

Integrable Wavelet Transforms with Abelian Dilation Groups

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Abstract. We consider a class of semidirect products $G = \mathbb{R}^n \rtimes H$, with H a suitably chosen abelian matrix group. The choice of H ensures that there is a wavelet inversion formula, and we are looking for criteria to decide under which conditions there exists a wavelet such that the associated reproducing kernel is integrable.

It is well-known that the existence of integrable wavelet coefficients is related to the question whether the unitary dual of G contains open compact sets. Our main general result reduces the latter problem to that of identifying compact open sets in the quotient space of all orbits of maximal dimension under the dual action of H on \mathbb{R}^n . This result is applied to study integrability for certain families of dilation groups; in particular, we give a characterization valid for connected abelian matrix groups acting in dimension three.

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1. Introduction

Before we describe the aims of this paper, we need to introduce some terminology and notations. Let $H < \mathrm{GL}(n, \mathbb{R})$ denote a closed matrix group. The affine group generated by H and all translations is the semidirect product $G = \mathbb{R}^n \rtimes H$. Elements of G are denoted by pairs $(x, h) \in \mathbb{R}^n \times H$, with group law $(x, h)(y, g) = (x + hy, hg)$. G acts unitarily on $L^2(\mathbb{R}^n)$ via the *quasiregular representation* π , defined by

$$\pi(x, h)f(y) = |\det(h)|^{-1/2}f(h^{-1}(y - x)) .$$

Given $\eta \in L^2(\mathbb{R}^n)$, the associated *continuous wavelet transform* is the mapping

$$V_\eta : L^2(\mathbb{R}^n) \rightarrow C_b(G) , \quad V_\eta f(x, h) = \langle f, \pi(x, h)\eta \rangle .$$

Here $C_b(G)$ denotes the space of bounded continuous functions on G . Whenever V_η is an isometry into $L^2(G)$ (defined with respect to left Haar measure), we call

η *admissible*. In turn, the group H will be called *admissible* if there exists an admissible vector for π .

Admissible vectors and groups have been studied extensively in the past twenty years, with varying degrees of generality. A small and subjective sample of the literature is [21, 3, 10, 18, 11, 12].

The property we intend to investigate in this paper is *integrability* of the quasi-regular representation. We call a representation integrable if there is an admissible vector η such that $\Delta_G^{-1/2}V_\eta\eta$ is in $L^1(G)$. By slight abuse of notation we will call such vectors η *integrable admissible vectors*. A major motivation for studying integrability in this context is the coorbit theory due to Feichtinger and Gröchenig [8, 13], which heavily relies on integrability. Coorbit spaces associated to a quasi-regular representation can be interpreted as a family of smoothness spaces induced by the action of the group. While the initial definition also relied on irreducibility of the representation, recent work (e.g. [6]) suggests that a meaningful theory can also be developed for reducible representations, as long as integrability (or somewhat weaker conditions) are retained.

Besides these connections to function spaces, integrability is also a representation-theoretic property of independent interest. For any admissible vector η , isometry of V_η implies that $V_\eta\eta = \widetilde{V_\eta\eta} = V_\eta\eta * V_\eta\eta$. Here we used the notation $\tilde{f}(x) = \overline{f(x^{-1})}$ for any function f on G . If, in addition, $F = \Delta_G^{-1/2}V_\eta\eta \in L^1(G)$, this implies that F is a *projection in the group algebra* $L^1(G)$, i.e. it fulfills $F = F * F = F^*$, where $F^*(x) = \Delta_G(x)^{-1}\tilde{F}(x^{-1})$. The question which group algebras contain projections has been studied, e.g., in [2, 14, 15, 16, 24, 25]. A necessary representation-theoretic condition for the existence of projections in L^1 hails from the following observation: If $F \in L^1(G)$ is a projection, then $\text{supp}(\widehat{F}) \subset \widehat{G}$ is a compact open set. (Note that here, as in the following, our definition of compactness does not include the Hausdorff property.) Here \widehat{F} is the group Fourier transform of F , obtained by integrating F against irreducible representations, and \widehat{G} denotes the unitary dual of G , i.e. the equivalence classes of irreducible representations of G . The topology on \widehat{G} is the Fell topology, defined in terms of uniform convergence of matrix coefficients on compact subsets of G , see [7, 9] for details. Thus, a necessary (but usually not sufficient) condition for the existence of a projection in L^1 is the existence of a compact open set in the dual.

For the setting studied here, the most significant precursors are [14, 16]; for a unified account of the results see also [17]. The paper [14] contains a full characterization of integrability for the case $H = \exp(\mathbb{R}A)$ (unlike our paper, [14] does not impose any condition on the spectrum of A). Briefly, the quasiregular representation associated to such a group is integrable iff the group is strictly contractive; or more technically, iff the signs of $\text{Re}\lambda$ coincide, for all eigenvalues λ of A .

In the following, we shall mostly concentrate on the following class of matrix groups: H is an abelian connected matrix group satisfying $\text{spec}(h) \subset \mathbb{R}^+$, for all $h \in H$. The chief reason for focussing on this class is that there exist easily checked admissibility criteria for it: H is admissible iff $\det|_H \neq 1$, and in addition, the dual action of H is free almost everywhere, see [4, Theorem 23]. Here, the dual action is defined by $(h, x) \mapsto h^{-T}x$, where $h^{-T} = (h^T)^{-1}$. We let $d = \dim(H)$.

The following theorem is the chief general result of this paper. Before we formulate it, we need some additional notation. Orbits under H^T are submanifolds of \mathbb{R}^n . If a matrix group H is fixed, we define, for $i \in \mathbb{N}$:

$$\mathcal{O}_i = \{x \in \mathbb{R}^n : \dim(H^T x) = i\}.$$

Clearly, $\mathcal{O}_i = \emptyset$ whenever $i > d$. For the class of groups that we consider here, the dual action is free almost everywhere iff $\mathcal{O}_d \neq \emptyset$. The “only if”-part is clear since the dimension of every free orbit equals the group dimension. For the “if”-part, note that if \mathcal{O}_d is nonempty, then it is Zariski-open, and its complement has measure zero. Furthermore, the action of H on \mathcal{O}_d is free: Since group dimension and orbit dimension coincide, the fixed groups associated to \mathcal{O}_d are discrete. But they are also connected, hence trivial.

Finally, we endow the orbit space with a pseudo-image of Lebesgue measure.

Theorem 1.1. *Let H be a d -dimensional connected, admissible abelian matrix group satisfying $\text{spec}(h) \subset \mathbb{R}^+$ for all $h \in H$. If π is integrable, there exists a compact conull set $C \subset \mathcal{O}_d/H^T$.*

The necessary compactness condition simplifies considerably, if more is known about the orbit space.

Corollary 1.2. *Let H be a d -dimensional connected, admissible abelian matrix group satisfying $\text{spec}(h) \subset \mathbb{R}^+$ for all $h \in H$. Assume in addition that \mathcal{O}_d/H^T is Hausdorff. Then, if π is integrable, \mathcal{O}_d/H^T is compact.*

Proof. If \mathcal{O}_d/H^T is Hausdorff, then every compact open subset C is closed as well, and thus it is a union of connected components of \mathcal{O}_d/H^T . All connected components of \mathcal{O}_d are open, thus of positive Lebesgue measure. Hence all compact components of \mathcal{O}_d/H^T have positive measure, as well. A compact conull subset C , if it exists, therefore must be all of \mathcal{O}_d/H^T . ■

A subset $C \subset \mathbb{R}^d$ is called a *topological section* for an H^T -invariant subset $U \subset \mathbb{R}^d$ if the map $H \times C \rightarrow U, (h, c) \mapsto h^T c$ is a homeomorphism. Note that the existence of a topological section implies that H acts freely on U . Furthermore, the natural bijection $C \rightarrow U/H^T$ is a homeomorphism as well, implying in particular that the quotient space is Hausdorff.

Corollary 1.3. *Let H be a d -dimensional connected, admissible abelian matrix group satisfying $\text{spec}(h) \subset \mathbb{R}^+$ for all $h \in H$. If there exists a topological section $C \subset \mathcal{O}_d$ for the dual action, then π is integrable iff C is compact.*

Proof. If C is a topological section, then the canonical bijection $C \rightarrow \mathcal{O}_d/H^T$ is a homeomorphism. In particular, the latter space is Hausdorff, and it is compact iff C is compact. Thus one direction is immediate from the previous corollary. The other direction follows from [16, Proposition 3.1]. ■

2. Topological properties of dual orbit spaces and integrability

In this subsection we collect some general observations concerning dual orbit spaces. We first recall an observation from [16]:

Lemma 2.1. *Let $U \subset \mathbb{R}^n$ be arbitrary. Then U/H^T is compact iff there exists a compact set $C \subset U$ with $H^T C = U$.*

Proof. Compare the proof of [16, Corollary 4.2]. ■

We further note that, as a quotient space of a second countable space, \mathbb{R}^n/H^T is second countable as well, and so are all its subspaces. It follows that a subset is compact iff it is sequentially compact. Our discussion of concrete cases in the following sections will mostly rely on this observation.

Our next aim is to generalize a known sufficient condition for the existence of integrable admissible vectors. It has been demonstrated in [16, Proposition 3.1] how to construct, given an open conull subset U having a topological section with compact closure inside U , admissible vectors that give rise to an integrable projection. Our aim is to replace the topological section by a weaker notion, a *topological quasi-section*. For this purpose, we borrow some notation introduced by Richard Palais [22]: Given $Y, Z \subset \mathbb{R}^n$, we let

$$((Y, Z)) = \{h \in H : h^T Y \cap Z \neq \emptyset\} .$$

Definition 2.2. Let $U \subset \mathbb{R}^n$ be open and H^T -invariant. A subset $C \subset U$ is called *topological quasi-section* (for U) if it is open, with $H^T C = U$, and such that $((C, C))$ is relatively compact.

The following lemma is clear, but useful to note.

Lemma 2.3. *Let $U \subset \mathbb{R}^n$ be open and H^T -invariant. Assume that $C \subset U$ is a topological section. Let $V \subset H$ be a relatively compact open neighborhood of the identity. Then $VC \subset U$ is a topological quasi-section.*

We already noted that the existence of a topological section implies that the action is free. In the case of a compact quasi-section, a weakened version of this observation applies

Lemma 2.4. *Let $U \subset \mathbb{R}^n$ be open and H^T -invariant. If U admits a topological quasi-section, then H_ξ is compact for all $\xi \in U$.*

Proof. If C is a topological quasi-section, it is sufficient to prove compactness of H_ξ , for all $\xi \in C$. But this follows from the fact that $((C, C))$ contains the stabilizer. ■

Lemma 2.5. *Let $U \subset \mathbb{R}^n$ be open and H^T -invariant. If there exists a topological quasi-section in U , then for all compact sets $C_1, C_2 \subset U$, $((C_1, C_2))$ is relatively compact.*

Proof. Denote the quasi-section by C . Then, by compactness of the C_i , we have $C_i \subset F_i C$, for suitable finite $F_i \subset H$ ($i = 1, 2$). A straightforward computation shows that $((C_1, C_2)) \subset F_2((C, C))F_1^{-1}$, which is relatively compact. ■

We are particularly interested in compact orbit spaces. In this setting, the notion of a quasi-section becomes particularly useful. In order to properly appreciate the usefulness of the following quite simple result, recall that the computation of cross-sections can be quite cumbersome. What is worse, the failure to produce a *topological* section usually does not provide any clue regarding the existence of such a section.

By contrast, as the following proposition shows, there is no “clever” way of picking quasi-sections, when the orbit space is compact: Either any relatively compact open set meeting every orbit will do, or none. Since topological sections give rise to quasi-sections, this provides a useful necessary condition for existence of topological sections.

Proposition 2.6. *Let $U \subset \mathbb{R}^n$ be open and H^T -invariant. Assume that U/H^T is compact, and let $C \subset U$ denote an open set with compact closure in U , such that $H^T C = U$. Then U has a topological quasi-section iff C is a topological quasi-section, i.e., iff $((C, C))$ is relatively compact.*

In particular, if $((C, C))$ is not relatively compact, then U does not have a topological section.

Proof. The “if”-direction is clear. The other direction follows from Lemma 2.5. The statement concerning topological sections now follows from Lemma 2.3. ■

The following result provides the extension of [16, Proposition 3.1].

Proposition 2.7. *Let $H < \text{GL}(n, \mathbb{R})$ be a closed matrix group. Suppose there exists an open H^T -invariant set $U \subset \mathbb{R}^n$ that admits a topological quasi-section with compact closure in U . Then the restriction π_U of the quasi-regular representation to the invariant closed subspace $\mathcal{H}_U = \{f \in L^2(\mathbb{R}^n) : \widehat{f} \cdot 1_U = \widehat{f}\}$ is integrable.*

Proof. We adapt the proof from [16]. Let $C \subset U$ denote the relatively compact quasi-section. Pick a relatively compact open set W with $\overline{C} \subset W \subset U$. Let $\phi \in C_c^\infty(U)$ with $1_{\overline{C}} \leq \phi \leq 1_W$. For $\xi \in U$, define

$$\sigma(\xi) = \int_H |\phi(h^T \xi)|^2 dh .$$

Then σ is a well-defined smooth function on U : Let $\xi \in U$ be arbitrary, and let $U' \subset U$ denote a compact neighborhood of ξ . Then for all $\xi' \in U'$,

$$\sigma(\xi') = \int_{((U', \overline{W}))} |\phi(h^T \xi')|^2 dh .$$

and Lemma 2.5 ensures that the integration domain is compact. In particular, σ is well-defined, and compactness of the integration domain allows to interchange differentiation and integration, which shows that σ is smooth.

Furthermore, for every $\xi \in U$, the map $h \mapsto \phi(h^T \xi)$ is a positive continuous function, and not identically zero: There exists $h \in H$ with $h^T \xi \in C$, and thus $|\phi(h^T \xi)|^2 \geq 1$ by choice of ϕ . Hence $\sigma(\xi) > 0$.

But then $\widehat{g} = \phi/|\sigma|^{1/2}$ is a smooth, compactly supported function on U , which we extend trivially to all of \mathbb{R}^n . We let $g \in L^2(\mathbb{R}^n)$ denotes its inverse Fourier transform; by construction, g is in fact a Schwartz function. Letting $\widehat{g}_h : \xi \mapsto \widehat{g}(h^T \xi)$, we find that $\widehat{g}\widehat{g}_h = 0$ whenever $h \notin C_{W,W}$, and Lemma 2.5 implies that this set is relatively compact.

Using the properties of \widehat{g} collected thus far, the proof of [16, Proposition 3.1] is seen to go through. Since our notation somewhat differs from that in [16], we include a short sketch. We first rewrite the wavelet transform as

$$V_g g(x, h) = |\det(h)|^{1/2} \left(\widehat{g} \cdot \overline{\widehat{g}_h} \right)^\vee(x).$$

It follows that we can estimate

$$\|\Delta_G^{-1/2} V_g g\|_{L^1} \leq \int_H \left\| \left(\widehat{g} \cdot \overline{\widehat{g}_h} \right)^\vee \right\|_{L^1} |\det(h)|^{-1/2} \Delta_G^{-1/2}(h) dh.$$

The vector-valued mapping

$$h \mapsto \widehat{g} \cdot \overline{\widehat{g}_h} \in C_c^\infty(U)$$

is continuous with respect to the Schwartz topology, implying that the mapping

$$h \mapsto \left\| \left(\widehat{g} \cdot \overline{\widehat{g}_h} \right)^\vee \right\|_{L^1} |\det(h)|^{-1/2} \Delta_G^{-1/2}(h)$$

is continuous as well, and we noted above that it is also compactly supported. This implies $\Delta_G^{-1/2} V_g g \in L^1(G)$. \blacksquare

Note that the proof shows that we may replace the factor $\Delta_G^{-1/2}$ by any weight function on H that is bounded on compact sets.

Unlike for topological sections, the existence of quasi-sections does not imply that the group action is free. In particular, we can apply the proposition to the case where U is an open dual orbit with compact stabilizer. The subrepresentations π_U corresponding to such orbits are precisely the discrete series subrepresentations [12, Corollary 21]. Hence we obtain the following result.

Corollary 2.8. *Let $H < \mathrm{GL}(n, \mathbb{R})$ be a closed matrix group. Every discrete series subrepresentation of the quasi-regular representation π is integrable.*

While the notion of quasi-section may seem somewhat ad hoc, a result by Palais shows that the existence of such a quasi-section naturally connects to well-studied topological properties of the orbit space. Recall for the following that *properness* of the dual action on an H^T -invariant subset U means that the inverse image of any compact subset $C \subset U \times U$ under the mapping $H \times U \rightarrow U \times U$, $(h, \xi) \mapsto (h^T \xi, \xi)$ is compact.

Proposition 2.9. *Let $U \subset \mathbb{R}^n$ be open and H^T -invariant. Then the following statements are equivalent:*

- (a) *There exists a topological quasi-section $C \subset U$ with compact closure in U .*
- (b) *The orbit space U/H^T is compact and the dual action of H on U is proper.*

If any of these conditions hold, U/H^T is Hausdorff and metrizable.

Proof. First assume that (a) holds. Then U/H^T is clearly compact. We first note that by Lemma 2.5, we have that $((C_1, C_2))$ is relatively compact, for all open relatively compact subsets C_1, C_2 of U . In the parlance of [22], this means that C_1, C_2 are *relatively thin*, and since this holds for all open, relatively compact neighborhoods of arbitrary points, it follows that each point in U has a *small neighborhood* (again in the terminology of [22]). Now implication (1) \Rightarrow (3) of [22, Theorem 1.2.9] yields that the dual action on U is proper, and we have concluded (b).

Conversely, assume that (b) holds. Pick a compact set $C \subset U$ with $H^T C = U$. Pick $W \supset C$ open, with closure compactly contained in U . Then implication (3) \Rightarrow (4) of [22, Theorem 1.2.9] implies that $((W, W))$ is relatively compact, i.e., W is a topological quasi-section.

The Hausdorff property of U/H^T follows by [22, Theorem 1.2.9], (3) \Rightarrow (2), and we already noted that U/H^T is second countable. Metrizable now follows from Urysohn's metrization theorem. ■

Remark 2.10. The results of this section provide a partial converse to Theorem 1.1: Let H be a d -dimensional admissible abelian matrix group, and assume that there exists an open conull H -invariant set $U \subset \mathcal{O}_d$ such that U/H^T is compact. If in addition the dual action of H on U is proper, then Propositions 2.7 and 2.9 imply that π is integrable.

It is currently unknown whether the gap between this statement and Theorem 1.1 can actually be closed: Proposition 4.1 (b) below presents a matrix group H for which the dual action has a conull open compact subset of the orbit space, but the properness condition is violated. For this example, the question of integrability of the associated quasi-regular representation is currently open.

3. Proof of Theorem 1.1

Throughout this section, H denotes an abelian matrix group fulfilling the conditions of Theorem 1.1. In particular, the dual action of H is free almost everywhere, and thus $d = \dim(H) \leq n$. Furthermore, H is connected and all elements $h \in H$ have positive spectrum.

Let us first comment on the proof strategy: The chief representation-theoretic tool employed in [14, 16] was the Mackey machine. However, while there are useful results providing topological information (such as continuity of induction), in general the Mackey machine only yields a full description of the dual *as a set*, and comprehensive information on the topological structure is not easily obtained this way.

For this reason we choose an alternative approach, which is available because the semidirect products under consideration turn out to be *exponential*. In fact, G is a completely solvable Lie group (by Lemma 3.1 below). Thus \widehat{G} is homeomorphic to the coadjoint orbit space of G : Denote by \mathfrak{g} the Lie algebra of G , and by \mathfrak{g}^* its dual. G acts on \mathfrak{g}^* by the coadjoint representation Ad^* , and there exists a canonical bijection $\kappa : \mathfrak{g}^*/\text{Ad}^*(G) \rightarrow \widehat{G}$, the Kirillov-Bernat correspondence. Furthermore, the bijection is in fact a homeomorphism, when we endow the orbit space with the quotient topology; see [19] for details. This observation allows one to employ the methods developed by Currey and Penney to derive sufficient information on the coadjoint orbit space for the proof of Theorem 1.1. A further important ingredient of the proof is the close relationship between dual and coadjoint orbits.

3.1. Adjoint and coadjoint representation of G .

We identify elements of the Lie algebra \mathfrak{g} with pairs $(v, X)^T \in \mathbb{R}^n \times \mathfrak{h}$. Accordingly, \mathfrak{g}^* is canonically isomorphic to $\mathbb{R}^n \times \mathfrak{h}^*$; i.e. we have

$$\langle (\xi, Y^*), (x, X) \rangle = \langle \xi, x \rangle + \langle Y^*, X \rangle.$$

In order to use block matrix calculus, we will write elements of the product spaces as column vectors, i.e. $\begin{pmatrix} w \\ X \end{pmatrix} \in \mathbb{R}^n \times \mathfrak{h}$, $\begin{pmatrix} \xi \\ Y^* \end{pmatrix} \in \mathbb{R}^n \times \mathfrak{h}^*$, etc.

The following formulae are conveniently verified by realizing that the mapping $(x, h) \mapsto \begin{pmatrix} h & x \\ 0 & 1 \end{pmatrix}$ is an isomorphism onto a closed group of $(n+1) \times (n+1)$ -matrices. Let $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ and $\text{Ad}^* : G \rightarrow \text{GL}(\mathfrak{g}^*)$ denote the adjoint and coadjoint representations of the group. Let ad and ad^* denote their Lie algebra counterparts.

We then obtain the following formulae, using that the adjoint representation of H is trivial:

$$\text{Ad}(x, h) \begin{pmatrix} w \\ Y \end{pmatrix} = (-\text{Ad}(h)(Y) \cdot x + hw, \text{Ad}(h)Y)^T = \begin{pmatrix} h & \varphi_x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} w \\ Y \end{pmatrix}, \quad (1)$$

with $\varphi_x : \mathfrak{h} \rightarrow \mathbb{R}^n$, $Y \mapsto -Y \cdot x$. For the coadjoint representation, we obtain

$$\text{Ad}^*(x, h) \begin{pmatrix} \xi \\ Y^* \end{pmatrix} = \begin{pmatrix} h^{-T} & 0 \\ \varphi_{-h^{-1}x}^* & 1 \end{pmatrix} \cdot \begin{pmatrix} \xi \\ Y^* \end{pmatrix}, \quad (2)$$

with $\varphi_y^* : \mathbb{R}^n \rightarrow \mathfrak{h}^*$ the map dual to φ_y .

We next turn to the Lie algebra representations. Here we obtain

$$\text{ad}(x, Z) \begin{pmatrix} w \\ Y \end{pmatrix} = \begin{pmatrix} Z & \varphi_x \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} w \\ Y \end{pmatrix}, \quad (3)$$

and finally

$$\text{ad}^*(x, Z) \begin{pmatrix} \xi \\ Y^* \end{pmatrix} = \begin{pmatrix} -Z^T & 0 \\ \varphi_{-x}^* & 0 \end{pmatrix} \cdot \begin{pmatrix} \xi \\ Y^* \end{pmatrix}. \quad (4)$$

From these formulae, we can derive a number of simple but important observations.

Lemma 3.1. *G is completely solvable.*

Proof. By Lemma 3, we find $\text{spec}(ad(x, Z)) \subset \{0\} \cup \text{spec}(Z)$. But the assumption that $\text{spec}(\mathbb{R}Z) \subset \mathbb{R}^+$ implies that $\text{spec}(Z) \subset \mathbb{R}$, and thus $\text{spec}(ad(x, Z)) \subset \mathbb{R}$, for all $(x, Z) \in \mathfrak{g}$. But this is equivalent to complete solvability of G . ■

Recall that the Kirillov-Bernat correspondence κ is characterized as follows. For $(\xi, Y^*) \in \mathfrak{g}^*$, suppose that \mathfrak{p} is a subalgebra of \mathfrak{g} such that (a) $[\mathfrak{p}, \mathfrak{p}] \subset \ker(\xi, Y^*)$, and (b) \mathfrak{p} is a maximal subalgebra with respect to the property (a). Then one has

$$\dim \text{Ad}^*(G)(\xi, Y^*) = 2 \dim(\mathfrak{g}/\mathfrak{p}). \tag{5}$$

For each $(\xi, Y^*) \in \mathfrak{g}^*$, there is a subalgebra \mathfrak{p} satisfying (a), (b), and also (c) $(\xi, Y^*) + \mathfrak{p}^\perp \subset \text{Ad}^*(G)(\xi, Y^*)$. (Here $\mathfrak{p}^\perp = \{(\zeta, X^*) \in \mathfrak{g}^* : \mathfrak{p} \subset \ker(\zeta, X^*)\}$.) Then (ξ, Y^*) defines a unitary character $\chi = \chi_{(\xi, Y^*)}$ on $P = \exp \mathfrak{p}$ by

$$\chi(\exp(x, X)) = e^{i\langle (\xi, Y^*), (x, X) \rangle},$$

and the unitary representation $\pi((\xi, Y^*), \mathfrak{p}) := \text{ind}_P^G(\chi)$ is irreducible. The unitary equivalence class $[\pi((\xi, Y^*), \mathfrak{p})]$ of $\pi((\xi, Y^*), \mathfrak{p})$ is independent of the choice of \mathfrak{p} , as well as independent of the choice of $(\zeta, X^*) \in \text{Ad}^*(G)(\xi, Y^*)$, and one has $\kappa(\text{Ad}^*(G)(\xi, Y^*)) = [\pi((\xi, Y^*), \mathfrak{p})]$.

The following lemma clarifies the relationship between coadjoint orbits, dual orbits, and the dual \widehat{G} , which is particularly transparent for the class of groups that we study here. For the more general setting, the interested reader is referred to [1]. The formulation of the lemma requires two more pieces of notation: We let $\mathfrak{h}_\xi = \{X \in \mathfrak{h} : X^T \xi = 0\}$; i.e., \mathfrak{h}_ξ is the Lie algebra of the dual stabilizer H_ξ . Its annihilator is given by $\mathfrak{h}_\xi^\perp = \{Y^* \in \mathfrak{h}^* : \langle Y^*, \mathfrak{h}_\xi \rangle = \{0\}\}$.

Lemma 3.2. *(a) For all $(\xi, Y^*) \in \mathfrak{g}^*$ we have*

$$\text{Ad}^*(G)(\xi, Y^*) = (H^T \xi) \times (Y^* + \mathfrak{h}_\xi^\perp) \tag{6}$$

(b) The projection map $P : \mathfrak{g}^ \rightarrow \mathbb{R}^n$ intertwines the coadjoint and dual actions. It therefore induces a continuous map $\overline{P} : \mathfrak{g}^*/\text{Ad}^*(G) \rightarrow \mathbb{R}^n/H^T$.*

(c) Let $\mathcal{O}_i^ \subset \mathfrak{g}^*$ denote the union of orbits of dimensions i . Then $\overline{P} : \mathcal{O}_{2d}^*/\text{Ad}^*(G) \rightarrow \mathcal{O}_d/H^T$ is a homeomorphism.*

(d) Let $m : \mathcal{O}_d/H^T \rightarrow \widehat{G}$ be the Mackey injection defined by $m(H^T \xi) = [\text{ind}_{\mathbb{R}^n}^G(\chi_\xi)]$. Then the restriction of $m \circ \overline{P}$ to $\mathcal{O}_{2d}^/\text{Ad}^*(G)$ coincides with the restriction of κ to $\mathcal{O}_{2d}^*/\text{Ad}^*(G)$*

Proof. To verify (6), put $\mathfrak{p}_{(\xi, Y^*)} = \mathbb{R}^n \times \mathfrak{h}_\xi$; using (2) we see that $\mathfrak{p}_{(\xi, Y^*)}$ is a subalgebra satisfying conditions (a), (b), and (c) with respect to (ξ, Y^*) . Hence

$$\dim \text{Ad}^*(G)(\xi, Y^*) = 2 \dim \mathfrak{g}/\mathfrak{p}_\xi = 2 \dim \mathfrak{h}/\mathfrak{h}_\xi = 2 \dim H^T \xi.$$

A direct calculation again using (2) shows that

$$\text{Ad}^*(G)(\xi, Y^*) \subset (H^T \xi) \times (Y^* + \mathfrak{h}_\xi^\perp),$$

and it is clear that $\dim (H^T \xi) \times (Y^* + \mathfrak{h}_\xi^\perp) = 2 \dim H^T \xi$, so (6) follows.

For part (c), we have seen that the coadjoint orbit dimension of (ξ, Y^*) is twice the dual orbit dimension of ξ . In particular, $(\xi, Y^*) \in \mathcal{O}_{2d}^*$ iff $\xi \in \mathcal{O}_d$. Furthermore, H acts freely on \mathcal{O}_d , which for $\xi \in \mathcal{O}_d$ implies $\mathfrak{h}_\xi^\perp = \mathfrak{h}^*$, and thus $\text{Ad}^*(G)(\xi, Y^*) = (H^T \xi) \times \mathfrak{h}^*$. Hence \bar{P} is bijective on $\mathcal{O}_{2d}^*/\text{Ad}^*(G)$, and thus a homeomorphism.

Finally, the above shows that for $(\xi, Y^*) \in \mathcal{O}_{2d}^*$, the subalgebra $\mathfrak{p} = \mathbb{R}^n$ satisfies conditions (a), (b) and (c) for the Kirillov-Bernat correspondence. The characters on \mathbb{R}^n defined by (ξ, Y^*) and ξ are obviously the same, giving the equality of the induced representations $\pi((\xi, Y^*), \mathfrak{p}) = \text{ind}_{\mathbb{R}^n}^G(\chi_\xi)$ so

$$\kappa(\text{Ad}^*(G)(\xi, Y^*)) = [\pi((\xi, Y^*), \mathfrak{p})] = [\text{ind}_{\mathbb{R}^n}^G(\chi_\xi)] = m(H \cdot \xi). \quad \blacksquare$$

3.2. A layering of \mathfrak{g}^* .

The reduction to dual orbits of maximal dimension will rely on a corresponding result for the space of coadjoint orbits. The following lemma contains the vital parts of the proof. For the most part, it is a rephrasing of the main result of [5], adapted to the our specific needs.

Lemma 3.3. *There exists a decomposition of \mathfrak{g}^* into $\text{Ad}^*(G)$ -invariant, disjoint sets $(\Omega_j^*)_{j=1, \dots, p}$ with the following properties:*

- (i) For all $1 \leq k \leq p$: $\bigcup_{j \leq k} \Omega_j^*$ is open.
- (ii) For all $1 \leq j \leq p$: All coadjoint orbits in Ω_j^* have the same dimension d_j . The sequence d_1, \dots, d_p is nonincreasing.
- (iii) For all $1 \leq j \leq p$: $\Omega_j^*/\text{Ad}^*(G)$ is Hausdorff.
- (iv) For all $1 \leq j \leq p$ we have that $\Omega_j^* = P^{-1}(P(\Omega_j^*))$.
- (v) Whenever $d_j < 2d$, all connected components of Ω_j^* are noncompact.

Proof. The proof relies on the explicit construction of cross-sections for the coadjoint orbits of G , as described in [5]. The starting point for this construction is a Jordan-Hölder basis of \mathfrak{g} . In our setting, this can be obtained with particular ease: By [4, Theorem 4], there exists a basis of \mathbb{R}^n with respect to which all elements of H are upper triangular. We denote this basis by X_1, \dots, X_n , and complement it by a basis X_{n+1}, \dots, X_{n+d} of \mathfrak{h} . It is then straightforward to check that

$$\mathfrak{g}_j = \text{span}(X_i : i \leq j),$$

defines an increasing sequence of ideals \mathfrak{g}_j in \mathfrak{g} . Hence X_1, \dots, X_{d+n} is the desired Jordan-Hölder basis.

Consequently, the dual basis $X_1^*, \dots, X_{d+n}^* \subset \mathfrak{g}^*$ is a Jordan-Hölder basis for the dual, with the usual, implicit convention that this time, the subspaces

$$V_j = \text{span}(X_i^* : i > j)$$

are submodules. It follows that G acts on \mathfrak{g}^*/V_j , and the construction of the layering will be obtained from data associated to the actions of G on the various quotient spaces.

Given $(\xi, Y^*)^T \in \mathfrak{g}^*$ and $1 \leq j \leq n + d$, we let

$$L_j(\xi, Y^*) = \{(w, X) \in \mathfrak{g} : \text{ad}^*(w, X)(\xi, Y^*)^T \in V_j\} .$$

By definition, $L_j(\xi, Y^*)$ is therefore the Lie algebra of the fixed group of $(\xi, Y^*)^T + V_j$ with respect to the action of G .

Now, given $L_j(\xi, Y^*)$, for $j = 1, \dots, n + d$, R. Penney and the first author describe in [5] how to associate an index set $\mathfrak{j}(\xi, Y^*)$ and an ordering scheme $\alpha(\xi, Y^*)$ for $\mathfrak{j}(\xi, Y^*)$, in such a way that the layering is obtained by picking suitable tuples \mathfrak{j}, α and letting

$$\Omega_{\mathfrak{j}, \alpha} = \{(\xi, Y^*) : \mathfrak{j}(\xi, Y^*) = \mathfrak{j}, \alpha(\xi, Y^*) = \alpha\} . \tag{7}$$

Here $\mathfrak{j}(\xi, Y^*)$ is the set of *jump indices* for (ξ, Y^*) , indicating when the dimension of $\text{Ad}^*(G)(\xi, Y^*) + V_{j+1}$ exceeds that of $\text{Ad}^*(G)(\xi, Y^*) + V_j$. Hence, the cardinality of the jump set is equal to orbit dimension.

In addition, one can number the nonempty layers as $\Omega_1, \dots, \Omega_p$ in such a way that property (i) is guaranteed (see [5, Main Theorem (c)]). In each layer, the dimensions in the coadjoint orbits is constant (see [5, Main Theorem (b)]). The property that the orbit dimension is nondecreasing follows since the layering is a refinement of the layering described in [23], as pointed out on [5, p. 311]. One can check that the coarser layering has this property by consulting the definition of the total ordering given directly after [23, Corollary 1.1.2].

Furthermore, there exists for each layer Ω_i a cross-section $\Sigma_i \subset \Omega_i$ and a diffeomorphism $\Theta_i : \Sigma_i \times W_i \rightarrow \Omega_i$ (with a suitably chosen vector space W_i) such that for each $(\xi, Y^*) \in \Sigma_i$ we have $\Theta_i((\xi, Y^*), W_i) = \text{Ad}^*(G)(\xi, Y^*)$ [5, Main Theorem (f)]. But this easily implies that $\Omega_i/\text{Ad}^*(G)$ is homeomorphic to Σ_i , in particular, it is Hausdorff. This takes care of (iii).

For the proof of part (iv), since $\text{ad}^*(x, Z)(\xi^*, Y^*) = \text{ad}^*(x, Z)(\xi^*, 0)$ (see (4)), it follows that $L_j(\xi, Y^*) = L_j(\xi, 0)$, for all $1 \leq j \leq n + d$. But this implies $\mathfrak{j}(\xi, Y^*) = \mathfrak{j}(\xi, 0)$, and the same for α , and (iv) follows.

Now (v) is an easy consequence of (iv) and of (6): Assume that $(\xi, Y^*)^T \in \Omega_j$, and $d_j < 2d$. Then $d_j = 2k$ with $k < d$, and thus $\text{codim}(\mathfrak{h}_\xi^\perp) \geq 1$. Let $V \subset \mathfrak{h}^*$ denote a complement of \mathfrak{h}_ξ^\perp . Then the map

$$V \ni Z^* \mapsto (H^T \xi) \times (Z^* + \mathfrak{h}_\xi^\perp)$$

defines a homeomorphism onto the closed subset $\overline{P}^{-1}(\{H^T \xi\})$ of $\Omega_i^*/\text{Ad}^*(G)$. This set contains the coadjoint orbit of $(\xi, Y^*)^T$, and it is connected and noncompact. Hence the connected component in $\Omega_i^*/\text{Ad}^*(G)$ containing that orbit is not compact. ■

The properties of the lemma will allow us to “peel off” all layers in the coadjoint orbit space that consist of orbits of less than maximal dimension.

Theorem 3.4. *Let $C \subset \mathfrak{g}^*/\text{Ad}^*(G)$ denote an open compact set. Then $C \subset \mathcal{O}_{2d}^*$.*

Proof. Let $\Omega_1^*, \dots, \Omega_p^*$ denote the layering from Lemma 3.3. Note that Ω_p^* consists of the zero-dimensional orbits. Suppose that $C \subset \mathfrak{g}^*/\text{Ad}^*(G)$ is compact and open. Then $C_p = C \cap \Omega_p^*/\text{Ad}^*(G)$ is a compact subset of $\Omega_p^*/\text{Ad}^*(G)$, which is Hausdorff by part (iii) of Lemma 3.3. Furthermore, it is open, hence it contains every connected components of $\Omega_p^*/\text{Ad}^*(G)$ it intersects. But these are noncompact, by Lemma 3.3.(v). Thus $C_p \cap \Omega_p^*/\text{Ad}^*(G)$ is empty.

If $d_{p-1} < d$, we can repeat the argument, using that $\Omega_{p-1}^* \subset \bigcup_{j \leq p-1} \Omega_j^*$ is closed, to obtain that $C \cap \Omega_{p-1}^*/\text{Ad}^*(G) = \emptyset$. Thus one concludes iteratively $C \cap \Omega_j^*/\text{Ad}^*(G) = \emptyset$, until $d_j = d$. ■

3.3. Proof of Theorem 1.1.

Suppose that $\eta \in L^2(\mathbb{R}^n)$ is an admissible vector with $F = \Delta_G^{-1/2} V_\eta \eta \in L^1(G)$. Denote by \widehat{F} the group Fourier transform. Then $\text{supp}(\widehat{F}) \subset \widehat{G}$ is a compact open set. Let $\kappa : \mathfrak{g}^*/\text{Ad}^*(G) \rightarrow \widehat{G}$ denote the Kirillov-Bernat correspondence. It is a homeomorphism [19], and thus Theorem 3.4 implies that $\text{supp}(\widehat{F}) = \kappa(C)$ for some compact open subset $C \subset \mathcal{O}_{2d}^*/\text{Ad}^*(G)$. By Lemma 3.2, $\overline{P}(C)$ is a compact open subset of \mathcal{O}_d/H^T .

Now let $\mathcal{H} = V_\eta((\mathbb{R}^n)) \subset L^2(G)$. The measure class on the Borel space \widehat{G} for the quasiregular representation π is supported on $\mathcal{O}_{2d}^*/\text{Ad}^*(G) = \mathcal{O}_d/H^T$, and is the push forward of the Plancherel measure on $\widehat{\mathbb{R}^n}$ (see for example [20, Theorem 7.1]). Hence the restriction of the regular representation of G to \mathcal{H} is quasiequivalent with the regular representation. Since convolution by F is projection of $L^2(G)$ onto \mathcal{H} , then the support of \widehat{F} must be conull. This entails that C is conull, and hence $\overline{P}(C) \subset \mathbb{R}^n$ must be conull as well.

4. The 3×3 case.

In this section we specialize to the case where $n = 3$, with the aim of proving a characterization. The proposition lists each connected abelian $H < GL(3, \mathbb{R})$ for which \mathbb{R}^3/H^T contains a compact open subset. Of particular interest is case (b) for $\alpha\beta \neq 0$, providing an example of a conull compact open subset of the orbit space where Proposition 2.7 is not applicable. Even though this problem seems amenable to direct calculations, the question whether an integrable admissible vector exists for this case is currently open.

Proposition 4.1. *Let \mathfrak{h} be a commutative Lie subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ having dimension $d \geq 2$, and suppose that there is a basis of \mathbb{R}^3 for which one of the following cases holds.*

(a) $\mathfrak{h}^T = (A, B)_{\mathbb{R}}$ where

$$A = \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \\ & & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}.$$

with $\alpha \neq 0$.

(b) $\mathfrak{h}^T = (A, B)_{\mathbb{R}}$ where

$$A = \begin{bmatrix} 1 & & \\ & 0 & \\ & & \alpha \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & \beta \end{bmatrix}.$$

with at least one of α or β positive.

(c) $\mathfrak{h}^T = (A, X, Z)_{\mathbb{R}}$ where

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} 0 & & \\ & 0 & \\ 1 & & 0 \end{bmatrix}.$$

(d) $\mathfrak{h}^T = (A, B, X)_{\mathbb{R}}$ where

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}, \quad \text{and} \quad X = \begin{bmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{bmatrix}.$$

(e) $\mathfrak{h}^T = (A, B, C)_{\mathbb{R}}$ where

$$A = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}.$$

Then \mathcal{O}_d/H^T is compact. Otherwise, \mathcal{O}_d/H^T contains no conull, compact open subset.

In case (b), there exists a topological section for \mathcal{O}_d/H^T iff $\alpha\beta = 0$, whereas for $\alpha\beta \neq 0$ there does not even exist a topological quasi-section. In all remaining cases, \mathcal{O}_2/H^T admits a topological section.

If none of the above cases applies, then π is not integrable. Conversely, π is integrable in all listed cases possessing a topological section.

Note that when $d = 3$, \mathcal{O}_d consists of open orbits, and it is easy to check directly that the groups generated in examples (c), (d), and (e) above generate finitely many open orbits. We elaborate on this in the cases below, but the main focus of the proof will be on the two-dimensional case.

We first decompose \mathbb{R}^3 into generalized eigenspaces for \mathfrak{h} . Let λ be a complex linear form on \mathfrak{h} , and put

$$E_\lambda = \cap_{A \in \mathfrak{h}} \ker_{\mathbb{C}^3}(A - \lambda(A)I)^2$$

where each A is extended to an endomorphism of \mathbb{C}^3 as usual. If $E_\lambda \neq \{0\}$, then we say that λ is root for \mathfrak{h} . Since \mathfrak{h} is commutative, there is a finite set \mathcal{R} of roots such that

$$\mathbb{C}^3 = \bigoplus_{\lambda \in \mathcal{R}} E_\lambda.$$

Accordingly we write $v = \sum_{\lambda \in \mathcal{R}} v^{(\lambda)}$. If $\lambda \in \mathcal{R}$ then $\bar{\lambda} \in \mathcal{R}$ also, and for $v \in \mathbb{R}^3$, $v^{(\bar{\lambda})} = \overline{v^{(\lambda)}}$. We then form a set $\{\lambda_1, \dots, \lambda_p\}$ by choosing exactly one member of each conjugate pair in \mathcal{R} . If λ is real, put $V_j = E_{\lambda_j} \cap \mathbb{R}^3$, and otherwise put $V_j = E_{\lambda_j}$. Then the map $\iota : \mathbb{R}^3 \rightarrow V_1 \oplus \dots \oplus V_p$ defined by $\iota(v) = \sum_{j=1}^p v^{(\lambda_j)}$ is an \mathbb{R} -linear isomorphism.

The preceding generalized eigenspace decomposition can be carried out for any commutative Lie algebra \mathfrak{h} in $\mathfrak{gl}(n, \mathbb{R})$. We now make some observations particular to the situation at hand, where $n = 3$. First, note that if λ_j is not real, then $V_j = \mathbb{C} \simeq \mathbb{R}^2$ and $p = 2$. Second, observe that if there is a non-zero nilpotent element in \mathfrak{h} , then $p \leq 2$ and all roots are real. Let $\mathfrak{n} = \bigcap_{j=1}^p \ker \lambda_j$ denote the subspace of nilpotent elements of \mathfrak{h} . In light of these observations we itemize several cases.

Case 0: $p = 1$ and $\lambda_1 = 0$. Here $\mathfrak{h} = \mathfrak{n}$ is a Lie subalgebra of the Heisenberg Lie algebra, and our assumption that \mathfrak{h} is commutative then requires that $d = 2$.

Case 1: $p = 1$ and $\lambda_1 \neq 0$. Here the dimension of \mathfrak{n} is one or two, and λ_1 is real. Suppose that $\dim \mathfrak{n} = 2$; then for any basis of \mathbb{R}^3 there must be $X \in \mathfrak{n}$ for which $x_{21} \neq 0$, and then the assumption that \mathfrak{h} is commutative implies that for all $Y \in \mathfrak{n}$, $y_{32} = 0$. This is example (d). It is easily seen that in this case \mathcal{O}_3 is the union of two open orbits.

Case 2: $p = 2$ and λ_1 and λ_2 are linearly dependent over \mathbb{R} . Here one of the generalized eigenspaces has dimension one, and it follows that the dimension of \mathfrak{n} is one, so $d = 2$.

Case 3: $p = 2$ and λ_1 and λ_2 are linearly independent over \mathbb{R} . In this case, if $\mathfrak{n} \neq \{0\}$, then both roots are real, and we are necessarily in example (d) above, whence \mathcal{O}_3 is the union of four open orbits. Otherwise there can be a complex root.

Case 4: $p = 3$. Here we must have $\mathfrak{n} = \{0\}$, and the dimension of the real span of the roots is d . If $d = 3$ then we clearly obtain example (e), where \mathcal{O}_3 is a union of eight open orbits. Otherwise, with λ_1 and λ_2 linearly independent, the question of compactness of \mathcal{O}_2/H^T will depend upon how λ_3 is written as a linear combination of λ_1 and λ_2 .

The following lemma allows us to assume that all roots are non-vanishing, and in particular, to eliminate Case 0.

Lemma 4.2. *Suppose that $\lambda = 0$ for some $\lambda \in \mathcal{R}$. Then \mathbb{R}^3/H^T contains no non-empty compact open subset.*

Proof. We have a generalized eigenspace V_0 in which the action of H^T is unipotent. Choose a basis e_1, \dots, e_{n_0} for V_0 for which every element of H^T has a lower triangular matrix, and let U be the span of e_2, \dots, e_{n_1} together with any other generalized eigenspaces (if $p = 1$ and $n_1 = 1$ then $U = \{0\}$). Then U is H^T -invariant. Let $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^3/U = \mathbb{R}e_1$ be the canonical map. Then π is continuous, open, and H^T -invariant, and so gives a well-defined, continuous open map $\bar{\pi} : \mathbb{R}^3/H^T \rightarrow \mathbb{R}e_1$. If C is a compact open subset of \mathbb{R}^3/H^T , then $\bar{\pi}(C)$ is compact and open in \mathfrak{R}_1 , hence empty. Thus C is empty. ■

Next we prove two lemmas for Case 1. When an ordered basis $\{e_1, e_2, e_3\}$ for \mathbb{R}^3 is chosen, let $p_i : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the linear (coordinate) function dual to e_i .

Lemma 4.3. *Suppose that $d = 2$ and we are in Case 1, and write $\mathfrak{h}^T = (A, X)_{\mathbb{R}}$ where $\lambda_1(A) = 1$ and $\lambda_1(X) = 0$. Choose a basis for \mathbb{R}^3 for which \mathfrak{h}^T is lower triangular, and write*

$$X = \begin{bmatrix} 0 & & \\ x_{21} & 0 & \\ x_{31} & x_{32} & 0 \end{bmatrix}$$

Put

$$\Omega_2 = \{v : p_2(Xv) \neq 0\}$$

and

$$\Omega_3 = \{v : p_2(Xv) = 0, p_3(Xv) \neq 0\}.$$

Then the sets Ω_2 and Ω_3 are H^T -invariant and $\mathcal{O}_2 = \Omega_2 \cup \Omega_3$. When Ω_b is nonempty, then

$$\Sigma_b = \{v \in \Omega_b : p_b(v) = 0 \text{ and } p_b(Xv) = \pm 1\}$$

is a topological section for the H^T -orbits in Ω_b .

Proof. We have $\Omega_2 \neq \emptyset$ if and only if $x_{21} \neq 0$. In this case $p_2(Xv) = x_{21}v_1$, and so $\Omega_2 = \{v : v_1 \neq 0\}$. Thus Ω_2 is H -invariant with

$$p_2(Xe^{sA}v) = e^{s\lambda_1(A)}p_2(Xv).$$

Now suppose that $v_1 = 0$. If $p_3(Xv) \neq 0$ for some v then $p = 1$ (i.e., there is only one generalized Jordan block) and $p_3(e^{sA}Xv) = e^{s\lambda_1(A)}p_3(Xv)$ for all v .

Observe that if Ω_b is nonempty, then the restriction of the map $\phi : \Omega_b \rightarrow \mathbb{R}^* \times \mathbb{R}$ defined by

$$\phi(v) = (p_b(Xv), v_b)$$

to each H -orbit has rank 2. Thus $\Omega_b \subset \mathcal{O}_2$. If $v \in \mathcal{O}_2$ then the action of $e^{\mathbb{R}X}$ is non-trivial and so $v \in \Omega_b, b = 2$ or 3 .

To see that Σ_b is a section, let $v \in \Omega_b$. Note that by definition of b , $p_b(X^2v) = 0$, so for any $t \in \mathbb{R}$,

$$p_b(e^{tX}v) = p_b(v) + tp_b(Xv).$$

Put $s = -\ln |p_b(Xv)|$, and

$$t = -\frac{p_b(v)}{p_b(Xv)}.$$

Put $v^* = e^{tX+sA}v$. Then $p_b(Xv^*) = p_b(Xe^{sA}v) = e^s p_b(Xv) = \pm 1$, and

$$p_b(v^*) = p_b(v) + t p_b(Xv) = 0.$$

Thus v^* belongs to Σ_b . On the other hand if $v^* \in \Sigma_b$ and $v = e^{tX+sA}v \in \Sigma_b$ also, then $p_b(Xv) = p_b(Xe^{sA}v^*) = e^s p_b(Xv^*)$ implies $s = 0$ and

$$0 = p_b(v) = p_b(e^{tX}v) = t p_b(Xv)$$

shows that $t = 0$. Thus Σ_b meets each orbit at exactly one point. Furthermore, the explicit computations determining s, t and finally v^* from v show that the map $v \mapsto v^*$ is continuous. This implies that the section is topological. ■

Lemma 4.4. *Suppose that $d = 2$ and we are in Case 1 and that \mathcal{O}_2 consists of only one layer: $\mathcal{O}_2 = \Omega_a = \Omega$. Then \mathcal{O}_2/H^T does not contain a non-empty compact open subset.*

Proof. Let Σ be the topological section for \mathcal{O}_2 constructed above, let $\sigma : \mathcal{O}_2/H^T \rightarrow \Sigma$ be the natural homeomorphism, and let C be a compact open subset of \mathcal{O}_2/H^T . Then $\sigma(C)$ is a compact and open subset of Σ . Since Σ carries the relative topology from \mathbb{R}^3 , then $\sigma(C)$ is closed and bounded. Now if $\sigma(C)$ were nonempty, then it would contain a component of Σ . But the components of Σ are unbounded, so this is impossible, and so $\sigma(C)$, and hence C , are empty. ■

We now prove the proposition.

Proof of Proposition 4.1. We proceed on a case by case basis, with the assumption that $d = 2$.

Case 0. We have already established in Lemma 4.2 that \mathcal{O}_2/H^T does not contain an open compact subset, and since \mathfrak{h} is conjugate to a Lie subalgebra of the Heisenberg Lie algebra, Theorem 1.1 applies to show that π is not integrable.

Case 1. First recall that here \mathfrak{n} is nontrivial, and all roots are real. Hence Theorem 1.1 applies once again, and yields non-integrability whenever \mathcal{O}_2/H^T does not contain nontrivial compact open subsets. Here λ is real and $\mathfrak{h} = (A, X)_{\mathbb{R}}$ where $\lambda(A) = 1$ and $\lambda(X) = 0$. Choose a basis for \mathbb{R}^3 for which A has Jordan form, and for which X is lower triangular. Put $Y = A - \lambda(A)I$. We distinguish several subcases.

(a) Suppose that $Y = 0$. If \mathcal{O}_2 consists of only one layer, then confer the preceding lemma. Suppose then that \mathcal{O}_2 is the union of the non-empty sets Ω_2 and Ω_3 . Then $\Omega_2 \neq \emptyset$ implies $x_{21} \neq 0$, while $\Omega_3 \neq \emptyset$ implies that for some v , $v_1 = 0$ and $p_3(Xv) \neq 0$, hence $x_{32} \neq 0$.

Let U be an open H^T -invariant subset of \mathcal{O}_2 that is conull for Lebesgue measure. Then $U \cap \Omega_2$ is conull and H^T -invariant, hence $U \cap \Sigma_2$ is conull for the Plancherel measure class, where $\Sigma_2 = \{[\pm 1, 0, v_3] : v_3 \in \mathbb{R}\}$. But in this case the

Plancherel measure class is that of Lebesgue measure on each component of Σ_2 , so in particular, the set

$$E = \{z \in \mathbb{R} : [1, 0, z]^t \in U\}$$

is conull for Lebesgue measure. Now choose a sequence $z(k) \in E$ such that $z(k) \rightarrow \infty$ and put $v(k) = [1, 0, \text{sign}(x_{21}x_{32})z(k)]^t, k = 1, 2, 3, \dots$. We claim that the sequence $H^T v(k)$ has no convergent subsequence in \mathcal{O}_2/H^T .

Suppose that the claim is false. Let $v(j) = [1, 0, \text{sign}(x_{21}x_{32})z(j)]^t$ be a subsequence of $v(k)$ for which $H^T v(j)$ converges to $H^T v$, for some $v \in \mathcal{O}_2$. Since $v(k)$ is not convergent in Σ_2 , then $v \notin \Sigma_2$, and $H^T v \in \Omega_3/H^T$. Hence we may assume that $v = [0, \pm 1, 0]^t \in \Sigma_3$. It follows from the definition of the quotient topology that we have $s_j, t_j \in \mathbb{R}$ such that $e^{s_j A + t_j X} v(j) \rightarrow v$; that is,

$$e^{s_j} [\pm 1, x_{21}t_j, \text{sign}(x_{21}x_{32})z(j) + x_{31}t_j + x_{21}x_{32}t_j^2/2] \rightarrow [0, \pm 1, 0]^t.$$

Thus $e^{s_j} \rightarrow 0$ while $e^{s_j} x_{21}t_j \rightarrow \pm 1$, and

$$e^{s_j} (\text{sign}(x_{21}x_{32})z(j) + x_{31}t_j + x_{21}x_{32}t_j^2/2) \rightarrow 0. \tag{8}$$

Now $e^{s_j} \rightarrow 0$ and $e^{s_j} x_{21}t_j \rightarrow \pm 1$ implies that $|t_j| \rightarrow +\infty$, and $e^{s_j} t_j^2 \rightarrow +\infty$. But then

$$\text{sign}(x_{21}x_{32}) e^{s_j} (\text{sign}(x_{21}x_{32})z(j) + x_{31}t_j + x_{21}x_{32}t_j^2/2) \rightarrow +\infty$$

contradicting (8). Thus the claim is proved, and U is not compact.

(b) Suppose that $Y \neq 0, Y^2 = 0$, that is, that either

$$A = \begin{bmatrix} \parallel & & \\ 1 & \lambda & \\ & & \lambda \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \lambda & & \\ & \lambda & \\ & 1 & \lambda \end{bmatrix}.$$

Then X has the form

$$X = \begin{bmatrix} 0 & & \\ x_{21} & 0 & \\ x_{31} & & 0 \end{bmatrix} \quad \text{or} \quad X = \begin{bmatrix} 0 & & \\ & 0 & \\ x_{31} & x_{32} & 0 \end{bmatrix}$$

respectively. If $x_{21} \neq 0$ (in the first case) then $\mathcal{O}_2 = \Omega_2$; otherwise, $\mathcal{O}_2 = \Omega_3$. In any case Lemma 4.4 applies.

(c) Suppose that $Y^2 \neq 0$, that is, that

$$A = \begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix}.$$

Then X has the form

$$X = \begin{bmatrix} 0 & & \\ x_{21} & 0 & \\ x_{31} & x_{32} & 0 \end{bmatrix}$$

where $b = x_{21} = x_{32}$. If $b = 0$ then $\mathcal{O}_2 = \Omega_3$ and Lemma 4.4 applies. Suppose then that $b \neq 0$. We have $\mathcal{O}_2 = \Omega_2 \cup \Omega_3$ and both are non-empty.

We assume $b = 1$, and write $c = x_{31}$. We compute for any $v \in \mathbb{R}^3$,

$$e^{sA+tX}v = e^{\lambda s} [v_1, v_2 + (s+t)v_1, v_3 + (s+t)v_2 + (ct + (1/2)(s+t)^2)v_1].$$

The argument now proceeds in a manner similar to that of Case 1(a). Let U be an open H -invariant subset of \mathcal{O}_2 that is conull for Lebesgue measure. As before we find that the set

$$E = \{z \in \mathbb{R} : [1, 0, z]^t \in U\}$$

is conull for Lebesgue measure. Now choose a sequence $z(k) \in E$ such that $z(k) \rightarrow \infty$ and put $v(k) = [1, 0, z(k)]^t, k = 1, 2, 3, \dots$. We claim that the sequence $H^T v(k)$ has no convergent subsequence in \mathcal{O}_2/H^T . Supposing the claim is false, then as before this leads us to the setup:

$$e^{s_j A + t_j X} v(j) \longrightarrow v$$

where $v(j)$ is a subsequence of $v(k)$ and where $v = [0, \pm 1, 0]^t \in \Sigma_3$. More explicitly this means that

$$e^{\lambda s_j} [1, s_j + t_j, z(j) + ct_j + (1/2)(s_j + t_j)^2] \longrightarrow [0, \pm 1, 0].$$

Thus $e^{\lambda s_j} \rightarrow 0$, and $e^{\lambda s_j}(s_j + t_j) \rightarrow \pm 1$, and

$$e^{\lambda s_j}(z(j) + ct_j + (1/2)(s_j + t_j)^2) \longrightarrow 0. \quad (9)$$

The first two relations imply that $e^{\lambda s_j}(s_j + t_j)^2 \rightarrow +\infty$, and moreover (since $e^{\lambda s_j} s_j \rightarrow 0$) that $e^{\lambda s_j} t_j \rightarrow \pm 1$. But $e^{\lambda s_j} z(j) > 0$ and so $e^{\lambda s_j}(s_j + t_j)^2 \rightarrow +\infty$ and $e^{\lambda s_j} t_j \rightarrow \pm 1$ implies that

$$e^{\lambda s_j}(z(j) + ct_j + (1/2)(s_j + t_j)^2) \longrightarrow +\infty$$

contradicting (9). Therefore the set U is not compact.

Case 2: Here we have two dependent roots λ_1 and λ_2 , both real. One of V_1 or V_2 has dimension two; assume it is V_1 . Then we have a basis for \mathbb{R}^3 and $X \in \mathfrak{h}$ with matrix form

$$X = \begin{bmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{bmatrix}.$$

By Lemma 4.2 we may assume that both λ_1 or λ_2 are non-vanishing. Hence we can choose a basis for \mathbb{R}^3 such that $\mathfrak{h} = (A, X)_{\mathbb{R}}$ where X is as above, and where

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \alpha \end{bmatrix}$$

for some $\alpha \neq 0$. Here it is easily seen that $\mathcal{O}_2 = \Omega_2 = \{v : v_1 \neq 0\}$, for which we have a global, connected and noncompact topological section $\Sigma_2 = \{v : v_1 = \pm 1, v_2 = 0, v_3 \in \mathbb{R}\}$. The Plancherel measure, when transferred to the section

Σ_2 , is again seen to be equivalent to Lebesgue measure with respect to the third coordinate. Hence, the same reasoning as in the previous cases shows that \mathcal{O}_2/H^T cannot contain a conull compact open subset.

Again, non-integrability follows from Theorem 1.1.

Case 3: $p = 2$, λ_1 and λ_2 are independent over \mathbb{R} . Write $\mathfrak{h} = (A, B)_{\mathbb{R}}$ where $\lambda_2(B) = 1$, $\lambda_2(A) = \lambda_1(B) = 0$, and where $\lambda_1(A) = c \in \mathbb{C}$.

(a) Suppose that $\Re c \neq 0$. We have $\iota : \mathbb{R}^3 \rightarrow \mathbb{C} \times \mathbb{R}$ defined by $\iota([v_1, v_2, v_3]^t = [v_1 + iv_2, v_3]^t$, and via this realization we can write

$$A = \begin{bmatrix} c & \\ & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$$

It is clear that $\mathcal{O}_2 = \{v : v_1 + iv_2 \neq 0, v_3 \neq 0\}$ and that a topological section for \mathcal{O}_2 is $\Sigma = \{v : v_1^2 + v_2^2 = 1, |v_3| = 1\}$. This section is compact so \mathcal{O}_2/H is compact.

(b) Suppose that $c = i$. Let $\pi : \mathcal{O}_2 \rightarrow \mathbb{C}^*$ be the map $\pi(v) = v_1 + iv_2$. Then π is continuous and open and $\pi(e^{sA+tB}v) = e^{is}(v_1 + iv_2)$; hence we have the continuous open mapping $\bar{\pi} : \mathcal{O}_2/H \rightarrow \mathbb{C}^*/H = (0, +\infty)$. If K is a compact open subset of \mathcal{O}_2/H then $\bar{\pi}(K)$ is a compact open subset of $(0, +\infty)$, hence empty, so K is empty.

For non-integrability of π , we observe that the block-diagonal structure of H allows to write

$$\mathbb{R}^3 \rtimes H \cong (\mathbb{R}^2 \rtimes \text{SO}(2)) \times (\mathbb{R} \rtimes \mathbb{R}')$$

It follows that the dual space of $G = \mathbb{R}^3 \rtimes H$ is the cartesian product of the unitary duals of the factors. In particular, projecting a nonempty open compact subset of \widehat{G} onto the first component would yield a nonempty open compact subset in the dual of $\mathbb{R}^2 \rtimes \text{SO}(2)$, contradicting Theorem 1.1 of [14]. Hence there are no integrable projections in $L^1(G)$.

Case 4: Here each λ_j is real and the action of H^T is semi-simple. Two of the roots are independent, and we can write $\mathfrak{h} = (A, B)_{\mathbb{R}}$ where

$$A = \begin{bmatrix} 1 & & \\ & 0 & \\ & & \alpha \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & \beta \end{bmatrix}.$$

By Lemma 4.2 we may assume that one of α or β are nonzero. We can write $\mathcal{O}_2 = \Omega_{12} \cup \Omega_{13} \cup \Omega_{23}$ as a disjoint union, where

$$\begin{aligned} \Omega_{12} &= \{v \in \mathcal{O}_2 : v_1 v_2 \neq 0\}, \\ \Omega_{13} &= \{v \in \mathcal{O}_2 : v_2 = 0, v_1 v_3 \neq 0\}, \\ \Omega_{23} &= \{v \in \mathcal{O}_2 : v_1 = 0, v_2 v_3 \neq 0\}. \end{aligned}$$

Note that if $\alpha = 0$, then $\Omega_{23} = \emptyset$ and if $\beta = 0$ then $\Omega_{13} = \emptyset$. Topological sections in each layer respectively are

$$\Sigma_{12} = \{[\pm 1, \pm 1, v_3] : v_3 \in \mathbb{R}\}, \Sigma_{13} = \{[\pm 1, 0 \pm 1]\}, \Sigma_{23} = \{[0, \pm 1, \pm 1]\}.$$

We insert a lemma for this particular case.

Lemma 4.5. *Suppose that \mathfrak{h} has the form above. Then the following are equivalent.*

- (i) \mathcal{O}_2/H^T is compact.
- (ii) \mathcal{O}_2/H^T contains a compact open conull subset.
- (iii) At least one of α or β is positive.

Proof. (i) implies (ii) is trivial. To prove (ii) implies (iii), suppose that \mathcal{U} is an H^T -invariant open conull subset of \mathcal{O}_2 . Then $\mathcal{U} \cap \Omega_{12} \neq \emptyset$, and it follows that there is an open conull subset Z of \mathbb{R} such that $\{\pm 1\} \times \{\pm 1\} \times Z \subset \Sigma_{12} \cap \mathcal{U}$. Suppose that neither α nor β are positive, and hence one is negative. Assume that $\beta < 0$. Put

$$v(k) = [1, 1, z(k)]$$

where $z(k) \in Z$ and $z(k) \rightarrow \infty$. We claim that no subsequence of $(H^T v(k))$ is convergent. Suppose the claim is false; let $s_j, t_j \in \mathbb{R}$ and $v(k_j) = v(j)$ be a subsequence that converges to a point v in \mathcal{O}_2 . Since $z(j) \rightarrow \infty$ and Σ_{12} is a topological section for Ω_{12} , then $v \notin \Omega_{12}$. Hence we may assume that $v \in \Sigma_{13} \cup \Sigma_{23}$. Now we have

$$[e^{s_j}, e^{t_j}, e^{\alpha s_j + \beta t_j} z(j)] \rightarrow v.$$

with $v \in \Sigma_{13} \cup \Sigma_{23}$. If $v \in \Sigma_{13}$, then $v = [1, 0, \pm 1]^t$. Hence $e^{s_j} \rightarrow 1$ and $e^{t_j} \rightarrow 0$, and so $e^{\alpha s_j + \beta t_j} \rightarrow +\infty$ hence $e^{\alpha s_j + \beta t_j} z(j) \rightarrow +\infty$, contradicting the convergence above. If $v \in \Sigma_{23}$, then $v = [0, 1, \pm 1]^t$. Hence $e^{s_j} \rightarrow 0$ and $e^{t_j} \rightarrow 1$. Now if $\alpha < 0$ then $e^{\alpha s_j + \beta t_j} \rightarrow +\infty$ while if $\alpha = 0$, then $e^{\alpha s_j + \beta t_j} \rightarrow 1$. In either case $e^{\alpha s_j + \beta t_j} z(j) \rightarrow +\infty$, contradicting the convergence above. If $\beta = 0$ then $\alpha < 0$ and the proof is similar. Thus (ii) implies (iii).

Now suppose (iii). Let $v(k)$ be any sequence in \mathcal{O}_2 . In order to show convergence of a subsequence of orbits, we may first assume that each $v(k)$ belongs to one of the three sections above, and then by passing to a subsequence, that every $v(k)$ belongs to one of these sections. Now if $v(k) \in \Sigma_{13}$ for all k , or $v(k) \in \Sigma_{23}$ for all k , then since these are finite sets, it is obvious there is a convergent subsequence. Suppose then that $v(k)$ belongs to Σ_{12} for all k .

Thus $v(k) = [\pm 1, \pm 1, v_3(k)]$ for each k . If the sequence $v_3(k)$ of real numbers is bounded, then again it is clear that we have a convergent subsequence, so, passing to a subsequence, we may assume that $v(k) \rightarrow +\infty$ or $v(k) \rightarrow -\infty$. Now suppose that $\beta > 0$. Then we can choose $t_k \in \mathbb{R}, k = 1, 2, 3, \dots$ so that

$$e^{\beta t_k} v_3(k) \rightarrow \pm 1.$$

Then $e^{t_k} \rightarrow 0$ and so

$$e^{t_k B} v(k) \rightarrow [\pm 1, 0, \pm 1].$$

Similarly, if $\alpha > 0$, then we can choose $s_k \in \mathbb{R}, k = 1, 2, 3, \dots$ so that

$$e^{s_k A} v(k) \rightarrow [0, \pm 1, \pm 1].$$

This completes the proof. ■

Having obtained the preceding lemma, the characterization of the cases admitting open compact conull subsets of the dual orbit space is complete. Furthermore, we already obtained a topological section for the case (a), and the finite open orbit cases (c) through (e) are also taken care of. Hence it remains to take a closer look at the existence of quasi-sections for case (b).

If $\alpha = 0$, one easily verifies that the set $\Sigma = \{v \in \mathbb{R}^3 : v_1 = 1, v_2^2 + v_3^2 = 1\}$ is a topological section; and the case $\beta = 0$ is similar.

Hence it remains to consider the case $\alpha\beta \neq 0$; w.l.o.g. $\alpha > 0$. Our assumptions imply that

$$\mathcal{O}_2 = \{v \in \mathbb{R}^3 : |v_i| \neq 0 \text{ for at least two indices } i\}.$$

Hence, letting for $i = 1, 2, 3$ and $\rho > 1$

$$C_i(\rho) = \{v \in \mathbb{R}^3 : |v_i| \leq \rho, \forall j \neq i : \frac{1}{\rho} \leq |v_j| \leq \rho\},$$

defines a compact set $C_i(\rho) \subset \mathcal{O}_2$. It is therefore sufficient, in view of Lemma 2.5, to prove that $((C_i(\rho), C_k(\rho)))$ is not relatively compact, for some pair (i, k) of indices.

First assume that $\beta > 0$, and consider $((C_1(\rho), C_2(\rho)))$. We then need to determine the set of pairs $(s, t) \in \mathbb{R}^2$ such that, with suitable $v \in \mathbb{R}^3$, the following inequalities hold:

$$\begin{aligned} |v_1| \leq \rho, & \quad \frac{1}{\rho} \leq |v_2| \leq \rho, & \quad \frac{1}{\rho} \leq |v_3| \leq \rho \\ \frac{1}{\rho} \leq e^s |v_1| \leq \rho, & \quad e^t |v_2| \leq \rho, & \quad \frac{1}{\rho} \leq e^{\alpha s + \beta t} |v_3| \leq \rho. \end{aligned}$$

These inequalities are solvable with suitable v iff

$$-2 \log(\rho) \leq s, \quad t \leq 2 \log(\rho), \quad -2 \log(\rho) \leq \alpha s + \beta t \leq 2 \log(\rho).$$

Given any $s \geq 0$, we may pick $t = -\alpha s / \beta < 0 \leq 2 \log(\rho)$ so that $\alpha s + \beta t = 0$. Hence all three inequalities are satisfied, which means that there exists $v \in C_1(\rho)$ such that $\exp(sA + tB)v \in C_2(\rho)$, i.e. $\exp(sA + tB) \in ((C_1(\rho), C_2(\rho)))$. In particular, $((C_1(\rho), C_2(\rho)))$ is not relatively compact.

For the case $\beta < 0$, we first note that a similar reasoning as above leads to the following necessary and sufficient conditions for $\exp(sA + tB)$ to be in $((C_2(\rho), C_3(\rho)))$:

$$-2 \log(\rho) \leq s \leq 2 \log(\rho), \quad -2 \log(\rho) \leq t, \quad \alpha s + \beta t \leq 2 \log(\rho).$$

This can be fulfilled by picking $s = 0$ and $t \geq 0$ arbitrary, showing that $((C_2(\rho), C_3(\rho)))$ is not relatively compact.

Again, Theorem 1.1 applies to yield non-integrability whenever \mathcal{O}_2/H^T contains no nonempty open compact set. ■

5. Combining diagonal and nilpotent generators

In this section we consider a particular class of abelian dilation groups. Recall that a one-parameter group infinitesimally generated by a diagonal matrix with strictly positive entries allows integrable admissible vectors. In this section we investigate what happens when we include a second infinitesimal generator, assumed to be nilpotent. The following result shows that this destroys integrability when $n > 2$.

Proposition 5.1. *Let $H = \exp \mathfrak{h}$, where \mathfrak{h} is the real span of two non-zero commuting endomorphisms of \mathbb{R}^n , one of which is nilpotent, and the other diagonalizable with real eigenvalues. Suppose that $n \geq 3$, and let U be an H -invariant, open, conull subset of \mathcal{O}_2 . Then U/H^T is not compact. It follows that there are no integrable admissible vectors for H .*

Observe that if $n = 2$ and H is as in the proposition, then \mathfrak{h}^T is the real span of X and A where

$$X = \begin{bmatrix} 0 & \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

In this case $\mathcal{O}_2 = \mathbb{R}^2 \setminus \{v : v_1 = 0\}$ is the union of two open orbits, and \mathcal{O}_2/H is compact.

Recall that H acts on $\widehat{\mathbb{R}}^n$ by $h \cdot \xi = (h^T)^{-1}\xi$, when $\widehat{\mathbb{R}}^n$ is identified with \mathbb{R}^n . We make this identification and describe \mathfrak{h} in terms of the transposes of its elements. Fix $A \in \mathfrak{h}$ diagonalizable with real eigenvalues, and write $\mathbb{R}^n = \bigoplus_{\lambda \in \mathcal{E}} W_\lambda$ where \mathcal{E} is the set of distinct eigenvalues of A , and W_λ is the eigenspace for the eigenvalue λ .

Given an eigenspace W , put $\tilde{W} = \bigoplus \{W_\lambda : W_\lambda \neq W\}$, and let v^W be the projection of $v \in \mathbb{R}^n$. Fix a non-zero nilpotent element $X \in \mathfrak{h}^T$. Each W_λ is X -invariant, and we choose a basis $\{e_1^{(\lambda)}, \dots, e_{n_\lambda}^{(\lambda)}\}$ for W_λ consisting of λ -eigenvectors of A , and for which the restriction of X to W_λ is of the form

$$X|_{W_\lambda} = \begin{bmatrix} 0 & & \cdots & 0 \\ \epsilon_2 & 0 & & \cdots & 0 \\ & \epsilon_3 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \epsilon_{n_\lambda} & 0 \end{bmatrix}$$

and $\epsilon_i = 0$ or 1 . For each $\lambda \in \mathcal{E}$ and $1 \leq i \leq n_\lambda$ let $p_i^{(\lambda)} : \mathbb{R}^n \rightarrow \mathbb{R}$ denote the coordinate function dual to the basis element $e_i^{(\lambda)}$. When $W = W_\lambda$ is fixed we denote $p_i^{(\lambda)}$ by p_i . The following shows that we can assume that all eigenvalues are non-zero.

Lemma 5.2. *Suppose that A has 0 as an eigenvalue. Then \mathbb{R}^n/H^T contains no nonempty compact open subset.*

Proof. Let $T = W_0$ and T_1 the span of $e_2^{(0)}, \dots, e_{n_0}^{(0)}$ in T . Let $\pi : \mathbb{R}^n \rightarrow T/T_1$ be the canonical map defined by $\pi(v) = v^{(\lambda)} + T_1$. Then π is continuous and

open. Now T_1 is H -invariant, and the natural action of H on T/T_1 defined by $\exp(sA+tX)\pi(v) = \pi(\exp(sA+tX)v)$ is trivial. Thus $\bar{\pi}(Hv) = H\pi(v) = \pi(v)$ is a well-defined map $\bar{\pi} : \mathbb{R}^n/H \rightarrow T/T_1$ that is continuous and open. Let $K \subset \mathbb{R}^n/H^T$ be compact and open. Then $\bar{\pi}(K)$ is a compact and open subset of $T/T_1 \simeq \mathbb{R}$ and hence empty. Therefore K is empty. ■

For the remainder of this section we assume that $0 \notin \mathcal{E}$. For each eigenspace W let $\mathcal{O}_2(W) = \{v \in \mathbb{R}^n : \dim Hv^W = 2\}$. We have the following.

Corollary 5.3. *One has*

$$\mathcal{O}_2 = \cup_{\lambda \in \mathcal{E}} \mathcal{O}_2(W_\lambda).$$

Proof. Let $v \in \mathcal{O}_2$. Then necessarily $Xv \neq 0$. Hence for some eigenspace $W = W_\lambda$, $W \not\subset \ker X$. Let $b = \min\{2 \leq i \leq n_\lambda : p_i(Xv) \neq 0\}$. Then

$$p_{b-1}(e^{sA+tX}v) = e^{\lambda s} p_{b-1}(v) \tag{10}$$

and

$$p_b(e^{sA+tX}v) = e^{\lambda s}(p_b(v) + tp_b(Xv)). \tag{11}$$

Since $\lambda \neq 0$ then $\dim Hv = 2$ and $v \in \mathcal{O}_2(W)$. ■

Fix an eigenspace $W = W_\lambda$ for which $W \not\subset \ker X$. Recall that we have chosen a basis for W for which $X|_W$ has the form

$$X|_W = \begin{bmatrix} 0 & & \cdots & 0 \\ \epsilon_2 & 0 & & \cdots & 0 \\ & \epsilon_3 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & \epsilon_{n_\lambda} & 0 \end{bmatrix}$$

and not all ϵ_i are zero. Note that $p_1(Xv) = 0$ and

$$p_i(Xv) = \epsilon_i p_{i-1}(v), 2 \leq i \leq n_\lambda. \tag{12}$$

A stratification of orbits in $\mathcal{O}_2(W)$ can be defined as follows. For each $v \in \mathcal{O}_2(W)$ put

$$b(v^W) = \min\{2 \leq i \leq n_\lambda : p_i(Xv) \neq 0\}.$$

For $2 \leq b \leq n_\lambda$ put $\Omega_b = \{v \in \mathbb{R}^n : b(v^W) = b\}$, and let $B = \{2 \leq i \leq n_\lambda : \epsilon_i \neq 0\}$. Then for each $1 \leq b \leq n_\lambda$, the relation (12) shows that Ω_b is non-empty if and only if $b \in B$, and the relations (10) and (11) show that Ω_b is included in $\mathcal{O}_2(W)$. Thus we have

$$\mathcal{O}_2(W) = \cup_{b \in B} \Omega_b.$$

If $v \in \Omega_b$, then $p_b(X^p v) \neq 0$ for all $p \geq 2$, and hence for any $t \in \mathbb{R}$,

$$p_i(Xe^{tX}v) = 0 \text{ for all } i < b, \text{ and } p_b(Xe^{tX}v) = p_b(Xv) \neq 0.$$

Thus Ω_b is $e^{\mathbb{R}X}$ -invariant. A section for the $e^{\mathbb{R}X}$ -orbits in Ω_b is given as follows.

Lemma 5.4. *The set*

$$L_b = \Omega_b \cap \{v \in \mathbb{R}^n : p_b(v) = 0\}$$

is a topological section for the $e^{\mathbb{R}X}$ orbits in Ω_b .

Proof. We have

$$p_b(e^{tX}v) = p_b(v) + tp_b(Xv) = p_b(v) + tp_{b-1}(v)$$

holds for all $v \in \Omega_b$. Let $v \in \Omega_b$, and define

$$t = -\frac{p_b(v)}{p_b(Xv)}.$$

Then $e^{tX}v \in L_b$. If $v \in L_b$ and also $e^{tX}v \in L_b$, then

$$0 = p_b(e^{tX}v) = p_b(v) + tp_b(Xv) = t p_b(Xv)$$

implies $t = 0$ and $e^{tX}v = v$. ■

Of particular interest is the “minimal layer” Ω_a , where $a = \min B$. Note that $\Omega_a = \{v \in \mathbb{R}^n : p_a(Xv) \neq 0\}$ is open and conull in \mathbb{R}^n .

Lemma 5.5. *Let $b \in B$. Then Ω_b is H -invariant, and the set*

$$\Sigma_b = \{v \in L_b : p_b(Xv) = \pm 1\}$$

is a topological section for the H -orbits in Ω_b .

Proof. Let $v \in \Omega_b$. For $b' \in B$, if $b' < b$ then we have $p_{b'}(Xe^{sA}v) = e^{\lambda s} p_{b'}(Xv) = 0$, while $p_b(Xe^{sA}v) = e^{\lambda s} p_b(Xv) \neq 0$. Hence $e^{sA}v \in \Omega_b$ and Ω_b is H -invariant. To see that Σ_b is a section, choose $t \in \mathbb{R}$ as above so that $e^{tX}v \in L_b$, and put $s = -\ln |p_b(Xe^{tX}v)|$. Then $e^{sA+tX}v \in L_b$ and

$$|p_b(Xe^{sA+tX}v)| = |e^{\lambda s} p_b(Xe^{tX}v)| = e^s |p_b(Xe^{tX}v)| = 1.$$

so $e^{sA+tX}v \in \Sigma_b$ and thus $H\Sigma_b = \Omega_b$. If $v \in \Sigma_b$ and $e^{sA+tX}v \in \Sigma_b$ also, then the preceding calculations show that $t = s = 0$. ■

Note that by Corollary 5.3, each $v \in \mathcal{O}_2$ belongs to some layer $\Omega_b \subset \mathcal{O}_2(W)$. On the other hand it is possible that \mathcal{O}_2 coincides with a minimal layer Ω_a : in this case there is only one eigenspace W that is not included in the kernel of X , and for the matrix of X in W , only one subdiagonal entry ϵ_a is non-zero. Applying Lemma 5.5 we obtain the following.

Corollary 5.6. *Suppose that $n \geq 3$ and $\mathcal{O}_2 = \Omega_a$. Then \mathcal{O}_2/H^T does not contain a nonempty compact open subset.*

Proof. Since Σ_a is a topological section for \mathcal{O}_2/H^T , then its components are homeomorphic with the components of \mathcal{O}_2/H , and any closed and open subset of \mathcal{O}_2/H is homeomorphic with a closed and open subset of Σ_a . Any non-empty closed and open subset of Σ_a must contain a component. Now since Ω_a is the minimal layer then the condition $p_a(Xv) = \pm 1$ implies that $v \in \Omega_a$. Thus by Lemma 5.5 we have

$$\Sigma_a = \{v \in V_a : p_a(Xv) = \pm 1\}$$

and Σ_a is the union of two components. Since $n \geq 3$, then each component is an unbounded affine hyperplane in V_a . Hence any non-empty closed and open subset of \mathcal{O}_2/H^T is not compact. But \mathcal{O}_2/H^T is Hausdorff, so any compact open subset must be a closed and open subset that is bounded, hence must be empty. ■

We are now ready to prove the proposition.

Proof of Proposition 5.1. Let U be an open H -invariant subset of \mathcal{O}_2 that is conull for Lebesgue measure. Let W be an eigenspace that is not included in $\ker X$ and $\Omega = \Omega_a$ be a minimal layer in $\mathcal{O}_2(W)$ constructed above. Then $U \cap \Omega$ is open, conull, and H -invariant. If $\mathcal{O}_2 = \Omega$, then by Corollary 5.6 we are done, so we suppose that \mathcal{O}_2 contains other layers Ω_b . We scale the generator A so that the eigenvalue for the eigenspace W is 1.

Let $\Sigma = \Sigma_a$ be the topological section for Ω , so that $\Omega = \{v \in \mathbb{R}^n : p_a(Xv) \neq 0\}$ and $\Sigma = \{v \in V_a : p_a(Xv) = \pm 1, p_a(v) = 0\}$. Put $\Sigma_+ = \{v \in \Sigma : p_a(Xv) = 1\}$; by the relation (12) we have

$$\Sigma_+ = \{v \in V_a : p_{a-1}(v) = 1, p_a(v) = 0\}.$$

Put $q = \dim \tilde{W}$. We identify Σ_+ with $\mathbb{R}^{a-2} \times \mathbb{R}^{n_1-a} \times \mathbb{R} \times \mathbb{R}^q$, via the coordinate functions $p_1, \dots, p_{a-2}, p_{a+1}, \dots, p_{n_1-1}$ and p_{n_1} for W , and the coordinate functions for all other $W' \neq W$. Denote the corresponding $n - 2$ dimensional Lebesgue measure $m_{a-2} \times m_{n_1-a-1} \times m_1 \times m_q$ on Σ_+ by m . (If $a = 2$, then m_{a-2} is point mass measure.)

Now $U \cap \Omega_+$ is conull in Ω_+ , where $\Omega_+ = H\Sigma_+$, and the measure m is equivalent with the push forward of Lebesgue measure on $\Omega_+ = \{v \in \Omega : p_a(Xv) > 0\}$. Hence $m(\Sigma_+ \setminus U) = 0$. It follows that there is $x \in \mathbb{R}^{a-2}, y \in \mathbb{R}^{n_1-a-1}$, and $w \in \mathbb{R}^q$, such that

$$(\Sigma_+ \cap U)^{(x,y,w)} = \{z \in \mathbb{R} : (x, y, z, w) \in \Sigma_+ \cap U\}$$

is m_1 -conull in \mathbb{R} .

Hence there is a sequence $z(j) \in (\Sigma \cap U)^{(x,y,w)}$ such that $z(j) \rightarrow +\infty$. Define the sequence $v(j)$ in Σ_+ using the coordinates indicated above by

$$v(j) = (x, y, z(j), w), \quad j = 1, 2, 3, \dots$$

Note that all coordinates of $v(j)$ are fixed except $p_{n_1}(v(j)) = z(j)$, and the elements x, y and w determine all other coordinates of $v(j)$ except $p_{a-1}(v(j)) = 1, p_a(v(j)) = 0$. We claim that $Hv(j)$ has no convergent subsequence in \mathcal{O}_2/H .

Suppose the contrary, that for some $v \in \mathcal{O}_2$, and for some subsequence $Hv(k)$ of $Hv(j)$, $Hv(k) \rightarrow Hv$. Since $z(k) \rightarrow +\infty$ then $v(k)$ is not convergent

in Σ and Σ is a topological section in Ω , therefore $Hv \notin \Omega/H^T$. By definition of the quotient topology on \mathcal{O}_2/H^T , we have $s(k), t(k) \in \mathbb{R}, k = 1, 2, 3, \dots$ such that

$$v'(k) := e^{s_k A + t_k X} v(k) \rightarrow v.$$

Now $v \notin \Omega$ means that $p_a(Xv) = 0$ and so

$$p_a(Xv'(k)) \rightarrow 0. \tag{13}$$

Recall that $p_a(Xe^{t_k X} v(k)) = p_a(Xv(k))$ and $v(k) \in \Sigma_a$, hence $|p_a(Xe^{t_k X} v(k))| = 1$. Hence

$$p_a(Xv'(k)) = e^{s_k} p_a(Xe^{t_k X} v(k)) = e^{s_k} p_a(Xv(k)) = \pm e^{s_k}$$

and so (13) implies

$$e^{s_k} \rightarrow 0.$$

For some eigenspace $W = W_\lambda$ and layer $\Omega_b \subset \mathcal{O}_2(W_\lambda)$, we have $v \in \Omega_b$; in fact we may assume that $v \in \Sigma_b$. Hence we have

$$p_b(v) = 0, \quad p_b(Xv) = \pm 1. \tag{14}$$

Suppose that $\lambda < 0$. Then $e^{\lambda s_k} \rightarrow \infty$, but

$$e^{\lambda s_k} p_2(Xv(k)) = p_2(Xv'(k)) \rightarrow p_2(Xv).$$

Hence $p_2(Xv(k)) = p_2(Xv) = 0$. Now if $b = 2$, then this would already be the desired contradiction in light of (14). Otherwise, $p_3((X^p v(k)) = 0$ for all k and $p \geq 1$, so now

$$e^{\lambda s_k} p_3(Xv(k)) = p_3(Xv'(k)) \rightarrow p_3(Xv)$$

and if $b = 3$ we arrive at a contradiction. Continuing in this way we obtain $p_b(Xv) = 0$ while also $p_b(Xv) = \pm 1$, a contradiction.

Now suppose that $\lambda > 0$, so that $e^{\lambda s_k} \rightarrow 0$. Since $p_b(v) = 0$ then

$$p_b(v'(k)) \rightarrow 0 \tag{15}$$

and since $|p_b(Xv)| = 1$, then

$$|p_b(Xv'(k))| \rightarrow 1. \tag{16}$$

We show that the preceding relations are not compatible. First, rewrite (15), (16) as

$$e^{\lambda s_k} p_b(e^{t_k X} v(k)) \rightarrow 0 \tag{17}$$

and

$$e^{\lambda s_k} |p_b(Xe^{t_k X} v(k))| \rightarrow 1. \tag{18}$$

To compute $p_b(e^{t_k X} v(k))$ and $p_b(Xe^{t_k X} v(k))$, note that for each $p \geq 1$, $p_b(X^p v(k))$ is independent of k , and put $c_p = p_b(X^p v(k))$. Then

$$\begin{aligned} p_b(Xe^{t_k X} v(k)) &= p_b(Xv(k) + t_k X^2 v(k) + \frac{1}{2!} t_k^2 X^3 v(k) + \dots) \\ &= c_1 + c_2 t_k + \frac{c_3}{2!} t_k^2 + \dots \end{aligned}$$

and since $e^{s_k} \rightarrow 0$, then the relation (18) shows that

$$\left| c_1 + c_2 t_k + \frac{c_3}{2!} t_k^2 + \cdots \right| \rightarrow \infty.$$

This shows that not all c_p are zero and $|t_k| \rightarrow +\infty$. Let $N = \max\{p : c_p \neq 0\}$; then $N \geq 1$ and we have

$$e^{s_k} \left| c_1 + c_2 t_k + \cdots + \frac{c_N}{(N-1)!} t_k^{N-1} \right| \rightarrow 1$$

so

$$\begin{aligned} e^{s_k} |t_k|^N & \left| \frac{c_1}{t_k^{N-1}} + \frac{c_2}{t_k^{N-2}} + \cdots + \frac{c_N}{(N-1)!} \right| \\ & = |t_k| e^{s_k} \left| c_1 + c_2 t_k + \cdots + \frac{c_N}{(N-1)!} t_k^{N-1} \right| \rightarrow +\infty \end{aligned}$$

hence

$$e^{s_k} |t_k|^N \rightarrow +\infty.$$

Similarly we compute that

$$p_b(e^{t_k X} v(k)) = c_0 + c_1 t_k + \frac{c_2}{2!} t_k^2 + \cdots + \frac{c_N}{N!} t_k^N.$$

But now the relation (17) gives

$$\begin{aligned} e^{s_k} |t_k|^N \left| \frac{c_0}{t_k^N} + \frac{c_1}{t_k^{N-1}} + \cdots + \frac{c_N}{N!} \right| & = e^{s_k} \left| c_0 + c_1 t_k + \cdots + \frac{c_N}{N!} t_k^N \right| \\ & = e^{s_k} |p_b(e^{t_k X} v(k))| \rightarrow 0 \end{aligned}$$

and hence $e^{s_k} |t_k|^N \rightarrow 0$, a contradiction. ■

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