Spin Norm, K-Types, and Tempered Representations

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Abstract.   We extend the notion spin norm slightly to a real reductive Lie group G in the Harish-Chandra class. Let K be a maximal compact subgroup of G. In this setting, the spin norm of any K-type π is still bounded from below by its lambda norm. We establish a bijection between irreducible tempered ($\mathfrak{g}$, K)-modules with nonzero Dirac cohomology and those K-types whose spin norm equals their lambda norm.
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1.   Introduction

Let G be a real reductive Lie group in the Harish-Chandra class [3]. That is, the real reductive Lie group G satisfies the following conditions:

  (i) G has only a finite number of connected components;

 (ii) the derived group $[G, G]$ has finite center;

(iii) the adjoint action $Ad(g)$ of any $g \in G$ is an inner automorphism of $\mathfrak{g} = (\mathfrak{g}_0)_\mathbb{C}$, where $\mathfrak{g}_0$ is the Lie algebra of G.

Let θ be a Cartan involution of G. We assume that the group $K = G^\theta$ of fixed points of θ is a maximal compact subgroup of G. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition of $\mathfrak{g}_0$. As usual, we drop the subscript for the complexification.

Based on the ideas of Parthasarathy [6, 7], Vogan [12], Huang and Pandžić [4], the notion spin norm was introduced in [1] for the classification of irreducible unitary representations of a *connected* real reductive Lie group with non-vanishing Dirac cohomology. In this paper, we will slightly extend spin norm to G, which is not necessarily connected. Then we will show in Proposition 2.5 that the spin

norm of any K-type (i.e, any irreducible representation of K) is lower bounded by its lambda norm, which was introduced by Vogan [10].

Thus it is natural to ask: For which K-types does their spin norm equal their lambda norm? This elementary question does not seem to bear a neat answer. Instead, we interpret it in another setting, and our main result says that picking up these K-types from $\widehat{K}$ amounts to finding out all irreducible tempered $(\mathfrak{g}, K)$-modules with nonzero Dirac cohomology in $\widehat{G}^{\text{temp,o}}$. Here $\widehat{K}$ denotes the set of irreducible representations of K (up to equivalence), and $\widehat{G}^{\text{temp,o}}$ denotes the set of irreducible tempered $(\mathfrak{g}, K)$-modules with real infinitesimal character (up to equivalence).

The bridge for our interpretation is the following result due to Vogan [11], noted by Trapa [9].

Theorem 1.1. *Let X be any irreducible tempered $(\mathfrak{g}, K)$-module with real infinitesimal character. Then X has a unique lowest K-type which occurs with multiplicity one. Moreover, the map*

$$\phi : \widehat{G}^{\text{temp,o}} \to \widehat{K}$$

defined by taking the lowest K-type, is a well-defined bijection.

Let $\widehat{G}^{\text{temp,d}}$ denote the equivalence classes of irreducible tempered $(\mathfrak{g}, K)$-modules with nonzero Dirac cohomology. Then, as we shall explain in Section 2.1, any $X \in \widehat{G}^{\text{temp,d}}$ necessarily has real infinitesimal character. Thus

$$\widehat{G}^{\text{temp,d}} \subseteq \widehat{G}^{\text{temp,o}}.$$

On the other hand, put

$$\widehat{K}_e = \{\pi \in \widehat{K} \mid \|\pi\|_{\text{spin}} = \|\pi\|_{\text{lambda}}\}.$$

Of course

$$\widehat{K}_e \subseteq \widehat{K}.$$

Now the precise statement of our main result is as follows.

Theorem 1.2. *The map ϕ in Theorem 1.1 restricts to a bijection from $\widehat{G}^{\text{temp,d}}$ to $\widehat{K}_e$.*

The paper is organized as follows: we collect necessary preliminaries on Dirac cohomology, lambda norm and spin norm in Section 2. Then we review the classification of irreducible $(\mathfrak{g}, K)$-modules in Section 3. After these preparations, the main result will be proved in Section 4.

2.   Preliminaries

Let θ be a Cartan involution of a real reductive Lie group G in Harish-Chandra class. We assumed that $K = G^\theta$ is a maximal compact subgroup of G. Let

$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition of the Lie algebra. We drop the subscript for the complexification. Thus $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of the complexified Lie algebra $\mathfrak{g}$. Now let us recall necessary preliminaries.

2.1. Dirac cohomology.

We fix a maximal torus T of K. Let $\mathfrak{a}_0 = Z_{\mathfrak{p}_0}(\mathfrak{t}_0)$ and let A be the corresponding analytic subgroup of G. Then $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a fundamental Cartan subalgebra of $\mathfrak{g}_0$. We fix a nondegenerate invariant symmetric bilinear form B on $\mathfrak{g}$. Its restriction to $\mathfrak{k}$, $\mathfrak{p}$, etc., will also be denoted by B.

We denote by $\Delta(\mathfrak{g}, \mathfrak{h})$ (resp., $\Delta(\mathfrak{g}, \mathfrak{t})$) the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (resp., $\mathfrak{t}$). The root system of $\mathfrak{k}$ with respect to $\mathfrak{t}$ is denoted by $\Delta(\mathfrak{k}, \mathfrak{t})$. The Weyl groups corresponding to these root systems will be denoted by $W(\mathfrak{g}, \mathfrak{h})$, $W(\mathfrak{g}, \mathfrak{t})$ and $W(\mathfrak{k}, \mathfrak{t})$. Throughout this paper, we fix compatible choices of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$ and $\Delta^+(\mathfrak{g}, \mathfrak{t})$ so that a positive root in $\Delta(\mathfrak{g}, \mathfrak{h})$ restricts to a positive root in $\Delta(\mathfrak{g}, \mathfrak{t})$. As usual, we write ρ_c as the half sum of roots in $\Delta^+(\mathfrak{k}, \mathfrak{t})$.

Fix an orthonormal basis $Z_1, \cdots, Z_n$ of $\mathfrak{p}_0$ with respect to the inner product induced by B. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ (with respect to B). The Dirac operator $D \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is defined as

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i.$$

It is easy to check that D does not depend on the choice of the orthonormal basis Z_i and it is K-invariant for the diagonal action of K given by adjoint actions on both factors. This algebraic version was introduced by Vogan [12].

Let $\widetilde{K}$ be the subgroup of $K \times \mathrm{Pin}(\mathfrak{p}_0)$ consisting of all pairs (k, s) such that $\mathrm{Ad}(k) = p(s)$, where $\mathrm{Ad} : K \to \mathrm{O}(\mathfrak{p}_0)$ is the adjoint action, and $p : \mathrm{Pin}(\mathfrak{p}_0) \to \mathrm{O}(\mathfrak{p}_0)$ is the pin double covering map. If X is a $(\mathfrak{g}, K)$-module, and if S_G denotes a spin module for $C(\mathfrak{p})$, then $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ acts on $X \otimes S_G$ in the obvious fashion, while $\widetilde{K}$ acts on X through K and on S_G through the pin group $\mathrm{Pin}(\mathfrak{p}_0)$. Now the Dirac operator acts on $X \otimes S_G$, and the Dirac cohomology of X is defined as the $\widetilde{K}$-module

$$H_D(X) = \mathrm{Ker}\, D / (\mathrm{Im}\, D \cap \mathrm{Ker} D). \tag{1}$$

By setting the linear functionals on $\mathfrak{t}$ to be zero on $\mathfrak{a}$, we embed $\mathfrak{t}^*$ as a subspace of $\mathfrak{h}^*$. The following result is a tiny extension of the Vogan conjecture [12] proved by Huang and Pandžić [4] to disconnected groups, see Theorem 2.2 of [2].

Theorem 2.1. *Let G be a real reductive Lie group in Harish-Chandra class. Let X be an irreducible $(\mathfrak{g}, K)$-module with infinitesimal character Λ. Suppose that $\widetilde{\pi}$ is an irreducible $\widetilde{K}$-module in the Dirac cohomology $H_D(X)$ with a highest weight μ. Then Λ is conjugate to $\mu + \rho_c$ under $W(\mathfrak{g}, \mathfrak{h})$.*

In particular, if $H_D(X)$ is nonzero, then Λ must be *real* in the sense of Definition 5.4.1 of [10].

2.2. Lambda norm.

Recall that the analytic Weyl group is defined by

$$W(K, T) = N_K(T)/Z_K(T).$$

It acts on the root system $\Delta(\mathfrak{k}, \mathfrak{t})$, and we define

$$R(G) := \{r \in W(K, T) \mid r\Delta^+(\mathfrak{k}, \mathfrak{t}) = \Delta^+(\mathfrak{k}, \mathfrak{t})\}. \tag{2}$$

The notation is chosen to reflect that $R(G)$ is closely related to the R-group of Knapp and Stein. Recall from Lemma 0.1 of [8] that if π is a K-type, then the collection of highest weights of π as $\mathfrak{k}$ module is a single orbit of $R(G)$ on $\widehat{T} \subset i\mathfrak{t}_0^*$. Here $\widehat{T}$ is the abelian group of characters of T.

Now given any K-type π, take a highest weight μ of it. Then $\mu \in i\mathfrak{t}_0^*$ is dominant integral for $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Choose a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ making $\mu + 2\rho_c$ dominant. Write ρ for the half sum of roots in $\Delta^+(\mathfrak{g}, \mathfrak{t})$. Let P denote the projection to the dominant chamber corresponding to $\Delta^+(\mathfrak{g}, \mathfrak{t})$. Then $\|P(\mu + 2\rho_c - \rho)\|$ turns out to be independent of the choices of μ and $\Delta^+(\mathfrak{g}, \mathfrak{t})$, cf. Section 1 and Corollary 2.4 of [8].

Now let us recall the notions lambda norm and lowest K-type due to Vogan [10].

Definition 2.2. For any $\pi \in \widehat{K}$, we define its *lambda norm* by

$$\|\pi\|_{\mathrm{lambda}} = \|P(\mu + 2\rho_c - \rho)\|, \tag{3}$$

where μ is any highest weight of π. For any irreducible admissible $(\mathfrak{g}, K)$-module X, we define

$$\|X\|_{\mathrm{lambda}} = \min \|\pi\|_{\mathrm{lambda}}, \tag{4}$$

where π runs over all the K-types occurring in X. We call π a *lowest K-type* of X if it occurs in X and $\|\pi\|_{\mathrm{lambda}} = \|X\|_{\mathrm{lambda}}$.

2.3. Spin norm.

The following definition was introduced in [1] for *connected* real reductive Lie groups. It works also in the current setting.

Definition 2.3. For any $\pi \in \widehat{K}$, we define its *spin norm* by

$$\|\pi\|_{\mathrm{spin}} = \min \|\gamma + \rho_c\|, \tag{5}$$

where γ runs over all the highest weights of the $\widetilde{K}$-types in $\pi \otimes S_G$. For any irreducible admissible $(\mathfrak{g}, K)$-module X, we define

$$\|X\|_{\mathrm{spin}} = \min \|\pi\|_{\mathrm{spin}}, \tag{6}$$

where π runs over all the K-types occurring in X. We call π a *spin lowest K-type* of X if it occurs in X and $\|\pi\|_{\mathrm{spin}} = \|X\|_{\mathrm{spin}}$.

As a direct consequence of Theorem 2.4 of [2], we have

Proposition 2.4. *For any irreducible unitary $(\mathfrak{g}, K)$-module X with infinitesimal character Λ, let π be any K-type occurring in X. Then*

(a) $\|X\|_{\mathrm{spin}} \geq \|\Lambda\|$, *and equality holds if and only if $H_D(X)$ is non-zero.*

(b) $\|\pi\|_{\mathrm{spin}} \geq \|\Lambda\|$, *and equality holds if and only if π contributes to $H_D(X)$.*

(c) *If $H_D(X) \neq 0$, it is exactly the spin lowest K-types of X that contribute to $H_D(X)$.*

Here we say a K-type π of an irreducible unitary $(\mathfrak{g}, K)$-module X *contributes to* $H_D(X)$ if a $\widetilde{K}$-type of $\pi \otimes S_G$ occurs in $H_D(X)$.

Proposition 2.5. *For any K-type π, we have*

$$\|\pi\|_{\mathrm{spin}} \geq \|\pi\|_{\mathrm{lambda}}.$$

Proof. This follows directly from Proposition 3.8 of [1] and the knowledge of $R(G)$. ∎

3. Classification of irreducible $(\mathfrak{g}, K)$ modules

Let $\Pi_a(G)$ be the set of equivalence classes of irreducible $(\mathfrak{g}, K)$-modules. In this section, we review the classification of $\Pi_a(G)$ according to [10]. For any dominant $\mu \in i\mathfrak{t}_0^*$, let us write $\lambda_a(\mu)$ for $P(\mu + 2\rho_c - \rho)$. Given $\lambda \in \Lambda_a := \{\lambda_a(\mu) \mid \mu \in \widehat{T} \text{ dominant}\}$, as in [8] we define:

$$
\begin{aligned}
B_a^\lambda(G) &= \{\delta \in \widehat{K} \mid \delta \text{ has a highest weight } \mu \text{ such that } \lambda_a(\mu) = \lambda\}, \\
\Pi_a^\lambda(G) &= \{X \in \Pi_a(G) \mid X \text{ has a lowest } K \text{ type in } B_a^\lambda(G)\}, \\
G(\lambda) &= \text{isotropy group at } \lambda \text{ for } G \text{ action};
\end{aligned}
$$

this is a subgroup of G containing T. It is easy to check that $G(\lambda)$ is a reductive Lie group in Harish-Chandra class, with Cartan involution $\theta|_{G(\lambda)}$. Moreover, the set of roots of $\mathfrak{t}$ in the complexified Lie algebra of $G(\lambda)$ is

$$\Delta(\mathfrak{g}(\lambda), \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \mid \langle \alpha, \lambda \rangle = 0\}.$$

We construct a θ-stable parabolic subalgebra $\mathfrak{q}(\lambda) = \mathfrak{g}(\lambda) + \mathfrak{u}(\lambda)$ by requiring

$$\Delta(\mathfrak{u}(\lambda), \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \mid \langle \alpha, \lambda \rangle > 0\}. \tag{7}$$

The first step of the classification of $\Pi_a(G)$ is achieved by reducing it to the classification of $\Pi_a(G(\lambda))$. Note that $G(\lambda)$ has the special property of being quasisplit (see Theorem 3.3 below).

Theorem 3.1. (Theorem 6.5.12 of [10] or Theorem 2.9 of [8]) *Suppose $\lambda \in \Lambda_a$, put $S = dim\,(\mathfrak{u}(\lambda) \cap \mathfrak{k})$, with other notations as above.*

(i) *The cohomological parabolic induction functor $\mathcal{L}_S^K(\lambda)$ implements a bijection from $B_a^{\lambda - \rho(\mathfrak{u}(\lambda))}(G(\lambda))$ to $B_a^\lambda(G)$.*

(ii) *Take any* $Z_a \in \Pi_a^{\lambda-\rho(\mathfrak{u}(\lambda))}(G(\lambda))$, *pick up one of its lowest* $G(\lambda) \cap K$ *types* $\pi_{G(\lambda)} \in B_a^{\lambda-\rho(\mathfrak{u}(\lambda))}(G(\lambda))$. *Then* $\pi = \mathcal{L}_S^K(\lambda)(\pi_{G(\lambda)})$ *must occur as a lowest* K *type of* $\mathcal{L}_S(\lambda)(Z_a)$. *Let* X *be the unique irreducible subquotient of* $\mathcal{L}_S(\lambda)(Z_a)$ *containing the* K *type* π. *Then the map* $Z_a \mapsto X$ *gives a natural bijection from* $\Pi_a^{\lambda-\rho(\mathfrak{u}(\lambda))}(G(\lambda))$ *to* $\Pi_a^\lambda(G)$.

The bijection from $\Pi_a^{\lambda-\rho(\mathfrak{u}(\lambda))}(G(\lambda))$ to $\Pi_a^\lambda(G)$ described in the theorem above preserves tempered representations.

Theorem 3.2.   (Corollary 4.4 of [8]) *In the setting of Theorem 3.1, suppose* $Z_a \in \Pi_a^{\lambda-\rho(\mathfrak{u}(\lambda))}(G(\lambda))$ *is an irreducible tempered* $(\mathfrak{g}(\lambda), G(\lambda) \cap K)$-*module, then* $\mathcal{L}_S(\lambda)(Z_a)$ *is an irreducible tempered* $(\mathfrak{g}, K)$ *module in* $\Pi_a^\lambda(G)$. *Conversely, every irreducible tempered* $(\mathfrak{g}, K)$-*module in* $\Pi_a^\lambda(G)$ *arises in this way.*

Now it remains to classify $\Pi_a(G(\lambda))$, which can be realized via real parabolic induction.

Theorem 3.3.   (Theorem 4.4.8 of [10] or Proposition 4.1 of [8]) *Suppose that* $\lambda \in \Lambda_a$, *and that* $G(\lambda) = G$. *Then* G *is quasisplit; that is, there is a Borel subgroup* $B^q = T^q A^q N^q$ *of* G *such that the Levi factor* $H^q := T^q A^q$ *is a maximally split Cartan subgroup of* G. *Replacing* B^q *by a conjugate, we may assume that the identity component of* T^q *is contained in the maximal torus* T. *Then* λ *vanishes on the orthogonal complement of* $\mathfrak{t}^q$ *in* $\mathfrak{t}$. *Suppose* $\pi \in B_a^\lambda(G)$, *and* $X \in \Pi_a^\lambda(G)$ *is an irreducible* $(\mathfrak{g}, K)$-*module with lowest* K *type* π. *Let* δ *be a* T^q *module appearing in the restriction of* π *to* T^q. *Then there is a character* $\nu \in \widehat{A^q}$ *so that* X *is a subquotient of the principal series* $\mathrm{Ind}_{T^q A^q N^q}^G(\delta \otimes \nu)$. *In particular, the infinitesimal character of* X *has Harish-Chandra parameter* $(\lambda, \nu) \in (\mathfrak{t}^q)^* + (\mathfrak{a}^q)^*$.

Finally, we recall a result of Knapp and Stein [5].

Theorem 3.4.   (Knapp and Stein [5] or Corollary 4.4.11 of [10]) *In the setting of Theorem 3.3, suppose that* $\nu \in \widehat{A^q}$ *is unitary. Then the principal series* $\mathrm{Ind}_{T^q A^q N^q}^G(\delta \otimes \nu)$ *is the sum of* $|R_\delta(\nu)|$ *inequivalent irreducible representations.*

In the above theorem, $R_\delta(\nu)$ is a finite group, and we refer the reader to Definition 4.4.9 of [10] for its precise meaning.

4.   Proof of the main result

Proof of Theorem 1.2. Let X be an irreducible tempered $(\mathfrak{g}, K)$-module with real infinitesimal character. Then by Theorem 1.1, X has a unique lowest K type π which occurs with multiplicity one. Take a highest weight $\mu \in \widehat{T} \subset i\mathfrak{t}_0^*$ of π, and let $\lambda = P(\mu + 2\rho_c - \rho) \in i\mathfrak{t}_0^*$. As in (7), we use λ to define a θ-stable parabolic subalgebra $\mathfrak{q}(\lambda) = \mathfrak{g}(\lambda) \oplus \mathfrak{u}(\lambda)$.

Firstly, by Theorems 3.1 and 3.2, $G(\lambda)$ is quasisplit, and there exists an

irreducible tempered $(\mathfrak{g}(\lambda), G(\lambda) \cap K)$-module Y such that

$$X \cong \mathcal{L}_S(\lambda)(Y)(\pi), \tag{8}$$

where $S = \dim(\mathfrak{u}(\lambda) \cap \mathfrak{k})$, $\mathcal{L}_S(\lambda)(\cdot)$ is the cohomological parabolic induction functor, and $\mathcal{L}_S(\lambda)(Y)(\pi)$ denotes the unique irreducible subquotient of $\mathcal{L}_S(\lambda)(Y)$ containing the K-type π. Actually, Y contains a lowest $G(\lambda) \cap K$ type $\pi_{G(\lambda)}$ such that $\mathcal{L}_S^K(\lambda)(\pi_{G(\lambda)}) = \pi$.

Secondly, by Theorem 3.3, Y can be realized as a *subquotient* of the principal series

$$\mathrm{Ind}_{T^q A^q N^q}^{G(\lambda)}(\delta \otimes \nu).$$

Here $H^q = T^q A^q$ is a maximally split Cartan subgroup of $G(\lambda)$ such that the identity component of T^q is contained in T; $\delta \in \widehat{T^q}$ occurs in the restriction of $\pi_{G(\lambda)}$ to T^q; and $\nu \in \widehat{A^q}$.

Now on one hand, since Y is tempered, the continuous parameter $\nu \in \widehat{A^q}$ is purely imaginary (cf. Corollary 4.4 of [8]); while one the other hand, by Theorems 3.1 and 3.3, the infinitesimal character of X is $\Lambda = (\lambda, \nu)$, which is assumed to be real. Thus we conclude that $\nu = 0$. Note that by Theorem 3.4, the principal series $\mathrm{Ind}_{T^q A^q N^q}^{G(\lambda)}(\delta \otimes 0)$ is a direct sum of finitely many inequivalent irreducible representations. Thus we see that Y actually is the irreducible *submodule* of $\mathrm{Ind}_{T^q A^q N^q}^{G(\lambda)}(\delta \otimes 0)$ containing the $G(\lambda) \cap K$ type $\pi_{G(\lambda)}$.

Take any K type π' of X, by Proposition 2.5, we have

$$\|\pi'\|_{\mathrm{spin}} \geq \|\pi'\|_{\mathrm{lambda}} \geq \|\pi\|_{\mathrm{lambda}} = \|\lambda\| = \|\Lambda\|.$$

Thus by Proposition 2.4, we see that only the unique lowest K type π of X can possibly contribute to $H_D(X)$. Thus $H_D(X) \neq 0$ if and only if π contributes to $H_D(X)$, if and only if

$$\|\pi\|_{\mathrm{spin}} = \|\Lambda\| = \|\lambda\| = \|\pi\|_{\mathrm{lambda}}.$$

The first equality above follows from Proposition 2.4. We conclude that the map ϕ in Theorem 1.1 restricts to a bijection from $\widehat{G}^{\mathrm{temp,d}}$ to $\widehat{K}_e$. ∎

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