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Commutators of Small Elements in Compact Semisimple Groups and Lie Algebras

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Abstract. We prove openness at the identity element of the commutator map of compact real semisimple Lie algebras and compact semisimple groups.
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1. Openness of the commutator map in a compact semisimple Lie algebra

In this section, we investigate the openness of the commutator map of a (real, compact) semisimple Lie algebra $\mathfrak{g}$ at $(0,0)$. We say that a map f between two topological spaces X and Y is open at a point $x \in X$ if for all neighbourhoods U of x, the image $f(U)$ is a neighbourhood of $f(x)$. Thus we are interested in understanding whether all sufficiently small elements arise as commutators of elements out of a prescribed neighborhood of 0.

Throughout this section, G will be a semisimple Lie group and $\mathfrak{g}$ its Lie algebra; on $\mathfrak{g}$ we consider the Killing form $\kappa_{\mathfrak{g}}$. In the following Lemma we characterize pairs of maximal toral subalgebras of $\mathfrak{su}_n$ which are orthogonal to each other with respect to the Killing form. If $u_1, \ldots, u_n$ is an orthonormal basis of $\mathbb{C}^n$ then we denote by $\mathfrak{t}_u$ the set of elements in $\mathfrak{su}_n$ which are diagonal with respect to this basis.

Lemma 1.1. *Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be two orthonormal bases of $\mathbb{C}^n$. Then $\mathfrak{t}_u$ is orthogonal to $\mathfrak{t}_v$ if and only if*

$$|u_i \cdot v_h| = |u_j \cdot v_k|, \tag{1}$$

for all i, j, h, k.

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Proof. Define $U_{ij} \in \mathfrak{su}_n$ by

$$U_{ij}(u_h) = \begin{cases} \sqrt{-1}\ u_i & \text{if } h = i; \\ -\sqrt{-1}\ u_j & \text{if } h = j; \\ 0 & \text{otherwise.} \end{cases}$$

and similarly define V_{ij} using the orthonormal basis v_i. Then the operators U_{ij} span $\mathfrak{t}_u$ and the operators V_{ij} span $\mathfrak{t}_v$. Easy computations show that

$$\operatorname{Tr}\left(U_{ij} V_{hk}\right) = |u_i \cdot v_k|^2 + |u_j \cdot v_h|^2 - |u_i \cdot v_h|^2 - |u_j \cdot v_k|^2.$$

Hence $\mathfrak{t}_u$ is orthogonal to $\mathfrak{t}_v$ if and only if

$$|u_i \cdot v_k|^2 + |u_j \cdot v_h|^2 = |u_i \cdot v_h|^2 + |u_j \cdot v_k|^2$$

for all i, j, h, k. Hence if (1) is verified, then the two subalgebras are orthogonal. Vice versa, assume they are orthogonal. Then, summing over h, we see that the above equalities imply

$$|u_i \cdot v_k|^2 = |u_j \cdot v_k|^2$$

for all i, j, and summing over j we get

$$|u_i \cdot v_h|^2 = |u_i \cdot v_k|^2$$

proving the claim. ∎

Lemma 1.2. *Let G be compact, $\mathfrak{t}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then there exists a maximal toral subalgebra of $\mathfrak{g}$ orthogonal to $\mathfrak{t}$.*

Proof. We first analyse the case $\mathfrak{g} = \mathfrak{su}_n$. Set $\zeta = e^{\frac{2\pi\sqrt{-1}}{n}}$, and let $u_1, \ldots, u_n$ be an orthonormal basis of $\mathbb{C}^n$ such that $\mathfrak{t} = \mathfrak{t}_u$. For $j = 1, \ldots, n$ define

$$v_j = \frac{1}{\sqrt{n}}(u_1 + \zeta^j u_2 + \zeta^{2j} u_2 + \cdots + \zeta^{(n-1)j} u_n).$$

Then $v_1, \ldots, v_n$ is an orthonormal basis of $\mathbb{C}^n$ and $|u_i \cdot v_j| = 1/\sqrt{n}$ for all i, j. Hence $\mathfrak{t}_u$ is orthogonal to $\mathfrak{t}_v$. For $\mathfrak{g}$ not isomorphic to $\mathfrak{su}_n$ we proceed by induction on the rank of $\mathfrak{g}$. If $\mathfrak{g}$ is not simple the claim follows immediately by induction, so we assume that $\mathfrak{g}$ is simple.

Let $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{t}_{\mathbb{C}}$) be the complexification of $\mathfrak{g}$ (resp. $\mathfrak{t}_{\mathbb{C}}$), denote by Φ the associated root system and choose a simple basis $\Delta \subset \Phi$. Let ω_α, $\alpha \in \Delta$, be the corresponding fundamental weights, and θ be the highest root of Φ. Since $\mathfrak{g}$ is not of type A there exists a simple root α such that $\theta = \omega_\alpha$ or $\theta = 2\omega_\alpha$, see [3, Planches II-IX]. Let Ψ be the root system generated by $\Delta \setminus \{\alpha\}$.

We can choose a standard Chevalley basis $h_\alpha, \alpha \in \Delta$, and $x_\alpha, \alpha \in \Phi$, such that elements

$$k_\alpha = \sqrt{-1}\ h_\alpha, \qquad u_\alpha = x_\alpha - x_{-\alpha}, \ \text{ and } \ v_\alpha = \sqrt{-1}\ (x_\alpha + x_{-\alpha})$$

are a basis of $\mathfrak{g}$, see [12, Thm. 6.11, Formula (6.12)]. Notice that the subspace orthogonal to $\mathfrak{t}$ is the linear span of the elements u_α and v_α. Define

$$\mathfrak{h} = \langle k_\beta, u_\beta, v_\beta : \beta \in \Psi \rangle.$$

This is the semisimple part of the maximal Levi subalgebra associated with α. In particular the claim is true for $\mathfrak{h}$. Let $\mathfrak{s}$ be a maximal toral subalgebra orthogonal to the maximal toral subalgebra of $\mathfrak{h}$ given by $\mathfrak{t} \cap \mathfrak{h}$.

Notice also that we have $[u_\theta, \mathfrak{h}] = [u_{-\theta}, \mathfrak{h}] = 0$. Hence, for dimensional reasons, $\mathfrak{s} \oplus \mathbb{R}u_\theta$ is a maximal toral subalgebra of $\mathfrak{g}$ orthogonal to $\mathfrak{t}$. ∎

We can now prove the following fact.

Theorem 1.3. *Let G be compact. Then the commutator map*

$$\mathrm{comm}_\mathfrak{g} : \mathfrak{g} \times \mathfrak{g} \ni (x, y) \mapsto [x, y] \in \mathfrak{g}$$

is open at $(0,0)$.

Proof. We need to prove that if U is a neighbourhood of 0 then $\mathrm{comm}_\mathfrak{g}(U \times U)$ contains a neighbourhood of 0. Notice first that G being compact we can assume that U is G-stable under the adjoint action.

Choose now a regular element $x \in U$ (an element is said to be regular if its centralizer is a toral subalgebra) and let $\mathfrak{t}$ be its centralizer. Let $\mathfrak{m}$ be the orthogonal of $\mathfrak{t}$. Then $\mathrm{ad}_x : \mathfrak{m} \longrightarrow \mathfrak{m}$ is a linear isomorphism. Hence there exists a G-stable neighbourhood V of 0 in $\mathfrak{g}$ such that $\mathrm{ad}_x(\mathfrak{m} \cap U) \supset \mathfrak{m} \cap V$.

Consider now $\psi : G \times (\mathfrak{m} \cap U) \to \mathfrak{g}$ given by $\psi(g, y) = \mathrm{Ad}_g[x, y]$. It is clear that the image of ψ is contained in $\mathrm{comm}_\mathfrak{g}(U \times U)$ and that the image of ψ contains $G \cdot (V \cap \mathfrak{m})$. Finally from the previous lemma we have $G.(V \cap \mathfrak{m}) = V$. Hence $\mathrm{comm}_\mathfrak{g}(U \times U) \supset V$, thus proving the theorem. ∎

Corollary 1.4. *The function* $\mathrm{comm}_\mathfrak{g}$ *is surjective.*

Proof. The image of the bilinear map $\mathrm{comm}_\mathfrak{g}$ is a scalar multiplication invariant neighbourhood of 0. ∎

An alternate proof, due to K.-H. Neeb, of Corollary 1.4 by means of Kostant's Convexity Theorem can be found in [9, p. 653].

2. Openness of the commutator map in a Lie group

We will denote by

$$\mathrm{Comm}_G : G \times G \ni (X, Y) \mapsto XYX^{-1}Y^{-1} \in G$$

the commutator map of a group G. In this section, G is a Lie group and $\mathfrak{g}$ is the corresponding Lie algebra. We will show that Comm_G is open at $(\mathrm{id}, \mathrm{id})$ as soon as the corresponding infinitesimal commutator map $\mathrm{comm}_\mathfrak{g} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is open at $(0,0)$.

Lemma 2.1. *The map* $\mathrm{comm}_{\mathfrak{g}}$ *is open at* $(0,0)$ *if and only if the map*

$$C_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \ni (x,y) \mapsto x - \exp(\mathrm{ad}_y)x \in \mathfrak{g}$$

is open at $(0,0)$.

Proof. The map

$$\phi : \mathfrak{g} \times \mathfrak{g} \ni (x,y) \mapsto \left(\frac{\exp(\mathrm{ad}_y) - 1}{\mathrm{ad}_y}(x), y \right) \in \mathfrak{g} \times \mathfrak{g}$$

is smooth, and has invertible differential in $(0,0)$, so it is a local diffeomorphism. However, the composition of $\mathrm{comm}_{\mathfrak{g}} \circ \phi$ equals $C_{\mathfrak{g}}$. ∎

Remark 2.2. Lemma 2.1 immediately generalizes to the case of more than one commutator. Let $\mathrm{comm}^n_{\mathfrak{g}} : \mathfrak{g}^{2n} \to \mathfrak{g}$ be the map $(x_1, y_1, \dots, x_n, y_n) \mapsto [x_1, y_1] + \dots + [x_n, y_n]$, and denote by $C^n_{\mathfrak{g}} : \mathfrak{g}^{2n} \to \mathfrak{g}$ the map

$$(x_1, y_1, \dots, x_n, y_n) \mapsto \sum_{i=1}^{n} x_i - \exp(\mathrm{ad}_{y_i})x_i.$$

Then $\mathrm{comm}^n_{\mathfrak{g}}$ and $C^n_{\mathfrak{g}}$ coincide up to a local diffeomorphism of $\mathfrak{g}^{2n}$, hence openness at $(0, \dots, 0)$ of the former map is equivalent to that of the latter.

In order to deal with openness of the commutator map in a group, we are going to use the following variant [15, Rem. 4.2] of the Baker-Campbell-Hausdorff formula.

Lemma 2.3. *There exist, on a neighbourhood of* $(0,0) \in \mathfrak{g} \times \mathfrak{g}$*, analytic functions*

$$P, Q : \mathfrak{g} \times \mathfrak{g} \to G, \qquad P(0,0) = Q(0,0) = \mathrm{id},$$

satisfying

$$\exp(a+b) = \exp(P(a,b).a)\exp(Q(a,b).b),$$

for all a, b.

Proposition 2.4. *The group commutator map* Comm_G *is open at* $(\mathrm{id}, \mathrm{id})$ *as soon as the infinitesimal commutator map* $\mathrm{comm}_{\mathfrak{g}}$ *is open at* $(0,0)$.

Proof. Let us apply Lemma 2.3 to $a = x, b = -\exp(\mathrm{ad}_y)x$. Using the notation introduced above, we set $P = P(a,b), Q = Q(a,b)$ and obtain

$$\begin{aligned} \exp(x - \exp(\mathrm{ad}_y)x) &= \exp(P.x)\exp(-Q.\exp(\mathrm{ad}_y)x) \\ &= P\exp(x)P^{-1}(Q\exp(y))\exp(-x)(Q\exp(y))^{-1} \\ &= ABA^{-1}B^{-1}, \end{aligned}$$

where $A = P\exp(x)P^{-1}, B = Q\exp(y)P^{-1}$.

Let U be a (G-stable) neighbourhood of $0 \in \mathfrak{g}$ on which exp restricts to a diffeomorphism. The map $\psi : \mathfrak{g} \times \mathfrak{g} \to G$ defined by

$$(x, y) \mapsto Q(x, \exp(\mathrm{ad}_y)x) \exp(y) P(x, \exp(\mathrm{ad}_y)x)^{-1}$$

is analytic, hence continuous. We may then find a neighbourhood U' of 0 in $\mathfrak{g}$, which we assume to be G-stable and contained in U, such that $\psi(U' \times U') \subset \exp U$. If x, y lie in U', then A and $B = \psi(x, y)$ lie in $\exp U$; moreover, the composition $\exp \circ C_{\mathfrak{g}}$ maps (x, y) to $ABA^{-1}B^{-1}$ and is open at $(0, 0)$.

We thus conclude that all elements in a suitable neighbourhood of $\mathrm{id} \in G$ arise as commutators of elements from $\exp U$. ∎

Theorem 2.5. *If G is a compact semisimple Lie group, then Comm_G is open at $(\mathrm{id}, \mathrm{id})$.*

Proof. It follows immediately from Proposition 2.4 and Theorem 1.3. ∎

Remark 2.6. Once again, it is easy to use Remark 2.2 in order to adapt the proof of Proposition 2.4 so as to show that

$$\mathrm{Comm}_G^n : G^{2n} \ni (X_1, Y_1, \ldots, X_n, Y_n) \mapsto (X_1 Y_1 X_1^{-1} Y_1^{-1}) \cdots (X_n Y_n X_n^{-1} Y_n^{-1}) \in G$$

is open at $(\mathrm{id}, \ldots, \mathrm{id})$ if and only if $\mathrm{comm}_{\mathfrak{g}}^n$ is open at $(0, \ldots, 0)$.

Notice that when G is a semisimple (but not necessarily compact) Lie group, it is then possible to give a direct proof that Comm_G^2 is open at $(\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id})$ and that $\mathrm{comm}_{\mathfrak{g}}^2$ is open at $(0, 0, 0, 0)$. Let us briefly sketch the proof of this claim for Comm_G^2, the proof for the map $\mathrm{comm}_{\mathfrak{g}}^2$ being completely analogous.

Let U be an open neighborhood of id in G. Recall that the set of semisimple regular elements is an open and dense subset of G. Let $g \in U$ be a regular semisimple element and let $\mathfrak{t} = \mathfrak{g}^g$ the Cartan subalgebra of $\mathfrak{g}$ of all points fixed by g. The restriction of the Killing form to $\mathfrak{t} \times \mathfrak{t}$ is non degenerate and the image of Ad_g is equal to $\mathfrak{t}^{\perp}$, the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$. If $x \in \mathfrak{t}$ then the centralizer of x is different from $\mathfrak{t}$ if and only if x belongs to the union of a finite number number of hyperplanes $H_1, \ldots, H_m$ of $\mathfrak{t}$. Denote by $S_1, \ldots, S_m$ the connected subgroups of G whose Lie algebra equal $H_1, \ldots, H_m$, respectively; then we can find a regular semisimple element $h \in U$ such that $h \notin \bigcup S_i$. Set $\mathfrak{s} = \mathfrak{g}^h$. Then $\mathfrak{t} \cap \mathfrak{s} = \{0\}$ which implies $\mathfrak{t}^{\perp} + \mathfrak{s}^{\perp} = \mathfrak{g}$. Now consider the map

$$D : U \times U \longrightarrow G \text{ defined by } D(u, v) = [g, u] \cdot [h, v].$$

The differential of D in $(\mathrm{id}, \mathrm{id})$ is given by $(x, y) \mapsto Ad_g(x) + Ad_h(y)$ hence its image equals $\mathfrak{t}^{\perp} + \mathfrak{s}^{\perp} = \mathfrak{g}$. In particular $D(U \times U)$ contains a neighborhood of id in G. Finally notice that by construction $\mathrm{Comm}^2(U \times U \times U \times U) \supset D(U \times U)$.

3. Openness of the commutator map in a compact semisimple group

In this section we extend the previous openness result to the the case of arbitrary compact semisimple groups; we are indebted to Karl Hofmann for suggesting the strategy we outline here. Recall that a compact connected group G is called semisimple if $G = G'$. The following result is well known, see e.g., [8, Thm. 6.55].

Theorem 3.1 (Gotô [7]). *If G is a connected compact semisimple group, then* Comm_G *is surjective.*

We aim to use Gotô's result in order to generalize Theorem 2.5 to the non-Lie case. We are going to need the following description of semisimple compact groups in terms of simple Lie groups.

Lemma 3.2. *Let G be a connected compact semisimple group. Then there exists a family $\{S_j, j \in J\}$ of simple (simply connected) compact Lie groups and an open continuous homomorphism $f : \prod_{j\in J} S_j \to G$.*

Proof. This is [8, Cor. 9.20], along with the observation that the surjective homomorphism f is necessarily open by the open mapping theorem for (locally) compact groups. ∎

Theorem 3.3. *If G is a connected semisimple compact group, then* Comm_G *is open at* $(\mathrm{id}, \mathrm{id})$.

Proof. The statement is clear when G is isomorphic to a product $\prod_{j\in J} S_j$ of connected simple compact Lie groups. Indeed, in this case one may produce a neighbourhood basis for the identity element by considering products $U = \prod_{j\in J} U_j$, where $U_j \subset S_j$ is an open neighbourhood of id whenever j lies in a finite subset $F \subset J$, and $U_j = S_j$ when $j \notin F$.

Now, if $j \notin F$, then $\mathrm{Comm}_{S_j}(S_j, S_j) = S_j$ by Gotô's theorem; if instead $j \in F$, then $\mathrm{Comm}_{S_j}(U_j, U_j)$ contains a neighbourhood of the identity element by Theorem 2.5. Thus, $\mathrm{Comm}_G(U, U) = \prod_j \mathrm{Comm}_{S_j}(U_j, U_j)$ is open in G, for all choices of U in a neighbourhood basis for the identity. We conclude that Comm_G is open at $(\mathrm{id}, \mathrm{id})$. The general case follows now immediately from Lemma 3.2. ∎

4. Background of the commutator functions

Let G be a connected semisimple (Lie or algebraic) group. Then G equals its derived subgroup and it is expected that, in many cases, every element of G arises as a single commutator. The problem of understanding under what conditions this claim holds, or at least every element can be expressed as a product of a uniformly bounded quantity of commutators, has been investigated at length.

The fact that every element in a semisimple compact group is a commutator dates back to Gotô [7], whereas counterexamples are easy to construct in non compact cases — for instance, in $\mathrm{SL}_2(\mathbb{R})$, $-\mathrm{Id}$ does not arise as a commutator. Later, Thompson [16] provided a classification of all groups of the form $\mathrm{SL}_n(k)$, where k is an arbitrary field, containing elements that are not commutators.

Connected semisimple algebraic groups are treated in the complex case in [13], and in [14] over an algebraically closed field of any characteristic. More recently, Đoković showed [6] that in real semisimple groups, whose maximal compact connected subgroups are again semisimple, every element is a product of at most two commutators.

Many variations on the topic have also been considered. To name just a few, Brown considered the analogous statement in the case of simple Lie algebras [4]; Borel studied instead maps $G^n \to G$ induced by nontrivial group words in n letters, showing [2] that they yield dominant maps.

The usual proofs that in a compact semisimple Lie group every element is a commutator provide little information towards proving openness of the commutator map, as they proceed by expressing each element in the group as the commutator between an element lying in a torus and some expression, which is typically "far" off the identity, depending on a nontrivial Coxeter element chosen in the associated Weyl group. Openness of the commutator maps in semisimple compact Lie groups and Lie algebras appears to be a natural fact but we could not find an explicit reference in the literature, with the single exception of [11, Problem 8.15] for the Lie algebra case in type A.

We should finally mention that, in order to ensure openness at zero of the commutator map in a compact Lie algebra, it is not sufficient to prove surjectivity. Indeed, establishing whether the surjectivity of a bilinear map implies its openness at zero was a classical problem answered negatively by Cohen and Horowitz [5, 10].

We would like to thank Alessandro Berarducci for drawing our attention to this problem, which arises from his work on definable groups. Our openness statement is equivalent to the claim that every element belonging to the infinitesimal neighbourhood of the identity (which is a perfect subgroup) in the nonstandard version of a compact semisimple Lie group is a commutator. This issue was considered by Berarducci, Peterzil and Pillay—see in particular the comments after [1, Prop. 2.14]—in connection with the question whether a finite central extension of a group definable in an o-minimal structure M is interpretable in M. A positive answer to the latter question would also imply that a finite central extension of a compact Lie group has an induced Lie structure making the extension a topological cover.

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