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Normalisers of Abelian Ideals of a Borel Subalgebra and $\mathbb{Z}$-Gradings of a Simple Lie Algebra

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Abstract. Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{Ab}$ the poset of all abelian ideals of a fixed Borel subalgebra of $\mathfrak{g}$. If $\mathfrak{a} \in \mathfrak{Ab}$, then the normaliser of $\mathfrak{a}$ is a standard parabolic subalgebra of $\mathfrak{g}$. We give an explicit description of the normaliser for a class of abelian ideals that includes all maximal abelian ideals. We also elaborate on a relationship between abelian ideals and $\mathbb{Z}$-gradings of $\mathfrak{g}$ associated with their normalisers.
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Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$, where $\mathfrak{t}$ is a fixed Cartan subalgebra and $\mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t}$ is a fixed Borel subalgebra. A subspace $\mathfrak{a} \subset \mathfrak{b}$ is an *abelian ideal* if $[\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{a}$ and $[\mathfrak{a}, \mathfrak{a}] = 0$. Then $\mathfrak{a} \subset \mathfrak{u}$. The general theory of abelian ideals of $\mathfrak{b}$ is based on their relations with the so-called *minuscule elements* of the affine Weyl group $\widehat{W}$, which is due to D. Peterson (see Kostant's account in [6]). The subsequent development has lead to a number of spectacular results of combinatorial and representation-theoretic nature, see e.g. [2, 3, 4, 7, 8, 10, 13, 14].

The normaliser of $\mathfrak{a}$ in $\mathfrak{g}$, denoted $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$, contains $\mathfrak{b}$, i.e., it is a *standard* parabolic subalgebra of $\mathfrak{g}$. In this note, we study the normalisers of abelian ideals using the corresponding minuscule elements of $\widehat{W}$ and $\mathbb{Z}$-gradings of $\mathfrak{g}$.

Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$, Δ^+ the set of positive roots corresponding to $\mathfrak{u}$, Π the set of simple roots in Δ^+, and θ the highest root in Δ^+. Then W is the Weyl group and $\mathfrak{g}_\gamma$ is the root space for $\gamma \in \Delta$. We write $\mathfrak{Ab} = \mathfrak{Ab}(\mathfrak{g})$ for the set of all abelian ideals of $\mathfrak{b}$ and think of $\mathfrak{Ab}$ as poset with respect to inclusion. Since $\mathfrak{a} \in \mathfrak{Ab}$ is a sum of certain root spaces of $\mathfrak{u}$, we often identify such an $\mathfrak{a}$ with the corresponding subset $I = I_{\mathfrak{a}}$ of Δ^+.

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Let $\mathfrak{Ab}^o$ denote the set of nonzero abelian ideals and Δ_l^+ the set of long positive roots. In [8, Sect. 2], we defined a surjective mapping $\tau : \mathfrak{Ab}^o \to \Delta_l^+$ and studied its fibres. If $\tau(\mathfrak{a}) = \mu$, then $\mu \in \Delta_l^+$ is called the *rootlet* of $\mathfrak{a}$. Letting $\mathfrak{Ab}_\mu = \tau^{-1}(\mu)$, we get a partition of $\mathfrak{Ab}^o$ parameterised by Δ_l^+. Each fibre $\mathfrak{Ab}_\mu$ is a sub-poset of $\mathfrak{Ab}$. By [8, Sect. 3], the poset $\mathfrak{Ab}_\mu$ has a unique minimal and unique maximal element for any $\mu \in \Delta_l^+$. These are denoted by $\mathfrak{a}(\mu)_{min}$ and $\mathfrak{a}(\mu)_{max}$, respectively. The corresponding sets of positive roots are $I(\mu)_{\min}$ and $I(\mu)_{\max}$. The abelian ideals of the form $\mathfrak{a}(\mu)_{min}$ (resp. $\mathfrak{a}(\mu)_{max}$) will be referred to as the *root-minimal* (resp. *root-maximal*). The set of globally maximal abelian ideals coincides with $\{\mathfrak{a}(\alpha)_{max} \mid \alpha \in \Pi_l\}$, where $\Pi_l = \Delta_l^+ \cap \Pi$ [8, Cor. 3.8].

If $\mathfrak{p} \supset \mathfrak{b}$, then a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ is said to be *standard*, if $\mathfrak{l} \supset \mathfrak{t}$. Set $\mathfrak{p}[\mu]_{min} = \mathfrak{n}_\mathfrak{g}(\mathfrak{a}(\mu)_{min})$ and $\mathfrak{p}[\mu]_{max} = \mathfrak{n}_\mathfrak{g}(\mathfrak{a}(\mu)_{max})$. Write $\Pi[\mu]_{min}$ for the simple roots of the standard Levi subalgebra of $\mathfrak{p}[\mu]_{min}$, and likewise for 'max'. Our main results are the following:

I. We explicitly describe $\Pi[\mu]_{min}$ for any root-minimal ideal $\mathfrak{a}(\mu)_{min}$. The answer is given in terms of the element $w_\mu \in W$ that takes θ to μ and has minimal possible length, see Theorem 2.3. The elements w_μ have already been considered in [8], and we also provide here new properties of them. Furthermore, if θ is fundamental and $\alpha_\theta \in \Pi$ is such that $(\theta, \alpha_\theta) \neq 0$, then α_θ is long and we prove that $\Pi \setminus \Pi[\alpha_\theta]_{min}$ consists of the simple roots that are adjacent to α_θ in the Dynkin diagram (Proposition 2.4).

II. We give a new characterisation of normalisers of arbitrary $\mathfrak{b}$-stable subspaces of $\mathfrak{u}$ (Theorem 3.3) and then explicitly describe the normalisers of the globally maximal abelian ideals, i.e., we determine $\Pi[\alpha]_{max}$ for all $\alpha \in \Pi_l$ (Theorem 3.9). This is based on a relationship between $\mathfrak{a}(\alpha)_{min}$ and $\mathfrak{a}(\alpha)_{max}$ for $\alpha \in \Pi_l$ [10, Theorem 4.7], which allows us to retrieve information on $\Pi[\alpha]_{max}$ from that on $\Pi[\alpha]_{min}$.

III. In Section 4, we relate $\mathfrak{a} \in \mathfrak{Ab}(\mathfrak{g})$ to the $\mathbb{Z}$-grading of $\mathfrak{g}$ corresponding to $\mathfrak{n}_\mathfrak{g}(\mathfrak{a})$. Let $\mathfrak{Par}(\mathfrak{g})$ denote the set of all standard parabolic subalgebras of $\mathfrak{g}$. By Peterson's theorem [6], $\#\mathfrak{Ab}(\mathfrak{g}) = 2^{\mathsf{rk}\,\mathfrak{g}}$, hence the sets $\mathfrak{Ab}(\mathfrak{g})$ and $\mathfrak{Par}(\mathfrak{g})$ are equipotent. There is the natural mapping $f_1 : \mathfrak{Ab}(\mathfrak{g}) \to \mathfrak{Par}(\mathfrak{g})$ that takes $\mathfrak{a}$ to $\mathfrak{n}_\mathfrak{g}(\mathfrak{a})$. By [12], f_1 is a bijection if and only if $\mathfrak{g} = \mathfrak{sl}_{n+1}$ or $\mathfrak{sp}_{2n}$. Using the $\mathbb{Z}$-grading associated with $\mathfrak{p} \in \mathfrak{Par}(\mathfrak{g})$, we define here the natural mapping $f_2 : \mathfrak{Par}(\mathfrak{g}) \to \mathfrak{Ab}(\mathfrak{g})$ and prove that f_2 is a bijection if and only if $\mathfrak{g} = \mathfrak{sl}_{n+1}$ or $\mathfrak{sp}_{2n}$; furthermore, $f_2 = f_1^{-1}$ for these two series (Theorem 4.5). We say that $\mathfrak{a} \in \mathfrak{Ab}$ is *reflexive*, if $(f_2 \circ f_1)(\mathfrak{a}) = \mathfrak{a}$. Then all abelian ideals for $\mathfrak{sl}_{n+1}$ and $\mathfrak{sp}_{2n}$ are reflexive. We also prove that $\mathfrak{a}(\alpha)_{min}$ and $\mathfrak{a}(\alpha)_{max}$ $(\alpha \in \Pi_l)$ are always reflexive and characterise them in terms of the corresponding $\mathbb{Z}$-gradings (see Theorem 4.2 and Remark 4.6). Finally, we conjecture that the sets $\mathsf{Im}(f_1 \circ f_2)$ and $\mathsf{Im}(f_2 \circ f_1)$ are always equipotent and the maps f_1 and f_2 induce the mutually inverse bijections between them.

We refer to [1, 5] for standard results on root systems and (affine) Weyl groups.

1. Preliminaries on minuscule elements and normalisers of abelian ideals

We equip Δ^+ with the usual partial ordering '$\preccurlyeq$'. This means that $\mu \preccurlyeq \nu$ if $\nu-\mu$ is a non-negative integral linear combination of simple roots. If M is a subset of Δ^+, then $\min(M)$ and $\max(M)$ are the minimal and maximal elements of M with respect to "$\preccurlyeq$".

Any $\mathfrak b$-stable subspace $\mathfrak c \subset \mathfrak u$ is a sum of certain root spaces in $\mathfrak u$, i.e., $\mathfrak c = \bigoplus_{\gamma\in I_{\mathfrak c}} \mathfrak g_\gamma$. The relation $[\mathfrak b, \mathfrak c] \subset \mathfrak c$ is equivalent to that $I = I_{\mathfrak c}$ is an *upper ideal* of the poset $(\Delta^+, \preccurlyeq)$, i.e., if $\nu \in I$, $\gamma\in\Delta^+$, and $\nu \preccurlyeq \gamma$, then $\gamma \in I$. We mostly work in the combinatorial setting, so that a $\mathfrak b$-ideal $\mathfrak c \subset \mathfrak u$ is being identified with the corresponding upper ideal I of Δ^+. The property of being abelian additionally means that $\gamma'+\gamma'' \notin \Delta^+$ for all $\gamma', \gamma'' \in I$.

We recall below the notion of a minuscule element of $\widehat W$ and their relation to abelian ideals. We have $\Pi = \{\alpha_1, \dots, \alpha_n\}$, the vector space $\mathfrak t_{\mathbb R} = V = \oplus_{i=1}^n \mathbb R\alpha_i$, the Weyl group W generated by simple reflections $s_1, \dots, s_n$, and a W-invariant inner product $(\ ,\ )$ on V. Letting $\widehat V = V \oplus \mathbb R\delta \oplus \mathbb R\lambda$, we extend the inner product $(\ ,\ )$ on $\widehat V$ so that $(\delta, V) = (\lambda, V) = (\delta,\delta) = (\lambda,\lambda) = 0$ and $(\delta,\lambda)=1$. Set $\alpha_0 = \delta-\theta$, where θ is the highest root in Δ^+. Then

$\widehat\Delta = \{\Delta + k\delta \mid k \in \mathbb Z\}$ is the set of affine (real) roots;

$\widehat\Delta^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geqslant 1\}$ is the set of positive affine roots;

$\widehat\Pi = \Pi \cup \{\alpha_0\}$ is the corresponding set of affine simple roots;

$\mu^\vee = 2\mu/(\mu,\mu)$ is the coroot corresponding to $\mu \in \widehat\Delta$.

For each $\alpha_i \in \widehat\Pi$, let $s_i = s_{\alpha_i}$ denote the corresponding reflection in $GL(\widehat V)$. That is, $s_i(x) = x - (x,\alpha_i)\alpha_i^\vee$ for any $x \in \widehat V$. The affine Weyl group, $\widehat W$, is the subgroup of $GL(\widehat V)$ generated by the reflections $s_0, s_1, \dots, s_n$. The extended inner product $(\ ,\ )$ on $\widehat V$ is $\widehat W$-invariant. The *inversion set* of $w \in \widehat W$ is $\mathcal N(w) = \{\nu \in \widehat\Delta^+ \mid w(\nu) \in -\widehat\Delta^+\}$. Note that if $w \in W \subset \widehat W$, then $\mathcal N(w) \subset \Delta^+$.

Following Peterson, we say that $w \in \widehat W$ is *minuscule*, if

$$\mathcal N(w) = \{-\gamma+\delta \mid \gamma \in I_w\} \text{ for some } I_w \subset \Delta.$$

One then proves that (i) $I_w \subset \Delta^+$, (ii) I_w is (the set of roots of) an abelian ideal, and (iii) the assignment $w \mapsto I_w$ yields a bijection between the minuscule elements of $\widehat W$ and the abelian ideals, see [6], [2, Prop. 2.8]. Conversely, if $\mathfrak a \in \mathfrak{Ab}$ and $I = I_{\mathfrak a}$, then $w_{\mathfrak a} \in \widehat W$ stands for the corresponding minuscule element. Clearly, $\dim\mathfrak a = \#I_{\mathfrak a} = \#\mathcal N(w_{\mathfrak a})$.

Given $\mathfrak a \in \mathfrak{Ab}^o$ and $w_{\mathfrak a} \in \widehat W$, the *rootlet* of $\mathfrak a$ is defined by

$$\tau(\mathfrak a) = w_{\mathfrak a}(\alpha_0) + \delta = w_{\mathfrak a}(2\delta-\theta).$$

By [8, Prop. 2.5], we have $\tau(\mathfrak a) \in \Delta_l^+$ and every $\mu \in \Delta_l^+$ occurs in this way.

Let $\mathfrak l$ be the standard Levi subalgebra of $\mathfrak p = \mathfrak n_{\mathfrak g}(\mathfrak a)$ and $\Pi(\mathfrak l) \subset \Pi$ the set of simple roots of $\mathfrak l$. By [9, Theorem 2.8], the set $\Pi(\mathfrak l)$ is determined by $w_{\mathfrak a}$ as follows:

$$\alpha \in \Pi(\mathfrak l) \iff w_{\mathfrak a}(\alpha) \in \widehat\Pi.$$

(Actually, this result of [9] has been proved for any $\mathfrak{b}$-stable subspace $\mathfrak{c} \subset \mathfrak{u}$ in place of $\mathfrak{a}$. To this end, one also needs a more general theory of elements of $\widehat{W}$ associated with arbitrary $\mathfrak{b}$-stable subspaces of $\mathfrak{u}$ [2].)

An advantage of our situation is that, for the root-minimal abelian ideals $\mathfrak{a} = \mathfrak{a}(\mu)_{min}$, there is a simple formula for $w_{\mathfrak{a}}$, which allows us to describe the corresponding normaliser in terms of μ. We also need the following facts:

- $\#\tau^{-1}(\mu) = 1$ (i.e., $\mathfrak{a}(\mu)_{min} = \mathfrak{a}(\mu)_{max}$) if and only if $(\theta, \mu) \neq 0$ [8, Theorem 5.1].
- $\mathfrak{a}$ is root-minimal if and only if $I_{\mathfrak{a}} \subset \mathcal{H} := \{\gamma \in \Delta^+ \mid (\gamma, \theta) \neq 0\}$ [8, Theorem 4.3];

In what follows, it will be important to distinguish the cases whether θ is fundamental or not, and whether $(\theta, \mu) = 0$ or not. Recall that θ is fundamental if and only if Δ is not of type $\mathbf{A}_n$ or $\mathbf{C}_n$. One also has

$$\#(\Pi \cap \mathcal{H}) = \begin{cases} 2 & \text{for } \mathbf{A}_n \\ 1 & \text{for all other types.} \end{cases}$$

For the classical series, we use the standard notation and numbering for Π, which seems to be the same in all sources. For instance, for $\mathbf{A}_n$, we have $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, \dots, n)$, whence $\Pi \cap \mathcal{H} = \{\alpha_1, \alpha_n\}$. For $\mathbf{E}_6$, our numbering is $\begin{smallmatrix} 1-2-3-4-5 \\ \quad\;\; | \\ \quad\;\; 6 \end{smallmatrix}$; hence $\Pi \cap \mathcal{H} = \{\alpha_6\}$.

For $\gamma \in \Delta$ and $\alpha \in \Pi$, $[\gamma : \alpha]$ stands for the coefficient of α in the expression of γ via Π.

2. Normalisers of the root-minimal abelian ideals

In this section, we describe normalisers of the root-minimal abelian ideals for all $\mu \in \Delta_l^+$.

There is a unique element of minimal length in W taking θ to μ [8, Theorem 4.1], which is denoted by w_μ. The ideal $\mathfrak{a}(\mu)_{min}$ is completely determined by w_μ. Namely, $w_\mu s_0 \in \widehat{W}$ is the minuscule element corresponding to $\mathfrak{a}(\mu)_{min}$ [8, Theorem 4.2]. We begin with two useful properties of the elements w_μ.

Lemma 2.1. *If $\beta \in \Pi$ and $(\beta, \mu) = 0$, then $w_\mu^{-1}(\beta) \in \Pi$ and $(w_\mu^{-1}(\beta), \theta) = 0$.*

Proof. It is known that $\mathcal{N}(w_\mu^{-1}) = \{\gamma \in \Delta^+ \mid (\gamma, \mu^\vee) = -1\}$ [8, Theorem 4.1(2)]. Therefore $w_\mu^{-1}(\beta) \in \Delta^+$. Assume that $w_\mu^{-1}(\beta) = \gamma_1 + \gamma_2$ is a sum of positive roots. Then $\beta = w_\mu(\gamma_1) + w_\mu(\gamma_2)$. Without loss of generality, one may assume that $-\nu_1 := w_\mu(\gamma_1)$ is negative. Then $\nu_1 \in \mathcal{N}(w_\mu^{-1})$, hence $(-\nu_1, \mu^\vee) = 1$. Consequently, $(\gamma_1, \theta^\vee) = 1$. On the other hand, $0 = (\mu, \beta) = (\theta, \gamma_1 + \gamma_2)$ and therefore $(\theta, \gamma_2) < 0$, which is impossible. Thus, $w_\mu^{-1}(\beta)$ must be simple and $(w_\mu^{-1}(\beta), \theta) = (\beta, \mu) = 0$. ■

Lemma 2.2. *Suppose that θ is fundamental and $\alpha_\theta \in \Pi$ is not orthogonal to θ. If $(\theta, \mu) > 0$ and $\theta \neq \mu$, then $w_\mu^{-1}(\theta) = \theta - \alpha_\theta$; or, equivalently, $w_\mu(\alpha_\theta) = \mu - \theta$.*

Proof. It is well known and easily verified that α_θ is long and $[\theta : \alpha_\theta] = 2$ (cf. also Theorem 4.1(ii)). If $\mu \in \mathcal{H} \setminus \{\theta\}$, then $[\mu : \alpha_\theta] = 1$. By [11, Section 1], multiplicities of the simple reflections in any reduced expression of w_μ are the same, and they are determined by the coefficients of $\theta - \mu$. In particular, s_{α_θ} occurs only once, since $[\theta - \mu : \alpha_\theta] = 1$ and α_θ is long. Moreover, the reduced expressions of w_μ are in a bijections with the "root paths" connecting θ with μ inside Δ_l^+. Since θ is fundamental, the passage $\theta \rightsquigarrow s_{\theta_\alpha}(\theta)$ is the only step down from θ inside Δ_l^+. Hence any root path leading to μ starts with this step. Therefore, every reduced expression of w_μ begins with s_{α_θ}, and one can write $w_\mu = w' s_{\alpha_\theta}$, where w' does not contain factors s_{α_θ}. Therefore, $w_\mu^{-1}(\theta) = s_{\alpha_\theta} w'^{-1}(\theta) = s_{\alpha_\theta}(\theta) = \theta - \alpha_\theta$. ■

Remark. This is a generalisation of [11, Lemma 4.3], where the similar assertion is proved for $\mu = \alpha_\theta$.

Recall that $\Pi[\mu]_{min} \subset \Pi$ is the set of simple roots for the standard Levi subalgebra of $\mathfrak{p}[\mu]_{min}$. Since θ is not fundamental if and only if $\Delta = \mathbf{A}_n$ or $\mathbf{C}_n$, the following result covers all the possibilities for μ.

Theorem 2.3. *For any $\mu \in \Delta_l^+$, the set $\Pi[\mu]_{min}$ has the following description.*

(i) $\Pi[\mu]_{min} \cap \theta^\perp = \{w_\mu^{-1}(\beta) \mid \beta \in \Pi \ \& \ (\beta, \mu) = 0\}$
$= \{\alpha \in \Pi \mid w_\mu(\alpha) \in \Pi \ \& \ (\alpha, \theta) = 0\}$.

(ii) *If $(\mu, \theta) = 0$, then $\Pi[\mu]_{min} = \{w_\mu^{-1}(\beta) \mid \beta \in \Pi \ \& \ (\beta, \mu) = 0\}$. In particular, $\Pi[\mu]_{min} \subset \theta^\perp$.*

(iii) *Suppose that $(\mu, \theta) \neq 0$ (i.e., $\mu \in \mathcal{H}$) and $\mu \neq \theta$.*

a) if θ is fundamental, then $\Pi[\mu]_{min} = \{\alpha_\theta\} \cup \{w_\mu^{-1}(\beta) \mid \beta \in \Pi \ \& \ (\beta, \mu) = 0\}$, where α_θ is the only simple root such that $(\theta, \alpha_\theta) \neq 0$;

b) if $\Delta = \mathbf{C}_n$, then there is no such long roots μ;

c) if $\Delta = \mathbf{A}_n$ and $\mu = \alpha_1 + \dots + \alpha_i = \gamma_i$ ($i < n$) or $\alpha_j + \dots + \alpha_n = \tilde{\gamma}_j$ ($j > 1$), then
$\Pi[\gamma_i]_{min} = \{\alpha_n\} \cup \{w_{\gamma_i}^{-1}(\beta) \mid \beta \in \Pi \ \& \ (\beta, \gamma_i) = 0\} = \Pi \setminus \{\alpha_1, \alpha_i\}$ *and*
$\Pi[\tilde{\gamma}_j]_{min} = \{\alpha_1\} \cup \{w_{\tilde{\gamma}_j}^{-1}(\beta) \mid \beta \in \Pi \ \& \ (\beta, \tilde{\gamma}_j) = 0\} = \Pi \setminus \{\alpha_j, \alpha_n\}$.

(iv) *If $\mu = \theta$, then $\Pi[\theta]_{min} = \{\beta \in \Pi \mid (\beta, \theta) = 0\}$.*

Proof. Since $w_\mu s_0 \in \widehat{W}$ is the minuscule element corresponding to $I(\mu)_{min}$, the general theory of normalisers of $\mathfrak{b}$-stable subspaces of $\mathfrak{u}$ asserts that

$$\alpha \in \Pi[\mu]_{min} \iff w_\mu s_0(\alpha) \in \widehat{\Pi}, \tag{2·1}$$

see [9, Theorem 2.8]. Here one has to distinguish two possibilities:

(1) $w_\mu s_0(\alpha) \in \Pi$;

(2) $w_\mu s_0(\alpha) = \alpha_0 = \delta - \theta$.

• Suppose that $w_\mu s_0(\alpha) = \beta \in \Pi$. Then $w_\mu^{-1}(\beta) = s_0(\alpha) \in \Delta$. Hence $s_0(\alpha) = \alpha$ and therefore $(\theta, \alpha) = 0$ and $(\beta, \mu) = (w_\mu(\alpha), w_\mu(\theta)) = 0$. Thus, if $\alpha \in \Pi[\mu]_{min}$ satisfies ***(1)***, then $w_\mu(\alpha) = \beta \in \Pi$ and $(\beta, \mu) = (\theta, \alpha) = 0$.

Conversely, if $\beta \in \Pi$ and $(\beta, \mu) = 0$, then Lemma 2.1 shows that $\alpha := w_\mu^{-1}(\beta) \in \Pi$ and $(\alpha, \theta) = 0$. Hence ***(1)*** is satisfied for μ and α.

- Suppose that $w_\mu s_0(\alpha) = \alpha_0 = \delta - \theta$. Then $w_\mu^{-1}(\delta - \theta) = s_0(\alpha)$. Therefore, $\alpha \in \Pi_l$ and $s_0(\alpha) \neq \alpha$, i.e., $(\alpha, \theta) \neq 0$. More precisely, $\delta - w_\mu^{-1}(\theta) = \delta - (\theta - \alpha)$, hence $w_\mu^{-1}(\theta) = \theta - \alpha$. The last equality can be rewritten as $\theta = \mu - w_\mu(\alpha)$. Therefore, $(\mu, \theta) \neq 0$ and $\mu \neq \theta$. Hence equality ***(2)*** can only occur for $\mu \in \mathcal{H} \setminus \{\theta\}$ and $\alpha \in \mathcal{H}$. Furthermore, if θ is fundamental, then one must have $\alpha = \alpha_\theta$. By Lemma 2.2, the equality $w_\mu^{-1}(\theta) = \theta - \alpha_\theta$ is then satisfied and we conclude that $\alpha_\theta \in \Pi[\mu]_{min}$.

This proves parts (i),(ii),(iiia).

Parts (iiib) is clear, and (iiic) is obtained by a direct calculation.

(iv) Here $\mathfrak{a}(\theta)_{min} = \mathfrak{g}_\theta$, and the assertion is obvious. ■

Theorem 2.3 provides a complete description of $\Pi[\mu]_{min}$ for all $\mu \in \Delta_l^+$. But for some long simple roots, the assertion can be made even more precise.

Proposition 2.4. *If θ is fundamental and $(\theta, \alpha_\theta) \neq 0$, then*
$$\Pi[\alpha_\theta]_{min} = \{\alpha_\theta\} \cup \{\beta \in \Pi \mid (\beta, \alpha_\theta) = 0\}.$$
Therefore, $\Pi \setminus \Pi[\alpha_\theta]_{min}$ consists of the simple roots that are adjacent to α_θ in the Dynkin diagram.

Proof. By Theorem 2.3(iii), we have $\Pi[\alpha_\theta]_{min} = \{\alpha_\theta\} \cup \{w_{\alpha_\theta}^{-1}(\beta) \mid \beta \in \Pi \ \& \ (\beta, \alpha_\theta) = 0\}$. Therefore, we are to prove that $w_{\alpha_\theta}^{-1}$ permutes the simple roots orthogonal to α_θ. If $\beta \in \Pi$ and $(\beta, \alpha_\theta) = 0$, then we already know that $w_{\alpha_\theta}^{-1}(\beta) \in \Pi$. Next, using Lemma 2.2 with $\mu = \alpha_\theta$, we obtain

$$(w_{\alpha_\theta}^{-1}(\beta), \alpha_\theta) = (\beta, w_{\alpha_\theta}(\alpha_\theta)) = (\beta, \alpha_\theta - \theta) = -(\beta, \theta).$$

Since $\beta \neq \alpha_\theta$ and θ is fundamental, this must be zero. ■

The minuscule elements for the root-maximal abelian ideals do not admit a simple formula. Therefore, we cannot explicitly describe $\mathfrak{p}[\mu]_{max}$ for all $\mu \in \Delta^+$. However, if $\mu \in \Pi_l$, then $\mathfrak{a}(\mu)_{min}$ is closely related to $\mathfrak{a}(\mu)_{max}$, and such a situation is considered in the next section.

3. Normalisers of some root-maximal abelian ideals

We begin with a new property of the normaliser of an arbitrary $\mathfrak{b}$-stable subspace of $\mathfrak{u}$. Let $\mathfrak{c} \subset \mathfrak{u}$ be such a subspace and $I_\mathfrak{c}$ the corresponding set of positive roots. Being a standard parabolic subalgebra, $\mathfrak{n}_\mathfrak{g}(\mathfrak{c})$ is fully determined by the simple roots of the standard Levi subalgebra or, equivalently, by the set of simple roots α such that $\mathfrak{g}_{-\alpha} \not\in \mathfrak{n}_\mathfrak{g}(\mathfrak{c})$. The following is proved in [12, Theorem 3.2].

Theorem 3.1. *For any $\mathfrak{b}$-stable subspace $\mathfrak{c} \subset \mathfrak{u}$ and $\alpha \in \Pi$, we have*

$$\mathfrak{g}_{-\alpha} \not\in \mathfrak{n}_\mathfrak{g}(\mathfrak{c}) \ \Leftrightarrow \ \exists \gamma \in \min(I_\mathfrak{c}) \ \textit{ such that } \ \gamma - \alpha \in \Delta^+ \cup \{0\}.$$

The point of this result is that it suffices to test only the *minimal* roots of $I_{\mathfrak{c}}$. Note that if $\gamma-\alpha$ is a root, then $\gamma-\alpha\in\Delta^+\setminus I_{\mathfrak{c}}$. Our new observation is that it is equally suitable to test only the *maximal* roots of $\Delta^+\setminus I_{\mathfrak{c}}$. To this end, we first provide an auxiliary assertion.

Lemma 3.2. *Suppose that $\mu\in\Delta^+$ and $\alpha,\tilde\alpha$ are different simple roots. If $\mu+\alpha,\mu+\tilde\alpha\in\Delta$, then $\mu+\alpha+\tilde\alpha\in\Delta$.*

Proof. As $\alpha,\tilde\alpha\in\Pi$, we have $(\tilde\alpha,\alpha)\leqslant 0$. Furthermore, $(\mu+\alpha)-(\mu+\tilde\alpha)\not\in\Delta$, hence

$$(\mu+\alpha,\mu+\tilde\alpha)=(\mu,\mu)+(\mu,\alpha)+(\mu,\tilde\alpha)+(\alpha,\tilde\alpha)\leqslant 0.$$

Since $(\mu,\mu)>0$, the sum contains at least one negative summand.

- If $(\mu,\alpha)<0$, then $(\mu+\tilde\alpha,\alpha)<0$ and we are done.
- If $(\mu,\tilde\alpha)<0$, then $(\mu+\alpha,\tilde\alpha)<0$ and we are done.
- If $(\mu,\alpha)=(\mu,\tilde\alpha)=0$, then $\mu,\alpha,\tilde\alpha$ are short roots. Then $(\mu+\alpha,\mu+\tilde\alpha)=(\mu,\mu)-(\alpha,\tilde\alpha)\geqslant(\mu,\mu)-\frac{1}{2}(\mu,\mu)>0$, which shows that this case is impossible. ∎

Theorem 3.3. *Suppose that $\mathfrak{c}\subset\mathfrak{u}$ is $\mathfrak{b}$-stable and $\mathfrak{c}\ne\mathfrak{u}$. For $\alpha\in\Pi$, we have*

$$\mathfrak{g}_{-\alpha}\not\in\mathfrak{n}_{\mathfrak{g}}(\mathfrak{c})\ \Leftrightarrow\ \exists\gamma\in\max(\Delta^+\setminus I_{\mathfrak{c}})\ \ \text{such that}\ \ \gamma+\alpha\in\Delta\ \ \text{(and hence}\ \ \gamma+\alpha\in I_{\mathfrak{c}}\text{)}.$$

Proof. The implication "$\Leftarrow$" is obvious.
"$\Rightarrow'$". If $\mathfrak{g}_{-\alpha}\not\in\mathfrak{n}_{\mathfrak{g}}(\mathfrak{c})$, then there is $\mu\in\min(I_{\mathfrak{c}})$ such that $\mu-\alpha\in(\Delta^+\setminus I_{\mathfrak{c}})\cup\{0\}$.

- If $\mu-\alpha\in\max(\Delta^+\setminus I_{\mathfrak{c}})$, then $\gamma=\mu-\alpha$, and we are done;
- If $\mu-\alpha$ is nonzero and not maximal in $\Delta^+\setminus I_{\mathfrak{c}}$, then there is an $\tilde\alpha\in\Pi$ such that $\mu-\alpha+\tilde\alpha\in\Delta^+\setminus I_{\mathfrak{c}}$. Applying Lemma 3.2 to $\mu-\alpha$ shows that $\mu+\tilde\alpha$ is a root and then automatically, $\mu+\tilde\alpha\in I_{\mathfrak{c}}$. Thus, the pair $\{\mu-\alpha,\mu\}$ can be replaced with the "higher" pair $\{\mu-\alpha+\tilde\alpha,\mu+\tilde\alpha\}$. Eventually, we obtain a pair whose lower root is maximal in $\Delta^+\setminus I_{\mathfrak{c}}$.
- If $\mu=\alpha$, then $I_{\mathfrak{c}}$ contains all positive roots with nonzero coefficient of α. Since $\Delta^+\setminus I_{\mathfrak{c}}\ne\varnothing$, there exists a $\nu\in\Delta^+\setminus I_{\mathfrak{c}}$ such that $\nu+\alpha$ is a root, necessarily in $I_{\mathfrak{c}}$. If $\nu\not\in\max(\Delta^+\setminus I_{\mathfrak{c}})$, then we can perform the induction procedure of the previous paragraph. ∎

In the setting of abelian ideals, there is a special case in which $\max(\Delta^+\setminus I_{\mathfrak{c}})$ is related to the *minimal* roots of another ideal.

Proposition 3.4 ([10, Theorem 4.7])**.** *For any $\tilde\alpha\in\Pi_l$, one has*

$$\gamma\in\min(I(\tilde\alpha)_{min})\ \Longleftrightarrow\ \theta-\gamma\in\max(\Delta^+\setminus I(\tilde\alpha)_{max}).$$

In particular, if $\mathsf{rk}\,\Delta>1$ (i.e., $I(\tilde\alpha)_{min}\ne\{\theta\}$), then $\max(\Delta^+\setminus I(\tilde\alpha)_{max})\subset\mathcal{H}\setminus\{\theta\}$.

In the rest of this section, we only consider the abelian ideals with rootlet $\tilde\alpha\in\Pi_l$. Using Theorem 3.3 and Proposition 3.4, we are going to compare the normalisers $\mathfrak{p}[\tilde\alpha]_{max}=\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}(\tilde\alpha)_{max})$ and $\mathfrak{p}[\tilde\alpha]_{min}=\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}(\tilde\alpha)_{min})$. We write $\mathcal{S}[\tilde\alpha]_{max}$ and $\mathcal{S}[\tilde\alpha]_{min}$, respectively, for the simple roots that do **not** belong to their standard Levi subalgebras. In other words, $\mathcal{S}[\tilde\alpha]_{min}:=\Pi\setminus\Pi[\tilde\alpha]_{min}$, and likewise for 'max'.

Theorem 3.5. *For any* $\tilde\alpha\in\Pi$, *we have* $\mathcal{S}[\tilde\alpha]_{max}\subset\mathcal{S}[\tilde\alpha]_{min}$ *and thereby* $\mathfrak{p}[\tilde\alpha]_{max}\supset\mathfrak{p}[\tilde\alpha]_{min}$.

Proof. If $\mathfrak{g}\ne\mathfrak{sl}_2$, then $[\mathfrak{u},\mathfrak{u}]\ne 0$. Hence $\mathfrak{a}(\tilde\alpha)_{max}\ne\mathfrak{u}$, i.e., $I(\tilde\alpha)_{max}\ne\Delta^+$. Therefore, $\alpha\in\mathcal{S}[\tilde\alpha]_{max}$ if and only if there exists $\gamma\in\max(\Delta^+\setminus I(\tilde\alpha)_{max})$ such that $\gamma+\alpha\in I(\tilde\alpha)_{max}$ (Theorem 3.3). Then $\gamma\in\mathcal{H}\setminus\{\theta\}$ (Proposition 3.4) and hence $\gamma+\alpha\in\mathcal{H}\cap I(\tilde\alpha)_{max}=I(\tilde\alpha)_{min}$ [10, Proposition 3.2]. By Proposition 3.4, we have $\nu:=\theta-\gamma\in\min(I(\tilde\alpha)_{min})$ and $\nu-\alpha=\theta-(\gamma+\alpha)$ is either a root or zero. In both cases, applying Theorem 3.1 to ν, we conclude that $\alpha\in\mathcal{S}[\tilde\alpha]_{min}$. ■

Actually, there is a more precise statement.

Theorem 3.6. *Excluding the case in which* Δ *is of type* $\mathbf{A}_n$ *with* $\tilde\alpha=\alpha_1$ *or* α_n, *we have* $\mathcal{S}[\tilde\alpha]_{max}=\mathcal{S}[\tilde\alpha]_{min}\cap\theta^\perp$.

Proof. **1.** Suppose that $\alpha\in\mathcal{S}[\tilde\alpha]_{\max}$ and $\gamma\in\max(\Delta^+\setminus I(\tilde\alpha)_{max})$ is such that $\gamma+\alpha\in I(\tilde\alpha)_{max}$. As explained in the previous proof, we then have $\nu=\theta-\gamma\in\min(I(\tilde\alpha)_{min})\subset\mathcal{H}$ and $\nu-\alpha\in\Delta^+\cup\{0\}$. Consider these two possibilities for $\nu-\alpha$.

(i) $\nu=\alpha$. Then $\alpha\in I(\tilde\alpha)_{min}$, which is only possible if $\tilde\alpha=\alpha$, since $I(\tilde\alpha)_{min}\subset\{\mu\in\Delta^+\mid\mu\succcurlyeq\tilde\alpha\}$ [10, Proposition 3.4]. Therefore $\tilde\alpha=\alpha$, $\tilde\alpha\in\mathcal{H}$, and $[\theta:\tilde\alpha]=1$. All this only occurs for Δ of type $\mathbf{A}_n$ with $\tilde\alpha=\alpha_1$ or α_n.

(ii) $\nu-\alpha\in\Delta^+$. Then $\nu-\alpha\in\mathcal{H}$, since $(\nu-\alpha)+(\gamma+\alpha)=\theta$. That is both ν and $\nu-\alpha$ belong to $\mathcal{H}\setminus\{\theta\}$. Hence $(\theta,\alpha)=0$.

2. Conversely, assume that $\alpha\in\mathcal{S}[\tilde\alpha]_{min}\cap\theta^\perp$. That is, $(\theta,\alpha)=0$ and for some $\nu\in\min(I(\tilde\alpha)_{min})$, we have $\nu-\alpha\in\Delta^+\cup\{0\}$.

For $\nu=\alpha$, we argue as in part **1**(i). If $\nu-\alpha\in\Delta^+$, then both $\gamma=\theta-\nu$ and $\gamma+\alpha$ are roots, and $\gamma\in\max(\Delta^+\setminus I(\tilde\alpha)_{max})$ in view of Proposition 3.4. Hence $\alpha\in\mathcal{S}[\tilde\alpha]_{max}$. ■

Remark 3.7. Recall that $\mathfrak{a}(\tilde\alpha)_{min}=\mathfrak{a}(\tilde\alpha)_{max}$ if and only if $(\tilde\alpha,\theta)\ne 0$, i.e., $\tilde\alpha\in\mathcal{H}$ [8, Theorem 5.1(i)]. If this is the case (and $\Delta\ne\mathbf{A}_n$), then Theorem 3.6 implies that $\mathcal{S}[\tilde\alpha]_{max}=\mathcal{S}[\tilde\alpha]_{min}\subset\theta^\perp$. In the distinguished case of $(\mathbf{A}_n,\alpha_1$ or $\alpha_n)$, we have $\mathfrak{a}(\alpha_1)_{min}=\mathfrak{a}(\alpha_1)_{max}$ and $\mathcal{S}[\alpha_1]_{min}=\{\alpha_1\}$, whereas $\Pi\cap\mathcal{H}=\{\alpha_1,\alpha_n\}$.

Corollary 3.8. *If* $I(\tilde\alpha)_{min}\ne I(\tilde\alpha)_{max}$, *then* $\mathfrak{p}[\tilde\alpha]_{min}\ne\mathfrak{p}[\tilde\alpha]_{max}$.

Proof. Since $I(\tilde\alpha)_{min}\ne I(\tilde\alpha)_{max}$, we have $(\tilde\alpha,\theta)=0$. Then $\Pi[\tilde\alpha]_{min}\subset\theta^\perp$ by Theorem 2.3(ii). Then $\mathcal{S}[\tilde\alpha]_{min}\supset\Pi\cap\mathcal{H}$, and $\mathcal{S}[\tilde\alpha]_{max}\cap\mathcal{H}=\varnothing$ in view of Theorem 3.6. That is, $\mathcal{S}[\tilde\alpha]_{min}\ne\mathcal{S}[\tilde\alpha]_{max}$. ■

Combining Theorems 2.3 and 3.6 yields a complete description of the normaliser for the maximal abelian ideals $\mathfrak{a}(\tilde\alpha)_{max}$, which turns out to be more uniform than that for $\mathfrak{a}(\tilde\alpha)_{min}$. In the rest of the section, we write $\tilde w$ in place of $w_{\tilde\alpha}$.

Theorem 3.9. (i) *Excluding the case in which* Δ *is of type* $\mathbf{A}_n$ *with* $\tilde\alpha=\alpha_1$

or α_n*, we have*

$$\Pi[\tilde\alpha]_{max} = (\Pi\cap\mathcal{H})\bigsqcup\{\tilde w^{-1}(\beta)\mid \beta\in\Pi \ \&\ (\beta,\tilde\alpha)=0\}.$$

(ii) *In particular, if* $(\theta,\tilde\alpha)=0$*, then* $\Pi[\tilde\alpha]_{max}=(\Pi\cap\mathcal{H})\sqcup\Pi[\tilde\alpha]_{min}$*;*

(iii) *In particular, if* θ *is fundamental and* $(\theta,\tilde\alpha)\neq 0$*, then* $\Pi[\tilde\alpha]_{max}=\Pi[\tilde\alpha]_{min}=\{\tilde\alpha\}\sqcup\{\beta\in\Pi\mid(\beta,\tilde\alpha)=0\}$.

Let us say that $\beta\in\Pi$ is *admissible* (for $\tilde\alpha$) if $(\beta,\tilde\alpha)=0$. It follows from Theorem 2.3 that an admissible root always gives rise to a simple root of the Levi subalgebra of $\mathfrak{p}[\tilde\alpha]_{min}$. Furthermore, if θ is fundamental and $(\tilde\alpha,\theta)\neq 0$, then $\tilde\alpha$ also belongs to $\Pi[\tilde\alpha]_{min}$.

Example 3.10. (1) $\Delta=\mathbf{A}_n$, $\tilde\alpha=\alpha_2$. Here $\tilde w=s_1s_3\dots s_n$ and the admissible roots are $\alpha_4,\dots,\alpha_n$. One has $\tilde w^{-1}(\alpha_i)=\alpha_{i-1}$ for them. Hence $\Pi[\alpha_2]_{min}=\{\alpha_3,\alpha_4,\dots,\alpha_{n-1}\}$ and $\mathcal{S}[\alpha_2]_{min}=\{\alpha_1,\alpha_2,\alpha_n\}$. Then $\mathcal{S}[\alpha_2]_{max}=\{\alpha_2\}$.

More generally, for $\tilde\alpha=\alpha_i$ $(2\leqslant i\leqslant n-1)$, one obtains $\mathcal{S}[\alpha_i]_{min}=\{\alpha_1,\alpha_i,\alpha_n\}$ and $\mathcal{S}[\alpha_i]_{max}=\{\alpha_i\}$.

(2a) $\Delta=\mathbf{D}_4$, $\tilde\alpha=\alpha_1$. Here $\tilde w=s_2s_3s_4s_2$ and the admissible roots are α_3,α_4. One has $\tilde w^{-1}(\alpha_3)=\alpha_4$ and $\tilde w^{-1}(\alpha_4)=\alpha_3$. Hence $\mathcal{S}[\alpha_1]_{min}=\{\alpha_1,\alpha_2\}$ and $\mathcal{S}[\alpha_1]_{max}=\{\alpha_1\}$.

(2b) $\Delta=\mathbf{D}_4$, $\tilde\alpha=\alpha_2$. There is no admissible roots here, hence $\tilde w$ is not really needed. Since $(\alpha_2,\theta)\neq 0$, we have $\mathcal{S}[\alpha_2]_{min}=\mathcal{S}[\alpha_2]_{max}=\{\alpha_1,\alpha_3,\alpha_4\}=\Pi\setminus(\Pi\cap\mathcal{H})$.

(3) $\Delta=\mathbf{C}_n$, $\tilde\alpha=\alpha_n$ (the only long simple root). Here $\tilde w=s_{n-1}\dots s_2s_1$ and the admissible roots are $\alpha_1,\dots,\alpha_{n-2}$. One has $\tilde w^{-1}(\alpha_i)=\alpha_{i+1}$ for them. Hence $\Pi[\alpha_n]_{min}=\{\alpha_2,\alpha_3,\dots,\alpha_{n-1}\}$ and $\mathcal{S}[\alpha_n]_{min}=\{\alpha_1,\alpha_n\}$. Then $\mathcal{S}[\alpha_n]_{max}=\{\alpha_n\}$.

(4a) $\Delta=\mathbf{E}_6$, $\tilde\alpha=\alpha_3$. Here $\tilde w=s_6s_4s_2s_5s_3s_1s_2s_4s_3s_6$ and the admissible roots are α_1,α_5. One has $\tilde w^{-1}(\alpha_1)=\alpha_4$ and $\tilde w^{-1}(\alpha_5)=\alpha_2$. Hence $\mathcal{S}[\alpha_3]_{min}=\{\alpha_1,\alpha_3,\alpha_5,\alpha_6\}$ and $\mathcal{S}[\alpha_3]_{max}=\{\alpha_1,\alpha_3,\alpha_5\}$.

(4b) $\Delta=\mathbf{E}_6$, $\tilde\alpha=\alpha_2$. Here $\tilde w=s_3s_6s_4s_5s_3s_1s_2s_4s_3s_6$ and the admissible roots are $\alpha_4,\alpha_5,\alpha_6$. One has $\tilde w^{-1}(\alpha_4)=\alpha_3$, $\tilde w^{-1}(\alpha_5)=\alpha_2$ and $\tilde w^{-1}(\alpha_6)=\alpha_5$. Hence $\mathcal{S}[\alpha_2]_{min}=\{\alpha_1,\alpha_4,\alpha_6\}$ and $\mathcal{S}[\alpha_2]_{max}=\{\alpha_1,\alpha_4\}$.

4. Normalisers of abelian ideals and $\mathbb{Z}$-gradings

In this section, we elaborate on a relationship between the abelian ideals, their normalisers and the associated $\mathbb{Z}$-gradings. Any subset $S\subset\Pi$ gives rise to a $\mathbb{Z}$-grading of $\mathfrak{g}$. Set $\deg(\alpha)=\begin{cases}0, & \alpha\in\Pi\setminus S\\ 1, & \alpha\in S\end{cases}$, and extend it to the whole of Δ by linearity. Then the $\mathbb{Z}$-grading $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}(i)$ is defined by the requirement that $\mathfrak{t}\subset\mathfrak{g}(0)$ and $\mathfrak{g}_\gamma\subset\mathfrak{g}(\deg(\gamma))$ for any $\gamma\in\Delta$. Set $\mathfrak{g}({\geqslant}j)=\bigoplus_{i\geqslant j}\mathfrak{g}(i)$. If we wish to make the dependance on S explicit, then we write $\mathfrak{g}(i;S)$ and $\mathfrak{g}({\geqslant}j;S)$.

Let $\mathfrak{p}$ be a standard parabolic subalgebra, $\mathfrak{l}$ the standard Levi subalgebra of $\mathfrak{p}$, and $\Pi(\mathfrak{l})$ the set of simple roots of $\mathfrak{l}$. Then $S=S(\mathfrak{p})=\Pi\setminus\Pi(\mathfrak{l})$ determines the $\mathbb{Z}$-grading *associated with* $\mathfrak{p}$, and we also write $\mathfrak{p}=\mathfrak{p}(S)$. In this case, $\mathfrak{g}(0;S)=\mathfrak{l}$, $\mathfrak{g}({\geqslant}0;S)=\mathfrak{p}$, and $\mathfrak{g}({\geqslant}1;S)$ is the nilradical of $\mathfrak{p}$.

The *height* of a $\mathbb{Z}$-grading is the maximal i such that $\mathfrak{g}(i) \neq \{0\}$. For $S = \Pi \setminus \Pi(\mathfrak{l})$, we also say that it is the *height of* $\mathfrak{p}(S)$, denoted $\mathsf{ht}(\mathfrak{p}(S))$. It is easily seen that $\mathsf{ht}(\mathfrak{p}(S)) = \deg(\theta) = \sum_{\alpha\in S}[\theta : \alpha]$. Clearly, if $j \geqslant [\mathsf{ht}(\mathfrak{p})/2] + 1$, then $\mathfrak{g}(\geqslant j)$ is an abelian ideal of $\mathfrak{b}$.

Convention. If $(\theta, \tilde\alpha) \neq 0$, then $I(\tilde\alpha)_{min} = I(\tilde\alpha)_{max}$. In this case, we omit the subscripts 'min' and 'max' from the notation for all relevant objects; that is, we merely write $\mathfrak{p}[\tilde\alpha]$, $\mathcal{S}[\tilde\alpha]$, etc.

Theorem 4.1. *Suppose that θ is fundamental, with the corresponding $\alpha_\theta \in \Pi$.*

(i) $\mathcal{S}[\alpha_\theta] = \{\beta \in \Pi \setminus \{\alpha_\theta\} \mid (\beta, \alpha_\theta) \neq 0\}$, *the set of all simple roots adjacent to* α_θ*;*

(ii) α_θ *is long,* $[\theta : \alpha_\theta] = 2$*, and* $\mathsf{ht}(\mathfrak{p}[\alpha_\theta]) = 3$*;*

(iii) $\mathfrak{a}(\alpha_\theta) = \mathfrak{g}(\geqslant 2; \mathcal{S}[\alpha_\theta])$.

Proof. (i) It is already proved in Proposition 2.4.
(ii) If θ is fundamental, then $(\theta, \alpha_\theta^\vee) = 1 = (\alpha_\theta, \theta^\vee)$. Hence α_θ is necessarily long. Furthermore,

$$(\theta, \theta) = (\theta, \sum_{\alpha\in\Pi}[\theta : \alpha]\alpha) = [\theta : \alpha_\theta](\theta, \alpha_\theta) = \frac{1}{2}[\theta : \alpha_\theta](\theta, \theta).$$

Hence $[\theta : \alpha_\theta] = 2$. Finally,

$$1 = (\theta, \alpha_\theta^\vee) = 2[\theta : \alpha_\theta] - \sum_{\beta \text{ adjacent}} [\theta : \beta],$$

where the sum ranges over the simple roots β adjacent to α_θ in the Dynkin diagram. Therefore, $3 = \sum_{\beta \text{ adjacent}}[\theta : \beta] = \mathsf{ht}(\mathfrak{p}[\alpha_\theta])$.

(iii) A general description of the minimal roots for all root-minimal ideals $\mathfrak{a}(\mu)_{min}$ is provided in [8, Prop. 4.6]. In the situation with $\mu = \alpha_\theta$, this yields

$$\min(I(\alpha_\theta)) = \{w_{\alpha_\theta}^{-1}(\alpha_\theta + \beta_i) \mid \beta_i \in \Pi \ \& \ \beta_i \text{ is adjacent to } \ \alpha_\theta\}.$$

Set $\nu_i = w_{\alpha_\theta}^{-1}(\alpha_\theta + \beta_i) = \theta + w_{\alpha_\theta}^{-1}(\beta_i)$ and write $\nu_i = m\alpha_\theta + \sum_j m_j\beta_j + (\text{others})$. Then $m = 1$, since $m = (\nu_i, \theta^\vee) = (\theta + w_{\alpha_\theta}^{-1}(\beta_i), \theta^\vee) = 2 - 1 = 1$. Next, using Lemma 2.2 with $\mu = \alpha_\theta$, we obtain

$$(\nu_i, \alpha_\theta^\vee) = (\theta + w_{\alpha_\theta}^{-1}(\beta_i), \alpha_\theta^\vee) = 1 + (\beta_i, \alpha_\theta^\vee - \theta^\vee) = 1 - 1 = 0.$$

On the other hand,

$$(\nu_i, \alpha_\theta^\vee) = 2m - \sum_j m_j.$$

Therefore, $\sum_j m_j = 2$ and all minimal roots belong to $\mathfrak{g}(2; \mathcal{S}[\alpha_\theta])$. Since $\mathfrak{g}(\geqslant 2; \mathcal{S}[\alpha_\theta])$ is an abelian ideal and $\mathfrak{a}(\alpha_\theta)$ is maximal abelian, we must have $\mathfrak{g}(\geqslant 2; \mathcal{S}[\alpha_\theta]) = \mathfrak{a}(\alpha_\theta)$. ■

Theorem 4.1 is a particular case of the following general assertion.

Theorem 4.2.

(i) *For any* $\tilde\alpha \in \Pi_l$ *and* $n_{\tilde\alpha} := [\theta : \tilde\alpha]$*, we have* $\mathsf{ht}(\mathfrak p[\tilde\alpha]_{max}) = 2n_{\tilde\alpha} - 1$ *and* $\mathfrak a(\tilde\alpha)_{max} = \mathfrak g(\geqslant n_{\tilde\alpha}; \mathcal S[\tilde\alpha]_{max})$.

(ii) *If* $(\tilde\alpha, \theta) = 0$ *(and hence* $\mathcal S[\tilde\alpha]_{max} \ne \mathcal S[\tilde\alpha]_{min}$*), then* $\mathsf{ht}(\mathfrak p[\tilde\alpha]_{min}) = 2n_{\tilde\alpha} + 1$ *and* $\mathfrak a(\tilde\alpha)_{min} = \mathfrak g(\geqslant n_{\tilde\alpha} + 1; \mathcal S[\tilde\alpha]_{min})$.

Proof. Our proof for both parts consists of a case-by-case verification. Using explicit information on $\min(I(\tilde\alpha)_{min})$ and $\min(I(\tilde\alpha)_{max})$ or results of Section 3, we explicitly determine $\mathcal S[\tilde\alpha]_{min}$ and $\mathcal S[\tilde\alpha]_{max}$. This yields the associated $\mathbb Z$-gradings and height of all parabolics involved. The minimal roots of $I(\tilde\alpha)_{min}$ can be determined with the help of [8, Prop. 4.6], whereas the minimal roots of $I(\tilde\alpha)_{max}$ ("generators") are indicated in [12, Tables I,II]. Then one verifies that the sets $\min(I(\tilde\alpha)_{min})$ and $\min(I(\tilde\alpha)_{max})$ always coincide with the set of minimal roots of $\mathfrak g(\geqslant n_{\tilde\alpha} + 1; \mathcal S[\tilde\alpha]_{min})$ and $\mathfrak g(\geqslant n_{\tilde\alpha}; \mathcal S[\tilde\alpha]_{max})$, respectively. ■

Remark 4.3. We can directly explain the following outcome of Theorem 4.2:

If $(\tilde\alpha, \theta) = 0$, *then* $\mathsf{ht}(\mathfrak p[\tilde\alpha]_{min}) = \mathsf{ht}(\mathfrak p[\tilde\alpha]_{max}) + 2$.

For, by Theorem 3.9(ii), we know that $\mathcal S[\tilde\alpha]_{min} = (\Pi \cap \mathcal H) \cup \mathcal S[\tilde\alpha]_{max}$. Hence

$$\mathsf{ht}(\mathfrak p[\tilde\alpha]_{min}) - \mathsf{ht}(\mathfrak p[\tilde\alpha]_{max}) = \sum_{\beta \in \Pi\cap\mathcal H} n_\beta.$$

If θ is fundamental, then $\Pi \cap \mathcal H = \{\alpha_\theta\}$ and $n_{\alpha_\theta} = 2$ (Theorem 4.1(ii)). For $\mathbf A_n$, we have $\Pi \cap \mathcal H = \{\alpha_1, \alpha_n\}$ and $n_{\alpha_1} + n_{\alpha_n} = 2$. This does not apply to $\mathbf C_n$, where $(\tilde\alpha, \theta) \ne 0$ for the unique long simple root $\tilde\alpha$.

Example 4.4. If $n_{\tilde\alpha} = 1$, then $I(\tilde\alpha)_{max} = \{\gamma \in \Delta^+ \mid [\gamma : \tilde\alpha] = 1\}$ and $\mathfrak p[\tilde\alpha]_{max}$ is the maximal parabolic subalgebra with $\mathcal S[\tilde\alpha]_{max} = \{\tilde\alpha\}$. Here $\mathsf{ht}(\mathfrak p[\tilde\alpha]_{max}) = 1$. Hence Theorem 4.2(i) is satisfied here. Furthermore, if θ is fundamental and $(\theta, \alpha_\theta) \ne 0$, then $\tilde\alpha \ne \alpha_\theta$ (because $n_{\alpha_\theta} = 2$), $(\theta, \tilde\alpha) = 0$, and $\mathcal S[\tilde\alpha]_{min} = \{\tilde\alpha, \alpha_\theta\}$, see Theorem 3.9(ii). Therefore $\mathsf{ht}(\mathfrak p[\tilde\alpha]_{min}) = 3$, and I can prove *a priori* that $\mathfrak a(\tilde\alpha)_{min} = \mathfrak g(\geqslant 2; \{\tilde\alpha, \alpha_\theta\})$. (As this is not a decisive step, the proof is omitted.)

That is, in principle, there is a better proof of Theorem 4.2 if $n_{\tilde\alpha} = 1$ or $\tilde\alpha = \alpha_\theta$.

Now, we consider arbitrary abelian ideals of $\mathfrak b$. Let $\mathfrak{Par}(\mathfrak g, \mathfrak b) = \mathfrak{Par}(\mathfrak g)$ be the set of all standard parabolic subalgebras of $\mathfrak g$. If $\mathfrak a \in \mathfrak{Ab}(\mathfrak g)$, then $\mathfrak n_{\mathfrak g}(\mathfrak a) \in \mathfrak{Par}(\mathfrak g)$. It is proved in [12] that the assignment $\mathfrak a \mapsto f_1(\mathfrak a) = \mathfrak n_{\mathfrak g}(\mathfrak a)$ sets up a bijection $\mathfrak{Ab}(\mathfrak g) \xrightarrow{f_1} \mathfrak{Par}(\mathfrak g)$ if and only if Δ is of type $\mathbf A_n$ or $\mathbf C_n$ (i.e., θ is not fundamental).

Here we extend that observation by looking at a natural mapping in the opposite direction. For $\mathfrak p \in \mathfrak{Par}(\mathfrak g)$ and the associated $\mathbb Z$-grading, we set

$$f_2(\mathfrak p) = \mathfrak g(\geqslant [\mathsf{ht}(\mathfrak p)/2] + 1) \in \mathfrak{Ab}(\mathfrak g).$$

This mapping occurs implicitly in Theorem 4.2, where $\mathsf{ht}(\mathfrak p)$ appears to be always odd.

Theorem 4.5.

(i) *If Δ is of type $\mathbf{A}_n$ or $\mathbf{C}_n$, then $f_2 : \mathfrak{Par}(\mathfrak{g}) \to \mathfrak{Ab}(\mathfrak{g})$ is a bijection. Moreover, $f_2 = f_1^{-1}$;*

(ii) *If θ is fundamental, then f_2 is* **not** *a bijection. In fact, there is a uniform construction of two different $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{Par}(\mathfrak{g})$ such that $f_2(\mathfrak{p}_1) = f_2(\mathfrak{p}_2)$.*

Proof. (i) First, we recall the (slightly modified) construction of the bijection f_1 for $\mathbf{A}_n$. For $\mathfrak{a} \in \mathfrak{Ab}(\mathfrak{sl}_{n+1})$, let $\min(I_{\mathfrak{a}}) = \{\gamma_1, \dots, \gamma_k\}$ with $\gamma_t = \alpha_{i_t} + \alpha_{i_{t+1}} + \dots + \alpha_{j_t}$, where $i_t \leqslant j_t$. Assuming that $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_k$, we actually obtain the restrictions

$$1 \leqslant i_1 < i_2 < \dots < i_k \leqslant j_1 < \dots < j_k \leqslant n$$

and thereby the bijection between $\mathfrak{Ab}(\mathfrak{sl}_{n+1})$ and the subsets of $[n] = \{1, \dots, n\}$. Here one obtains a subset of odd (resp. even) cardinality if $i_k = j_1$ (resp. $i_k < j_1$). Moreover, if $\mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$, then it follows from Theorem 3.1 that $S = S(\mathfrak{p}) = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}, \alpha_{j_1}, \dots, \alpha_{j_k}\}$, modulo the possible coincidence of i_k and j_1.

Suppose that $\mathfrak{p} \in \mathfrak{Par}(\mathfrak{sl}_{n+1})$ and $\#S$ is odd, $S \sim \{t_1, t_2, \dots, t_{2k-1}\} \subset [n]$, with $t_1 < \dots < t_{2k-1}$. Then $\mathsf{ht}(\mathfrak{p}) = 2k-1$ and the minimal roots of $\mathfrak{g}(\geqslant k; S)$ are in a bijection with the shortest intervals of $[n]$ that contain k elements of S. Therefore, these minimal roots are

$$\gamma_1 = \alpha_{t_1} + \alpha_{t_2} + \dots + \alpha_{t_k}, \quad \gamma_2 = \alpha_{t_2} + \alpha_{t_3} + \dots + \alpha_{t_{k+1}}, \\ \dots, \quad \gamma_k = \alpha_{t_k} + \alpha_{t_{k+1}} + \dots + \alpha_{t_{2k-1}},$$

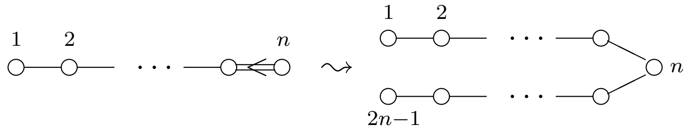
and it is immediate that, for the abelian ideal $\mathfrak{a} = f_2(\mathfrak{p})$ generated by $\gamma_1, \dots, \gamma_k$, we have $f_1(\mathfrak{a}) = \mathfrak{p}$.

If $\#S$ is even, $S \sim \{t_1, t_2, \dots, t_{2k}\} \subset [n]$, then $\mathsf{ht}(\mathfrak{p}) = 2k$ and the minimal roots of $\mathfrak{g}(\geqslant k+1; S)$ are

$$\gamma_1 = \alpha_{t_1} + \alpha_{t_2} + \dots + \alpha_{t_{k+1}}, \quad \gamma_2 = \alpha_{t_2} + \alpha_{t_3} + \dots + \alpha_{t_{k+2}}, \\ \dots, \quad \gamma_k = \alpha_{t_k} + \alpha_{t_{k+1}} + \dots + \alpha_{t_{2k}}.$$

Here again one obtains $\mathfrak{a} = f_2(\mathfrak{p})$ such that $f_1(\mathfrak{a}) = \mathfrak{p}$.

We omit the part related to $\mathbf{C}_n$, since it goes along the same lines, using the explicit description of f_1 given in [12, Theorem 3.3]. The point is that the unfolding $\mathbf{C}_n \rightsquigarrow \mathbf{A}_{2n-1}$ (see picture below) yields the identification of $\mathfrak{Ab}(\mathfrak{sp}_{2n})$ and $\mathfrak{Par}(\mathfrak{sp}_{2n})$ with the symmetric (with respect to the middle) subsets of $[2n-1]$, and one can use a symmetrised version of the previous argument.



(ii) Our goal is to produce two different subsets $S_1, S_2 \subset \Pi$ such that $\mathfrak{p}(S_1)$ and $\mathfrak{p}(S_2)$ give rise to the same abelian ideal. Below we use Theorem 4.1 and its proof.

As usual, α_θ is the only simple root that is not orthogonal to θ. Let

S_1 be the set of all simple roots adjacent to α_θ and $S_2 = S_1 \cup \{\alpha_\theta\}$. Then $\mathfrak{p}(S_1) = \mathfrak{p}[\alpha_\theta]$, $\mathsf{ht}(\mathfrak{p}[\alpha_\theta]) = 3$, and $\mathfrak{a}(\alpha_\theta) = \mathfrak{g}(\geqslant 2; S_1)$. Since $n_{\alpha_\theta} = 2$, we have $\mathsf{ht}(\mathfrak{p}(S_2)) = 2 + \mathsf{ht}(\mathfrak{p}[\alpha_\theta]) = 5$ and $\mathfrak{g}(\geqslant 3; S_2)$ is an abelian ideal. The proof of Theorem 4.1 shows that if $\nu_i \in \min(I(\alpha_\theta))$, then $[\nu_i : \alpha_\theta] = 1$ and $\sum_{\beta\in S_1}[\nu : \beta] = 2$. Hence $\mathfrak{g}_{\nu_i} \in \mathfrak{g}(3; S_2)$ and $\mathfrak{a}(\alpha_\theta) \subset \mathfrak{g}(\geqslant 3; S_2)$. As $\mathfrak{a}(\alpha_\theta)$ is maximal abelian, one has the equality and therefore $f_2(\mathfrak{p}(S_1)) = f_2(\mathfrak{p}(S_2))$. ■

Remark 4.6 (Some speculations). Set $\mathcal{F} = f_1 \circ f_2$ and $\tilde{\mathcal{F}} = f_2 \circ f_1$. We say that $\mathfrak{a} \in \mathfrak{Ab}(\mathfrak{g})$ is *reflexive*, if $\tilde{\mathcal{F}}(\mathfrak{a}) = \mathfrak{a}$; likewise, $\mathfrak{p} \in \mathfrak{Par}(\mathfrak{g})$ is *reflexive*, if $\mathcal{F}(\mathfrak{p}) = \mathfrak{p}$. It is easily seen that $\mathcal{F}(\mathfrak{p}) \supset \mathfrak{p}$ for all $\mathfrak{p}$, while it can happen that $\tilde{\mathcal{F}}(\mathfrak{a}) \not\supset \mathfrak{a}$ for some $\mathfrak{a}$ (e.g. if $\mathfrak{g} = \mathbf{E}_6$).

For $\mathfrak{sl}_{n+1}$ and $\mathfrak{sp}_{2n}$, all abelian ideals are reflexive, whereas this is certainly not the case for the other simple types. However, Theorem 4.2 implies that the ideals $\mathfrak{a}(\tilde\alpha)_{min}$ and $\mathfrak{a}(\tilde\alpha)_{max}$ ($\tilde\alpha \in \Pi_l$) are always reflexive. It might be interesting to explicitly determine all reflexive abelian ideals.

Our calculations with $\mathfrak{g}$ up to rank 4 suggest that it also might be true that (the restrictions of) f_1 and f_2 induce the mutually inverse bijections between $\mathsf{Im}(\tilde{\mathcal{F}}) \subset \mathfrak{Ab}(\mathfrak{g})$ and $\mathsf{Im}(\mathcal{F}) \subset \mathfrak{Par}(\mathfrak{g})$; in particular, $\#\mathsf{Im}(\mathcal{F}) = \#\mathsf{Im}(\tilde{\mathcal{F}})$. But the equality $\#\mathsf{Im}(f_1) = \#\mathsf{Im}(f_2)$ is false in general (e.g. for $\mathfrak{g} = \mathfrak{so}_9$).

We also conjecture that $\mathsf{Im}(\mathcal{F}) = \{\mathfrak{p} \mid \mathcal{F}(\mathfrak{p}) = \mathfrak{p}\}$ and $\mathsf{Im}(\tilde{\mathcal{F}}) = \{\mathfrak{a} \mid \tilde{\mathcal{F}}(\mathfrak{a}) = \mathfrak{a}\}$; in other words, $\mathcal{F}^2 = \mathcal{F}$ and $\tilde{\mathcal{F}}^2 = \tilde{\mathcal{F}}$ in the rings of endomorphisms of the finite sets $\mathfrak{Par}(\mathfrak{g})$ and $\mathfrak{Ab}(\mathfrak{g})$, respectively.

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