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Visible Actions on Spherical Nilpotent Orbits in Complex Simple Lie Algebras

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Abstract. This paper studies nilpotent orbits in complex simple Lie algebras from the viewpoint of strongly visible actions in the sense of T. Kobayashi. We prove that the action of a maximal compact group consisting of inner automorphisms on a nilpotent orbit is strongly visible if and only if it is spherical, namely, admitting an open orbit of a Borel subgroup. Further, we find a concrete description of a slice in the strongly visible action. As a corollary, we clarify a relationship among different notions of complex nilpotent orbits: actions of Borel subgroups (sphericity); multiplicity-free representations in regular functions; momentum maps; and actions of compact subgroups (strongly visible actions).
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1. Introduction

This paper studies nilpotent orbits, and bridges the two notions, "spherical varieties" studied by D. Panyushev [19, 20] and "visible actions" introduced by T. Kobayashi [10]. We shall prove that

"spherical nilpotent orbits = visible nilpotent orbits",

and give some structural results ("slice" coming from the right-hand side).

T. Kobayashi established a new theory on multiplicity-freeness for unitary representations of Lie groups by introducing the notion of visible actions. We recall briefly from [10, 11, 14] the *propagation theory of multiplicity-freeness property.* Let G be a Lie group, and $\mathcal{V}$ a G-equivariant Hermitian holomorphic vector bundle over a complex manifold D. Then, we have a natural action of G on the space $\mathcal{O}(D,\mathcal{V})$ of holomorphic sections, which is not necessary irreducible when G does not act transitively on D. Suppose we are given a unitary representation $\mathcal{H}$ of G from which there exists a continuous injective homomorphism to $\mathcal{O}(D,\mathcal{V})$. In general, $\mathcal{H}$ may not be multiplicity-free even if each fiber $\mathcal{V}_x$ is multiplicity-free

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as a representation of the isotropy subgroup G_x $(x \in D)$. However, $\mathcal{H}$ becomes multiplicity-free whenever G acts on the base space D strongly visibly and each fiber is multiplicity-free. This is the propagation theory in [14], which yields a unified explanation of multiplicity-freeness for various kinds of multiplicity-free representations which have been studied by different approaches (see [10, 11, 14]).

A holomorphic action of a Lie group G on a connected complex manifold D is called *strongly visible* if there exist a real submanifold S in D and an anti-holomorphic diffeomorphism σ on D such that the following conditions are satisfied:

$$D = G \cdot S, \tag{V.1}$$

$$\sigma|_S = \mathrm{id}_S, \tag{S.1}$$

$$\sigma \text{ preserves each } G\text{-orbit in } D. \tag{S.2}$$

We say that the submanifold S is a *slice*. It is automatically totally real, namely, $J_x(T_xS) \cap T_xS = \{0\}$ for any $x \in S$ (see [11, Remark 3.3.3]). Here, J stands for the complex structure of D. We are particularly interested in a slice of minimal dimension, namely, which coincides with the codimension of generic G-orbits in D.

We remark that the original definition [11, Definition 3.3.1] of strongly visible actions is wider slightly, namely, it allows a complex manifold D containing a non-empty G-invariant open set satisfying (V.1)–(S.2). For the propagation theory of multiplicity-freeness property, this wider definition is sufficient. However, since we shall see that these two definitions are equivalent for G_u-actions on complex nilpotent orbits, we adopt the above definition for simplicity for the rest of this paper.

Strongly visible actions arise from many different geometric settings (cf. [10, 11, 23]). Recently, a classification theory of strongly visible actions has been developed for Hermitian symmetric spaces [13], generalized flag varieties [12, 27, 28, 29, 30], and linear spaces [22, 24].

This paper deals with a new case where a complex manifold is a nilpotent orbit in a complex simple Lie algebra. In order to state our main results, we fix notation. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and $G_{\mathbb{C}} := \mathrm{Int}\,\mathfrak{g}$ the inner automorphism group of $\mathfrak{g}$. We denote by $\mathcal{O}_X$ a $G_{\mathbb{C}}$-orbit through $X \in \mathfrak{g}$. Then we have a $G_{\mathbb{C}}$-isomorphism of complex manifolds $\mathcal{O}_X \simeq G_{\mathbb{C}}/(G_{\mathbb{C}})_X$ where $(G_{\mathbb{C}})_X$ stands for the isotropy subgroup at X. We say that $\mathcal{O}_X$ is a *nilpotent* orbit if $X \in \mathfrak{g}$ is a nilpotent element, and is *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $\mathcal{O}_X$. Let G_u be a compact real form of $G_{\mathbb{C}}$. We prove:

Theorem 1.1. *If $\mathcal{O}_X$ is nilpotent and spherical, then the G_u-action on $\mathcal{O}_X$ is strongly visible.*

The idea of our proof of Theorem 1.1 is based on the *induction theorem of strongly visible actions* which is first formulated by Kobayashi [11, Theorem 20] for Type A group. We generalize this idea for arbitrary complex simple Lie groups. For this, we choose an $\mathfrak{sl}_2$-triple $\{H, X, Y\}$ containing X as a nilpositive element. The semisimple element H defines the $\mathbb{Z}$-grading of $\mathfrak{g}$, denoted by $\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}(m)$.

Then the complex subalgebra $\mathfrak{l} := \mathfrak{g}(0)$ is reductive. Let $L_{\mathbb{C}}$ be an analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{l}$. Taking a conjugation if necessary, we may and do assume that $L_u := L_{\mathbb{C}} \cap G_u$ is a compact real form of $L_{\mathbb{C}}$. We set a nilpotent subalgebra by $\mathfrak{n} := \bigoplus_{m \geq 2} \mathfrak{g}(m)$. Then, the nilpotent orbit $\mathcal{O}_X$ can be realized via the following map:

$$G_u \times_{L_u} \mathfrak{n} \to \overline{\mathcal{O}_X}, \quad (g, Z) \mapsto g \cdot Z, \tag{1.1}$$

in particular, the closure $\overline{\mathcal{O}_X}$ is equal to $G_u \cdot \mathfrak{n} = \{g \cdot Z : g \in G_u, Z \in \mathfrak{n}\}$. In this setting, we generalize the induction theorem of strongly visible actions:

Theorem 1.2 (see Theorem 4.1)**.** *If the L_u-action on $\mathfrak{n}$ is strongly visible, then the G_u-action on $\mathcal{O}_X$ is strongly visible.*

Theorem 1.2 means that the strong visibility for non-linear action on $\mathcal{O}_X$ is induced from the strong visibility for linear action on $\mathfrak{n}$ via (1.1). We then can apply the previous results [22, 24] for the classification of linear visible actions to the L_u-action on $\mathfrak{n}$, and thus prove:

Theorem 1.3. *If $\mathcal{O}_X$ is spherical, then the L_u-action on $\mathfrak{n}$ is strongly visible.*

Therefore, Theorem 1.1 follows from Theorems 1.2 and 1.3.

Our proof of Theorem 1.3 applies a case-by-case analysis by using Panyushev [19]. Moreover, we give an explicit description of a slice and an anti-holomorphic diffeomorphism for the L_u-action on $\mathfrak{n}$ when $\mathcal{O}_X$ is spherical.

Together with the earlier results [19, 32, 31], we summarize:

Corollary 1.4. *The following five conditions on a nilpotent orbit $\mathcal{O}_X$ in a complex simple Lie algebra $\mathfrak{g}$ are equivalent:*

(i) *$\mathcal{O}_X$ is spherical.*

(ii) *The height of $\mathcal{O}_X$ equals two or three.*

(iii) *The space of regular functions on $\mathcal{O}_X$ is multiplicity-free as a representation of $G_{\mathbb{C}} = \operatorname{Int} \mathfrak{g}$.*

(iv) *The L_u-action on the nilpotent subalgebra $\mathfrak{n}$ is strongly visible.*

(v) *The G_u-action on $\mathcal{O}_X$ is strongly visible.*

Here, the height of $\mathcal{O}_X$ is defined by the maximum of $m \in \mathbb{Z}$ satisfying $\mathfrak{g}(m) \neq \{0\}$ (see Definition 5.2).

The equivalence between (i) and (ii) is proved by Panyushev [19]. The equivalence (i) $\Leftrightarrow$ (iii) is due to Vinberg–Kimelfeld [32] and Vinberg [31]. The implication (v) $\Rightarrow$ (iii) is a special case of the propagation theory of multiplicity-freeness property by Kobayashi [10, 11, 14]. The implication (i) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) hold by Theorems 1.3 and 1.2, respectively.

$$\begin{array}{ccccc} \text{(ii)} & \overset{[19]}{\Leftrightarrow} & \text{(i)} & \overset{[32,\,31]}{\Leftrightarrow} & \text{(iii)} \\ & \text{Theorem 1.3} & \Downarrow & & \Uparrow\ [10,\,11,\,14] \\ & & \text{(iv)} & \underset{\text{Theorem 1.2}}{\Rightarrow} & \text{(v)} \end{array}$$

Corollary 1.4 has some connection with "small infinite-dimensional representations" of complex reductive Lie groups $G_{\mathbb{C}}$. If $\overline{\mathcal{O}_\pi}$ is the associated variety of an admissible representation π of $G_{\mathbb{C}}$ (see [33]), then the G_u-type in π is asymptotically the same with the G_u-type in the space of regular functions in $\overline{\mathcal{O}_\pi}$ by [15, Proposition 3.3].

This paper is organized as follows. In Section 2, we review basic notion of nilpotent orbits in complex semisimple Lie algebras. In Section 3, we explain a key theorem for the proof of Theorem 1.3 (Theorem 3.5), namely, properties of our choice of a slice and an anti-holomorphic diffeomorphism in the strongly visible L_u-action on $\mathfrak{n}$ if $\mathcal{O}_X$ is spherical. In Section 4, we show the induction theorem of strongly visible actions, namely, Theorem 1.2. In Section 5, we give a proof of Theorem 3.5.

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2. Preliminaries

In this section, we review structural theories on nilpotent orbits in complex semisimple Lie algebras which is based on [4].

2.1. $\mathfrak{sl}_2$-triple of nilpotent orbit. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and $G_{\mathbb{C}} := \operatorname{Int}\mathfrak{g}$ the inner automorphism group of $\mathfrak{g}$. An element $X \in \mathfrak{g}$ is called a nilpotent element if $\operatorname{ad}(X) \in \operatorname{End}(\mathfrak{g})$ satisfies $\operatorname{ad}(X)^N = 0$ for some $N \in \mathbb{N}$, and called a semisimple element if $\operatorname{ad}(X)$ is diagonalizable. We write $\mathcal{N}$ for the cone of nilpotent elements of $\mathfrak{g}$ and $\mathcal{S}$ for the set of semisimple elements of $\mathfrak{g}$.

We denote by $\mathcal{O}_X = G_{\mathbb{C}} \cdot X$ the $G_{\mathbb{C}}$-orbit through $X \in \mathfrak{g}$. An orbit $\mathcal{O}_X$ is called a nilpotent (resp. semisimple) orbit if $X \in \mathcal{N}$ (resp. $X \in \mathcal{S}$), from which $\mathcal{O}_X \subset \mathcal{N}$ (resp. $\mathcal{O}_X \subset \mathcal{S}$). We write $\mathcal{N}^*/G_{\mathbb{C}}$ ($\mathcal{N}^* := \mathcal{N} \setminus \{0\}$) and $\mathcal{S}/G_{\mathbb{C}}$ for the set of non-zero nilpotent orbits and of semisimple orbits, respectively.

Suppose we are given $X \in \mathcal{N}^*$. By Jacobson–Morozov, there exist $H, Y \in \mathfrak{g}$ such that $\{H, X, Y\}$ forms an $\mathfrak{sl}_2$-triple, namely,

$$[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = H.$$

Then, H is called a neutral element, X a nilpositive element, and Y a nilnegative element. We remark that $H \in \mathcal{S}$ and $X, Y \in \mathcal{N}$.

Two $\mathfrak{sl}_2$-triples $\{H, X, Y\}, \{H', X', Y'\}$ in $\mathfrak{g}$ are said to be conjugate if there exists $g \in G_{\mathbb{C}}$ such that $H' = g \cdot H$, $X' = g \cdot X$, and $Y' = g \cdot Y$. Clearly, two elements $X, X' \in \mathcal{N}^*$ are conjugate if two $\mathfrak{sl}_2$-triples $\{H, X, Y\}, \{H', X', Y'\}$ in $\mathfrak{g}$ are conjugate. The opposite is also true by Kostant [16]. Then, this gives rise to the following bijection

$$\mathcal{N}^*/G_{\mathbb{C}} \xrightarrow{\sim} \{\mathfrak{sl}_2\text{-triples in } \mathfrak{g}\}/G_{\mathbb{C}}, \quad \mathcal{O}_X \mapsto \{H, X, Y\}. \tag{2.1}$$

Moreover, it follows from Mal'cev [18] that $\{H, X, Y\}$ and $\{H', X', Y'\}$ are conjugate if the corresponding neutral elements H, H' are conjugate. This implies that the map

$$\{\mathfrak{sl}_2\text{-triples in } \mathfrak{g}\}/G_{\mathbb{C}} \to \mathcal{S}/G_{\mathbb{C}}, \quad \{H, X, Y\} \mapsto \mathcal{O}_H \tag{2.2}$$

is injective. Hence, composing (2.1) and (2.2) yields the injective map from the set of nilpotent orbits to that of semisimple ones:

$$\Phi : \mathcal{N}^*/G_{\mathbb{C}} \to \mathcal{S}/G_{\mathbb{C}}, \quad \mathcal{O}_X \mapsto \mathcal{O}_H. \tag{2.3}$$

2.2. $\mathbb{Z}$-grading of $\mathfrak{g}$. In this subsection, we consider the semisimple transformation $\mathrm{ad}(H)$ on $\mathfrak{g}$ for $H \in \Phi(\mathcal{O}_X)$.

By the general theory on finite-dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$, all $\mathrm{ad}(H)$-eigenvalues are integers. We write $\mathfrak{g}(m)$ for the $\mathrm{ad}(H)$-eigenspace with eigenvalue m, namely,

$$\mathfrak{g}(m) := \{Z \in \mathfrak{g} : \mathrm{ad}(H)Z = mZ\} \quad (m \in \mathbb{Z}). \tag{2.4}$$

Then, $\mathfrak{g}$ is decomposed into the finite sum of $\mathrm{ad}(H)$-eigenspaces as

$$\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}(m). \tag{2.5}$$

Since the inclusion $[\mathfrak{g}(m), \mathfrak{g}(n)] \subset \mathfrak{g}(m+n)$ holds for $m, n \in \mathbb{Z}$, the decomposition (2.5) defines a $\mathbb{Z}$-grading of $\mathfrak{g}$.

Let us take another element $H' \in \Phi(\mathcal{O}_X)$ and consider the $\mathrm{ad}(H')$-eigenspace $\mathfrak{g}'(m)$. We note that $H' = g_0 \cdot H$ for some $g_0 \in G_{\mathbb{C}}$. Then, we have:

Lemma 2.1. *$\mathfrak{g}'(m) = g_0 \cdot \mathfrak{g}(m) = \{g_0 \cdot Z : Z \in \mathfrak{g}(m)\}$ for any $m \in \mathbb{Z}$.*

Proof. For $Z \in \mathfrak{g}(m)$, we observe $g_0 \cdot (\mathrm{ad}(H)Z)$ as follows. First, by the definition of $\mathfrak{g}(m)$, we have

$$g_0 \cdot (\mathrm{ad}(H)Z) = g_0 \cdot (mZ) = m(g_0 \cdot Z). \tag{2.6}$$

Second, in view of $g_0 \cdot [H, Z] = [g_0 \cdot H, g_0 \cdot Z]$, we express as

$$g_0 \cdot (\mathrm{ad}(H)Z) = \mathrm{ad}(g_0 \cdot H)(g_0 \cdot Z) = \mathrm{ad}(H')(g_0 \cdot Z). \tag{2.7}$$

Comparing (2.6) and (2.7), we obtain $\mathrm{ad}(H')(g_0 \cdot Z) = m(g_0 \cdot Z)$, from which $g_0 \cdot Z \in \mathfrak{g}'(m)$. Hence, we have shown $g_0 \cdot \mathfrak{g}(m) \subset \mathfrak{g}'(m)$.

Similarly, we have $g_0^{-1} \cdot \mathfrak{g}'(m) \subset \mathfrak{g}(m)$, and then $\mathfrak{g}'(m) \subset g_0 \cdot \mathfrak{g}(m)$.

Therefore, we have proved $\mathfrak{g}'(m) = g_0 \cdot \mathfrak{g}(m)$. ■

Lemma 2.1 shows that the $\mathbb{Z}$-grading (2.5) is determined by $\mathcal{O}_X$, particularly, independent on the choice of semisimple elements of $\Phi(\mathcal{O}_X)$.

2.3. Nilpotent subalgebra. We define a parabolic subalgebra $\mathfrak{q}$ arising from the $\mathbb{Z}$-grading (2.5) by

$$\mathfrak{q} := \bigoplus_{m \geq 0} \mathfrak{g}(m) \tag{2.8}$$

with Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ where

$$\mathfrak{l} := \mathfrak{z}_{\mathfrak{g}}(H) = \mathfrak{g}(0) \tag{2.9}$$

is the Levi subalgebra and $\mathfrak{u} = \bigoplus_{m>0} \mathfrak{g}(m)$ is the nilradical. Then, $\mathfrak{l}$ is a complex reductive Lie algebra.

Let $\mathfrak{n}$ be a nilpotent subalgebra defined by

$$\mathfrak{n} := \bigoplus_{m \geq 2} \mathfrak{g}(m). \tag{2.10}$$

Clearly, $[\mathfrak{q}, \mathfrak{n}] \subset \mathfrak{n}$. As $X \in \mathfrak{g}(2)$, we have $[\mathfrak{q}, X] \subset \mathfrak{n}$. Further, the opposite inclusion also holds by the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, from which we obtain:

Lemma 2.2. $\mathfrak{n} = [\mathfrak{q}, X]$.

2.4. Realization of $\mathcal{O}_X$ via momentum map. Let $Q_{\mathbb{C}}$ be a parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$, which acts on $\mathfrak{n}$. We set

$$\mathfrak{n}^\circ := Q_{\mathbb{C}} \cdot X. \tag{2.11}$$

By Lemma 2.2, $\mathfrak{n}^\circ$ is an open set in $\mathfrak{n}$, in particular, its closure $\overline{\mathfrak{n}^\circ}$ is equal to $\mathfrak{n}$.

We define a $G_{\mathbb{C}}$-equivariant smooth map φ from the holomorphic vector bundle $G_{\mathbb{C}} \times_{Q_{\mathbb{C}}} \mathfrak{n}$ on the flag manifold $G_{\mathbb{C}}/Q_{\mathbb{C}}$ to $\mathfrak{g}$ by

$$\varphi : G_{\mathbb{C}} \times_{Q_{\mathbb{C}}} \mathfrak{n} \to \mathfrak{g}, \ (x, Z) \mapsto x \cdot Z. \tag{2.12}$$

Lemma 2.3. $\mathcal{O}_X = \varphi(G_{\mathbb{C}} \times_{Q_{\mathbb{C}}} \mathfrak{n}^\circ)$.

Proof. It follows from (2.11) that

$$\varphi(G_{\mathbb{C}} \times_{Q_{\mathbb{C}}} \mathfrak{n}^\circ) = \varphi(G_{\mathbb{C}} \times_{Q_{\mathbb{C}}} (Q_{\mathbb{C}} \cdot X)) = G_{\mathbb{C}} \cdot X = \mathcal{O}_X. \tag{2.13}$$

Hence, Lemma 2.3 has been proved. ■

Next, let $L_{\mathbb{C}}$ be a Levi subgroup of $Q_{\mathbb{C}}$ with Lie algebra $\mathfrak{l}$. We take a compact real form $L_u := L_{\mathbb{C}} \cap G_u$ of $L_{\mathbb{C}}$. Then, the inclusion map $G_u \hookrightarrow G_{\mathbb{C}}$ induces a biholomorphic diffeomorphism $G_u/L_u \simeq G_{\mathbb{C}}/Q_{\mathbb{C}}$. This gives rise to an isomorphism as a G_u-equivariant holomorphic vector bundle as follows:

$$G_u \times_{L_u} \mathfrak{n} \simeq G_{\mathbb{C}} \times_{Q_{\mathbb{C}}} \mathfrak{n}. \tag{2.14}$$

We use the same letter φ to denote the G_u-equivariant map $G_u \times_{L_u} \mathfrak{n} \to \mathfrak{g}$ via the isomorphism (2.14). Then, we have:

Proposition 2.4. $\mathcal{O}_X = G_u \cdot \mathfrak{n}^\circ$.

Proof. By Lemma 2.3 and (2.14), we have

$$\mathcal{O}_X = \varphi(G_u \times_{L_u} \mathfrak{n}^\circ) = G_u \cdot \mathfrak{n}^\circ,$$

from which Proposition 2.4 has been proved. ■

Remark 2.5 (cf. [11, Theorem 20]). We can regard φ as the restriction of the momentum map of the Hamiltonian G_u-action on the flag manifold G_u/L_u.

Let $\mathfrak{g}_u$ and $\mathfrak{l}_u$ be the Lie algebras of G_u and L_u, respectively. Identifying $\mathfrak{g}_u/\mathfrak{l}_u$ with $\mathfrak{u}$, the nilpotent subalgebra $\mathfrak{n}$ seems to be an L_u-submodule of $\mathfrak{g}_u/\mathfrak{l}_u$. Then, the principal vector bundle $G_u \times_{L_u} (\mathfrak{g}_u/\mathfrak{l}_u)$ is isomorphic to the cotangent bundle $T^*(G_u/L_u)$. Via the identification of $\mathfrak{g}$ with the dual space $\mathfrak{g}^*$ by the Killing form on $\mathfrak{g}$, the map

$$\psi : G_u \times_{L_u} \mathfrak{u} \to \mathfrak{g}, \quad (g, Z) \mapsto g \cdot Z$$

is essentially a momentum map of the Hamiltonian G_u-action on the flag manifold G_u/L_u. It turns out that $\varphi = \psi|_{G_u \times_{L_u} \mathfrak{n}}$.

3. Visible actions on nilpotent subalgebras

Let us retain the setting of Section 2. A nilpotent orbit $\mathcal{O}_X$ is written as $\mathcal{O}_X = G_u \cdot \mathfrak{n}^\circ$. In view of this realization, it is crucial for the study on the G_u-action on $\mathcal{O}_X$ to understand the L_u-action on $\mathfrak{n}^\circ$. Here, the closure of the $Q_{\mathbb{C}}$-orbit $\mathfrak{n}^\circ$ through X is equal to the nilpotent subalgebra $\mathfrak{n}$. In this section, we focus on the L_u-action on $\mathfrak{n}$.

3.1. Normal real form of complex simple Lie algebra. First, we define an anti-holomorphic involution of a complex simple Lie algebra.

For a real reductive Lie algebra $\mathfrak{g}'$, we denote by $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}'$ the real rank of $\mathfrak{g}'$. A real form $\mathfrak{g}'_{\mathbb{R}}$ of a complex reductive Lie algebra $\mathfrak{g}'$ is called *normal* if $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}'_{\mathbb{R}} = \operatorname{rank} \mathfrak{g}'$. Normal real forms of a complex simple Lie algebra exist and are unique up to isomorphism.

Let $\mathfrak{g}$ be a complex simple Lie algebra. We take a normal real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$. We denote by σ the complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_{\mathbb{R}}$, namely,

$$\sigma(X + \sqrt{-1}Y) = X - \sqrt{-1}Y \quad (X, Y \in \mathfrak{g}_{\mathbb{R}}). \tag{3.1}$$

Let $\mathfrak{k}_{\mathbb{R}}$ be a maximal compact subalgebra of $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ the corresponding Cartan decomposition. Then, the Lie algebra $\mathfrak{k}_{\mathbb{R}} + \sqrt{-1}\mathfrak{p}_{\mathbb{R}}$ is a σ-stable compact real form of $\mathfrak{g}$. Since compact real forms are unique up to isomorphism, we may and do assume that $\mathfrak{g}_u := \mathfrak{k}_{\mathbb{R}} + \sqrt{-1}\mathfrak{p}_{\mathbb{R}}$ is the Lie algebra of G_u by taking a conjugation of $\mathfrak{g}_{\mathbb{R}}$ if necessary.

3.2. σ-stability of subalgebra. Let $\mathfrak{a}_{\mathbb{R}}$ be a maximal abelian subspace in $\mathfrak{p}_{\mathbb{R}}$ and

$$\mathfrak{a} := \mathfrak{a}_{\mathbb{R}} + \sqrt{-1}\mathfrak{a}_{\mathbb{R}}. \tag{3.2}$$

Then, $\mathfrak{a}$ is σ-stable, and a Cartan subalgebra of $\mathfrak{g}$ since $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}} = \operatorname{rank} \mathfrak{g}$. Then, $\mathfrak{a}_{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$.

Let $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{a})$ be a root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We denote by $\mathfrak{g}_\alpha$ the root space corresponding to $\alpha \in \Delta$. Then, we have:

Lemma 3.1. *The root space $\mathfrak{g}_\alpha$ is σ-stable for any $\alpha \in \Delta$.*

Proof. According to (3.2), we write $A \in \mathfrak{a}$ as $A = A_1 + \sqrt{-1}A_2$ $(A_1, A_2 \in \mathfrak{a}_{\mathbb{R}})$. As $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$, we have $\sigma(A) = A_1 - \sqrt{-1}A_2$. Hence,

$$\alpha(\sigma(A)) = \alpha(A_1 - \sqrt{-1}A_2) = \alpha(A_1) - \sqrt{-1}\alpha(A_2).$$

We remark that all of the roots are real on $\mathfrak{a}_{\mathbb{R}}$ (cf. [8, Section 4]). This means that $\alpha(A_1), \alpha(A_2) \in \mathbb{R}$, and then, $\alpha(\sigma(A)) = \alpha(A_1) - \sqrt{-1}\alpha(A_2) = \overline{\alpha(A)}$. Since σ is anti-linear, we have

$$\sigma(\alpha(A)Z) = \overline{\alpha(A)}\sigma(Z) = \alpha(\sigma(A))\sigma(Z) \quad (Z \in \mathfrak{g}). \tag{3.3}$$

Let Z_α be a root vector of $\mathfrak{g}_\alpha$. It follows from (3.3) that

$$\sigma([A, Z_\alpha]) = \sigma(\alpha(A)Z_\alpha) = \alpha(\sigma(A))\sigma(Z_\alpha). \tag{3.4}$$

On the other hand, we have $\sigma([A, Z_\alpha]) = [\sigma(A), \sigma(Z_\alpha)]$. Replacing A with $\sigma(A)$ in the equality (3.4) shows

$$[A, \sigma(Z_\alpha)] = \alpha(A)\sigma(Z_\alpha). \tag{3.5}$$

This means that $\sigma(Z_\alpha)$ lies in $\mathfrak{g}_\alpha$. Hence, Lemma 3.1 has been proved. ■

We recall the well-known facts that all elements contained in a Cartan subalgebra are semisimple and that two Cartan subalgebras of $\mathfrak{g}$ are conjugate by $G_{\mathbb{C}}$. This means that $\Phi(\mathcal{O}_X) \cap \mathfrak{a} \neq \emptyset$. Hence, we take a semisimple element H in $\Phi(\mathcal{O}_X) \cap \mathfrak{a}$. In this setting, we have:

Lemma 3.2. *The $\operatorname{ad}(H)$-eigenspace $\mathfrak{g}(m)$ is σ-stable for any $m \in \mathbb{Z}$.*

Proof. Since $[H, Z_\alpha] = \alpha(H)Z_\alpha$ holds for any $Z_\alpha \in \mathfrak{g}_\alpha$, we have $\mathfrak{g}_\alpha \subset \mathfrak{g}(\alpha(H))$. This implies that the eigenspace $\mathfrak{g}(m)$ is written as $\mathfrak{g}(m) = \bigoplus_{\alpha \in \Delta_m} \mathfrak{g}_\alpha$ for some subset $\Delta_m \subset \Delta \cup \{0\}$. Hence, the σ-stability of $\mathfrak{g}(m)$ follows from Lemma 3.1. ■

The parabolic subalgebra $\mathfrak{q}$, the Levi subalgebra $\mathfrak{l}$, and the nilpotent subalgebra $\mathfrak{n}$, respectively, consist of some $\mathrm{ad}(H)$-eigenspaces. Then, the following lemma is an immediate consequence of Lemma 3.2.

Lemma 3.3. *The subalgebras $\mathfrak{q}$, $\mathfrak{l}$, and $\mathfrak{n}$ of $\mathfrak{g}$ are σ-stable.*

By Lemma 3.3, the restriction of $\sigma \in \mathrm{Aut}\,\mathfrak{g}$ to $\mathfrak{l}$ becomes an anti-holomorphic involution of $\mathfrak{l}$. Here, we set a real form of $\mathfrak{l}$ by $\mathfrak{l}_{\mathbb{R}} = \mathfrak{l}^{\sigma} = \mathfrak{l} \cap \mathfrak{g}_{\mathbb{R}}$. Then, we have:

Lemma 3.4. *The real form $\mathfrak{l}_{\mathbb{R}}$ of $\mathfrak{l}$ is normal.*

Proof. By definition, let us show the equality $\mathrm{rank}_{\mathbb{R}}\,\mathfrak{l}_{\mathbb{R}} = \mathrm{rank}\,\mathfrak{l}$. Here, the inequality $\mathrm{rank}_{\mathbb{R}}\,\mathfrak{l}_{\mathbb{R}} \leq \mathrm{rank}\,\mathfrak{l}$ holds in general. Then, it is sufficient to see $\mathrm{rank}_{\mathbb{R}}\,\mathfrak{l}_{\mathbb{R}} \geq \mathrm{rank}\,\mathfrak{l}$.

For this, we consider the maximal abelian $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{p}_{\mathbb{R}}$. The semisimple element $H \in \mathfrak{a}$ satisfies $[H, \mathfrak{a}] = \{0\}$, from which $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{a} \subset \mathfrak{l}$. Then, we obtain $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{l} \cap \mathfrak{p}_{\mathbb{R}} = \mathfrak{l}_{\mathbb{R}} \cap \mathfrak{p}_{\mathbb{R}}$. As $\mathfrak{l}_{\mathbb{R}} = (\mathfrak{l}_{\mathbb{R}} \cap \mathfrak{k}_{\mathbb{R}}) + (\mathfrak{l}_{\mathbb{R}} \cap \mathfrak{p}_{\mathbb{R}})$ is a Cartan decomposition of $\mathfrak{l}_{\mathbb{R}}$, this inclusion implies that $\mathrm{rank}_{\mathbb{R}}\,\mathfrak{l}_{\mathbb{R}} \geq \dim \mathfrak{a}_{\mathbb{R}}$. Since $\mathrm{rank}\,\mathfrak{g} \geq \mathrm{rank}\,\mathfrak{l}$ holds in general and $\mathfrak{g}_{\mathbb{R}}$ is a normal real form of $\mathfrak{g}$, we conclude

$$\mathrm{rank}_{\mathbb{R}}\,\mathfrak{l}_{\mathbb{R}} \geq \mathrm{rank}\,\mathfrak{g} \geq \mathrm{rank}\,\mathfrak{l}.$$

Therefore, Lemma 3.4 has been proved. ∎

3.3. Compatible condition. We lift the anti-linear involution σ on $\mathfrak{g}$ to an anti-holomorphic involution $\widetilde{\sigma}$ on the complex simple Lie group $G_{\mathbb{C}} = \mathrm{Int}\,\mathfrak{g}$. More precisely, we write $\widetilde{\sigma}(g) := \sigma g \sigma \in G_{\mathbb{C}}$ $(g \in G_{\mathbb{C}})$. Then, we have:

$$\sigma(g \cdot Z) = \widetilde{\sigma}(g) \cdot \sigma(Z) \quad (g \in G_{\mathbb{C}},\ Z \in \mathfrak{g}). \tag{3.6}$$

We say that $\widetilde{\sigma}$ is compatible with σ (see [14, (4.2.1)]).

In view of Lemma 3.3, the Levi subgroup $L_{\mathbb{C}}$ is $\widetilde{\sigma}$-stable, and the restriction of $\sigma \in \mathrm{Aut}\,\mathfrak{g}$ to $\mathfrak{n}$ induces an anti-holomorphic diffeomorphism on $\mathfrak{n}$. Hence, we obtain:

$$\sigma(k \cdot Z) = \widetilde{\sigma}(k) \cdot \sigma(Z) \quad (k \in L_{\mathbb{C}},\ Z \in \mathfrak{n}). \tag{3.7}$$

The compact real form G_u of $G_{\mathbb{C}}$ is $\widetilde{\sigma}$-stable because its Lie algebra $\mathfrak{g}_u$ is σ-stable. It follows from Lemma 3.3 that $\mathfrak{l}_u := \mathfrak{l} \cap \mathfrak{g}_u$ is σ-stable, from which $L_u = L_{\mathbb{C}} \cap G_u$ is $\widetilde{\sigma}$-stable.

3.4. Visible actions on nilpotent subalgebra. We are ready to state our theorem on the L_u-action on $\mathfrak{n}$ as follows.

Theorem 3.5. *If $\mathcal{O}_X$ is nilpotent and spherical, then one can find $S_0 \subset \mathfrak{n}$ such that the following properties are satisfied:*

(a) *S_0 is a real vector space.*

(b) $\mathfrak{n} = L_u \cdot S_0$.

(c) $\sigma|_{S_0} = \mathrm{id}_{S_0}$.

Our proof of Theorem 3.5 uses a case-by-case analysis for each spherical $\mathcal{O}_X$, which will be given in Section 5 separated from this section.

3.5. Proof of Theorem 1.3. Let us see that Theorem 1.3 follows from Theorem 3.5.

Proof of Theorem 1.3. Suppose that $\mathcal{O}_X$ is nilpotent and spherical. By Theorem 3.5, there exists a real vector subspace S_0 such that $\mathfrak{n} = L_u \cdot S_0$ and $\sigma|_{S_0} = \mathrm{id}_{S_0}$. We will verify that the data (S_0, σ) satisfies the definition of strongly visible actions (see Section 1).

The properties (b) and (c) coincide with (V.1) and (S.1), respectively. To see (S.2), we take $Z \in \mathfrak{n}$, and write $Z = k \cdot Z_0$ for some $k \in L_u$ and $Z_0 \in S_0$ according to (V.1). By the relation (3.7) and the property (c), we have

$$\sigma(Z) = \sigma(k \cdot Z_0) = \widetilde{\sigma}(k) \cdot \sigma(Z_0) = \widetilde{\sigma}(k) \cdot Z_0 = \widetilde{\sigma}(k)k^{-1} \cdot Z.$$

The element $\widetilde{\sigma}(k)k^{-1}$ lies in L_u, from which $\sigma(Z) \in L_u \cdot Z$. Hence, we have verified (S.2).

Consequently, Theorem 1.3 has been proved. ■

4. Induction theorem of strongly visible actions

In this section, we show Theorem 1.2. We again reformulate Theorem 1.2 as follows:

Theorem 4.1. *Let $\mathcal{O}_X$ be a nilpotent orbit, $\mathfrak{l} = \mathfrak{g}(0)$ and $\mathfrak{n} = \bigoplus_{m \geq 2} \mathfrak{g}(m)$ are defined by the $\mathfrak{sl}_2$-triple as in (2.9) and (2.10), respectively, and σ the anti-holomorphic involution on $\mathfrak{g}$ as in Section 3. If the L_u-action on $\mathfrak{n}$ is strongly visible with σ, then the G_u-action on $\mathcal{O}_X$ is strongly visible.*

Remark 4.2. This theorem generalizes *induction theorem of strongly visible actions* [11, Theorem 20] for Type A case. It produces new strongly visible actions out of known strongly visible actions (linear visible actions [22, 24]).

Suppose that the L_u-action on $\mathfrak{n}$ is strongly visible with σ as in Section 3. Then, one can take a real submanifold S_0 in $\mathfrak{n}$ such that $\mathfrak{n} = L_u \cdot S_0$, $\sigma|_{S_0} = \mathrm{id}_{S_0}$. We set

$$S := S_0 \cap \mathfrak{n}^\circ. \tag{4.1}$$

Then, S is a real submanifold of $\mathfrak{n}^\circ$ since $\mathfrak{n}^\circ$ is open in $\mathfrak{n}$.

Lemma 4.3. $\mathfrak{n}^\circ = L_u \cdot S$.

Proof. In view of the equality $\mathfrak{n}^\circ = \mathfrak{n} \cap \mathfrak{n}^\circ$, we have

$$\mathfrak{n}^\circ = (L_u \cdot S_0) \cap \mathfrak{n}^\circ = (L_u \cdot S_0) \cap (L_u \cdot \mathfrak{n}^\circ) = L_u \cdot (S_0 \cap \mathfrak{n}^\circ) = L_u \cdot S.$$

Hence, Lemma 4.3 has been proved. ∎

Combining Proposition 2.4 with Lemma 4.3, we have

$$\mathcal{O}_X = G_u \cdot \mathfrak{n}^\circ = G_u \cdot (L_u \cdot S) = (G_u L_u) \cdot S = G_u \cdot S.$$

Therefore, we have proved:

Proposition 4.4. *The submanifold S satisfies the condition (V.1) for the G_u-action on $\mathcal{O}_X$.*

Next, we define an anti-holomorphic diffeomorphism of $\mathcal{O}_X$, which arises from σ defined by (3.1) as follows.

Let Z be an element of $\mathcal{O}_X$, and write $Z = g \cdot Z_0$ for some $g \in G_u$ and $Z_0 \in S$ due to Proposition 4.4. It is obvious from $S \subset S_0$ and $\sigma|_{S_0} = \mathrm{id}_{S_0}$ that

$$\sigma|_S = \mathrm{id}_S\,. \tag{4.2}$$

Then, the relation (3.6) shows

$$\sigma(Z) = \sigma(g \cdot Z_0) = \widetilde{\sigma}(g) \cdot \sigma(Z_0) = \widetilde{\sigma}(g) \cdot Z_0. \tag{4.3}$$

Since G_u is $\widetilde{\sigma}$-stable (see Section 3), the element $\widetilde{\sigma}(g)$ lies in G_u. Then, the equality (4.3) means that $\sigma(Z) \in G_u \cdot S = \mathcal{O}_X$. Hence, $\mathcal{O}_X$ is σ-stable. This implies that the restriction of σ to $\mathcal{O}_X$ becomes an anti-holomorphic diffeomorphism on $\mathcal{O}_X$, which we use the same letter to denote.

Now, we give a proof of Theorem 4.1.

Proof of Theorem 4.1. It is clear that (V.1) and (S.1) hold by Proposition 4.4 and (4.2), respectively. Let $Z \in \mathcal{O}_X$ and write $Z = g \cdot Z_0 \in G_u \cdot S$. Then, we have

$$\sigma(Z) = \widetilde{\sigma}(g)g^{-1} \cdot (g \cdot Z_0) = \widetilde{\sigma}(g)g^{-1} \cdot Z \in G_u \cdot Z.$$

Hence, we have verified (S.2).

Therefore, Theorem 4.1 has been proved. ∎

5. Proof of Theorem 3.5

This section is devoted to the proof of Theorem 3.5.

First of all, we give a short summary of our proof. The Dynkin–Kostant theory explains that a nilpotent orbit $\mathcal{O}_X$ in a complex simple Lie algebra $\mathfrak{g}$ is characterized by the weighted Dynkin diagram, denoted by $\Omega(\mathcal{O}_X)$, which is the Dynkin diagram of $\mathfrak{g}$ with numerical labels. A classification of nilpotent orbits is given in terms of the weighted Dynkin diagrams. Moreover, $\mathcal{O}_X$ defines the height, denoted by $\mathrm{ht}(\mathcal{O}_X)$. D. Panyushev provides a criterion for $\mathcal{O}_X$ to be spherical by its height. Since $\mathrm{ht}(\mathcal{O}_X)$ can be calculated from $\Omega(\mathcal{O}_X)$, the table of spherical nilpotent orbits is obtained.

Under these theories, we apply case-by-case analysis on the L_u-action on $\mathfrak{n}$ for each spherical $\mathcal{O}_X$. Indeed, we clarify a semisimple element $H \in \Phi(\mathcal{O}_X) \cap \mathfrak{a}$ from $\Omega(\mathcal{O}_X)$ and express the $L_{\mathbb{C}}$-action on $\mathfrak{n}$. By using the classification of strongly visible linear actions, we verify the strong visibility for the L_u-action on $\mathfrak{n}$, and give an explicit description of S_0 satisfying (a)–(c).

5.1. Weighted Dynkin diagram of nilpotent orbit. In this subsection, we review the weighted Dynkin diagram corresponding to a nilpotent orbit $\mathcal{O}_X$ in a complex semisimple Lie algebra $\mathfrak{g}$. See [4] for survey on weighted Dynkin diagrams in complex semisimple Lie algebras.

Let us retain the setting of Sections 2 and 3. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ such that $\mathfrak{b}$ contains the Cartan subalgebra $\mathfrak{a}$ and is contained in the parabolic subalgebra $\mathfrak{q}$. We fix a positive system $\Delta^+ \equiv \Delta^+(\mathfrak{g}, \mathfrak{a})$ satisfying $\mathfrak{b} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and set a closed Weyl chamber by

$$\mathfrak{a}_+ = \{A \in \mathfrak{a} : \alpha(A) \geq 0 \ (\forall \alpha \in \Delta^+)\}.$$

As mentioned in Section 3, the intersection of $\mathfrak{a}$ and the semisimple orbit $\Phi(\mathcal{O}_X)$ is non-empty. In particular, $\Phi(\mathcal{O}_X) \cap \mathfrak{a}_+$ is a singleton set. Then, we define an injective map by

$$\Psi : \mathcal{N}^*/G_{\mathbb{C}} \to \mathfrak{a}_+, \quad \mathcal{O}_X \mapsto H \in \Phi(\mathcal{O}_X) \cap \mathfrak{a}_+. \tag{5.1}$$

Definition 5.1 (cf. [19]). We say that $\Psi(\mathcal{O}_X) \in \Phi(\mathcal{O}_X) \cap \mathfrak{a}_+$ is the *characteristic element* of $\mathcal{O}_X$.

For the rest of this paper, we fix H to be $H = \Psi(\mathcal{O}_X) \in \mathfrak{a}_+$.

We write $\alpha_1, \ldots, \alpha_r \in \Delta^+$ for the simple roots of Δ ($r := \operatorname{rank} \mathfrak{g} = \dim_{\mathbb{C}} \mathfrak{a}$). As $\mathfrak{b} \subset \mathfrak{q}$, the number $\alpha_j(H)$ is a non-negative integer. Moreover, it follows from the representation theory that $\alpha_j(H) \in \{0, 1, 2\}$ for $j = 1, 2, \ldots, r$. Then, we define the injective map as follows:

$$\Omega : \mathcal{N}^*/G_{\mathbb{C}} \to \{0, 1, 2\}^r, \quad \Omega(\mathcal{O}_X) := (\alpha_1(H), \ldots, \alpha_r(H)). \tag{5.2}$$

We label the node of the Dynkin diagram of $\mathfrak{g}$ corresponding to each simple root α_j with $\alpha_j(H)$. The Dynkin diagram with such labels is called the *weighted Dynkin diagram* of $\mathcal{O}_X$.

We recall from (2.1) that there is a one-to-one correspondence between nilpotent orbits and conjugacy classes of $\mathfrak{sl}_2$-triples. Thanks to Dynkin's work on $\mathfrak{sl}_2$-triples [5], the injective map (5.2) provides a characterization of nilpotent orbits in $\mathfrak{g}$ by corresponding weighted Dynkin diagrams.

Next, the $\mathbb{Z}$-grading (2.5) defined by $\mathcal{O}_X$ introduces a function on $\mathcal{N}^*/G_{\mathbb{C}}$ as follows:

$$\mathrm{ht} : \mathcal{N}^*/G_{\mathbb{C}} \to \mathbb{Z}, \quad \mathcal{O}_X \mapsto \max\{m \in \mathbb{Z} : \mathfrak{g}(m) \neq \{0\}\}. \tag{5.3}$$

Since $X \in \mathfrak{g}(2)$, we obtain $\mathrm{ht}(\mathcal{O}_X) \geq 2$ for any $\mathcal{O}_X \in \mathcal{N}^*/G_{\mathbb{C}}$.

Definition 5.2 ([19, Section 2]). We say that the positive integer $\mathrm{ht}(\mathcal{O}_X)$ is the *height* of a nilpotent orbit $\mathcal{O}_X$.

D. Panyushev gives a necessary and sufficient condition for a nilpotent orbit to be spherical.

Lemma 5.3 ([19, Theorem 3.1]). *For a nilpotent orbit $\mathcal{O}_X$, the following two conditions are equivalent:*

(i) $\mathcal{O}_X$ *is spherical.*

(ii) $\mathrm{ht}(\mathcal{O}_X) \leq 3$.

In view of Lemma 5.3, we consider how to calculate $\mathrm{ht}(\mathcal{O}_X)$ by the weighted Dynkin diagram $\Omega(\mathcal{O}_X)$. Let $\beta \in \Delta^+$ be the highest root. We write $\beta = k_1\alpha_1 + \cdots + k_r\alpha_r$ for some positive integers $k_1, \ldots, k_r$. Then, we have:

Lemma 5.4. $\mathrm{ht}(\mathcal{O}_X) = \beta(H) = k_1\alpha_1(H) + \cdots + k_r\alpha_r(H)$.

Proof. By the proof of Lemma 3.2, we formulate

$$\mathrm{ht}(\mathcal{O}_X) = \max\{m \in \mathbb{Z} : \mathfrak{g}(m) \neq \{0\}\} = \max\{\alpha(H) : \alpha \in \Delta\}. \tag{5.4}$$

Let $\alpha \in \Delta$ and write $\alpha = l_1\alpha_1 + \cdots + l_r\alpha_r$ $(l_1, \ldots, l_r \in \mathbb{Z})$. Since β is the highest root, the inequality $l_j \leq k_j$ holds for any $j = 1, 2, \ldots, r$. As $\alpha_j(H) \geq 0$ (see (5.2)), we estimate

$$\begin{aligned}\alpha(H) &= l_1\alpha_1(H) + \cdots + l_r\alpha_r(H)\\ &\leq k_1\alpha_1(H) + \cdots + k_r\alpha_r(H)\\ &= \beta(H).\end{aligned}$$

This means that

$$\max\{\alpha(H) : \alpha \in \Delta\} = \beta(H). \tag{5.5}$$

Combining (5.4) and (5.5), we get $\mathrm{ht}(\mathcal{O}_X) = \beta(H)$. ■

Applying Lemma 5.4 to the classification of complex nilpotent orbits [1, 2] (cf. [4]), we list all weighted Dynkin diagrams $\Omega(\mathcal{O}_X)$ with $\mathrm{ht}(\mathcal{O}_X) = 2, 3$ in the first and second columns of Tables 5.3 and 5.4 (see also [19]).

5.2. Visible linear actions. In this subsection, we recall the recent works on strongly visible linear actions, see [22, 24] for details.

Let $K_{\mathbb{C}}$ be a connected complex reductive Lie group and V a finite-dimensional vector space over $\mathbb{C}$. Suppose we are given a holomorphic representation of $K_{\mathbb{C}}$ on V. Then, we have naturally the representation of $K_{\mathbb{C}}$ on the polynomial ring $\mathbb{C}[V]$ defined by $f(v) \mapsto f(g^{-1} \cdot v)$. We say that the $K_{\mathbb{C}}$-action on V is a *multiplicity-free action*, or, V is a *multiplicity-free $K_{\mathbb{C}}$-space* if $\mathbb{C}[V]$ is multiplicity-free as a representation of $K_{\mathbb{C}}$.

Multiplicity-free actions are classified by Kac, Benson–Ratcliff, and Leahy [3, 7, 17] up to geometrically equivalences. Here, two holomorphic representations (π, V) and (π', V') of connected complex reductive Lie groups $K_{\mathbb{C}}$ and $K'_{\mathbb{C}}$, respectively, are *geometrically equivalent* if the image of π coincides with that of π' under some linear isomorphism from V to V'.

Let K_u be a compact real form of $K_{\mathbb{C}}$. Then, we have:

Lemma 5.5 ([22, 24]). *For a holomorphic representation of $K_{\mathbb{C}}$ on V, the followings are equivalent:*

(a) *The $K_{\mathbb{C}}$-action on V is a multiplicity-free action.*

(b) *The K_u-action on V is strongly visible.*

Lemma 5.5 gives a classification of strongly visible linear actions. During them, as we will see in the proof of Theorem 3.5, we need only eight series of multiplicity-free actions, which are listed in Table 5.1.

	$K_{\mathbb{C}}$	V
(1)	$\mathbb{C}^\times$	$\mathbb{C}$
(2)	$SL(p, \mathbb{C})$	$\mathbb{C}^p$
(3)	$SL(p, \mathbb{C}) \times \mathbb{C}^\times$	$\mathrm{Sym}(p, \mathbb{C})$
(4)	$SL(2p, \mathbb{C}) \times \mathbb{C}^\times$	$\mathrm{Alt}(2p, \mathbb{C})$
(5)	$SL(p, \mathbb{C}) \times SL(p, \mathbb{C}) \times \mathbb{C}^\times$	$M(p, \mathbb{C})$
(6)	$SO(p, \mathbb{C}) \times \mathbb{C}^\times$	$\mathbb{C}^p$
(7)	$E_6(\mathbb{C}) \times \mathbb{C}^\times$	$\mathfrak{J}_{\mathbb{C}}$
(8)	$SL(2p, \mathbb{C}) \times \mathbb{C}^\times$	$\mathrm{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^{2p}$

Table 5.1: Multiplicity-free $K_{\mathbb{C}}$-action on V

We pin down some restrictions on integer p in Table 5.1 as follows: In (2), (3), (5), and (6), $p \geq 2$; In (4) and (8), $p \geq 1$.

In (1), the multiplicative group $\mathbb{C}^\times = GL(1, \mathbb{C})$ acts on $\mathbb{C}$ as the standard complex multiplication. The special linear group $SL(p, \mathbb{C}) = \{g \in M(p, \mathbb{C}) : \det g = 1\}$ and the special complex orthogonal group $SO(p, \mathbb{C}) = \{g \in SL(p, \mathbb{C}) : {}^t g g = I_p\}$ act linearly on $\mathbb{C}^p$, respectively, where ${}^t g$ denotes the transposed matrix of g. In (3), $SL(p, \mathbb{C})$ acts on the space $\mathrm{Sym}(p, \mathbb{C})$ of complex symmetric matrices by $g \cdot A = gA\,{}^t g$. In (4), $SL(2p, \mathbb{C})$ acts on the space $\mathrm{Alt}(2p, \mathbb{C})$ of complex alternating matrices by $g \cdot A = gA\,{}^t g$. In (5), $SL(p, \mathbb{C}) \times SL(p, \mathbb{C})$ acts on $M(p, \mathbb{C})$ by $(g, h) \cdot A = gAh^{-1}$. In (7), $\mathfrak{J}_{\mathbb{C}} = \mathfrak{J} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Herm}(3, \mathfrak{C}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Herm}(3, \mathfrak{C}_{\mathbb{C}})$ is the complexified exceptional Jordan algebra, namely, it consists of Hermitian matrices of degree three whose entries are the complexified Cayley algebra $\mathfrak{C}_{\mathbb{C}} = \mathfrak{C} \otimes_{\mathbb{R}} \mathbb{C}$. Then, $\mathfrak{J}_{\mathbb{C}}$ is a vector space over $\mathbb{C}$ with dimension 27. We denote by $E_6(\mathbb{C})$ the connected and simply connected complex simple Lie group of exceptional type. Then, $E_6(\mathbb{C})$ acts on $\mathfrak{J}_{\mathbb{C}}$ as automorphisms.

In (3)–(7), the center $\mathbb{C}^\times$ of K_u acts on V as the scalar multiplication. In (8), the semisimple part $SL(2p, \mathbb{C})$ of K_u acts on $\mathbb{C}^{2p} \oplus \mathrm{Alt}(2p, \mathbb{C})$ by $g \cdot (v, A) = (gv, gA\,{}^t g)$, and the center $\mathbb{C}^\times$ acts by $s \cdot (v, A) = (s^3 v, s^2 A)$.

For $(K_{\mathbb{C}}, V)$ in Table 5.1, we present:

Lemma 5.6 ([22, 24]). *Let a multiplicity-free $K_{\mathbb{C}}$-action on V be one of cases in Table 5.1. Then, one can take a real vector subspace T and an anti-holomorphic diffeomorphism σ for the strongly visible K_u-action on V satisfying the following conditions:*

(a) $V = K_u \cdot T$.

(b) $\sigma|_T = \mathrm{id}_T$.

(c) *The dimension of the vector space T over $\mathbb{R}$ is equal to the rank of the semigroup of highest weights occurring in $\mathbb{C}[V]$.*

Indeed, we choose T and σ as in Table 5.2. Then, we can verify by [22, 24] that (T, σ) satisfies Lemma 5.6.

$(K_{\mathbb{C}}, V)$	T	σ	$\widetilde{\sigma}$
(1)	$\mathbb{R}$	σ_1	$\widetilde{\sigma}_0$
(2)	T_1	σ_1	$\widetilde{\sigma}_1$
(3)	D_p	σ_1	$\widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_0$
(4)	A_p	σ_1	$\widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_0$
(5)	D_p	σ_1	$\widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_0$
(6)	$D_{1,1}$	σ_2	$\widetilde{\sigma}_2 \boxtimes \widetilde{\sigma}_0$
(7)	D_3	σ_3	$\widetilde{\sigma}_3 \boxtimes \widetilde{\sigma}_0$
(8)	$A_p \oplus T_p$	$\sigma_1 \oplus \sigma_1$	$\widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_0$

Table 5.2: Our choice of T, σ, $\widetilde{\sigma}$ for the K_u-action on V

Here, let us explain the notation used in Table 5.2 as follows.

First, let $\{e_1, \ldots, e_N\}$ be the standard basis of $\mathbb{C}^N$. We define two real subspaces $T_{N'}, D_{1,1}$ in $\mathbb{C}^N$ by

$$T_{N'} := \mathbb{R}e_1 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_5 \oplus \cdots \oplus \mathbb{R}e_{2N'-1},$$
$$D_{1,1} := \mathbb{R}e_1 \oplus \sqrt{-1}\mathbb{R}e_2,$$

where $N' := \lfloor \frac{N+1}{2} \rfloor$ denotes the maximum of integers which are not greater than $\frac{N+1}{2}$. Second, we denote by D_N the real subspace of $M(N, \mathbb{C})$ consisting of diagonal matrices whose entries are all real, namely,

$$D_N := \{\mathrm{diag}(r_1, \ldots, r_N) \in M(N, \mathbb{C}) : r_1, \ldots, r_N \in \mathbb{R}\}.$$

Third, we set a real subspace A_p in $\mathrm{Alt}(2p, \mathbb{C})$ by

$$A_p := \{J(r_1, \ldots, r_p) \in \mathrm{Alt}(2p, \mathbb{C}) : r_1, \ldots, r_p \in \mathbb{R}\}.$$

where $J(r_1, \ldots, r_p) \in \mathrm{Alt}(2p, \mathbb{C})$ stands for the following block diagonal matrix

$$J(r_1, \ldots, r_p) := \mathrm{diag}(r_1 J_1, \ldots, r_p J_1), \quad J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

On the other hand, we denote by σ_1 and σ_2 the complex conjugations of $\mathbb{C}^N$ with respect to real forms V_1 and V_2, respectively, where $N' = \lfloor \frac{N+1}{2} \rfloor$, $N'' = \lfloor \frac{N}{2} \rfloor$ and

$$V_1 := \bigoplus_{i=1}^{N} \mathbb{R}e_i, \quad V_2 := \bigoplus_{i=1}^{N'} \mathbb{R}e_{2i-1} \oplus \bigoplus_{i=1}^{N''} \sqrt{-1}\mathbb{R}e_{2i}.$$

By using the coordinate with respect to the standard basis $\{e_1, \dots, e_N\}$, we write σ_1 and σ_2, respectively, as

$$\sigma_1(v) = \overline{v}, \quad \sigma_2(v) = I_a \overline{v} \quad (v \in \mathbb{C}^N)$$

where $\varepsilon_i := (-1)^{i+1}$ $(i = 1, 2, \dots, N)$ and

$$I_a := \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_N) \in M(N, \mathbb{C}). \tag{5.6}$$

For the standard basis $\{E_{ij} : 1 \leq i, j \leq N\}$ of $M(N, \mathbb{C})$, we also define the standard real form $\bigoplus_{1 \leq i,j \leq N} \mathbb{R}E_{ij} = M(N, \mathbb{R})$. With respect to $M(N, \mathbb{R})$, we define the complex conjugation of $M(N, \mathbb{C})$, which we use the same notation σ_1 to denote.

The definition of σ_3 for (7) will be given in the proof of Lemma 5.7.

Lemma 5.7. *Suppose we take T and σ as in Table 5.2 for a multiplicity-free $K_{\mathbb{C}}$-action on V listed in Table 5.1. Then, there exists an anti-holomorphic automorphism $\widetilde{\sigma}$ on $K_{\mathbb{C}}$ such that the followings are satisfied:*

(a) *$\widetilde{\sigma}$ stabilizes K_u.*

(b) $\operatorname{rank}_{\mathbb{R}} \operatorname{Lie}(K_{\mathbb{C}}^{\widetilde{\sigma}}) = \operatorname{rank} \operatorname{Lie}(K_{\mathbb{C}})$.

(c) $\sigma(k \cdot v) = \widetilde{\sigma}(k) \cdot \sigma(v)$ $(k \in K_{\mathbb{C}},\ v \in V)$.

Proof. Let us first consider $(K_{\mathbb{C}}, V)$ where the semisimple part of $K_{\mathbb{C}}$ is classical, namely, (1)–(6) and (8) of Table 5.1.

For convenience, we put $K_1 := SL(p, \mathbb{C})$ and $K_2 := SO(p, \mathbb{C})$. By using our matrix realization of $SL(p, \mathbb{C})$ and $SO(p, \mathbb{C})$, we define anti-holomorphic involutions $\widetilde{\sigma}_i$ on K_i $(i = 1, 2)$ by

$$\widetilde{\sigma}_1(k) := \overline{k} \quad (k \in K_1), \quad \widetilde{\sigma}_2(k) := I_a^{-1} \overline{k} I_a \quad (k \in K_2).$$

Then, the fixed point set $K_1^{\widetilde{\sigma}_1}$ coincides with $SL(p, \mathbb{R})$, and $K_2^{\widetilde{\sigma}_2}$ is isomorphic to the indefinite special orthogonal group $SO(\lfloor \frac{p}{2} \rfloor, \lfloor \frac{p+1}{2} \rfloor)$. Clearly, $\operatorname{rank}_{\mathbb{R}} \operatorname{Lie}(K_1^{\widetilde{\sigma}_1}) = p - 1 = \operatorname{rank} \operatorname{Lie}(K_1)$ and $\operatorname{rank}_{\mathbb{R}} \operatorname{Lie}(K_2^{\widetilde{\sigma}_2}) = \lfloor \frac{p}{2} \rfloor = \operatorname{rank} \operatorname{Lie}(K_2)$. Similarly, we define an anti-holomorphic involution $\widetilde{\sigma}_0$ on $\mathbb{C}^\times$ by $\widetilde{\sigma}_0(s) = \overline{s}$ $(s \in \mathbb{C}^\times)$. Then, $(\mathbb{C}^\times)^{\widetilde{\sigma}_0} = \mathbb{R}^\times$.

The right column of Table 5.2 gives our choice of anti-holomorphic involution $\widetilde{\sigma}$ on $K_{\mathbb{C}}$ for each strongly visible K_u-action on V. The direct computation shows that $\widetilde{\sigma}$ satisfies Lemma 5.7.

We give a short summary of the proof for (7), namely, $(K_{\mathbb{C}}, V) = (E_6(\mathbb{C})\times\mathbb{C}^{\times}, \mathfrak{J}_{\mathbb{C}})$. See [25] for detail.

The representation of a compact real form $E_6 \times \mathbb{T}$ of $K_{\mathbb{C}}$ on $\mathfrak{J}_{\mathbb{C}}$ is realized as the isotropy representation for non-compact Hermitian symmetric space $E_{7(-25)}/E_6 \cdot \mathbb{T}$ in the sense as follows. Here, we write $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ for the one-dimensional torus. Let θ be the Cartan decomposition of the Lie algebra $\mathfrak{e}_{7(-25)}$ corresponding to $\mathfrak{e}_6 + \sqrt{-1}\mathbb{R}$. We write $\mathfrak{e}_{7(-25)} = (\mathfrak{e}_6 + \sqrt{-1}\mathbb{R}) + \mathfrak{p}$ for the corresponding Cartan decomposition. Then, the representation of $(E_6 \times \mathbb{T})$ on $\mathfrak{J}_{\mathbb{C}}$ is geometrically equivalent to that on $\mathfrak{p}$.

We take an involutive automorphism σ_3 on $\mathfrak{e}_{7(-25)}$ commuting with θ such that $\mathfrak{e}_{7(-25)}^{\sigma_3} \simeq \mathfrak{su}^*(8)$, and extend σ_3 to an anti-linear involution on $e_7(\mathbb{C})$. As $\theta\sigma_3 = \sigma_3\theta$, σ_3 stabilizes both $e_6(\mathbb{C}) + \mathbb{C}$ and $\mathfrak{p}$. Then, we have $(e_6(\mathbb{C}) + \mathbb{C})^{\sigma_3} \simeq e_{6(6)} + \mathbb{R}$ (see [25]), which is a normal real form of $\mathfrak{e}_6(\mathbb{C}) + \mathbb{C}$.

Since $\operatorname{rank}_{\mathbb{R}} \mathfrak{su}^*(8) = \operatorname{rank}_{\mathbb{R}} \mathfrak{e}_{7(-25)} = 3$, a maximal abelian subspace $T \simeq D_3$ in $\mathfrak{p}^{\sigma_3}$ is also a maximal abelian in $\mathfrak{p}$. Then, the $(E_6 \times \mathbb{T})$-action on $\mathfrak{p}$ is strongly visible with slice T and anti-holomorphic diffeomorphism σ_3 on $\mathfrak{p}$ ([13, Lemma 2.4] and [11, Theorem 18]). Moreover, we lift $\sigma_3 \in \operatorname{Aut}(e_6(\mathbb{C}) + \mathbb{C})$ to an anti-holomorphic involution $\widetilde{\sigma}_3$ on $E_6(\mathbb{C}) \times \mathbb{C}^{\times}$. Then, Lemma 5.7 holds for D_3, σ_3 and $\widetilde{\sigma}_3$.

Therefore, Lemma 5.7 has been proved. ■

Concerning to our choice of $\widetilde{\sigma}$ as in Table 5.2, we denote by $\mu \boxtimes \mu'$ for involutions μ and μ' on complex Lie groups $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$, respectively, the involution on $H_{\mathbb{C}} \times H'_{\mathbb{C}}$ defined by $(\mu \boxtimes \mu')(h, h') = (\mu(h), \mu'(h'))$.

Remark 5.8. Lemma 5.7 holds for all multiplicity-free actions, on which we will discuss in forthcoming paper [25].

5.3. Procedure. In this subsection, we explain the procedure of our proof of Theorem 3.5.

The standard basis $\{e_1, \ldots, e_N\}$ of $\mathbb{C}^N$ defines the standard real form $V_{\mathbb{R}} = \mathbb{R}^N$ and the standard Hermitian inner product $\langle\cdot,\cdot\rangle$ satisfying $\langle e_i, e_j\rangle = \delta_{ij}$ $(1 \leq i, j \leq N)$. The dual $\mathfrak{a}_{\mathbb{R}}^*$ of the Cartan subalgebra $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ is realized as a subspace of $V_{\mathbb{R}}$, denoted by V. Let $\{E_1, \ldots, E_N\}$ be the dual basis of $\{e_1, \ldots, e_N\}$ with $\langle e_i, E_j\rangle = \delta_{ij}$ $(1 \leq i, j \leq N)$. Then, $\mathfrak{a}_{\mathbb{R}}$ is isomorphic to $V^* \subset V_{\mathbb{R}}^* = \mathbb{R}E_1 \oplus \cdots \oplus \mathbb{R}E_N$. Hence, $\mathfrak{a} = \mathfrak{a}_{\mathbb{R}} + \sqrt{-1}\mathfrak{a}_{\mathbb{R}} \simeq V^* + \sqrt{-1}V^*$.

We have seen in Lemma 3.1 that the complex conjugation σ with respect to $\mathfrak{g}_{\mathbb{R}}$ stabilizes the root space $\mathfrak{g}_\alpha$ $(\alpha \in \Delta)$. Then, we decompose $\mathfrak{g}_\alpha$ into the σ-eigenspaces as $\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{g}_{\mathbb{R}}) + (\mathfrak{g}_\alpha \cap \sqrt{-1}\mathfrak{g}_{\mathbb{R}})$, equivalently, $\mathfrak{g}_\alpha$ is the complexification of the real vector space $\mathfrak{g}_\alpha \cap \mathfrak{g}_{\mathbb{R}}$. As $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$, we have $\dim(\mathfrak{g}_\alpha \cap \mathfrak{g}_{\mathbb{R}}) = 1$. Thus, we express $\mathfrak{g}_\alpha \cap \mathfrak{g}_{\mathbb{R}} = \mathbb{R}E_\alpha$ for some root vector E_α.

Under this setting, we carry out for each spherical nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{g}$ as follows:

1. Specify the characteristic element $H \in \mathfrak{a}_+$ from the corresponding weighted Dynkin diagram $\Omega(\mathcal{O}_X)$ (see (5.2)).

2. Write $\mathfrak{l} = \mathfrak{g}(0)$, $\mathfrak{g}(2)$, and $\mathfrak{g}(3)$, respectively, as a direct sum of root spaces (see Lemma 3.2).

3. Verify that the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is a multiplicity-free action comparing with Table 5.1. Then, the L_u-action on $\mathfrak{n}$ is strongly visible (Lemma 5.5).

4. Give a slice S_0 for the L_u-action on $\mathfrak{n}$ explicitly by using Table 5.2. In particular, describe S_0 as $S_0 = \bigoplus_{\alpha \in \Delta^+(\mathcal{O}_X)} \mathbb{R} E_\alpha$ for some subset $\Delta^+(\mathcal{O}_X)$ in Δ^+.

Owing to Lemmas 5.6 and 5.7, Theorem 3.5 holds for the subspace S_0 which is constructed according to the above procedure.

5.4. Type $\mathbf{A}_{n-1}$. We begin with the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ for integer $n \geq 2$. In this case, $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(n, \mathbb{R})$. Then, $\mathfrak{a}_{\mathbb{R}}^* = \{a_1 e_1 + \cdots + a_n e_n : a_1, \ldots, a_n \in \mathbb{R},\ a_1 + \cdots + a_n = 0\}$. A root system $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{a})$ is $\Delta = \{\pm(e_i - e_j) : 1 \leq i < j \leq n\}$. We fix a positive system as $\Delta^+ = \{e_i - e_j : 1 \leq i < j \leq n\}$. The simple roots $\alpha_1, \ldots, \alpha_{n-1}$ are given by $\alpha_i = e_i - e_{i+1}$ $(1 \leq i \leq n-1)$. The highest root β is written as $\beta = e_1 - e_n = \alpha_1 + \cdots + \alpha_{n-1}$.

Let $\mathcal{O}_X$ be a nilpotent orbit with characteristic element $H = h_1 E_1 + \cdots + h_n E_n \in \mathfrak{a}_+$ where $h_1 + \cdots + h_n = 0$ and $h_1 \geq h_2 \geq \cdots \geq h_n$. Then, $\alpha_i(H) = h_i - h_{i+1}$ $(1 \leq i \leq n-1)$. Hence, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, m_2, \ldots, m_{n-1})$ is given by $(h_1 - h_2, h_2 - h_3, \ldots, h_{n-1} - h_n)$.

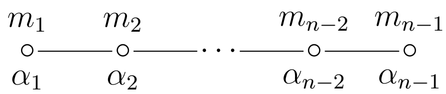

Figure 5.1: Weighted Dynkin diagram for $\mathfrak{sl}(n, \mathbb{C})$

A nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{sl}(n, \mathbb{C})$ is spherical if and only if $\Omega(\mathcal{O}_X)$ coincides with either (A) or (A$'$):

(A) $\Omega(\mathcal{O}_X) = (\underbrace{0, \ldots, 0}_{p-1}, 1, 0, \ldots 0, 1, \underbrace{0, \ldots, 0}_{p-1})$ for $1 \leq p < \frac{n}{2}$, namely, $m_p = m_{n-p} = 1$ and $m_i = 0$ for $i \neq p, n-p$.

(A$'$) $n = 2p$ and $\Omega(\mathcal{O}_X) = (\underbrace{0, \ldots, 0}_{p-1}, 2, \underbrace{0, \ldots, 0}_{p-1})$, namely, $m_p = 2$ and $m_i = 0$ for $i \neq p$.

For each case, it follows from Lemma 5.4 that its height $\mathrm{ht}(\mathcal{O}_X)$ equals two. Then, the nilpotent subalgebra $\mathfrak{n}$ coincides with $\mathfrak{g}(2)$.

Case (A). Let $\Omega(\mathcal{O}_X)$ satisfy Case (A) for $1 \leq p < \frac{n}{2}$. Since $h_p - h_{p+1} = h_{n-p} - h_{n-p+1} = 1$ and $h_i - h_{i+1} = 0$ $(i \neq p, n-p)$, H forms

$$H = (E_1 + \cdots + E_p) - (E_{n-p+1} + \cdots + E_n).$$

The nilpotent orbit $\mathcal{O}_X$ with the above $\Omega(\mathcal{O}_X)$ consists of complex matrices of degree n with Jordan type $(2^p, 1^{n-2p})$.

The Levi subalgebra $\mathfrak{l} = \mathfrak{g}(0)$ is given as follows:

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \le i < j \le p} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \bigoplus_{p+1 \le i < j \le n-p} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \bigoplus_{n-p+1 \le i < j \le n} \mathfrak{g}_{\pm(e_i - e_j)}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(n-2p, \mathbb{C}) \oplus \mathfrak{sl}(p, \mathbb{C}) \oplus \mathbb{C}^2.$$

Here, $\{(E_1 + \cdots + E_p) - \frac{p}{n-2p}(E_{p+1} + \cdots + E_{n-p}),\ \frac{p}{n-2p}(E_{p+1} + \cdots + E_{n-p}) - (E_{n-p+1} + \cdots + E_n)\}$ is a basis of the two-dimensional center $\mathbb{C}^2$.

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1 \le i,j \le p} \mathfrak{g}_{e_i - e_{n-p+j}}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to $M(p, \mathbb{C})$, from which

$$\mathfrak{n} \simeq M(p, \mathbb{C}). \tag{5.7}$$

The semisimple part $SL(p, \mathbb{C}) \times SL(n-2p, \mathbb{C}) \times SL(p, \mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts on $M(p, \mathbb{C})$ by

$$(g_1, h, g_2) \cdot A = g_1 A g_2^{-1},$$

and the center $(\mathbb{C}^\times)^2$ of $L_{\mathbb{C}}$ as the scalar multiplication as follows:

$$(s, t) \cdot A = stA.$$

Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the irreducible action of $SL(p, \mathbb{C}) \times SL(p, \mathbb{C}) \times \mathbb{C}^\times$ on $M(p, \mathbb{C})$. It follows from (5) of Table 5.1 that this action is a multiplicity-free action.

A compact real form L_u of $L_{\mathbb{C}}$ is isomorphic to $S(U(p) \times U(n-2p) \times U(p))$. Here, $SU(N)$ denotes the special unitary group $\{g \in SL(N, \mathbb{C}) : {}^t\overline{g}g = I_N\}$. We take the subset S_0 in $\mathfrak{n}$ as

$$S_0 := \bigoplus_{1 \le i \le p} \mathbb{R} E_{e_i - e_{n-p+i}}. \tag{5.8}$$

Then, S_0 is isomorphic to the slice D_p of Table 5.2 for the strongly visible $(SU(p) \times SU(p) \times \mathbb{T})$-action on $M(p, \mathbb{C})$. By Lemma 5.6, the vector space S_0 satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, we have verified Theorem 3.5 for Case (A).

Case (A′). In case of $n = 2p$ and $\Omega(\mathcal{O}_X)$ satisfies Case (A′), the characteristic element $H \in \mathfrak{a}$ is of the form

$$H = (E_1 + \cdots + E_p) - (E_{p+1} + \cdots + E_{2p}).$$

Then, $\mathfrak{l} \simeq \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(p, \mathbb{C}) \oplus \mathbb{C}$ and $\mathfrak{n} \simeq M(p, \mathbb{C})$. Hence, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is a multiplicity-free action. Therefore, the subset S_0 defined by (5.8) satisfies Theorem 3.5, from which we have verified for Case (A′).

5.5. Type $\mathbf{B}_n$. In this subsection, we give a proof of Theorem 3.5 for $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C})$ for a positive integer $n \geq 2$. In this case, $\mathfrak{g}_{\mathbb{R}}$ is isomorphic to $\mathfrak{so}(n+1,n)$. Then, we have $\mathfrak{a}_{\mathbb{R}}^* = V_{\mathbb{R}}$. A root system $\Delta \equiv \Delta(\mathfrak{g},\mathfrak{a})$ is $\Delta = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \sqcup \{\pm e_i : 1 \leq i \leq n\}$. We fix a positive system as $\Delta^+ = \{e_i \pm e_j : 1 \leq i < j \leq n\} \sqcup \{e_i : 1 \leq i \leq n\}$. The simple roots $\alpha_1, \dots, \alpha_n$ are given by $\alpha_i = e_i - e_{i+1}$ $(1 \leq i \leq n-1)$, and $\alpha_n = e_n$. The highest root β is written as $\beta = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$.

Let $\mathcal{O}_X$ be a nilpotent orbit with characteristic element $H = h_1E_1 + \cdots + h_nE_n \in \mathfrak{a}_+$ with $h_1 \geq \cdots \geq h_n \geq 0$. Then, we have $\alpha_i(H) = h_i - h_{i+1}$ $(1 \leq i \leq n-1)$, and $\alpha_n(H) = h_n$. Thus, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, \dots, m_{n-1}, m_n)$ is given by $(h_1 - h_2, \dots, h_{n-1} - h_n, h_n)$.

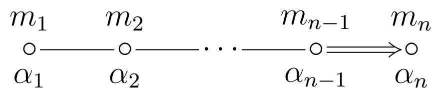

Figure 5.2: Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{so}(2n+1,\mathbb{C})$

A nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{so}(2n+1,\mathbb{C})$ is spherical if and only if $\Omega(\mathcal{O}_X)$ forms one of the following cases:

(B1) $\Omega(\mathcal{O}_X) = (2,0,\dots,0)$, namely, $m_1 = 2$ and $m_i = 0$ $(i \neq 1)$.

(B2) $\Omega(\mathcal{O}_X) = (\underbrace{0,\dots,0}_{2p-1},1,0,\dots,0)$ for $1 \leq p \leq \frac{n}{2}$, namely, $m_{2p} = 1$ and $m_i = 0$ $(i \neq 2p)$.

(B3) $\Omega(\mathcal{O}_X) = (1,\underbrace{0,\dots,0}_{2p-1},1,0,\dots,0)$ for $1 \leq p \leq \frac{n-1}{2}$, namely, $m_1 = m_{2p+1} = 1$ and $m_i = 0$ $(i \neq 1, 2p+1)$.

By Lemma 5.4, its height $\mathrm{ht}(\mathcal{O}_X)$ equals two for Cases (B1), (B2), and three for Case (B3).

Case (B1). Let us consider the case $\Omega(\mathcal{O}_X) = (2,0,\dots,0)$. Then, H is given by

$$H = 2E_1.$$

This $\mathcal{O}_X$ consists of complex matrices with Jordan type $(3, 1^{2n-2})$.

The Levi subalgebra $\mathfrak{l} = \mathfrak{g}(0)$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \leq i < j \leq n} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \bigoplus_{2 \leq i \leq n} \mathfrak{g}_{\pm e_i}.$$

Then,

$$\mathfrak{l} \simeq \mathfrak{so}(2n-1,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{2\le j\le n} \mathfrak{g}_{e_1\pm e_j} \oplus \mathfrak{g}_{e_1}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathbb{C}^{2n-1}$. As $\mathrm{ht}(\mathcal{O}_X) = 2$, the nilpotent subalgebra $\mathfrak{n}$ coincides with $\mathfrak{g}(2)$, namely,

$$\mathfrak{n} \simeq \mathbb{C}^{2n-1}.$$

The semisimple part $SO(2n-1,\mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts on $\mathbb{C}^{2n-1}$ as the standard action, namely,

$$g \cdot v = gv,$$

and its center $\mathbb{C}^\times$ acts as the scalar multiplication. This implies that the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SO(2n-1,\mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathbb{C}^{2n-1}$. It follows from (6) of Table 5.1 that this action is a multiplicity-free action.

A compact real form of $L_{\mathbb{C}}$ is isomorphic to $SO(2n-1) \times \mathbb{T}$. Here, $SO(N)$ denotes the special orthogonal group $\{g \in GL(N,\mathbb{R}) : {}^t gg = I_N\}$. We take the subset S_0 in $\mathfrak{n}$ as

$$S_0 := \mathbb{R}E_{e_1+e_2} \oplus \mathbb{R}E_{e_1-e_2}.$$

Then, S_0 is isomorphic to the slice $D_{1,1}$ of Table 5.2 for the strongly visible $(SO(2n-1) \times \mathbb{T})$-action on $\mathbb{C}^{2p-1}$. By Lemma 5.6, S_0 satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, we have verified Theorem 3.5 for Case (B1).

Case (B2). Let $\Omega(\mathcal{O}_X)$ satisfy $m_{2p} = 1$ and $m_i = 0$ $(i \neq 2p)$ for $1 \le p \le \frac{n}{2}$. Then,

$$H = E_1 + E_2 + \cdots + E_{2p}.$$

This $\mathcal{O}_X$ consists of complex matrices with Jordan type $(2^{2p}, 1^{2n-4p+1})$. In particular, $\mathcal{O}_X$ with $\Omega(\mathcal{O}_X) = (0,1,0,0,\ldots,0)$ $(p = 1)$ is the minimal nilpotent orbit.

The Levi subalgebra $\mathfrak{l} = \mathfrak{g}(0)$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1\le i<j\le 2p} \mathfrak{g}_{\pm(e_i-e_j)} \oplus \bigoplus_{2p+1\le i<j\le n} \mathfrak{g}_{\pm e_i\pm e_j} \oplus \bigoplus_{2p+1\le i\le n} \mathfrak{g}_{\pm e_i}$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(2p,\mathbb{C}) \oplus \mathfrak{so}(2n-4p+1,\mathbb{C}) \oplus \mathbb{C},$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1\le i<j\le 2p} \mathfrak{g}_{e_i+e_j}$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathrm{Alt}(2p,\mathbb{C})$. As $\mathrm{ht}(\mathcal{O}_X)=2$,

$$\mathfrak{n}=\mathfrak{g}(2)\simeq \mathrm{Alt}(2p,\mathbb{C}).$$

The semisimple part of the Levi subgroup $L_{\mathbb{C}}$ is isomorphic to $SL(2p,\mathbb{C})\times SO(2n-4p+1,\mathbb{C})$. Then, $SL(2p,\mathbb{C})$ acts on $\mathrm{Alt}(2p,\mathbb{C})$ by

$$g\cdot A = gA\,{}^t g,$$

and $SO(2n-4p+1,\mathbb{C})$ acts trivially. Further, its center $\mathbb{C}^\times$ acts as the scalar multiplication. This implies that the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the action of $SL(2p,\mathbb{C})\times\mathbb{C}^\times$ on $\mathrm{Alt}(2p,\mathbb{C})$. It follows from (4) of Table 5.1 that this action is a multiplicity-free action.

We take S_0 as

$$S_0 := \bigoplus_{1\le i\le p}\mathbb{R}E_{e_{2i-1}+e_{2i}}.$$

Then, S_0 is isomorphic to the slice A_p of Table 5.2 for the $(SU(2p)\times\mathbb{T})$-action on $\mathrm{Alt}(2p,\mathbb{C})$. By Lemma 5.6, Theorem 3.5 holds for Case (B2).

Case (B3). Let $\Omega(\mathcal{O}_X)$ satisfy $m_1=m_{2p+1}=1$ and $m_i=0$ for $1\le p\le \frac{n-1}{2}$. Then, $\mathrm{ht}(\mathcal{O}_X)=3$ and

$$H=2E_1+E_2+E_3+\cdots+E_{2p+1}.$$

This $\mathcal{O}_X$ consists of complex matrices with Jordan type $(3,2^{2p},1^{2n-4p-2})$.

We divide Case (B3) into two cases: $n\neq 2p+1$; and $n=2p+1$.

First, let us consider the general case $n\neq 2p+1$. Then, the Levi subalgebra $\mathfrak{l}=\mathfrak{g}(0)$ is given by

$$\mathfrak{l}=\mathfrak{a}\oplus\bigoplus_{2\le i<j\le 2p+1}\mathfrak{g}_{\pm(e_i-e_j)}\oplus\bigoplus_{2p+2\le i<j\le n}\mathfrak{g}_{\pm e_i\pm e_j}\oplus\bigoplus_{2p+2\le i\le n}\mathfrak{g}_{\pm e_i}.$$

This means that

$$\mathfrak{l}\simeq\mathfrak{sl}(2p,\mathbb{C})\oplus\mathfrak{so}(2n-4p-1,\mathbb{C})\oplus\mathbb{C}^2.$$

We note that $\{E_1, E_2+\cdots+E_{2p+1}\}$ is a basis of the center $\mathbb{C}^2$.

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2)=\mathfrak{g}_{e_1}\oplus\bigoplus_{2p+2\le j\le n}\mathfrak{g}_{e_1\pm e_j}\oplus\bigoplus_{2\le i<j\le 2p+1}\mathfrak{g}_{e_i+e_j}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathbb{C}^{2n-4p-1}\oplus\mathrm{Alt}(2p,\mathbb{C})$. Further, $\mathfrak{g}(3)$ is of the form

$$\mathfrak{g}(3)=\bigoplus_{2\le j\le 2p+1}\mathfrak{g}_{e_1+e_j}\simeq\mathbb{C}^{2p}.$$

Hence, $\mathfrak{n}$ is isomorphic to

$$\mathfrak{n}=\mathfrak{g}(2)\oplus\mathfrak{g}(3)\simeq\mathbb{C}^{2n-4p-1}\oplus\mathrm{Alt}(2p,\mathbb{C})\oplus\mathbb{C}^{2p}. \tag{5.9}$$

The semisimple part $SL(2p,\mathbb{C})\times SO(2n-4p-1,\mathbb{C})$ of $L_{\mathbb{C}}$ acts on $\mathbb{C}^{2n-4p-1}\oplus \mathrm{Alt}(2p,\mathbb{C})\oplus\mathbb{C}^{2p}$ by

$$(g,h)\cdot(v,A,w)=(hv,gA\,{}^tg,gw),$$

and its center $(\mathbb{C}^\times)^2$ acts by

$$(s,t)\cdot(v,A,w)=(sv,t^2A,stw).$$

Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the decomposable action consisting of the indecomposable $(SL(2p,\mathbb{C})\times\mathbb{C}^\times)$-action on $\mathrm{Alt}(2p,\mathbb{C})\oplus\mathbb{C}^{2p}$ ((8) of Table 5.1) and the irreducible $(SO(2n-4p-1,\mathbb{C})\times\mathbb{C}^\times)$-action on $\mathbb{C}^{2n-4p-1}$ ((6) of Table 5.1). Hence, this action is a multiplicity-free action.

Our slice for this action is the direct sum of the slices S_0' and S_0'' which are isomorphic to the slice $A_p\oplus T_p$ of Table 5.2 for the action of $SU(2p)\times\mathbb{T}$ on $\mathrm{Alt}(2p,\mathbb{C})\oplus\mathbb{C}^{2p}$ and $D_{1,1}$ for the action of $SO(2n-4p-1)\times\mathbb{T}$ on $\mathbb{C}^{2n-4p-1}$, respectively. In fact, we define

$$S_0':=\bigoplus_{1\le i\le p}\mathbb{R}E_{e_{2i}+e_{2i+1}}\oplus\bigoplus_{1\le j\le p}\mathbb{R}E_{e_1+e_{2j}}, \tag{5.10}$$

and

$$S_0'':=\mathbb{R}E_{e_1+e_{2p+2}}\oplus\mathbb{R}E_{e_1-e_{2p+2}}.$$

By Lemma 5.6, the subspace

$$S_0:=S_0'\oplus S_0''\simeq(D_{1,1}\oplus A_p)\oplus T_p \tag{5.11}$$

satisfies $\mathfrak{n}=L_u\cdot S_0$. Therefore, Theorem 3.5 has been proved for Case (B3) with $n\neq 2p+1$.

In the special case where $\mathfrak{g}=\mathfrak{so}(4p+3,\mathbb{C})$ and $\Omega(\mathcal{O}_X)=(1,0,\dots,0,1)$, The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l}=\mathfrak{a}\oplus\bigoplus_{2\le i<j\le 2p+1}\mathfrak{g}_{\pm(e_i-e_j)}\simeq\mathfrak{sl}(2p,\mathbb{C})\oplus\mathbb{C}^2.$$

The $\mathrm{ad}(H)$-eigenspaces $\mathfrak{g}(2),\mathfrak{g}(3)$ are written as

$$\mathfrak{g}(2)=\mathfrak{g}_{e_1}\oplus\bigoplus_{2\le i<j\le 2p+1}\mathfrak{g}_{e_i+e_j}\simeq\mathbb{C}\oplus\mathrm{Alt}(2p,\mathbb{C}),$$
$$\mathfrak{g}(3)=\bigoplus_{2\le j\le 2p+1}\mathfrak{g}_{e_1+e_j}\simeq\mathbb{C}^{2p}.$$

We take $S_0'\simeq A_p\oplus T_p$ as in (5.10) and S_0'' as

$$S_0'':=\mathbb{R}E_{e_1}\simeq\mathbb{R}.$$

By Lemma 5.6, the vector space

$$S_0:=S_0'\oplus S_0''\simeq(\mathbb{R}\oplus A_p)\oplus T_p$$

satisfies $\mathfrak{n} = L_u \cdot S_0$. Hence, Theorem 3.5 has been proved for Case (B3).

5.6. Type $\mathbf{C}_n$. In this subsection, we give a proof of Theorem 3.5 for $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. In this case, $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$. Then, $\mathfrak{a}_{\mathbb{R}}^* \simeq V_{\mathbb{R}}$. A root system $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{a})$ is $\Delta = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \sqcup \{\pm 2e_i : 1 \leq i \leq n\}$. We fix a positive system Δ^+ as $\Delta^+ = \{e_i \pm e_j : 1 \leq i < j \leq n\} \sqcup \{2e_i : 1 \leq i \leq n\}$. The simple roots $\alpha_1, \ldots, \alpha_n$ are given by $\alpha_i = e_i - e_{i+1}$ $(1 \leq i \leq n-1)$ and $\alpha_n = 2e_n$. The highest root β is written as $\beta = 2e_1 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$.

Let $\mathcal{O}_X$ be a nilpotent orbit with characteristic element $H = h_1 E_1 + \cdots + h_n E_n \in \mathfrak{a}_+$ with $h_1 \geq \cdots \geq h_n \geq 0$. Then, $m_i = h_i - h_{i+1}$ $(i = 1, 2, \ldots, n-1)$, and $m_n = 2h_n$. Hence, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, \ldots, m_n)$ is given by $\Omega(\mathcal{O}_X) = (h_1 - h_2, \ldots, h_{n-1} - h_n, 2h_n)$.

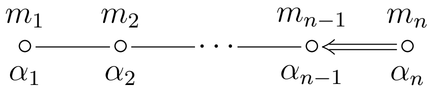

Figure 5.3: Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{sp}(n, \mathbb{C})$

A nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{sp}(n, \mathbb{C})$ is spherical if and only if $\Omega(\mathcal{O}_X)$ satisfies either Case (C) or Case (C′):

(C) $\Omega(\mathcal{O}_X) = (\underbrace{0, \ldots, 0}_{p-1}, 1, 0, \ldots, 0)$ for $1 \leq p < n$, namely, $m_p = 1$ and $m_i = 0$ $(i \neq p)$.

(C′) $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 2)$, namely, $m_n = 2$ and $m_i = 0$ $(i \neq n)$.

Then, the characteristic element $H \in \mathfrak{a}$ is written as

$$H = E_1 + \cdots + E_p \tag{5.12}$$

for each $\Omega(\mathcal{O}_X)$. It follows from Lemma 5.4 that the height of $\mathcal{O}_X$ equals two. Further, Such $\mathcal{O}_X$ consists of complex matrices with Jordan type $(2^p, 1^{2n-2p})$ $(1 \leq p \leq n)$. In particular, $\mathcal{O}_X$ with weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (1, 0, \ldots, 0)$ is the minimal nilpotent orbit.

First, let us consider the general $p \neq n$. The Levi subalgebra $\mathfrak{l} = \mathfrak{g}(0)$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq i < j \leq p} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \bigoplus_{p+1 \leq i < j \leq n} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \bigoplus_{p+1 \leq i \leq n} \mathfrak{g}_{\pm 2e_i}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sp}(n-p, \mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1 \leq i < j \leq p} \mathfrak{g}_{e_i + e_j} \oplus \bigoplus_{1 \leq i \leq p} \mathfrak{g}_{2e_i}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathrm{Sym}(p,\mathbb{C})$, from which

$$\mathfrak{n} \simeq \mathrm{Sym}(p,\mathbb{C}).$$

The action of the semisimple part $SL(p,\mathbb{C}) \times Sp(n-p,\mathbb{C})$ on $\mathrm{Sym}(p,\mathbb{C})$ is written as follows: $SL(p,\mathbb{C})$ acts by

$$g \cdot A = gA\,{}^t g,$$

and $Sp(n-p,\mathbb{C})$ acts trivially. Its center $\mathbb{C}^\times$ acts as the scalar multiplication. Then, the $L_\mathbb{C}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SL(p,\mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathrm{Sym}(p,\mathbb{C})$. It follows from (3) of Table 5.1 that this action is a multiplicity-free action.

We take the subset S_0 as

$$S_0 = \bigoplus_{1 \le i \le p} \mathbb{R} E_{2e_i}. \tag{5.13}$$

Then, S_0 is isomorphic to the slice D_p of Table 5.2 for the strongly visible $(SU(p) \times \mathbb{T})$-action on $\mathrm{Sym}(p,\mathbb{C})$. By Lemma 5.6, this S_0 satisfies $\mathfrak{n} = L_u \cdot S_0$.

In case of $p = n$, the Levi subalgebra $\mathfrak{l}$ is

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \le i < j \le n} \mathfrak{g}_{\pm(e_i - e_j)} \simeq \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C},$$

and $\mathfrak{g}(2)$ is

$$\mathfrak{g}(2) = \bigoplus_{1 \le i < j \le n} \mathfrak{g}_{e_i + e_j} \oplus \bigoplus_{1 \le i \le n} \mathfrak{g}_{2e_i} \simeq \mathrm{Sym}(n,\mathbb{C}).$$

Then, the $L_\mathbb{C}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SL(n,\mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathrm{Sym}(n,\mathbb{C})$. Hence, the equation $\mathfrak{n} = L_u \cdot S_0$ holds for the subset S_0 defined by (5.13)

Therefore, Theorem 3.5 has been verified for $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C})$

5.7. Type $\mathbf{D}_n$. In this subsection, we give a proof of Theorem 3.5 for $\mathfrak{g} = \mathfrak{so}(2n,\mathbb{C})$ for integer $n \ge 4$. In this case, $\mathfrak{g}_\mathbb{R}$ is isomorphic to $\mathfrak{so}(n,n)$. Then, $\mathfrak{a}_\mathbb{R}^* = V_\mathbb{R}$. A root system $\Delta \equiv \Delta(\mathfrak{g},\mathfrak{a})$ is $\Delta = \{\pm e_i \pm e_j : 1 \le i < j \le n\}$. We fix a positive system Δ^+ as $\Delta^+ = \{e_i \pm e_j : 1 \le i < j \le n\}$. The simple roots $\alpha_1, \dots, \alpha_n$ are given by $\alpha_i = e_i - e_{i+1}$ $(1 \le i \le n-1)$ and $\alpha_n = e_{n-1} + e_n$. The highest root β is written as $\beta = e_1 + e_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$.

Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{so}(2n,\mathbb{C})$ with characteristic element $H = h_1 E_1 + \cdots + h_n E_n \in \mathfrak{a}_+$ with $h_1 \ge \cdots \ge h_{n-1} \ge |h_n|$. Then, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, \dots, m_n)$ is given by $(h_1 - h_2, \dots, h_{n-1} - h_n, h_{n-1} + h_n)$.

A nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{so}(2n,\mathbb{C})$ is spherical if and only if $\Omega(\mathcal{O}_X)$ satisfies one of the following cases:

(D1) $\Omega(\mathcal{O}_X) = (2, 0, \dots, 0)$, namely, $m_1 = 2$ and $m_i = 0$ $(i \ne 1)$.

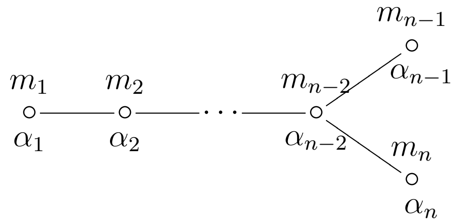

Figure 5.4: Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{so}(2n, \mathbb{C})$

(D2) $\Omega(\mathcal{O}_X) = (\underbrace{0, \ldots, 0}_{2p-1}, 1, 0, \ldots, 0)$ for $1 \leq p \leq \frac{n}{2}-1$, namely, $m_{2p} = 1$ and $m_i = 0$ $(i \neq 2p)$.

(D2′) $n = 2p+1$ and $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 1, 1)$, namely, $m_{2p} = m_{2p+1} = 1$ and $m_i = 0$ $(i \neq 2p, 2p+1)$.

(D2″) $n = 2p$ and $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 2)$, namely, $m_{2p} = 2$ and $m_i = 0$ $(i \neq 2p)$.

(D2‴) $n = 2p$ and $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 2, 0)$, namely, $m_{2p-1} = 2$ and $m_i = 0$ $(i \neq 2p-1)$.

(D3) $\Omega(\mathcal{O}_X) = (1, \underbrace{0, \ldots, 0}_{2p-1}, 1, 0, \ldots, 0)$ for $1 \leq p < \frac{n}{2} - 1$, namely, $m_1 = m_{2p+1} = 1$ and $m_i = 0$ $(i \neq 1, 2p+1)$.

(D3′) $n = 2p+2$ and $\Omega(\mathcal{O}_X) = (1, 0, 0, \ldots, 0, 1, 1)$, namely, $m_1 = m_{2p+1} = m_{2p+2} = 1$ and $m_i = 0$ $(i \neq 1, 2p+1, 2p+2)$.

By Lemma 5.4, the height of $\mathcal{O}_X$ equals two for Cases (D1)–(D2″), and three for Cases (D3), (D3′).

Case (D1). Let us consider the case $\Omega(\mathcal{O}_X) = (2, 0, \ldots, 0)$. Then, the characteristic element H is of the form

$$H = 2E_1.$$

This $\mathcal{O}_X$ consists of all complex matrices with Jordan type $(3, 1^{2n-3})$.

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \leq i < j \leq n} \mathfrak{g}_{\pm e_i \pm e_j}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{so}(2n-2, \mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{2 \leq j \leq n} \mathfrak{g}_{e_1 \pm e_j}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathbb{C}^{2n-2}$, from which

$$\mathfrak{n} \simeq \mathbb{C}^{2n-2}.$$

Similarly to Case (B1), it turns out that the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SO(2n-2,\mathbb{C}) \times \mathbb{C}^{\times})$-action on $\mathbb{C}^{2n-2}$. It follows from (6) of Table 5.1 that this action is a multiplicity-free action.

We take the subset S_0 in $\mathfrak{n}$ as

$$S_0 = \mathbb{R}E_{e_1+e_2} \oplus \mathbb{R}E_{e_1-e_2}.$$

Then, S_0 is isomorphic to the slice $D_{1,1}$ of Table 5.2 for the $(SO(2n-2) \times \mathbb{T})$-action on $\mathbb{C}^{2n-2}$. By Lemma 5.6, S_0 satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified for Case (D1).

Case (D2). Let $\Omega(\mathcal{O}_X)$ satisfy $m_{2p} = 1$ and $m_i = 0$ $(i \neq 2p)$ for $1 \leq p \leq \frac{n}{2} - 1$. Then,

$$H = E_1 + E_2 + \cdots + E_{2p}. \tag{5.14}$$

This $\mathcal{O}_X$ consists of all complex matrices with Jordan type $(2^{2p}, 1^{2n-4p})$. In particular, $\mathcal{O}_X$ with Jordan type $(2^2, 1^{2n-4})$ $(p = 1)$ is the minimal nilpotent orbit in $\mathfrak{so}(2n,\mathbb{C})$.

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1\leq i<j\leq 2p} \mathfrak{g}_{\pm(e_i-e_j)} \oplus \bigoplus_{2p+1\leq i<j\leq n} \mathfrak{g}_{\pm e_i \pm e_j}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(2p,\mathbb{C}) \oplus \mathfrak{so}(2n-4p,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1\leq i<j\leq 2p} \mathfrak{g}_{e_i+e_j}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathrm{Alt}(2p,\mathbb{C})$, from which

$$\mathfrak{n} \simeq \mathrm{Alt}(2p,\mathbb{C}).$$

The semisimple part $SL(2p,\mathbb{C}) \times SO(2n-4p,\mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts on $\mathrm{Alt}(2p,\mathbb{C})$ as follows: $SL(2p,\mathbb{C})$ by

$$g \cdot A = gA\,{}^t g,$$

and $SO(2n-4p,\mathbb{C})$ trivially. Its center $\mathbb{C}^{\times}$ acts as scalar multiplications. Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SL(2p,\mathbb{C}) \times \mathbb{C}^{\times})$-action on $\mathrm{Alt}(2p,\mathbb{C})$. It follows from (4) of Table 5.1 that this action is a multiplicity-free action.

We take the subset S_0 in $\mathfrak{n}$ as

$$S_0 := \bigoplus_{j=1}^{p} \mathbb{R} E_{e_{2j-1}+e_{2j}}. \tag{5.15}$$

Then, S_0 is isomorphic to our slice A_p of Table 5.2 for the $(SU(2p) \times \mathbb{T})$-action on $\mathrm{Alt}(2p, \mathbb{C})$. By Lemma 5.6, we have $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified for Case (D2).

Case (D2 ′). Let us consider the case where $\mathfrak{g} = \mathfrak{so}(4p+2, \mathbb{C})$ $(n = 2p+1)$ and $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 1, 1)$. This nilpotent orbit is the set of all complex matrices with Jordan type $(2^{2p}, 1^2)$. Then, the proof for Case (D2 ′) can be given similarly to Case (D2). In fact, the characteristic element H forms

$$H = E_1 + \cdots + E_{2p}$$

which is the same as in (5.14).

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq i < j \leq 2p} \mathfrak{g}_{\pm(e_i - e_j)} \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C}^2.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1 \leq i < j \leq 2p} \mathfrak{g}_{e_i + e_j} \simeq \mathrm{Alt}(2p, \mathbb{C}).$$

Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SL(2p, \mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathrm{Alt}(2p, \mathbb{C})$. Similarly to the previous case, $\mathfrak{n} = L_u \cdot S_0$ holds for the subset S_0 defined by (5.15). Hence, Theorem 3.5 for Case (D2 ′) has been verified.

Case (D2 ″). Let us treat the case where $\mathfrak{g} = \mathfrak{so}(4p, \mathbb{C})$ and $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 2)$. This nilpotent orbit $\mathcal{O}_X$ is very even, namely, $\mathcal{O}_X$ is the set of all complex matrices with Jordan type (2^{2p}). The characteristic element H forms the same as in (5.14). Then,

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq i < j \leq 2p} \mathfrak{g}_{\pm(e_i - e_j)} \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C},$$
$$\mathfrak{n} = \mathfrak{g}(2) = \bigoplus_{1 \leq i < j \leq 2p} \mathfrak{g}_{e_i + e_j} \simeq \mathrm{Alt}(2p, \mathbb{C}).$$

Similarly to Cases (D2) and (D2′), the equality $\mathfrak{n} = L_u \cdot S_0$ holds for S_0 defined by (5.15). Hence, Theorem 3.5 for Case (D2 ″) has been checked.

Case (D2 ‴). There are two weighted Dynkin diagrams corresponding to nilpotent orbits $\mathcal{O}_X$ in $\mathfrak{so}(4p, \mathbb{C})$ with Jordan type (2^{2p}). One is Case (D2 ″), the other is $\Omega(\mathcal{O}_X) = (0, \ldots, 0, 2, 0)$, namely, Case (D2 ‴). For the latter case, H is of the form

$$H = E_1 + \cdots + E_{2p-1} - E_{2p}$$

which is slightly different from the form in (5.14).

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \le i < j \le 2p-1} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \bigoplus_{1 \le i \le 2p-1} \mathfrak{g}_{\pm(e_i + e_{2p})}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1 \le i < j \le 2p-1} \mathfrak{g}_{e_i + e_j} \oplus \bigoplus_{1 \le i \le 2p-1} \mathfrak{g}_{e_i - e_{2p}}.$$

Then,

$$\mathfrak{n} \simeq \mathrm{Alt}(2p, \mathbb{C}).$$

Hence, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SL(2p,\mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathrm{Alt}(2p,\mathbb{C})$, which is a multiplicity-free action.

We take S_0 as

$$S_0 = \bigoplus_{1 \le i \le p-1} \mathbb{R} E_{e_{2j-1} + e_{2j}} \oplus \mathbb{R} E_{e_{2p-1} - e_{2p}}.$$

Then, S_0 is isomorphic to our slice A_p of Table 5.2 for the $(SU(2p) \times \mathbb{T})$-action via $\mathfrak{n} \simeq \mathrm{Alt}(2p,\mathbb{C})$. By Lemma 5.6, $\mathfrak{n} = L_u \cdot S_0$. Hence, Theorem 3.5 has been verified.

Case (D3). Let $\Omega(\mathcal{O}_X)$ satisfy $m_1 = m_{2p+1} = 1$ and $m_i = 0$ $(i \neq 1, 2p+1)$ for $1 \le p \le \frac{n}{2} - 1$. Then, H is of the form

$$H = 2E_1 + E_2 + \cdots + E_{2p+1}. \tag{5.16}$$

This $\mathcal{O}_X$ is the set of all complex matrices with Jordan type $(3, 2^{2p}, 1^{2n-4p-3})$.

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \le i < j \le 2p+1} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \bigoplus_{2p+2 \le i < j \le n} \mathfrak{g}_{\pm e_i \pm e_j},$$

from which $\mathfrak{l}$ is isomorphic to

$$\mathfrak{l} \simeq \mathfrak{sl}(2p, \mathbb{C}) \oplus \mathfrak{so}(2n - 4p - 2, \mathbb{C}) \oplus \mathbb{C}^2.$$

Here, $\{E_1, E_2 + \cdots + E_{2p+1}\}$ is a basis of the two-dimensional center $\mathbb{C}^2$.

Next, the $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{2 \le i < j \le 2p+1} \mathfrak{g}_{e_i + e_j} \oplus \bigoplus_{2p+2 \le j \le n} \mathfrak{g}_{e_1 \pm e_j}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to the direct sum $\mathrm{Alt}(2p,\mathbb{C})\oplus\mathbb{C}^{2n-4p-2}$. The eigenspace $\mathfrak{g}(3)$ forms

$$\mathfrak{g}(3)=\bigoplus_{2\le j\le 2p+1}\mathfrak{g}_{e_1+e_j}\simeq\mathbb{C}^{2p}.$$

This implies that

$$\mathfrak{n}\simeq(\mathrm{Alt}(2p,\mathbb{C})\oplus\mathbb{C}^{2n-4p-2})\oplus\mathbb{C}^{2p}.$$

The semisimple part $SL(2p,\mathbb{C})\times SO(2n-4p-2,\mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts on $\mathrm{Alt}(2p,\mathbb{C})\oplus\mathbb{C}^{2n-4p-2}\oplus\mathbb{C}^{2p}$ by

$$(g,h)\cdot(A,v,w)=(gA\,{}^tg,hv,gw),$$

and its center $(\mathbb{C}^\times)^2$ acts by

$$(s,t)\cdot(A,v,w)=(t^2A,sv,stw).$$

Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the decomposable action consisting of the indecomposable $(SL(2p,\mathbb{C})\times\mathbb{C}^\times)$-action on $\mathrm{Alt}(2p,\mathbb{C})\oplus\mathbb{C}^{2p}$ ((8) of Table 5.1) and the irreducible $(SO(2n-4p-2,\mathbb{C})\times\mathbb{C}^\times)$-action on $\mathbb{C}^{2n-4p-2}$ ((6) of Table 5.1). Hence, this action is a multiplicity-free action.

Here, we define

$$S_0':=\bigoplus_{1\le i\le p}\mathbb{R}E_{e_{2i}+e_{2i+1}}\oplus\bigoplus_{1\le j\le p}\mathbb{R}E_{e_1+e_{2j}},$$
$$S_0'':=\mathbb{R}E_{e_1+e_{2p+2}}\oplus\mathbb{R}E_{e_1-e_{2p+2}}$$

Then, S_0' is isomorphic to the slice $A_p\oplus T_p$ of Table 5.2 for the $(SU(2p)\times\mathbb{T})$-action on $\mathrm{Alt}(2p,\mathbb{C})\oplus\mathbb{C}^{2p}$, and S_0'' to the slice $D_{1,1}$ for the $(SO(2n-4p-2)\times\mathbb{T})$-action on $\mathbb{C}^{2n-4p-2}$. We set

$$S_0:=S_0'\oplus S_0''\simeq(A_p\oplus D_{1,1})\oplus T_p. \tag{5.17}$$

Then, it follows from Lemma 5.6 that $\mathfrak{n}=L_u\cdot S_0$. Therefore, Theorem 3.5 has been verified for Case (D3).

Case (D3′). Let us consider the case where $\mathfrak{g}=\mathfrak{so}(4p+4,\mathbb{C})$ and $\Omega(\mathcal{O}_X)=(1,0,\dots,0,1,1)$. Then, the characteristic element H forms the same as in (5.16). This $\mathcal{O}_X$ is the set of all nilpotent elements with Jordan type $(3,2^{2p},1)$.

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l}=\mathfrak{a}\oplus\bigoplus_{2\le i<j\le 2p+1}\mathfrak{g}_{\pm(e_i-e_j)}\simeq\mathfrak{sl}(2p,\mathbb{C})\oplus\mathbb{C}^3.$$

We note that $\{E_1,E_2+\cdots+E_{2p+1},E_{2p+2}\}$ is a basis of the three-dimensional center $\mathbb{C}^3$.

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2)=\bigoplus_{2\le i<j\le 2p+1}\mathfrak{g}_{e_i+e_j}\oplus\mathfrak{g}_{e_1\pm e_{2p+2}}\simeq\mathrm{Alt}(2p,\mathbb{C})\oplus\mathbb{C}^2.$$

Further, $\mathfrak{g}(3)$ is of the form

$$\mathfrak{g}(3) = \bigoplus_{2 \leq j \leq 2p+1} \mathfrak{g}_{e_1+e_j} \simeq \mathbb{C}^{2p}.$$

This implies that the nilpotent subalgebra $\mathfrak{n}$ is isomorphic to

$$\mathfrak{n} \simeq (\mathrm{Alt}(2p, \mathbb{C}) \oplus \mathbb{C}^2) \oplus \mathbb{C}^{2p}.$$

The subset S_0 in $\mathfrak{n}$ defined by (5.17) satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified.

5.8. Type $\mathbf{E}_6$. In Sections 5.8–5.12, we deal with $\mathfrak{g}$ of exceptional type. In this subsection, we give a proof of Theorem 3.5 for $\mathfrak{g} = \mathfrak{e}_6(\mathbb{C})$. In this case, $\mathfrak{g}_\mathbb{R} \simeq \mathfrak{e}_{6(6)}$.

Then, $\mathfrak{a}_\mathbb{R}^* = \{v \in \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_8 : \langle v, e_8 + e_7 \rangle = \langle v, e_7 - e_6 \rangle = 0\}$
$= \mathbb{R}(e_8 - e_7 - e_6) \oplus \mathbb{R}e_5 \oplus \cdots \oplus \mathbb{R}e_1$.

A root system $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{a})$ is given by $\Delta = \{\pm e_i \pm e_j : 1 \leq j < i \leq 5\} \sqcup \{\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{j=1}^5 (-1)^{n(j)} e_j) : \sum_{j=1}^5 n(j) = 0, 2, 4\}$, where $n(1), \ldots n(5) \in \{0, 1\}$. We fix a positive system Δ^+ of $\mathfrak{g}$ as follows: $\Delta^+ = \{e_i \pm e_j : 1 \leq j < i \leq 5\} \sqcup \{\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{j=1}^5 (-1)^{n(j)} e_j) : \sum_{j=1}^5 n(j) = 0, 2, 4\}$. The simple roots $\alpha_1, \ldots, \alpha_6$ are given by $\alpha_1 = \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1)$, $\alpha_2 = e_2 + e_1$, and $\alpha_i = e_{i-1} - e_{i-2}$ $(3 \leq i \leq 6)$. The highest root β is written as $\beta = \frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 + e_2 + e_1) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$.

Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{e}_6(\mathbb{C})$ with characteristic element $H = h_6(E_8 - E_7 - E_6) + h_5 E_5 + \cdots + h_1 E_1 \in \mathfrak{a}_+$. Then, $\alpha_1(H) = \frac{1}{2}(3h_6 - h_5 - h_4 - h_3 - h_2 + h_1)$, $\alpha_2(H) = h_2 + h_1$, and $\alpha_i(H) = h_{i-1} - h_{i-2}$ $(i = 3, 4, 5, 6)$. Hence, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, m_2, m_3, m_4, m_5, m_6)$ is given by $(\frac{1}{2}(3h_6 - h_5 - h_4 - h_3 - h_2 + h_1), h_2 + h_1, h_2 - h_1, h_3 - h_2, h_4 - h_3, h_5 - h_4)$.

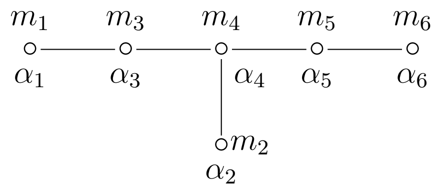

Figure 5.5: Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{e}_6(\mathbb{C})$

A nilpotent orbit $\mathcal{O}_X$ is spherical if and only if $\Omega(\mathcal{O}_X)$ satisfies one of the following cases:

($\mathrm{E}_6 1$) $\Omega(\mathcal{O}_X) = (0, 1, 0, 0, 0, 0)$.

($\mathrm{E}_6 2$) $\Omega(\mathcal{O}_X) = (1, 0, 0, 0, 0, 1)$.

($\mathrm{E}_6 3$) $\Omega(\mathcal{O}_X) = (0, 0, 0, 1, 0, 0)$.

By Lemma 5.4, the height of $\mathcal{O}_X$ equals two for Cases ($\mathrm{E}_6 1$), ($\mathrm{E}_6 2$), and three for Case ($\mathrm{E}_6 3$).

Case ($\mathbf{E}_6\mathbf{1}$). We consider the case $\Omega(\mathcal{O}_X) = (0,1,0,0,0,0)$. Then, $H \in \mathfrak{a}$ is written by

$$H = \frac{1}{2}(E_8 - E_7 - E_6 + E_5 + E_4 + E_3 + E_2 + E_1).$$

This $\mathcal{O}_X$ is the minimal nilpotent orbit with dimension 22.

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \le j < i \le 5} \mathfrak{g}_{\pm(e_i - e_j)} \oplus \bigoplus_{\sum_{j=1}^5 n(j) = 4} \mathfrak{g}_{\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{j=1}^5 (-1)^{n(j)} e_j)}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(6,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 + e_2 + e_1)} \simeq \mathbb{C}.$$

Hence, the semisimple part $SL(6,\mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts trivially on $\mathbb{C}$, and the center $\mathbb{C}^\times$ by

$$s \cdot z = s^2 z.$$

Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $\mathbb{C}^\times$-action on $\mathbb{C}$. It follows from (1) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace S_0 as

$$S_0 := \mathbb{R} E_{\frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 + e_2 + e_1)}.$$

Then, S_0 is isomorphic to the slice $\mathbb{R}$ of Table 5.1 for the $\mathbb{T}$-action on $\mathbb{C}$. By Lemma 5.6, $\mathfrak{n} = L_u \cdot S_0$. Hence, Theorem 3.5 has been verified for Case ($\mathrm{E}_6 1$).

Case ($\mathbf{E}_6\mathbf{2}$). Let $\Omega(\mathcal{O}_X)$ be $(1,0,0,0,0,1)$. Then, $H \in \mathfrak{a}$ is

$$H = E_8 - E_7 - E_6 + E_5.$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \le j < i \le 4} \mathfrak{g}_{\pm e_i \pm e_j}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{so}(8,\mathbb{C}) \oplus \mathbb{C}^2.$$

Here, $\{E_8 - E_7 - E_6, E_5\}$ is a basis of the two-dimensional center $\mathbb{C}^2$.

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{\sum_{j=1}^4 n(j) = 0,2,4} \mathfrak{g}_{\frac{1}{2}(e_8 - e_7 - e_6 + e_5) + \frac{1}{2}\sum_{j=1}^4 (-1)^{n(j)} e_j}.$$

Then, the semisimple part $Spin(8,\mathbb{C})$ of $L_{\mathbb{C}}$ acts on

$$\mathfrak{n} \simeq \mathbb{C}^8$$

as the half-spin representation. We know that this action is geometrically equivalent to the $SO(8,\mathbb{C})$-action on $\mathbb{C}^8$. On the other hand, the center $(\mathbb{C}^\times)^2$ of $L_{\mathbb{C}}$ acts on $\mathfrak{n}$ by

$$(s,t)\cdot v = s^3tv.$$

Hence, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SO(8,\mathbb{C})\times\mathbb{C}^\times)$-action on $\mathbb{C}^8$. It follows from (6) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace S_0 in $\mathfrak{n}$ as

$$S_0 = \mathbb{R}E_{\frac{1}{2}(e_8-e_7-e_6+e_5+e_4+e_3+e_2+e_1)} \oplus \mathbb{R}E_{\frac{1}{2}(e_8-e_7-e_6+e_5-e_4-e_3-e_2-e_1)}.$$

Then, S_0 is isomorphic to the slice $D_{1,1}$ of Table 5.2 for the $(SO(8)\times\mathbb{T})$-action on $\mathbb{C}^8$. By Lemma 5.6, $\mathfrak{n} = L_u\cdot S_0$. Hence, Theorem 3.5 for Case ($\mathrm{E}_6 2$) has been verified .

Case ($\mathbf{E}_6$3). Let $\Omega(\mathcal{O}_X) = (0,0,0,1,0,0)$. Then, H is written by

$$H = E_8 - E_7 - E_6 + E_5 + E_4 + E_3.$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{g}_{\pm\frac{1}{2}(e_8-e_7-e_6-e_5-e_4-e_3)\pm\frac{1}{2}(e_2-e_1)} \oplus \mathfrak{g}_{\pm e_2\pm e_1} \oplus \bigoplus_{3\le j<i\le 5} \mathfrak{g}_{\pm(e_i-e_j)}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{3\le j<i\le 5} \mathfrak{g}_{e_i+e_j} \oplus \bigoplus_{\sum_{j=3}^5 n(j)=1} \mathfrak{g}_{\frac{1}{2}(e_8-e_7-e_6+\sum_{j=3}^5(-1)^{n(j)}e_j)\pm\frac{1}{2}(e_2-e_1)}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to the vector space $M(3,\mathbb{C})$. Further, the eigenspace $\mathfrak{g}(3)$ forms

$$\mathfrak{g}(3) = \mathfrak{g}_{\frac{1}{2}(e_8-e_7-e_6+e_5+e_4+e_3)\pm\frac{1}{2}(e_2+e_1)},$$

which is isomorphic to $\mathbb{C}^2$. Hence,

$$\mathfrak{n} = \mathfrak{g}(2) \oplus \mathfrak{g}(3) \simeq M(3,\mathbb{C}) \oplus \mathbb{C}^2.$$

The semisimple part $SL(3,\mathbb{C})\times SL(2,\mathbb{C})\times SL(3,\mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts on $M(3,\mathbb{C})\oplus\mathbb{C}^2$ as follows: $SL(3,\mathbb{C})\times SL(3,\mathbb{C})$ acts by

$$(g_1,g_2)\cdot(A,v) = (g_1Ag_2^{-1}, v),$$

and $SL(2,\mathbb{C})$ acts by

$$h \cdot (A, v) = (A, hv).$$

The center $\mathbb{C}^\times$ acts on $M(3,\mathbb{C}) \oplus \mathbb{C}^2$ by

$$s \cdot (A, v) = (s^2 A, s^3 v).$$

Then, the $L_\mathbb{C}$-action on $\mathfrak{n}$ is geometrically equivalent to the decomposable action consisting of two irreducible actions; the $(SL(3,\mathbb{C}) \times SL(3,\mathbb{C}) \times \mathbb{C}^\times)$-action on $M(3,\mathbb{C})$ ((5) of Table 5.1); the $SL(2,\mathbb{C})$-action on $\mathbb{C}^2$ ((2) of Table 5.1). Thus, this action is a multiplicity-free action.

We take the subspace S_0' in $\mathfrak{g}(2)$ as

$$S_0' := \mathbb{R}E_{\frac{1}{2}(e_8-e_7-e_6+e_5+e_4-e_3+e_2-e_1)} \oplus \mathbb{R}E_{\frac{1}{2}(e_8-e_7-e_6+e_5-e_4+e_3-e_2+e_1)} \oplus \mathbb{R}E_{e_4+e_3}.$$

Then, S_0' is isomorphic to the slice D_3 of Table 5.2 for the $(SU(3) \times SU(3) \times \mathbb{T})$-action on $M(3,\mathbb{C})$. It follows from Lemma 5.6 that $\mathfrak{g}(2) = (SU(3) \times SU(3) \times \mathbb{T}) \cdot S_0'$. Similarly, the subspace S_0'' in $\mathfrak{g}(3)$ given by

$$S_0'' := \mathbb{R}E_{\frac{1}{2}(e_8-e_7-e_6+e_5+e_4+e_3+e_2+e_1)}$$

is isomorphic to T_1 of Table 5.1, and then $\mathfrak{g}(3) = SU(2) \cdot S_0''$. Thus,

$$S_0 := S_0' \oplus S_0'' \simeq D_3 \oplus T_1$$

satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 for Case ($\mathrm{E}_6 3$) has been verified.

5.9. Type $\mathbf{E}_7$. In this subsection, we give a proof of Theorem 3.5 for $\mathfrak{g} = \mathfrak{e}_7(\mathbb{C})$. In this case, $\mathfrak{g}_\mathbb{R} \simeq \mathfrak{e}_{7(7)}$. Then, $\mathfrak{a}_\mathbb{R}^* = \{v \in \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_8 : \langle v, e_8 + e_7 \rangle = 0\} = \mathbb{R}(e_8 - e_7) \oplus \mathbb{R}e_6 \oplus \cdots \oplus \mathbb{R}e_1$. A root system $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{a})$ is $\Delta = \{\pm e_i \pm e_j : 1 \leq j < i \leq 6\} \sqcup \{\pm(e_8 - e_7)\} \sqcup \{\pm\frac{1}{2}(e_8 - e_7 + \sum_{j=1}^6 (-1)^{n(j)} e_j) : \sum_{j=1}^6 n(j) = 1, 3, 5\}$ where $n(1), \ldots, n(6) \in \{0, 1\}$. We fix a positive system Δ^+ as $\Delta^+ = \{e_i \pm e_j : 1 \leq j < i \leq 6\} \sqcup \{e_8 - e_7\} \sqcup \{\frac{1}{2}(e_8 - e_7 + \sum_{j=1}^6 (-1)^{n(j)} e_j) : \sum_{j=1}^6 n(j) = 1, 3, 5\}$. The simple roots $\alpha_1, \ldots, \alpha_7$ are given by $\alpha_1 = \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1)$, $\alpha_2 = e_2 + e_1$, and $\alpha_{j+2} = e_{j+1} - e_j$ $(j = 1, \ldots, 5)$. The highest root is written as $\beta = e_8 - e_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$.

Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{e}_7(\mathbb{C})$ with characteristic element $H = h_7(E_8 - E_7) + h_6 E_6 + \cdots + h_1 E_1 \in \mathfrak{a}_+$. Then, $\alpha_1(H) = \frac{1}{2}(2h_7 - h_6 - h_5 - h_4 - h_3 - h_2 + h_1)$, $\alpha_2(H) = h_2 + h_1$, and $\alpha_i(H) = h_{i-1} - h_{i-2}$ $(i = 3, 4, 5, 6, 7)$. Hence, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, \ldots, m_7)$ is given by $(\frac{1}{2}(2h_7 - h_6 - h_5 - h_4 - h_3 - h_2 + h_1), h_2 + h_1, h_2 - h_1, h_3 - h_2, h_4 - h_3, h_5 - h_4, h_6 - h_5)$.

A nilpotent orbit $\mathcal{O}_X$ is spherical if and only if $\Omega(\mathcal{O}_X)$ satisfies one of the following cases:

($\mathrm{E}_7 1$) $\Omega(\mathcal{O}_X) = (1, 0, 0, 0, 0, 0, 0)$.

($\mathrm{E}_7 2$) $\Omega(\mathcal{O}_X) = (0, 0, 0, 0, 0, 1, 0)$.

($\mathrm{E}_7 3$) $\Omega(\mathcal{O}_X) = (0, 0, 0, 0, 0, 0, 2)$.

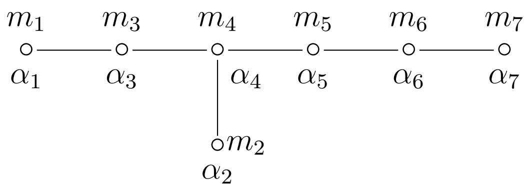

Figure 5.6: Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{e}_7(\mathbb{C})$

(E$_7$4) $\Omega(\mathcal{O}_X) = (0,0,1,0,0,0,0)$.

(E$_7$5) $\Omega(\mathcal{O}_X) = (0,1,0,0,0,0,1)$.

By Lemma 5.4, the height of $\mathcal{O}_X$ equals two for Cases (E$_7$1)–(E$_7$3) and three for Cases (E$_7$4), (E$_7$5).

Case (E$_7$1). Let $\Omega(\mathcal{O}_X) = (1,0,0,0,0,0,0)$. Then, H is of the form

$$H = E_8 - E_7.$$

This $\mathcal{O}_X$ is the minimal nilpotent orbit with dimension 34.

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq j < i \leq 6} \mathfrak{g}_{\pm e_i \pm e_j}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{so}(12,\mathbb{C}) \oplus \mathbb{C}.$$

The ad(H)-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \mathfrak{g}_{e_8 - e_7} \simeq \mathbb{C}.$$

Hence, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $\mathbb{C}^\times$-action on $\mathbb{C}$. It follows from (1) of Table 5.1 that this action is a multiplicity-free action.

We take

$$S_0 = \mathbb{R}E_{e_8 - e_7} \simeq \mathbb{R}.$$

By Lemma 5.6, S_0 satisfies $\mathfrak{n} = L_u \cdot S_0$. Hence, Theorem 3.5 for Case (E$_7$1) has been verified.

Case (E$_7$2). We consider the case $\Omega(\mathcal{O}_X) = (0,0,0,0,0,1,0)$. Then,

$$H = E_8 - E_7 + E_6 + E_5.$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq j < i \leq 4} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \mathfrak{g}_{\pm(e_6 - e_5)} \oplus \bigoplus_{\sum_{j=1}^4 n(j) = 1,3} \mathfrak{g}_{\pm \frac{1}{2}(e_8 - e_7 - e_6 - e_5 + \sum_{j=1}^4 (-1)^{n(j)} e_j)}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{so}(10,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \mathfrak{g}_{e_8-e_7} \oplus \mathfrak{g}_{e_6+e_5} \oplus \bigoplus_{\sum_{j=1}^{4} n(j)=1,3} \mathfrak{g}_{\frac{1}{2}(e_8-e_7+e_6+e_5)+\frac{1}{2}\sum_{j=1}^{4}(-1)^{n(j)}e_j}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathbb{C}^{10}$, from which

$$\mathfrak{n} \simeq \mathbb{C}^{10}.$$

The action of the semisimple part $SO(10,\mathbb{C}) \times SL(2,\mathbb{C})$ of the Levi subgroup $L_\mathbb{C}$ on $\mathbb{C}^{10}$ is given as follows: $SO(10,\mathbb{C})$ acts by the standard representation; $SL(2,\mathbb{C})$ acts trivially, and the action of the center $\mathbb{C}^\times$ is the scalar multiplication. Then, the $L_\mathbb{C}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SO(10,\mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathbb{C}^{10}$. It follows from (6) of Table 5.1 that this action is a multiplicity-free action.

We take a subspace S_0 as

$$S_0 := \mathbb{R}E_{e_8-e_7} \oplus \mathbb{R}E_{e_6+e_5}.$$

Then, S_0 is isomorphic to the slice $D_{1,1}$ of Table 5.2 for the $(SO(10) \times \mathbb{T})$-action via $\mathfrak{n} \simeq \mathbb{C}^{10}$. By Lemma 5.6, $\mathfrak{n} = L_u \cdot S_0$. Hence, Theorem 3.5 for Case ($\mathrm{E}_7 2$) has been verified.

Case ($\mathbf{E}_7\mathbf{3}$). Let $\Omega(\mathcal{O}_X)$ be $(0,0,0,0,0,0,2)$. Then,

$$H = E_8 - E_7 + 2E_6.$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1\le j<i\le 5} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \bigoplus_{\sum_{j=1}^{5} n(j)=0,2,4} \mathfrak{g}_{\pm\frac{1}{2}(e_8-e_7-e_6+\sum_{j=1}^{5}(-1)^{n(j)}e_j)}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{e}_6(\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1\le j\le 5} \mathfrak{g}_{e_6 \pm e_j} \oplus \mathfrak{g}_{e_8-e_7} \oplus \bigoplus_{\sum_{j=1}^{5} n(j)=1,3,5} \mathfrak{g}_{\frac{1}{2}(e_8-e_7+e_6+\sum_{j=1}^{5}(-1)^{n(j)}e_j)}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to the complexified Jordan algebra $\mathfrak{J}_\mathbb{C}$, from which

$$\mathfrak{n} \simeq \mathfrak{J}_\mathbb{C}.$$

This implies that the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(E_6(\mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathfrak{J}_{\mathbb{C}}$. It follows from (7) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace S_0 in $\mathfrak{n}$ as

$$S_0 = \mathbb{R}E_{e_8-e_7} \oplus \mathbb{R}E_{e_6+e_5} \oplus \mathbb{R}E_{e_6-e_5}.$$

Then, this is isomorphic to the slice D_3 of Table 5.2 for the $(E_6 \times \mathbb{T})$-action on $\mathfrak{J}_{\mathbb{C}}$. By Lemma 5.6, S_0 satisfies $\mathfrak{n} = L_u \cdot S_0$. Hence, Theorem 3.5 for Case ($\mathrm{E}_7$3) has been verified in this case.

Case ($\mathbf{E}_7$4). We consider the case $\Omega(\mathcal{O}_X) = (0,0,1,0,0,0,0)$. Then, the characteristic element H of $\mathcal{O}_X$ is expressed by

$$H = \frac{1}{2}(3E_8 - 3E_7 + E_6 + E_5 + E_4 + E_3 + E_2 - E_1).$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \le i \le 6} \mathfrak{g}_{\pm(e_i+e_1)} \oplus \bigoplus_{2 \le j < i \le 6} \mathfrak{g}_{\pm(e_i-e_j)} \oplus \mathfrak{g}_{\pm\frac{1}{2}(e_8-e_7-e_6-e_5-e_4-e_3-e_2+e_1)}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{\sum_{j=2}^6 n(j)=1} \mathfrak{g}_{\frac{1}{2}(e_8-e_7+\sum_{j=2}^6(-1)^{n(j)}e_j+e_1))} \oplus \bigoplus_{\sum_{j=2}^6 n(j)=2} \mathfrak{g}_{\frac{1}{2}(e_8-e_7+\sum_{j=2}^6(-1)^{n(j)}e_j-e_1))}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathrm{Alt}(6,\mathbb{C})$. Further, $\mathfrak{g}(3)$ forms

$$\mathfrak{g}(3) = \mathfrak{g}_{e_8-e_7} \oplus \mathfrak{g}_{\frac{1}{2}(e_8-e_7+e_6+e_5+e_4+e_3+e_2-e_1)} \simeq \mathbb{C}^2.$$

Thus,

$$\mathfrak{n} \simeq \mathrm{Alt}(6,\mathbb{C}) \oplus \mathbb{C}^2.$$

The semisimple part $SL(6,\mathbb{C}) \times SL(2,\mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts on $\mathrm{Alt}(6,\mathbb{C}) \oplus \mathbb{C}^2$ by

$$(g,h) \cdot (A,v) = (gA\,{}^tg, hv),$$

and its center $\mathbb{C}^\times$ acts by

$$s \cdot (A,v) = (s^2 A, s^3 v).$$

Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the decomposable action consisting of the $(SL(6,\mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathrm{Alt}(6,\mathbb{C})$ ((4) of Table 5.1) and the

$SL(2,\mathbb{C})$-action on $\mathbb{C}^2$ ((2) of Table 5.1). It follows from Lemma 5.6 that this action is a multiplicity-free action.

We take the subspace S_0' in $\mathfrak{g}(2)$ as

$$S_0' := \mathbb{R}E_{\frac{1}{2}(e_8-e_7+e_6+e_5+e_4+e_3-e_2+e_1)} \oplus \mathbb{R}E_{\frac{1}{2}(e_8-e_7+e_6+e_5-e_4-e_3+e_2-e_1)} \oplus \mathbb{R}E_{\frac{1}{2}(e_8-e_7-e_6-e_5+e_4+e_3+e_2-e_1)}.$$

Then, S_0' is isomorphic to the slice A_3 of Table 5.2 for the $(SU(6)\times\mathbb{T})$-action on $\mathrm{Alt}(6,\mathbb{C})$. By Lemma 5.6, $\mathfrak{g}(2) = (SU(6)\times\mathbb{T})\cdot S_0'$. Similarly, the subspace

$$S_0'' := \mathbb{R}E_{e_8-e_7}.$$

in $\mathfrak{g}(3)$ is isomorphic to T_1 and satisfies $\mathfrak{g}(3) = SU(2)\cdot S_0''$. Hence,

$$S_0 := S_0' \oplus S_0'' \simeq A_3 \oplus T_1$$

satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified for Case ($\mathrm{E}_7$4).

Case ($\mathbf{E}_7$5). Let $\Omega(\mathcal{O}_X)$ be $(0,1,0,0,0,0,1)$. Then,

$$H = \frac{1}{2}(3E_8 - 3E_7 + 3E_6 + E_5 + E_4 + E_3 + E_2 + E_1).$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1\le j<i\le 5} \mathfrak{g}_{\pm(e_i-e_j)} \oplus \bigoplus_{\sum_{j=1}^5 n(j)=4} \mathfrak{g}_{\pm\frac{1}{2}(e_8-e_7-e_6+\sum_{j=1}^5(-1)^{n(j)}e_j)}.$$

This implies that

$$\mathfrak{l} \simeq \mathfrak{sl}(6,\mathbb{C}) \oplus \mathbb{C}^2.$$

We note that $\{E_8 - E_7 + 2E_6,\ \frac{1}{2}(E_8 - E_7 - E_6 + E_5 + E_4 + E_3 + E_2 + E_1)\}$ is a basis of the two-dimensional center $\mathbb{C}^2$.

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1\le j\le 5} \mathfrak{g}_{e_6+e_j} \oplus \bigoplus_{\sum_{j=1}^5 n(j)=3} \mathfrak{g}_{\pm\frac{1}{2}(e_8-e_7+e_6+\sum_{j=1}^5(-1)^{n(j)}e_j)} \oplus \mathfrak{g}_{\frac{1}{2}(e_8-e_7-e_6+e_5+e_4+e_3+e_2+e_1)}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to the direct sum $\mathrm{Alt}(6,\mathbb{C})\oplus\mathbb{C}$. Further, $\mathfrak{g}(3)$ is of the form

$$\mathfrak{g}(3) = \mathfrak{g}_{e_8-e_7} \oplus \bigoplus_{\sum_{j=1}^5 n(j)=1} \mathfrak{g}_{\frac{1}{2}(e_8-e_7+e_6+\sum_{j=1}^5(-1)^{n(j)}e_j)} \simeq \mathbb{C}^6.$$

Hence, $\mathfrak{n}$ is isomorphic to

$$\mathfrak{n} \simeq \mathrm{Alt}(6,\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}^6.$$

The semisimple part $SL(6,\mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts on $\mathrm{Alt}(6,\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}^6$ by

$$g \cdot (A, z, v) = (gA\,{}^t g, z, gv),$$

and its center $(\mathbb{C}^\times)^2$ acts by

$$(s,t) \cdot (A, z, v) = (s^2 t A, s^2 t^3 z, s^3 t^2 v).$$

This implies that the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the decomposable action which consists of two actions: the indecomposable $(SL(6,\mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathrm{Alt}(6,\mathbb{C}) \oplus \mathbb{C}^6$ ((8) of Table 5.1); the $\mathbb{C}^\times$-action on $\mathbb{C}$ ((1) of Table 5.1). By Lemma 5.6, this action is a multiplicity-free action.

We take the subspace S_0' as

$$S_0' := \mathbb{R}E_{\frac{1}{2}(e_8-e_7+e_6+e_5+e_4-e_3-e_2-e_1)} \oplus \mathbb{R}E_{\frac{1}{2}(e_8-e_7+e_6-e_5-e_4+e_3+e_2-e_1)} \oplus \mathbb{R}E_{e_6+e_1},$$

and S_1' as

$$S_1' := \mathbb{R}E_{e_8-e_7} \oplus \mathbb{R}E_{\frac{1}{2}(e_8-e_7+e_6+e_5+e_4+e_3-e_2+e_1)} \oplus \mathbb{R}E_{\frac{1}{2}(e_8-e_7+e_6+e_5-e_4+e_3+e_2+e_1)}.$$

Then, the direct sum $S_0' \oplus S_1'$ is isomorphic to the slice $A_3 \oplus T_3$ of Table 5.2 for the $(SU(6) \times \mathbb{T})$-action on $\mathrm{Alt}(6,\mathbb{C}) \oplus \mathbb{C}^6$. Further, the subspace

$$S_0'' := \mathbb{R}E_{\frac{1}{2}(e_8-e_7-e_6+e_5+e_4+e_3+e_2+e_1)}$$

in $\mathfrak{g}(2)$ is isomorphic to the slice $\mathbb{R}$ for the $\mathbb{T}$-action on $\mathbb{C}$. We set

$$S_0 := (S_0' \oplus S_0'') \oplus S_1' \simeq (A_3 \oplus \mathbb{R}) \oplus T_3.$$

By Lemma 5.6, S_0 satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified for Case ($\mathrm{E}_7$5).

5.10. Type $\mathbf{E}_8$. In this subsection, we give a proof of Theorem 3.5 for $\mathfrak{g} = \mathfrak{e}_8(\mathbb{C})$. In this case, $\mathfrak{g}_{\mathbb{R}} \simeq \mathfrak{e}_{8(8)}$. Then, $\mathfrak{a}_{\mathbb{R}}^* = \mathbb{R}^8$. A root system of $\mathfrak{g}$ is given by $\Delta = \{\pm e_i \pm e_j : 1 \le j < i \le 8\} \sqcup \{\pm\frac{1}{2}(e_8 + \sum_{j=1}^7 (-1)^{n(j)} e_j) : \sum_{j=1}^7 n(j) = 0,2,4,6\}$ where $n(1),\dots,n(7) \in \{0,1\}$. We fix a positive system as $\Delta^+ = \{e_i \pm e_j : 1 \le j < i \le 8\} \sqcup \{\frac{1}{2}(e_8 + \sum_{j=1}^7 (-1)^{n(j)} e_j) : \sum_{j=1}^7 n(j) = 0,2,4,6\}$. The simple roots $\alpha_1,\dots,\alpha_8$ are given by $\alpha_1 = \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1)$, $\alpha_2 = e_2 + e_1$, and $\alpha_i = e_{i-1} - e_{i-2}$ $(3 \le i \le 8)$. The highest root β is written as $\beta = e_8 + e_7 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$.

Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{e}_8(\mathbb{C})$ with characteristic element $H = h_8 E_8 + \cdots + h_1 E_1 \in \mathfrak{a}_+$. Then, $\alpha_1(H) = \frac{1}{2}(h_8 - h_7 - h_6 - h_5 - h_4 - h_3 - h_2 + h_1)$, $\alpha_2(H) = h_2 + h_1$, and $\alpha_i(H) = h_{i-1} - h_{i-2}$ $(i = 3,4,\dots,8)$. Hence, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1,\dots,m_8)$ is given by $(\frac{1}{2}(h_8 - h_7 - h_6 - h_5 - h_4 - h_3 - h_2 + h_1), h_2 + h_1, h_2 - h_1, h_3 - h_2, h_4 - h_4, h_5 - h_4, h_6 - h_5, h_7 - h_6)$.

A nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{e}_8(\mathbb{C})$ is spherical if and only if $\Omega(\mathcal{O}_X)$ satisfies one of the following cases:

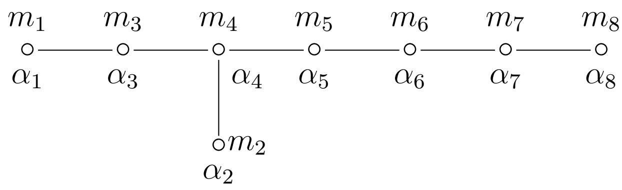

Figure 5.7: Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{e}_8(\mathbb{C})$

(E$_8$1) $\Omega(\mathcal{O}_X) = (0,0,0,0,0,0,0,1)$.
(E$_8$2) $\Omega(\mathcal{O}_X) = (1,0,0,0,0,0,0,0)$.
(E$_8$3) $\Omega(\mathcal{O}_X) = (0,0,0,0,0,0,1,0)$.
(E$_8$4) $\Omega(\mathcal{O}_X) = (0,1,0,0,0,0,0,0)$.

It follows from Lemma 5.4 that the height of $\mathcal{O}_X$ equals two for Cases (E$_8$1), (E$_8$2), and three for Cases (E$_8$3), (E$_8$4),

Case (E$_8$1). Let $\Omega(\mathcal{O}_X)$ be $(0,0,0,0,0,0,0,1)$. Then, $H \in \mathfrak{a}$ is given by

$$H = E_8 + E_7.$$

This $\mathcal{O}_X$ is the minimal nilpotent orbit in $\mathfrak{e}_8(\mathbb{C})$ with dimension 58.

The Levi subalgebra $\mathfrak{l} = \mathfrak{g}(0)$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq j < i \leq 6} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \mathfrak{g}_{\pm(e_8 - e_7)} \oplus \bigoplus_{\sum_{j=1}^{6} n(j) = 1,3,5} \mathfrak{g}_{\pm \frac{1}{2}(e_8 - e_7 + \sum_{j=1}^{6}(-1)^{n(j)} e_j)}.$$

This implies that

$$\mathfrak{l} \simeq \mathfrak{e}_7(\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is

$$\mathfrak{g}(2) = \mathfrak{g}_{e_8 + e_7} \simeq \mathbb{C}.$$

Thus, the action of the Levi subgroup $L_{\mathbb{C}}$ on $\mathfrak{n}$ is geometrically equivalent to the $\mathbb{C}^\times$-action on $\mathbb{C}$. It follows from (1) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace S_0 as

$$S_0 := \mathbb{R} E_{e_8 + e_7} \simeq \mathbb{R}.$$

Then, $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified in this case.

Case (E$_8$2). Let $\Omega(\mathcal{O}_X)$ be $(1,0,0,0,0,0,0,0)$. Then,

$$H = 2E_8.$$

The Levi subalgebra $\mathfrak{l} = \mathfrak{g}(0)$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1 \leq j < i \leq 7} \mathfrak{g}_{\pm e_i \pm e_j}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{so}(14,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1\leq j\leq 7} \mathfrak{g}_{e_8 \pm e_j}$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathbb{C}^{14}$, from which

$$\mathfrak{n} \simeq \mathbb{C}^{14}.$$

This implies that the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the $(SO(14,\mathbb{C})\times\mathbb{C}^{\times})$-action on $\mathbb{C}^{14}$. It follows from (6) of Table 5.1 that this action is a multiplicity-free space.

We take the subspace in $\mathfrak{n}$ as

$$S_0 := \mathbb{R}E_{e_8+e_7} \oplus \mathbb{R}E_{e_8-e_7}.$$

Then, S_0 is isomorphic to the slice $D_{1,1}$ of Table 5.2 for the $(SO(14)\times\mathbb{T})$-action on $\mathbb{C}^{14}$. By Lemma 5.6, $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 for Case ($\mathrm{E}_8 2$) has been verified.

Case ($\mathbf{E}_8$3). Let $\Omega(\mathcal{O}_X)$ be $(0,0,0,0,0,0,1,0)$. Then, H forms

$$H = 2E_8 + E_7 + E_6.$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{g}_{\pm(e_7-e_6)} \oplus \bigoplus_{1\leq j<i\leq 5} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \bigoplus_{\sum_{j=1}^5 n(j)=0,2,4} \mathfrak{g}_{\pm\frac{1}{2}(e_8-e_7-e_6+\sum_{j=1}^5(-1)^{n(j)}e_j)}.$$

This implies that

$$\mathfrak{l} \simeq \mathfrak{e}_6(\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \mathfrak{g}_{e_7+e_6} \oplus \bigoplus_{1\leq j\leq 5} \mathfrak{g}_{e_8\pm e_j} \oplus \bigoplus_{\sum_{j=1}^5 n(j)=0,2,4} \mathfrak{g}_{\frac{1}{2}(e_8+e_7+e_6+\sum_{j=1}^5(-1)^{n(j)}e_j)}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to the complexified Jordan algebra $\mathfrak{J}_{\mathbb{C}}$. Further,

$$\mathfrak{g}(3) = \mathfrak{g}_{e_8+e_7} \oplus \mathfrak{g}_{e_8+e_6} \simeq \mathbb{C}^2.$$

Hence,

$$\mathfrak{n} \simeq \mathfrak{J}_{\mathbb{C}} \oplus \mathbb{C}^2.$$

The semisimple part of the Levi subgroup $L_{\mathbb{C}}$ is isomorphic to $E_6(\mathbb{C}) \times SL(2,\mathbb{C})$, which acts on $\mathfrak{J}_{\mathbb{C}} \oplus \mathbb{C}^2$ diagonally, namely,

$$(g,h)\cdot(v,w) = (gv, hw).$$

Further. the center $\mathbb{C}^\times$ acts by

$$s\cdot(v,w) = (s^2 v, s^3 w).$$

This implies that the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the decomposable action consisting of the irreducible $(E_6(\mathbb{C})\times\mathbb{C}^\times)$-action on $\mathbb{C}^{27}$ ((7) of Table 5.1) and the irreducible $SL(2,\mathbb{C})$-action on $\mathbb{C}^2$ ((2) of Table 5.1). By Lemma 5.6, this action is a multiplicity-free action.

We take the subspace in $\mathfrak{g}(2)$ as

$$S_0' := \mathbb{R}E_{e_8+e_5} \oplus \mathbb{R}E_{\frac{1}{2}(e_8+e_7+e_6-e_5+e_4+e_3+e_2-e_1)} \oplus \mathbb{R}E_{\frac{1}{2}(e_8+e_7+e_6-e_5-e_4-e_3-e_2+e_1)}.$$

Then, S_0' is isomorphic to the slice D_3 for the $(E_6\times\mathbb{T})$-action on $\mathbb{C}^{27}$. By Lemma 5.6, $\mathfrak{g}(2) = (E_6\times\mathbb{T})\cdot S_0'$. Similarly, the subspace

$$S_0'' = \mathbb{R}E_{e_8+e_7}.$$

in $\mathfrak{g}(3)$ is isomorphic to T_1, and then $\mathfrak{g}(3) = SU(2)\cdot S_0''$. Hence,

$$S_0 := S_0' \oplus S_0'' \simeq D_3 \oplus T_1$$

satisfies $\mathfrak{n} = L_u \cdot S_0$, from which Theorem 3.5 for Case ($\mathrm{E}_8$3) has been verified.

Case ($\mathbf{E}_8$4). Let $\Omega(\mathcal{O}_X)$ be $(0,1,0,0,0,0,0,0)$. Then,

$$H = \frac{1}{2}(5E_8 + E_7 + E_6 + \cdots + E_1).$$

The Levi subalgebra $\mathfrak{l} = \mathfrak{g}(0)$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{1\le j<i\le 7} \mathfrak{g}_{\pm(e_i-e_j)} \oplus \bigoplus_{\sum_{j=1}^7 n(j)=6} \mathfrak{g}_{\pm\frac{1}{2}(e_8+\sum_{j=1}^7(-1)^{n(j)}e_j)}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(8,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \bigoplus_{1\le j\le 7} \mathfrak{g}_{e_8-e_j} \oplus \bigoplus_{\sum_{j=1}^7 n(j)=2} \mathfrak{g}_{\frac{1}{2}(e_8+\sum_{j=1}^7(-1)^{n(j)}e_j)}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathrm{Alt}(8,\mathbb{C})$. Moreover, the eigenspace $\mathfrak{g}(3)$ forms

$$\mathfrak{g}(3) = \bigoplus_{1\le j\le 7} \mathfrak{g}_{e_8+e_j} \oplus \mathfrak{g}_{\frac{1}{2}(e_8+e_7+e_6+e_5+e_4+e_3+e_2+e_1)} \simeq \mathbb{C}^8.$$

Hence,

$$\mathfrak{n} \simeq \mathrm{Alt}(8, \mathbb{C}) \oplus \mathbb{C}^8.$$

The semisimple part $SL(8, \mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts on $\mathrm{Alt}(8, \mathbb{C}) \oplus \mathbb{C}^8$ diagonally, namely,

$$g \cdot (A, v) = (gA\,{}^t g, gv),$$

and its center $\mathbb{C}^\times$ acts by

$$s \cdot (A, v) = (s^2 A, s^3 v).$$

Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the indecomposable $(SL(8, \mathbb{C}) \times \mathbb{C}^\times)$-action on $\mathrm{Alt}(8, \mathbb{C}) \oplus \mathbb{C}^8$. It follows from (8) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace $S_0' \subset \mathfrak{g}(2)$ as

$$\begin{aligned} S_0' := \mathbb{R}E_{e_8 - e_1} \oplus \mathbb{R}E_{\frac{1}{2}(e_8+e_7+e_6+e_5+e_4-e_3-e_2+e_1)} \\ \oplus \mathbb{R}E_{\frac{1}{2}(e_8+e_7+e_6-e_5-e_4+e_3+e_2+e_1)} \oplus \mathbb{R}E_{\frac{1}{2}(e_8-e_7-e_6+e_5+e_4+e_3+e_2+e_1)}, \end{aligned}$$

and $S_0'' \subset \mathfrak{g}(3)$ as

$$S_0'' := \mathbb{R}E_{e_8+e_7} \oplus \mathbb{R}E_{e_8+e_5} \oplus \mathbb{R}E_{e_8+e_3} \oplus \mathbb{R}E_{e_8+e_1}.$$

We set

$$S_0 := S_0' \oplus S_0''.$$

Then, S_0 is isomorphic to the slice $A_4 \oplus T_4$ of Table 5.2 for the $(SU(8) \times \mathbb{T})$-action on $\mathrm{Alt}(8, \mathbb{C}) \oplus \mathbb{C}^8$. It follows from Lemma 5.6 that $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified.

5.11. Type F_4. In this subsection, we give a proof of Theorem 3.5 for $\mathfrak{g} = \mathfrak{f}_4(\mathbb{C})$. In this case, $\mathfrak{g}_{\mathbb{R}} \simeq \mathfrak{f}_{4(4)}$. Then, $\mathfrak{a}_{\mathbb{R}}^* = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_4$. A root system Δ is $\Delta = \{\pm e_i \pm e_j : 1 \leq i < j \leq 4\} \sqcup \{\pm e_i : 1 \leq i \leq 4\} \sqcup \{\pm\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$. We fix a positive system Δ^+ as $\Delta^+ = \{e_i \pm e_j : 1 \leq i < j \leq 4\} \sqcup \{e_i : 1 \leq i \leq 4\} \sqcup \{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$. The simple roots $\alpha_1, \ldots, \alpha_4$ are given by $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, $\alpha_2 = e_4$, $\alpha_3 = e_3 - e_4$, and $\alpha_4 = e_2 - e_3$. The highest root $\beta = e_1 + e_2$ is written as $\beta = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$.

Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{f}_4(\mathbb{C})$ with characteristic element $H = h_1H_1 + \cdots + h_4H_4 \in \mathfrak{a}_+$. Then, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, \ldots, m_4)$ is given by $(\frac{1}{2}(h_1 - h_2 - h_3 - h_4), h_4, h_3 - h_4, h_2 - h_3)$.

A nilpotent orbit $\mathcal{O}_X$ is spherical if and only if $\Omega(\mathcal{O}_X)$ satisfies one of the following cases:

($\mathrm{F}_4$1) $\Omega(\mathcal{O}_X) = (0, 0, 0, 1)$.

($\mathrm{F}_4$2) $\Omega(\mathcal{O}_X) = (1, 0, 0, 0)$.

($\mathrm{F}_4$3) $\Omega(\mathcal{O}_X) = (0, 0, 1, 0)$.

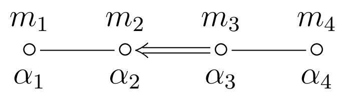

Figure 5.8: Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{f}_4(\mathbb{C})$

The height of $\mathcal{O}_X$ equals two for Cases (F$_4$1) and (F$_4$2), and three for Case (F$_4$3).

Case (F$_4$1). Let $\Omega(\mathcal{O}_X)$ be $(0,0,0,1)$. Then, $H \in \mathfrak{a}$ is given by

$$H = E_1 + E_2.$$

This $\mathcal{O}_X$ is the minimal nilpotent orbit with dimension 16.

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{g}_{\frac{1}{2}(\pm(e_1-e_2)\pm e_3 \pm e_4)} \oplus \mathfrak{g}_{\pm(e_1-e_2)} \oplus \mathfrak{g}_{\pm e_3 \pm e_4} \oplus \mathfrak{g}_{\pm e_3} \oplus \mathfrak{g}_{\pm e_4}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sp}(3,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \mathfrak{g}_{e_1+e_2} \simeq \mathbb{C}.$$

This implies that the action of the Levi subgroup $L_{\mathbb{C}}$ on $\mathfrak{n}$ is geometrically equivalent to the $\mathbb{C}^\times$-action on $\mathbb{C}$. It follows from (1) of Table 5.1 that this action is a multiplicity-free actions.

We take

$$S_0 = \mathbb{R}E_{e_1+e_2}.$$

Then, it follows from Lemma 5.6 that $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified for Case (F$_4$1).

Case (F$_4$2). Let $\Omega(\mathcal{O}_X) = (1,0,0,0)$. Then,

$$H = 2E_1.$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{2 \le i < j \le 4} \mathfrak{g}_{\pm e_i \pm e_j} \oplus \bigoplus_{2 \le i \le 4} \mathfrak{g}_{\pm e_i} \simeq \mathfrak{so}(7,\mathbb{C}) \oplus \mathbb{C}.$$

The eigenspace $\mathfrak{g}(2)$ is of the form:

$$\mathfrak{g}(2) = \mathfrak{g}_{e_1} \oplus \bigoplus_{2 \le i < j \le 4} \mathfrak{g}_{e_1 \pm e_j}.$$

Then,

$$\mathfrak{n} \simeq \mathbb{C}^7.$$

This implies that the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the irreducible $(SO(7,\mathbb{C})\times\mathbb{C}^\times)$-action on $\mathbb{C}^7$. It follows from (6) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace S_0 as

$$S_0=\mathbb{R}E_{e_1+e_2}\oplus\mathbb{R}E_{e_1-e_2}.$$

Then, this is isomorphic to the slice $D_{1,1}$ of Table 5.2 for the $(SO(7)\times\mathbb{T})$-action on $\mathbb{C}^7$. By Lemma 5.6, $\mathfrak{n}=L_u\cdot S_0$. Hence, Theorem 3.5 has been verified for Case ($\mathrm{F}_4$2).

Case ($\mathbf{F}_4$3). Let $\Omega(\mathcal{O}_X)$ be $(0,0,1,0)$. Then,

$$H=2E_1+E_2+E_3.$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l}=\mathfrak{a}\oplus\mathfrak{g}_{\pm(e_2-e_3)}\oplus\mathfrak{g}_{\pm e_4}\oplus\mathfrak{g}_{\pm\frac{1}{2}(e_1-e_2-e_3)\pm\frac{1}{2}e_4}.$$

This means that

$$\mathfrak{l}\simeq\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2)=\mathfrak{g}_{e_1}\oplus\mathfrak{g}_{e_2+e_3}\oplus\mathfrak{g}_{e_1\pm e_4}\oplus\mathfrak{g}_{\frac{1}{2}(e_1+e_2+e_3)\pm\frac{1}{2}e_4}.$$

Then, $\mathfrak{g}(2)$ is isomorphic to $\mathrm{Sym}(3,\mathbb{C})$. Further, $\mathfrak{g}(3)$ forms

$$\mathfrak{g}(3)=\mathfrak{g}_{e_1+e_2}\oplus\mathfrak{g}_{e_1+e_3}\simeq\mathbb{C}^2.$$

Hence,

$$\mathfrak{n}\simeq\mathrm{Sym}(3,\mathbb{C})\oplus\mathbb{C}^2.$$

The semisimple part $SL(3,\mathbb{C})\times SL(2,\mathbb{C})$ of the Levi subgroup $L_{\mathbb{C}}$ acts on $\mathrm{Sym}(3,\mathbb{C})\oplus\mathbb{C}^2$ by

$$(g,h)\cdot(A,v)=(gA\,{}^tg,hv),$$

and its center $\mathbb{C}^\times$ by

$$s\cdot(A,v)=(s^2A,s^3v).$$

Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the decomposable action consisting of the $(SL(3,\mathbb{C})\times\mathbb{C}^\times)$-action on $\mathrm{Sym}(3,\mathbb{C})$ ((3) of Table 5.1) and the $SL(2,\mathbb{C})$-action on $\mathbb{C}^2$ ((2) of Table 5.1). Hence, this action is a multiplicity-free action.

We take the subspace in $\mathfrak{g}(2)$ as

$$S_0':=\mathbb{R}E_{e_1+e_4}\oplus\mathbb{R}E_{e_1-e_4}\oplus\mathbb{R}E_{e_2+e_3}.$$

Then, S_0' is isomorphic to the slice D_3 of Table 5.2. By Lemma 5.6, $\mathfrak{g}(2) = (SU(3) \times \mathbb{T}) \cdot S_0'$. Similarly,

$$S_0'' := \mathbb{R}E_{e_1+e_2}$$

is isomorphic to T_1 of Table 5.1, and then $\mathfrak{g}(3) = SU(2) \cdot S_0''$. We set

$$S_0 = S_0' \oplus S_0'' \simeq D_3 \oplus T_1.$$

Then, the linear space S_0 satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified for Case ($\mathrm{F}_4 3$).

5.12. Type $\mathbf{G}_2$. In this subsection, we give a proof of Theorem 3.5 for $\mathfrak{g} = \mathfrak{g}_2(\mathbb{C})$. In this case, $\mathfrak{g}_{\mathbb{R}} \simeq \mathfrak{g}_{2(2)}$. Then, $\mathfrak{a}_{\mathbb{R}}^* = \{v \in \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3 : \langle v, e_1 + e_2 + e_3 \rangle = 0\}$. A root system $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{a})$ is $\Delta = \{\pm(e_i - e_j) : 1 \leq i < j \leq 3\} \sqcup \{\pm(-2e_1+e_2+e_3), \pm(e_1-2e_2+e_3), \pm(-e_1-e_2+2e_3)\}$. We fix a positive system Δ^+ as $\Delta^+ = \{e_1-e_2,\ -e_2+e_3,\ -e_1+e_3,\ -2e_1+e_2+e_3,\ e_1-2e_2+e_3,\ -e_1-e_2+2e_3\}$. The simple roots are $\alpha_1 := e_1 - e_2$ and $\alpha_2 := -2e_1 + e_2 + e_3$. The highest root $\beta = -e_1 - e_2 + 2e_3$ is written as $\beta = 3\alpha_1 + 2\alpha_2$.

Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{g}_2(\mathbb{C})$ with characteristic element $H = h_1E_1 + h_2E_2 + h_3E_3 \in \mathfrak{a}_+$ with $h_1 + h_2 + h_3 = 0$. Then, $\alpha_1(H) = h_1 - h_2$ and $\alpha_2(H) = -2h_1 + h_2 + h_3$. Hence, the weighted Dynkin diagram $\Omega(\mathcal{O}_X) = (m_1, m_2)$ is given by $(h_1 - h_2, -2h_1 + h_2 + h_3)$.

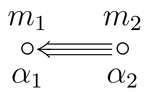

Figure 5.9: Weighted Dynkin diagram of $\mathcal{O}_X$ in $\mathfrak{g}_2$

A nilpotent orbit $\mathcal{O}_X$ is spherical if and only if $\Omega(\mathcal{O}_X)$ satisfies either Case ($\mathrm{G}_2 1$) or Case ($\mathrm{G}_2 2$):

($\mathrm{G}_2 1$) $\Omega(\mathcal{O}_X) = (0, 1)$.

($\mathrm{G}_2 2$) $\Omega(\mathcal{O}_X) = (1, 0)$.

It follows from Lemma 5.4 that the height of $\mathcal{O}_X$ equals two if $\Omega(\mathcal{O}_X) = (0, 1)$ (Case ($\mathrm{G}_2 1$)), and three if $\Omega(\mathcal{O}_X) = (1, 0)$ (Case ($\mathrm{G}_2 2$)).

Case ($\mathbf{G}_2 1$). Let $\Omega(\mathcal{O}_X)$ be $(0, 1)$. This $\mathcal{O}_X$ is the minimal nilpotent orbit with dimension six. Then, $H \in \mathfrak{a}$ is of the form

$$H = \frac{1}{3}(-E_1 - E_2 + 2E_3).$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{g}_{\pm(e_1-e_2)}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is

$$\mathfrak{g}(2) = \mathfrak{g}_{-e_1-e_2+2e_3} \simeq \mathbb{C}.$$

Then, the action of the Levi subgroup $L_{\mathbb{C}}$ on $\mathfrak{n}$ is geometrically equivalent to the $\mathbb{C}^\times$-action on $\mathbb{C}$. It follows from (1) of Table 5.1 that this action is a multiplicity-free action.

We take the subspace S_0 to be

$$S_0 = \mathbb{R}E_{-e_1-e_2+2e_3}.$$

Then, $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified for Case ($\mathrm{G}_2 1$).

Case ($\mathbf{G}_2\mathbf{2}$). Let $\Omega(\mathcal{O}_X)$ be $(1,0)$. Then,

$$H = -E_2 + E_3.$$

The Levi subalgebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{g}_{\pm(-2e_1+e_2+e_3)}.$$

This means that

$$\mathfrak{l} \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}.$$

The $\mathrm{ad}(H)$-eigenspace $\mathfrak{g}(2)$ is written as

$$\mathfrak{g}(2) = \mathfrak{g}_{-e_2+e_3} \simeq \mathbb{C}.$$

Further, the eigenspace $\mathfrak{g}(3)$ is

$$\mathfrak{g}(3) = \mathfrak{g}_{e_1-2e_2+e_3} \oplus \mathfrak{g}_{-e_1-e_2+2e_3} \simeq \mathbb{C}^2.$$

Then,

$$\mathfrak{n} \simeq \mathbb{C} \oplus \mathbb{C}^2.$$

The semisimple part $SL(2,\mathbb{C})$ of $L_{\mathbb{C}}$ acts on $\mathbb{C}$ trivially and on $\mathbb{C}^2$ as the standard representation. Its center $\mathbb{C}^\times$ acts by

$$s \cdot (z, v) = (s^2 z, s^3 v).$$

Then, the $L_{\mathbb{C}}$-action on $\mathfrak{n}$ is geometrically equivalent to the decomposable action consisting of the $\mathbb{C}^\times$-action on $\mathbb{C}$ ((1) of Table 5.1) and the $SL(2,\mathbb{C})$-action on $\mathbb{C}^2$ ((2) of Table 5.1). Hence, this action is a multiplicity-free action.

We take

$$S_0' := \mathbb{R}E_{-e_2+e_3} \simeq \mathbb{R}.$$

Then, S_0' satisfies $\mathfrak{g}(2) = \mathbb{T} \cdot S_0'$. Further, we take the subset S_0'' of $\mathfrak{g}(3)$ as

$$S_0'' := \mathbb{R}E_{-e_1-e_2+2e_3}.$$

Then, S_0'' is isomorphic to the slice T_1 for the $SU(2)$-action on $\mathbb{C}^2$. It follows from Lemma 5.6 that $\mathfrak{g}(3) = SU(2) \cdot S_0''$. Hence, we set

$$S_0 := S_0' \oplus S_0'' \simeq \mathbb{R} \oplus T_1.$$

Then, S_0 satisfies $\mathfrak{n} = L_u \cdot S_0$. Therefore, Theorem 3.5 has been verified.

5.13. Proof of Theorem 3.5.

Proof of Theorem 3.5. In Sections 5.4–5.12, we have given S_0 for the L_u-action on $\mathfrak{n}$ satisfying the properties (a)–(c) explicitly for each $\mathcal{O}_X$ in $\mathfrak{g}$. ∎

5.14. Corollary of proof for Theorem 3.5. Finally, we give two corollaries of the proof of Theorem 3.5 on $\mathcal{O}_X$ with $\mathrm{ht}(\mathcal{O}_X) = 2$.

The first corollary below is concerned to the property on the L_u-action on $\mathfrak{n} = \mathfrak{g}(2)$.

Corollary 5.9. *Let $\mathcal{O}_X$ be a a nilpotent orbit in a complex simple Lie algebra $\mathfrak{g}$. If $\mathrm{ht}(\mathcal{O}_X) = 2$, then we have:*

(1) *By replacing L_u with a locally isomorphic compact group K if necessary, the representation of L_u on $\mathfrak{n}$ is geometrically equivalent to the representation of K on the tangent space $T_{eK}(G/K)$ at the origin eK of some non-compact irreducible Hermitian symmetric space G/K.*

(2) *One can take a slice S for the strongly visible G_u-action on $\mathcal{O}_X$ satisfying $\dim_{\mathbb{R}} S = \operatorname{rank} G/K$.*

Proof. The first statement follows from the proof of Theorem 3.5 given in Sections 5.4–5.12 and [22]. The second one is an immediate consequence of [13]. ∎

A special case of height two nilpotent orbits is the minimal nilpotent orbit. Here, a nilpotent orbit $\mathcal{O}_X$ in the complex semisimple Lie algebra $\mathfrak{g}$ is called *minimal* if $\mathcal{O}_X$ is of minimal dimension in $\mathcal{N}^*/G_{\mathbb{C}}$. It is known that there exists uniquely the minimal nilpotent orbit in a complex simple Lie algebra $\mathfrak{g}$.

The second corollary gives a new characterization for a complex nilpotent orbit to be minimal by the nilpotent subalgebra $\mathfrak{n}$ as follows.

Corollary 5.10. *For a nilpotent orbit $\mathcal{O}_X$ in a complex simple Lie algebra $\mathfrak{g}$, the following two conditions are equivalent:*

(i) *$\mathcal{O}_X$ is minimal.*

(ii) $\dim_{\mathbb{C}} \mathfrak{n} = 1$.

Moreover, the G_u-action on the minimal $\mathcal{O}_X$ is strongly visible with one-dimensional slice.

$\mathfrak{g}$	$\Omega(\mathcal{O}_X) = (m_1, m_2, \ldots)$	$\mathfrak{l} = \mathfrak{g}(0)$	$\mathfrak{g}(2)$	$\mathfrak{g}(3)$	S_0
$\mathfrak{a}_{n-1}$	$m_p = m_{n-p} = 1$ $(p < \frac{n}{2})$	$\mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{sl}(n-2p,\mathbb{C}) \oplus \mathfrak{sl}(p,\mathbb{C}) \oplus \mathbb{C}^2$	$M(p,\mathbb{C})$	$\{0\}$	$\mathbb{R}^p$
$\mathfrak{a}_{2p-1}$	$m_p = 2$	$\mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{sl}(p,\mathbb{C}) \oplus \mathbb{C}$	$M(p,\mathbb{C})$	$\{0\}$	$\mathbb{R}^p$
$\mathfrak{b}_n$	$m_1 = 2$	$\mathfrak{so}(2n-1,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}^{2n-1}$	$\{0\}$	$\mathbb{R}^2$
$\mathfrak{b}_n$	$m_{2p} = 1$	$\mathfrak{sl}(2p,\mathbb{C}) \oplus \mathfrak{so}(2n-4p+1,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Alt}(2p,\mathbb{C})$	$\{0\}$	$\mathbb{R}^p$
$\mathfrak{b}_n$	$m_1 = m_{2p+1} = 1$ $(p < \frac{n-1}{2})$	$\mathfrak{sl}(2p,\mathbb{C}) \oplus \mathfrak{so}(2n-4p-1,\mathbb{C}) \oplus \mathbb{C}^2$	$\mathbb{C}^{2n-4p-1} \oplus \mathrm{Alt}(2p,\mathbb{C})$	$\mathbb{C}^{2p}$	$(\mathbb{R}^2 \oplus \mathbb{R}^p) \oplus \mathbb{R}^p$
$\mathfrak{b}_{2p+1}$	$m_1 = m_{2p+1} = 1$	$\mathfrak{sl}(2p,\mathbb{C}) \oplus \mathbb{C}^2$	$\mathbb{C} \oplus \mathrm{Alt}(2p,\mathbb{C})$	$\mathbb{C}^{2p}$	$(\mathbb{R} \oplus \mathbb{R}^p) \oplus \mathbb{R}^p$
$\mathfrak{c}_n$	$m_p = 1$ $(p < n)$	$\mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{sp}(n-p,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Sym}(p,\mathbb{C})$	$\{0\}$	$\mathbb{R}^p$
$\mathfrak{c}_n$	$m_n = 2$	$\mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Sym}(n,\mathbb{C})$	$\{0\}$	$\mathbb{R}^n$
$\mathfrak{d}_n$	$m_1 = 2$	$\mathfrak{so}(2n-2,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}^{2n-2}$	$\{0\}$	$\mathbb{R}^2$
$\mathfrak{d}_n$	$m_{2p} = 1$ $(p \leq \frac{n-2}{2})$	$\mathfrak{sl}(2p,\mathbb{C}) \oplus \mathfrak{so}(2n-4p,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Alt}(2p,\mathbb{C})$	$\{0\}$	$\mathbb{R}^p$
$\mathfrak{d}_{2p+1}$	$m_{2p} = m_{2p+1} = 1$	$\mathfrak{sl}(2p,\mathbb{C}) \oplus \mathbb{C}^2$	$\mathrm{Alt}(2p,\mathbb{C})$	$\{0\}$	$\mathbb{R}^p$
$\mathfrak{d}_{2p}$	$m_{2p} = 2$	$\mathfrak{sl}(2p,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Alt}(2p,\mathbb{C})$	$\{0\}$	$\mathbb{R}^p$
$\mathfrak{d}_{2p}$	$m_{2p-1} = 2$	$\mathfrak{sl}(2p,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Alt}(2p,\mathbb{C})$	$\{0\}$	$\mathbb{R}^p$
$\mathfrak{d}_n$	$m_1 = m_{2p+1} = 1$ $(p < \frac{n-2}{2})$	$\mathfrak{sl}(2p,\mathbb{C}) \oplus \mathfrak{so}(2n-4p-2,\mathbb{C}) \oplus \mathbb{C}^2$	$\mathbb{C}^{2n-4p-2} \oplus \mathrm{Alt}(2p,\mathbb{C})$	$\mathbb{C}^{2p}$	$(\mathbb{R}^2 \oplus \mathbb{R}^p) \oplus \mathbb{R}^p$
$\mathfrak{d}_{2p+2}$	$m_1 = m_{2p+1} = m_{2p+2} = 1$	$\mathfrak{sl}(2p,\mathbb{C}) \oplus \mathbb{C}^3$	$\mathbb{C}^2 \oplus \mathrm{Alt}(2p,\mathbb{C})$	$\mathbb{C}^{2p}$	$(\mathbb{R}^2 \oplus \mathbb{R}^p) \oplus \mathbb{R}^p$

Table 5.3: Spherical nilpotent orbits and slices for the L_u-action on $\mathfrak{n}$: $\mathfrak{g}$ is of classical type

$\mathfrak{g}$	$\Omega(\mathcal{O}_X)$	$\mathfrak{l} = \mathfrak{g}(0)$	$\mathfrak{g}(2)$	$\mathfrak{g}(3)$	S_0
$\mathfrak{e}_6$	$(0,1,0,0,0,0)$	$\mathfrak{sl}(6,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}$	$\{0\}$	$\mathbb{R}$
$\mathfrak{e}_6$	$(1,0,0,0,0,1)$	$\mathfrak{so}(8,\mathbb{C}) \oplus \mathbb{C}^2$	$\mathbb{C}^8$	$\{0\}$	$\mathbb{R}^2$
$\mathfrak{e}_6$	$(0,0,0,1,0,0)$	$\mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}$	$M(3,\mathbb{C})$	$\mathbb{C}^2$	$\mathbb{R}^3 \oplus \mathbb{R}$
$\mathfrak{e}_7$	$(1,0,0,0,0,0,0)$	$\mathfrak{so}(12,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}$	$\{0\}$	$\mathbb{R}$
$\mathfrak{e}_7$	$(0,0,0,0,0,1,0)$	$\mathfrak{so}(10,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}^{10}$	$\{0\}$	$\mathbb{R}^2$
$\mathfrak{e}_7$	$(0,0,0,0,0,0,2)$	$\mathfrak{e}_6(\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}^{27}$	$\{0\}$	$\mathbb{R}^3$
$\mathfrak{e}_7$	$(0,0,1,0,0,0,0)$	$\mathfrak{sl}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Alt}(6,\mathbb{C})$	$\mathbb{C}^2$	$\mathbb{R}^3 \oplus \mathbb{R}$
$\mathfrak{e}_7$	$(0,1,0,0,0,0,1)$	$\mathfrak{sl}(6,\mathbb{C}) \oplus \mathbb{C}^2$	$\mathbb{C} \oplus \mathrm{Alt}(6,\mathbb{C})$	$\mathbb{C}^6$	$(\mathbb{R} \oplus \mathbb{R}^3) \oplus \mathbb{R}^3$
$\mathfrak{e}_8$	$(0,0,0,0,0,0,0,1)$	$\mathfrak{e}_7(\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}$	$\{0\}$	$\mathbb{R}$
$\mathfrak{e}_8$	$(1,0,0,0,0,0,0,0)$	$\mathfrak{so}(14,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}^{14}$	$\{0\}$	$\mathbb{R}^2$
$\mathfrak{e}_8$	$(0,0,0,0,0,0,1,0)$	$\mathfrak{e}_6(\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}^{27}$	$\mathbb{C}^2$	$\mathbb{R}^3 \oplus \mathbb{R}$
$\mathfrak{e}_8$	$(0,1,0,0,0,0,0,0)$	$\mathfrak{sl}(8,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Alt}(8,\mathbb{C})$	$\mathbb{C}^8$	$\mathbb{R}^4 \oplus \mathbb{R}^4$
$\mathfrak{f}_4$	$(0,0,0,1)$	$\mathfrak{sp}(3,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}$	$\{0\}$	$\mathbb{R}$
$\mathfrak{f}_4$	$(1,0,0,0)$	$\mathfrak{so}(7,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}^7$	$\{0\}$	$\mathbb{R}^2$
$\mathfrak{f}_4$	$(0,0,1,0)$	$\mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Sym}(3,\mathbb{C})$	$\mathbb{C}^2$	$\mathbb{R}^3 \oplus \mathbb{R}$
$\mathfrak{g}_2$	$(0,1)$	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}$	$\{0\}$	$\mathbb{R}$
$\mathfrak{g}_2$	$(1,0)$	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{R} \oplus \mathbb{R}$

Table 5.4: Spherical nilpotent orbits and slices for the L_u-action on $\mathfrak{n}$: $\mathfrak{g}$ is of exceptional type

Proof. Table 5.5 gives a list of the minimal nilpotent orbit for each complex simple Lie algebra $\mathfrak{g}$. This shows that $\dim_{\mathbb{C}} \mathfrak{n} = \dim_{\mathbb{C}} \mathfrak{g}(2) = 1$ for minimal $\mathcal{O}_X$. By Theorem 3.5, the L_u-action on $\mathfrak{n}$ is strongly visible with slice $S_0 \simeq \mathbb{R}$. Hence, $S = S_0 \cap \mathfrak{n}^\circ \simeq \mathbb{R}^\times$ becomes a slice for the G_u-action on $\mathcal{O}_X$.

Conversely, suppose that $\dim_{\mathbb{C}} \mathfrak{n} = 1$. Then, the height of $\mathcal{O}_X$ has to be equal to two. Indeed, if $\mathrm{ht}(\mathcal{O}_X) = d > 2$, then $\mathfrak{n}$ contains the complex vector subspace $\mathfrak{g}(2) \oplus \mathfrak{g}(d)$. Obviously, $\dim_{\mathbb{C}}(\mathfrak{g}(2) \oplus \mathfrak{g}(d)) \geq 2$, from which we have $\dim_{\mathbb{C}} \mathfrak{n} \geq 2$. Let us assume that $\mathrm{ht}(\mathcal{O}_X) = 2$. From our case-by-case analysis on the L_u-action on $\mathfrak{n}$ (see also Tables 5.3 and 5.4), it turns out that $\dim_{\mathbb{C}} \mathfrak{n} \neq 1$ if $\mathcal{O}_X$ is not minimal. Therefore, we have proved that $\dim_{\mathbb{C}} \mathfrak{n} = 1$ only if $\mathcal{O}_X$ is minimal.

Consequently, Corollary 5.10 has been proved. ∎

$\mathfrak{g}$	$\Omega(\mathcal{O}_X)$	$\mathfrak{g}(2)$	S_0	Case
$\mathfrak{a}_{n-1}$	$(1, 0, \ldots, 0, 1)$	$\mathbb{C}$	$\mathbb{R}$	A
$\mathfrak{b}_n$	$(0, 1, 0, \ldots, 0)$	$\mathbb{C}$	$\mathbb{R}$	B2
$\mathfrak{c}_n$	$(1, 0, \ldots, 0)$	$\mathbb{C}$	$\mathbb{R}$	C
$\mathfrak{d}_n$	$(0, 1, 0, \ldots, 0)$	$\mathbb{C}$	$\mathbb{R}$	D2
$\mathfrak{e}_6$	$(0, 1, 0, 0, 0, 0)$	$\mathbb{C}$	$\mathbb{R}$	$E_6 1$
$\mathfrak{e}_7$	$(1, 0, 0, 0, 0, 0, 0)$	$\mathbb{C}$	$\mathbb{R}$	$E_7 1$
$\mathfrak{e}_8$	$(0, 0, 0, 0, 0, 0, 0, 1)$	$\mathbb{C}$	$\mathbb{R}$	$E_8 1$
$\mathfrak{f}_4$	$(0, 0, 0, 1)$	$\mathbb{C}$	$\mathbb{R}$	$F_4 1$
$\mathfrak{g}_2$	$(0, 1)$	$\mathbb{C}$	$\mathbb{R}$	$G_2 1$

Table 5.5: Minimal nilpotent orbit in $\mathfrak{g}$

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