffices to construct, for each m_0 in the domain U of ϕ , an open neighbourhood $V \subset U$ of m_0 such that $\phi|_V \in \text{pseud}(\nabla)$. Shrinking $U \ni m_0$ if necessary, we have $\phi = \beta \circ b$ for some local bisection $b: U \rightarrow G$ integrating \mathcal{F} . Let $g = b(m_0)$. Then there exists $\tilde{g} \in \tilde{G}$ such that $\Pi(\tilde{g}) = g$, whose source will be denoted $\tilde{m}_0 \in \tilde{M}$; we have $f(\tilde{m}_0) = m_0$. By (40) there exists an open neighbourhood \tilde{V} of \tilde{m}_0 , and a local bisection $\tilde{b}: \tilde{V} \rightarrow \tilde{G}$ integrating $\tilde{\mathcal{F}}$, such that $f: \tilde{M} \rightarrow M$ maps \tilde{V} diffeomorphically onto an open set $V \subset U$, and (41) above holds. Since \tilde{b} integrates $\tilde{\mathcal{F}}$, it follows from (39) (shrinking \tilde{V} above if necessary) that there exist $\tilde{\nabla}$ -parallel sections $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_k$ of $\tilde{\mathfrak{g}}$, and $t_1, t_2, \dots, t_k \in \mathbb{R}$, such that

$$(42) \quad \beta(\tilde{b}(\tilde{m})) = \Phi_{\#\tilde{X}_1}^{t_1} \circ \Phi_{\#\tilde{X}_2}^{t_2} \circ \dots \circ \Phi_{\#\tilde{X}_k}^{t_k} (\tilde{m}); \quad \tilde{m} \in \tilde{V}.$$

We may take the sections \tilde{X}_j to be globally defined ($\tilde{\nabla}$ is the canonical flat connection on a trivial bundle).

The following result, whose elementary proof is left to the reader, states that the image under $f: \tilde{M} \rightarrow M$ of an integral curve of a global vector field on \tilde{W} on \tilde{M} can be pieced together from integral curves of locally-defined vector fields on M that are f -related to \tilde{W} :

Lemma 6.1. *Let \tilde{W} be a globally defined vector field on \tilde{M} and suppose that $\tilde{m}_0 \in \text{dom } \Phi_{\tilde{W}}^t$. Then there exist:*

- open sets $\tilde{U}^1, \tilde{U}^2, \dots, \tilde{U}^n \subset \tilde{M}$ such that $\tilde{U}^n \subset \text{dom } \Phi_{\tilde{W}}^t$ is a neighbourhood of \tilde{m}_0 and $f: \tilde{M} \rightarrow M$ maps \tilde{U}^j diffeomorphically onto its image U^j ,
- local vector fields W^1, W^2, \dots, W^n defined respectively on U^1, U^2, \dots, U^n , and
- numbers $t^1, t^2, \dots, t^n \in \mathbb{R}$,

such that $\tilde{W}|_{\tilde{U}^i}$ is f -related to W^i and

$$f(\Phi_{\tilde{W}}^t(\tilde{m})) = \left(\Phi_{W^1}^{t^1} \circ \Phi_{W^2}^{t^2} \circ \dots \circ \Phi_{W^n}^{t^n} \right) (f(\tilde{m})), \quad \text{for all } \tilde{m} \in \tilde{U}^n.$$

With the help of this lemma (again shrinking \tilde{V} if necessary) one can find open subsets $\tilde{U}_i^j \subset \tilde{M}$ ($1 \leq i \leq k, 1 \leq j \leq n_i$) and local sections X_i^j of \mathfrak{g} , defined on $U_i^j := f(\tilde{U}_i^j) \subset M$, such that $\tilde{U}_k^{n_k} = \tilde{V}$, $\pi(\tilde{X}_i(\tilde{m})) = X_i^j(f(\tilde{m}))$ for all $\tilde{m} \in \tilde{U}_i^j$, and

$$(43) \quad f(\Phi_{\#\tilde{X}_i}^{t_i}(\tilde{m})) = \left(\Phi_{\#X_i^1}^{t_i^1} \circ \Phi_{\#X_i^2}^{t_i^2} \circ \dots \circ \Phi_{\#X_i^{n_i}}^{t_i^{n_i}} \right) (f(\tilde{m})); \quad \tilde{m} \in \tilde{U}_i^{n_i},$$

for $1 \leq i \leq k$.

Now let $\tilde{m} \in \tilde{V} = U_k^{n_k}$ be arbitrary. Then

$$\phi(f(\tilde{m})) = \beta(b(f(\tilde{m}))) = f(\beta(\tilde{b}(\tilde{m}))) = f\left(\Phi_{\#\tilde{X}_1}^{t_1} \circ \Phi_{\#\tilde{X}_2}^{t_2} \circ \dots \circ \Phi_{\#\tilde{X}_k}^{t_k} (\tilde{m}) \right),$$

where the second equality follows from (41), and the third from (42). Finally, employing (43) for each i between 1 and k , we obtain

$$\begin{aligned} \phi(f(\tilde{m})) = & \left(\Phi_{\#X_1^1}^{t_1^1} \circ \Phi_{\#X_1^2}^{t_1^2} \circ \cdots \circ \Phi_{\#X_1^{n_1}}^{t_1^{n_1}} \right. \\ & \circ \Phi_{\#X_2^1}^{t_2^1} \circ \Phi_{\#X_2^2}^{t_2^2} \circ \cdots \circ \Phi_{\#X_i^{n_2}}^{t_2^{n_2}} \circ \cdots \\ & \left. \circ \Phi_{\#X_k^1}^{t_k^1} \circ \Phi_{\#X_k^2}^{t_k^2} \circ \cdots \circ \Phi_{\#X_k^{n_k}}^{t_k^{n_k}} \right) (f(\tilde{m})). \end{aligned}$$

Since $\tilde{m} \in \tilde{V}$ is arbitrary, the formula holds with $f(\tilde{m})$ replaced with m , with $m \in V = f(\tilde{V})$ arbitrary. We conclude $\phi|_V \in \text{pseud}(\nabla)$. ■

Proof that formal integrations are integrations. To complete the proof of Theorem 2.6 it remains to show that any formal integration \mathcal{F}' of (\mathfrak{g}, ∇) is a bona fide integration, assuming the Lie groupoid G' supporting \mathcal{F}' is source-connected.

Let \mathcal{F} be the pseudoaction constructed in above. Then \mathcal{F} formally integrates (\mathfrak{g}, ∇) and, as we have just shown in the preceding sections, integrates (\mathfrak{g}, ∇) also:

$$(44) \quad \text{pseud}(\mathcal{F}) = \text{pseud}(\nabla).$$

Moreover, \mathcal{F} is supported by a Lie groupoid G that is source-simply-connected. It follows from Lie II for pseudoactions (Theorem 5.4) that there exists a Lie groupoid morphism $\Pi: G \rightarrow G'$ whose derivative is the identity on \mathfrak{g} and that is a morphism of pseudoactions $\mathcal{F} \rightarrow \mathcal{F}'$, and whence a covering of pseudoactions in the sense of Section 5, because we suppose G' is source-connected. Applying Proposition 5.5, we have $\text{pseud}(\mathcal{F}) = \text{pseud}(\mathcal{F}')$. Consequently, (44) implies $\text{pseud}(\mathcal{F}') = \text{pseud}(\nabla)$, i.e., \mathcal{F}' integrates (\mathfrak{g}, ∇) . ■

7. Integrating a complete twisted Lie algebra action

This section is devoted to the proof of Theorem 2.7, beginning with the untwisted case, first proven by Palais.

Integrating a complete Lie algebra action. Once again let \mathfrak{g}_0 be a finite-dimensional Lie algebra acting smoothly from the left on M , and denote the corresponding Lie algebra homomorphism $\mathfrak{g}_0 \rightarrow \Gamma(TM)$ by $\xi \mapsto \xi^\dagger$. The simply-connected Lie group with Lie algebra \mathfrak{g}_0 will be denoted by G_0 . We let ∇ denote the canonical flat connection on $\mathfrak{g}_0 \times M$.

Theorem 7.1 (Palais global integrability theorem [17]). *Suppose that ξ^\dagger is a complete vector field for every $\xi \in \mathfrak{g}_0$. Then there is a global action of G_0 on M whose infinitesimal generators are ξ^\dagger , $\xi \in \mathfrak{g}_0$. Moreover, if $\phi_g: M \rightarrow M$ denotes the global transformation corresponding to $g \in G_0$, then every $\phi \in \text{pseud}(\nabla)$ with connected domain U is of the form $\phi = \phi_g|_U$, for some $g \in G_0$.*

Recall here that $\text{pseud}(\nabla)$ is the pseudogroup of transformations generated by flows of infinitesimal generators.

Proof. According to Lie III and Lie I for pseudoactions (Theorems 2.6 and 5.6) there exists a pseudoaction \mathcal{F} , supported by a source-simply-connected Lie groupoid G , that both integrates and formally integrates $(\mathfrak{g}_0 \times M, \nabla)$. We will construct a global action of G_0 on M and a Lie groupoid isomorphism $G \rightarrow G_0 \times M$ that is also an isomorphism of pseudoactions $\mathcal{F} \rightarrow \mathcal{F}_0$. Here $G_0 \times M$ denotes the corresponding action groupoid and \mathcal{F}_0 the canonical pseudoaction on $G_0 \times M$. The theorem follows readily from the existence of this isomorphism.

Applying Lie II for pseudoactions (Theorem 5.4), there exists a Lie groupoid morphism $\Omega: G \rightarrow G_0$ integrating the canonical projection $\omega: \mathfrak{g}_0 \times M \rightarrow \mathfrak{g}_0$, and this morphism maps leaves of \mathcal{F} to points. A dimension count implies that the leaves of \mathcal{F} are connected components of fibres of Ω . Let $P \subset G$ denote an arbitrary source-fibre of G . Then as ω is a fibre-wise isomorphism, the restriction $\Omega: P \rightarrow G_0$ is a local diffeomorphism. The essential point is:

Lemma 7.2. *The restriction of $\Omega: G \rightarrow G_0$ to any source-fibre P is a diffeomorphism onto G_0 .*

Postponing the proof of the lemma, we construct an action of G_0 on M as follows. First note that the smooth map $\Omega \times \alpha: G \rightarrow G_0 \times M$ has full rank and so is a local diffeomorphism; here α denotes the source map. But from the surjectivity of $\alpha: G \rightarrow M$ and the lemma we readily see that $\Omega \times \alpha$ is bijective, and hence a diffeomorphism. Let $E: G_0 \times M \rightarrow G$ be the inverse diffeomorphism and — as in the proof of Palais' local integrability theorem (Theorem 4.1) — define $\phi_g(m) = \beta(E(g, m))$, where β is the target map. The proof that this defines an action of G_0 on M is the same as for the local result (read from Equation (15), replacing both Z and Z_{big} with G). This action makes $G_0 \times M$ into a Lie groupoid; it follows by construction that the diffeomorphism $\Omega \times \alpha: G \rightarrow G_0 \times M$ is compatible with the respective source and target maps. That $\Omega \times \alpha$ is Lie groupoid morphism now follows immediately from the fact that $\Omega: G \rightarrow G_0$ is. The derivative of $\Omega \times \alpha$ is just the identity on $\mathfrak{g}_0 \times M$. Since \mathcal{F}_0 formally integrates $(\mathfrak{g}_0 \times M, \nabla)$, it follows from Proposition 5.1 that Ω is in fact an isomorphism of pseudoactions $\Omega: \mathcal{F} \rightarrow \mathcal{F}_0$. ■

Proof of Lemma 7.2. For each $\xi \in \mathfrak{g}_0$, let ξ_c denote the corresponding constant section of $\mathfrak{g}_0 \times M$. Let ξ_P denote the restriction of the corresponding right-invariant vector field on G to one on P (in earlier notation, $\xi_P = \xi_c^R|_P$). Then, by Lemma 4.2, there exists through each point $g \in P$ an integral path of ξ_P , beginning at g and covering the integral path of $\# \xi_c$ beginning at $\beta(g)$; here $\beta: P \rightarrow M$ denotes the target projection, and is the map with respect to which 'covering' is to be understood. Since $\# \xi_c$ is complete, by hypothesis, it follows that ξ_P is complete also.

We now interpret the vector field completeness in terms of connections. Since $\Omega: G \rightarrow G_0$ is a groupoid morphism, ξ_P is Ω -related to the right-invariant vector field on G_0 generated by $\xi \in \mathfrak{g}_0$. Now let ∇ denote the canonical flat connection on G_0 with respect to which every right-invariant vector field is parallel, and pull ∇ back to a flat connection ∇^P on P using the local diffeomorphism $\Omega: P \rightarrow G_0$. Then, by construction, the geodesics of ∇^P are the integral curves

of ξ_P , which are complete. It follows from Proposition Section A given in the Appendix that the local diffeomorphism $\Omega: P \rightarrow G_0$ is a covering space. Since P is simply-connected (G is source-simply-connected) the lemma follows. ■

Proof of Theorem 2.7. We now prove our generalization of Palais' global integrability theorem to twisted Lie algebra actions.

Let (\mathfrak{g}, ∇) denote an arbitrary twisted Lie algebra action on M . According to Proposition 2.5, \mathfrak{g} can be identified with $\mathfrak{g}_0 \times_{\mu} M$, where $\mu: \Lambda \rightarrow \text{Aut}(\mathfrak{g}_0)$ is the monodromy representation, and \mathfrak{g}_0 is a Lie algebra acting on the universal cover \widetilde{M} . Since ∇ is complete, \mathfrak{g}_0 acts on \widetilde{M} by complete vector fields, and so Palais' global integrability theorem applies. According to the proof of that theorem above (read with M replaced with \widetilde{M}), there exists an action of G_0 on \widetilde{M} such that the canonical pseudoaction $\widetilde{\mathcal{F}}_0$ on the action groupoid $G_0 \times \widetilde{M}$ formally integrates $(\mathfrak{g}_0 \times \widetilde{M}, \widetilde{\nabla})$. Here $\widetilde{\nabla}$ denotes the canonical flat Cartan connection on $\mathfrak{g}_0 \times \widetilde{M}$.

On the other hand, Remark 2.4 tells us that $\mathfrak{g}_0 \times \widetilde{M}$ is naturally isomorphic to the formal pullback $\widetilde{\mathfrak{g}}$ of \mathfrak{g} , with the canonical action of Λ on $\widetilde{\mathfrak{g}}$ being represented by the action $\lambda \cdot (\xi, \widetilde{m}) = (\mu_{\lambda}\xi, \lambda(\widetilde{m}))$. According to the constructions of Section 6, (\mathfrak{g}, ∇) is formally integrated by a pseudoaction \mathcal{F} on G , where $G = \widetilde{G}/\Lambda$, \widetilde{G} is any source-simply-connected Lie groupoid formally integrating (and hence integrating) $\widetilde{\mathfrak{g}}$, and the action of Λ on \widetilde{G} is obtained by lifting its canonical representation on $\widetilde{\mathfrak{g}}$ using Lie II for Lie groupoids. But according to the preceding paragraph, we may take $\widetilde{G} = G_0 \times \widetilde{M}$, in which case we see that the action of Λ on $\widetilde{G} = G_0 \times \widetilde{M}$ must be given by $\lambda \cdot (g, \widetilde{m}) = (\nu_{\lambda}(g), \lambda(\widetilde{m}))$, where the homomorphism $\nu: \Lambda \rightarrow \text{Aut}(G_0)$ is the lift of the monodromy representation $\mu: \Lambda \rightarrow \text{Aut}(\mathfrak{g}_0)$ obtained by applying Lie II for Lie groups. So we have $G = \widetilde{G}/\Lambda = G_0 \times_{\nu} M$.

Recall also from Section 6, that the pseudoaction \mathcal{F} on G formally integrating (\mathfrak{g}, ∇) is obtained by dropping any pseudoaction $\widetilde{\mathcal{F}}$ on \widetilde{G} that formally integrates (and hence integrates) the pullback $(\widetilde{\mathfrak{g}}, \widetilde{\nabla})$ of (\mathfrak{g}, ∇) (such a drop being well-defined by construction). In the present case $\widetilde{G} = G_0 \times \widetilde{M}$, $\widetilde{\mathfrak{g}} = \mathfrak{g}_0 \times \widetilde{M}$, and $\widetilde{\nabla}$ is the canonical flat connection; so the uniqueness part of Lie I for pseudoactions (Theorem 5.6) implies $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_0$. It follows that the pseudoaction on $G = \widetilde{G}/\Lambda = G_0 \times_{\nu} M$ formally integrating (\mathfrak{g}, ∇) is the canonical one \mathcal{F}_0 . This completes the proof that (\mathfrak{g}, ∇) is formally integrated by a twisted Lie group action. ■

A. Technical details

Localization of multiplication in a Lie groupoid. We prove here the following result stated in Section 4

Proposition A.1. *Let $Z_{\text{big}} \subset G$ be a paracompact, Hausdorff, open neighbourhood of M in a Lie groupoid G over M . Then there exists a neighbourhood $Z \subset Z_{\text{big}}$ of M such that for all $h, g \in Z$ we have $hg \in Z_{\text{big}}$, whenever h and g are multipliable.*

In the proof we will make use of the following:

Lemma A.2. *Let U be a metric space and $\{U_i\}_{i \in I}$ an open cover of U . Then there exists a mapping $g \mapsto V_g$ of U into the collection of open subsets of U such that: (i) $g \in V_g$, and (ii) $V_{g_1} \cap V_{g_2} \neq \emptyset$ implies $V_{g_1} \cup V_{g_2} \subset U_i$, for some $i \in I$. Here $g, g_1, g_2 \in X$ are arbitrary.*

Proof of lemma. Since every metric space is paracompact, we may suppose, without loss of generality, that the cover $\{U_i\}_{i \in I}$ is locally finite. Then, for each $g \in U$, we can find $\epsilon_g > 0$ small enough that for all $i \in I$,

$$(45) \quad B_{3\epsilon_g}(g) \subset U_i \quad \text{whenever } g \in U_i.$$

Here $B_r(g)$ denotes the open ball of radius r centred at g . To show the sets $V_g := B_{\epsilon_g}(g)$ meet the requirements of the lemma, suppose $g_0 \in V_{g_1} \cap V_{g_2}$. Suppose $\epsilon_{g_2} \leq \epsilon_{g_1}$, and choose any $i \in I$ such that $g_1 \in U_i$. Then (45) immediately implies $V_{g_1} \subset U_i$ and it remains to show $V_{g_2} \subset U_i$. But we have

$$d(g_1, g_2) \leq d(g_1, g_0) + d(g_0, g_2) \leq \epsilon_{g_1} + \epsilon_{g_2} \leq 2\epsilon_{g_1}.$$

So, for any $g \in V_{g_2}$, we compute

$$d(g, g_1) \leq d(g, g_2) + d(g_1, g_2) \leq \epsilon_{g_2} + 2\epsilon_{g_1} \leq 3\epsilon_{g_1},$$

and conclude that $V_{g_2} \subset B_{3\epsilon_{g_1}}(g_1)$. It follows from (45) that $V_{g_2} \subset U_i$. \blacksquare

Proof of Proposition A.1. Using the continuity of multiplication, it is not hard to show that each point $m \in M$ has a neighbourhood U_m in Z_{big} such that $h, g \in U_m \Rightarrow hg \in Z_{\text{big}}$, whenever h and g are multipliable. The open neighbourhood $U := \cup_{m \in M} U_m$ of M is a paracompact, Hausdorff, smooth manifold, and is therefore metrizable. Applying the preceding lemma, we obtain a mapping $g \mapsto V_g$ from U to open subsets of U such that $g \in V_g$ and

$$V_{g_1} \cup V_{g_2} \subset U_n \text{ for some } n \in M \text{ whenever } V_{g_1} \cap V_{g_2} \neq \emptyset,$$

where g, g_1, g_2 are arbitrary. In particular, since $M \subset U$, we have $m \in V_m$ and

$$(46) \quad V_{m_1} \cup V_{m_2} \subset U_n \text{ for some } n \in M \text{ whenever } V_{m_1} \cap V_{m_2} \neq \emptyset,$$

where $m, m_1, m_2 \in M$ are arbitrary. We may also suppose that

$$(47) \quad \alpha(V_m) \subset V_m \cap M \quad \text{and} \quad \beta(V_m) \subset V_m \cap M; \quad m \in M,$$

where α and β are the source and target maps. For if not, replace each V_m with $V_m \cap \alpha^{-1}(V_m \cap M) \cap \beta^{-1}(V_m \cap M)$, and (46) still holds.

To show $Z := \cup_{m \in M} V_m$ satisfies the requirements of the proposition, suppose $h, g \in Z$ are multipliable. Then $h \in V_{m_1}$ and $g \in V_{m_2}$ for some $m_1, m_2 \in M$ with $m := \alpha(h) = \beta(g)$. Since $m = \alpha(h)$, (47) implies $m \in V_{m_1}$; since $m = \beta(g)$, (47) implies $m \in V_{m_2}$. So $V_{m_1} \cap V_{m_2} \neq \emptyset$, implying the existence of $n \in M$ such that $g, h \in U_n$, on account of (46). By the definition of U_n , we have $hg \in Z_{\text{big}}$. \blacksquare

Sufficient conditions for a local diffeomorphism to be a covering map.

We now turn to the proof of the following result used in the proof of Theorem 7.1:

Proposition A.3. *Let $\phi: \widetilde{M} \rightarrow M$ be a local diffeomorphism of smooth manifolds and suppose that M is connected and admits a linear connection ∇ such that the pullback $\widetilde{\nabla} := \phi^*\nabla$ is a complete connection on \widetilde{M} . Then $\phi: \widetilde{M} \rightarrow M$ is a smooth covering map (and, in particular, is surjective).*

Lemma A.4 (Lifting geodesics). *Assume the hypotheses of the proposition hold. Then for any geodesic $\gamma: [a, b] \rightarrow M$ for ∇ , and any point $\widetilde{m} \in \widetilde{M}$ satisfying $\phi(\widetilde{m}) = \gamma(t_0)$, for some $t_0 \in [a, b]$, there exists a unique geodesic $\widetilde{\gamma}: [a, b] \rightarrow \widetilde{M}$ for $\widetilde{\nabla}$ such that $\gamma = \phi \circ \widetilde{\gamma}$ and $\widetilde{\gamma}(t_0) = \widetilde{m}$.*

Proof of lemma. Since ϕ is a local diffeomorphism, there exists a unique tangent vector $v \in T_{\widetilde{m}}\widetilde{M}$ such that by $T\phi \cdot v = \dot{\gamma}(t_0)$. Let $\widetilde{\gamma}: [a, b] \rightarrow \widetilde{M}$ be the unique geodesic for $\widetilde{\nabla}$ such that $\dot{\widetilde{\gamma}}(t_0) = v$, which exists because $\widetilde{\nabla}$ is complete. Then $\gamma' := \phi \circ \widetilde{\gamma}$ must be a geodesic for ∇ and satisfy $\dot{\gamma}'(t_0) = \dot{\gamma}(t_0)$. By the uniqueness of geodesics with prescribed velocity at a point, we have $\gamma' = \gamma$, so that $\widetilde{\gamma}$ has the desired properties. ■

Proof of Proposition A.3. We show that ϕ evenly covers normal neighbourhoods for the connection ∇ . Indeed, let $m_0 \in M$ be arbitrary and let $B \subset M$ be a normal neighbourhood about m_0 for ∇ . That is, B is the diffeomorphic image of some open set $U \subset T_{m_0}M$ under the map $\exp_{m_0}: U \rightarrow M$ well-defined by the requirement that $\gamma(t) := \exp_{m_0}(t v)$ be a geodesic for ∇ satisfying $\gamma(0) = m_0$ and $\dot{\gamma}(0) = v$, for each choice of $v \in T_{m_0}M$.

We first suppose $\phi^{-1}(B)$ is non-empty. Later, we show that ϕ is surjective, so that this is no restriction. Let \widetilde{B} be any connected component of $\phi^{-1}(B)$. We must show the restriction $\phi: \widetilde{B} \rightarrow B$ is a homeomorphism. To this end, consider the discrete set $X := \widetilde{B} \cap \phi^{-1}(m_0)$. To see that X is non-empty let \widetilde{m} be any point in \widetilde{B} (non-empty by hypothesis) and let $\gamma: [0, 1] \rightarrow M$ be the geodesic joining $m_0 \in B$ to $\phi(\widetilde{m}) \in B$. Then if $\widetilde{\gamma}: [0, 1] \rightarrow \widetilde{M}$ is the lift of γ which ends at \widetilde{m} , whose existence is guaranteed by the lemma, then $\widetilde{\gamma}(0) \in X$, because $\widetilde{\gamma}^{-1}(0)$ and $\widetilde{\gamma}^{-1}(1)$ must lie in the same connected component of $\phi^{-1}(B)$.

In fact X has just one element. To see this construct, for each $x \in X$, a smooth right-inverse $s_x: B \rightarrow \widetilde{B}$ for the restriction $\phi: \widetilde{B} \rightarrow B$ as follows: If $m \in B$ is arbitrary and $\gamma: [0, 1] \rightarrow B$ is the unique geodesic joining m_0 to m , then $s_x(m) := \widetilde{\gamma}(1)$, where $\widetilde{\gamma}: [0, 1] \rightarrow \widetilde{B}$ is the lift of geodesic γ satisfying $\widetilde{\gamma}(0) = x$. Being a right-inverse, the derivative of s_x has full rank, i.e., is a local diffeomorphism. In particular, its image $s_x(B)$ is an open subset of \widetilde{B} . It is easy to see that the open sets $s_x(B)$, $x \in X$, are disjoint (use the uniqueness of geodesic lifts with prescribed end-point) and cover \widetilde{B} . But \widetilde{B} is *connected*, implying X has a single element, x_0 , say. Moreover, the single right-inverse $s_{x_0}: B \rightarrow \widetilde{B}$ must be surjective.

Being surjective, the smooth right-inverse $s_{x_0}: B \rightarrow \widetilde{B}$ is a two-sided inverse for the restriction $\phi: \widetilde{B} \rightarrow B$, which is accordingly a homeomorphism.

To summarise, every normal neighbourhood $B \subset M$ for ∇ is evenly covered by ϕ whenever it has non-trivial intersection with the image of ϕ . In particular, if a point $m_0 \in M$ is *not* in this image, then any normal neighbourhood of m_0

must intersect $\phi(\widetilde{M})$ in an empty set. So the complement of $\phi(\widetilde{M})$ is open in M . Since $\phi(\widetilde{M})$ is itself open (ϕ is a local diffeomorphism) and M is connected, we conclude that ϕ is surjective. ■

Acknowledgements. We acknowledge Marius Crainic, Rui Fernandes, Bas Janssens, Ercüment Ortaçgil, Maria Amelia Salazar and Ori Yudilevich for helpful discussions.

References

- [1] Blaom, A. D., *Geometric structures as deformed infinitesimal symmetries*. Trans. Amer. Math. Soc. **358** (2006), 3651–3671.
- [2] —, *Lie algebroids and Cartan’s method of equivalence*. Trans. Amer. Math. Soc. **364** (2012), 3071–3135.
- [3] —, *The infinitesimalization and reconstruction of locally homogeneous manifolds*. SIGMA **9** (2013), 074 (19 pages).
- [4] Cannas da Silva, A., and A. Weinstein, “Geometric Models for Noncommutative Algebras,” Berkeley Mathematics Lecture Notes **10**, American Mathematical Society, Providence, RI, 1999.
- [5] Cartan, É, *Le troisième théorème fondamental de Lie*. C. R. Acad. Sci. Paris Sér. I **190** (1930), 914–916, 1005–1007.
- [6] —, “La Topologie des Groupes de Lie,” Hermann, Paris, 1936.
- [7] Crainic, M. , and R. L. Fernandes, *Integrability of Lie brackets*, Ann. of Math. **157** (2003), 575–620.
- [8] —, *Lectures on integrability of Lie brackets* in “Lectures on Poisson geometry,” Geometry and Topology Monographs **17**, pages 1–107, Geometry and Topology Publications, Coventry, 2011.
- [9] Dazord, P., *Groupoïde d’holonomie et géométrie globale*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), 77–80.
- [10] Duistermaat, J. J., and J. A. C. Kolk, “Lie Groups,” Universitext. Springer-Verlag, Berlin, 2000.
- [11] Hermann, R., “Sophus Lie’s 1880 transformation group paper,” in Lie Groups: History, Frontiers and Applications, Vol. I., Mathematical Science Press, Brookline, Massachusetts, 1975. In part a translation of:
Lie, S., *Theorie der Transformations-Gruppen*, Math. Ann. **1** (1880), 441–528.
- [12] Kamber, F. W., and P. W. Michor, *Completing Lie algebra actions to Lie group actions*, Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 1–10.

- [13] Lang, S., “Differential and Riemannian Manifolds,” Graduate Texts in Mathematics **160**, Springer-Verlag, New York, third edition, 1995.
- [14] Mackenzie, K. C. H., “General Theory of Lie Groupoids and Lie Algebroids,” London Mathematical Society Lecture Note Series **213**, Cambridge University Press, Cambridge, 2005.
- [15] Michor, P. W., “Topics in Differential Geometry,” Graduate Studies in Mathematics **93**, American Mathematical Society, Providence, RI, 2008.
- [16] Olver, P. J., *Non-associative local Lie groups*, J. Lie Theory **6** (1996), 23–51.
- [17] Palais, R. S., *A Global Formulation of the Lie Theory of Transformation Groups*, Mem. Amer. Math. Soc., **22** (1957), iii+123 pages.
- [18] Salazar, A. S., “Pfaffian Groupoids,” PhD thesis, University of Utrecht, The Netherlands, 2013. Available at arxiv.org/abs/1306.1164.
- [19] Tang, Xiang, *Deformation quantization of pseudo-symplectic (Poisson) groupoids*, Geom. Funct. Anal. **16** (2006), 731–766.

Anthony D. Blaom
10 Huruhi Road
Waiheke Island, 1081
Auckland, New Zealand
anthony.blaom@gmail.com

Received October 28, 2014
and in final form October 16, 2015