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Construction of Primitive Representations of $\mathrm{U}(1,1)(\mathcal{O})$

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Abstract. Let $\mathcal{O}$ be the ring of integers of E, E being a ramified quadratic extension of a non-archimedean local field F of odd residual characteristic. In this paper, we construct a complete set of irreducible representations ρ of level $n+1$ of the quasi-split unitary group $\mathrm{U}(1,1)(\mathcal{O})$ (called primitive representations) such that every irreducible representation of the group has the form $\rho\otimes\chi$ for some character χ of $\mathcal{O}^\times$. We show that such representations only appear in level $n+1$ when n is even. Our approach is to consider $\mathrm{U}(1,1)(\mathcal{O})$ as a generalized special linear group $\mathrm{SL}_*^{-1}(2,\mathcal{O})$, *i.e.*, as the group of 2×2 matrices in $\mathrm{GL}(2,\mathcal{O})$ whose coefficients satisfy certain commutation relations involving the nontrivial element $*$ of the Galois group $\mathrm{Gal}(E/F)$. Considering $*=\mathrm{id}$ in the construction, we recover the irreducible representations of $\mathrm{SL}(2,\mathcal{O})$. Finally, we explicitly calculate the number and dimensions of the primitive representations so constructed.
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1. Introduction

Maximal compact subgroups have played an important role in representation theory of general linear groups defined over a non-archimedean local field E. These subgroups have been used among other things to describe the (irreducible) representations of the aforementioned groups. For instance, the supercuspidal representations of $\mathrm{GL}_2(E)$ were constructed as induced representations from compact modulo center subgroups [10]. Such construction is a very general procedure and it is used for $\mathrm{GL}_N(E)$ [8], [17].

Special attention has been given to the theory of representations of general and special linear groups over the ring of integers $\mathcal{O}$ of E. In fact, the case of $\mathrm{GL}_N(\mathcal{O})$ has been particularly considered by several authors, among other reasons, because of its role in the representation theory of $\mathrm{GL}_N(E)$ (see Aubert, Onn, Prasad and Stasinski [1], Singla [18], Hill [5], [6], [7]). For $n=2$, Stasinski [19] gives

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a complete description of the irreducible representations of $\mathrm{GL}_2(\mathcal{O})$ (recovering an unpublished result due to Kutzko). Continuing with the case $n = 2$, the theory of representations of the special linear group $\mathrm{SL}_2(\mathcal{O})$ was studied by Kloosterman [9], Tanaka [20] and Shalika [16], among others. In particular, in [16], all primitive representations of $\mathrm{SL}_2(\mathcal{O}/\mathcal{P}^n)$ were constructed for an arbitrary ring of p-adic integers $\mathcal{O}$ of odd residual characteristic with maximal ideal $\mathcal{P}$.

Diverse classical linear groups can be viewed as general (or special general) linear groups over certain rings. This point of view was taken by Pantoja and Soto-Andrade [14], [15], where they looked at different classical groups (even of high rank) as general linear groups of 2×2 matrices over a suitable ring of coefficients. This was accomplished with the help of the groups $\mathrm{SL}_*^{\varepsilon}(2, A)$ ($\varepsilon = \pm 1$) consisting of 2×2 matrices g with coefficients in a unitary involutive ring $(A, *)$, such that $g^* J g J^{-1} = I$, with $(g^*)_{ij} = g_{ji}^*$, I is the identity matrix and $J = \left(\begin{smallmatrix} 0 & 1 \\ \varepsilon 1 & 0 \end{smallmatrix}\right)$. In this way, the symplectic groups $\mathrm{Sp}(2n, F)$ and the split orthogonal groups $\mathrm{O}(n, n)$ were recovered by setting $A = \mathrm{M}_n(F)$ for $\varepsilon = 1$ and $\varepsilon = -1$, respectively, where F is a field and $*$ is the matrix transpose. Furthermore, several interesting groups may be obtained as generalized classical groups for different choices of unitary rings A with involution $*$. In this sense, considering specific groups like GL_* or SL_* is of interest not only because they are examples of a more general construction, but also because they provide an alternative approach to different situations.

As far as we know, the theory of representations of $\mathrm{U}(m, m)(\mathcal{O})$ (where $\mathcal{O}$ is the ring of integers of a field E) has not been explored yet and it is worth mentioning that this problem could be highly intrincate even in low rank. So accordingly, to attack the problem, it seems to be more appropriate to start with $\mathrm{U}(1,1)$ and $\mathrm{U}(2,2)$. We remark that these groups can be realized as $\mathrm{SL}_*^{\varepsilon}(2, A)$, where $A = \mathrm{M}_m(\mathcal{O})$, $\mathcal{O}$ is the ring of integers of E, E is a quadratic extension of a non-archimedean local field F and $*$ is the restriction of the nontrivial element of the Galois group $\mathrm{Gal}(E/F)$ of E/F. In this line of thought, it is important to directly know the construction of the irreducible representations of $\mathrm{U}(1,1)(\mathcal{O})$ and to tackle the general case. Subsequently, one could study the problem of classifying the irreducible representations of $\mathrm{U}(2,2)(\mathcal{O})$ by looking for an analogous notion of the regular representations given by Hill in [6].

There are several reasons to consider $\mathrm{U}(1,1)(\mathcal{O})$. For instance, Lansky and Raghuram [11] considered conductors and newforms for $\mathrm{U}(1,1)(E)$, giving examples that show the relevant role of newforms in the theory of automorphic forms. Moreover, there is a series of papers investigating the branching rules problem for irreducible representations of $\mathrm{GL}_2(k)$ or $\mathrm{SL}_2(k)$ and its restriction to $\mathrm{GL}_2(\mathcal{O})$ or $\mathrm{SL}_2(\mathcal{O})$ respectively. In this regard, the work of Nevins [12] on branching rules for supercuspidal representations of $\mathrm{SL}_2(k)$, where k is a p-adic field, is worth mentioning.

In this paper, we analyse the theory of representations of $K = \mathrm{SL}_*^{-1}(2, \mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a ramified quadratic extension E of a non-archimedean local field F of odd residual characteristic, and $*$ is an element of the Galois group $\mathrm{Gal}(E/F)$ of E/F. Using this approach, we investigate and construct, in a uniform manner, all primitive representations of K of level $n+1$, *i.e.*, representations of K such that every irreducible representation of the group has the form $\rho \otimes \chi$ for some character χ of $\mathcal{O}^\times$. Specifically, a primitive representation of K of level $n+1$ is an irreducible representation ρ of K such that the kernel of $\rho \otimes \chi \circ \det$ contains K_{n+1}, and not K_n for any character χ of $\mathcal{O}^\times$. Here K_l ($l \in \mathbb{N}$) denotes the kernel of the canonical map from K to $\mathrm{SL}_*^{-1}(2, \mathcal{O}/\mathcal{P}^l)$, where $\mathcal{P}$ is the maximal ideal of $\mathcal{O}$. We observe that if $* \neq \mathrm{id}$, then $K = \mathrm{SL}_*^{-1}(2, \mathcal{O})$ is the quasi-split unitary group $\mathrm{U}(1,1)(\mathcal{O})$. It is important to note that the primitive representations of K only appear for levels $n+1$, with n even (Propositions 4.2 and 4.3). On the other hand, if $* = \mathrm{id}$, the primitive representations of $\mathrm{SL}_*^{-1}(2, \mathcal{O})$ are only the irreducible representations of $\mathrm{SL}(2, \mathcal{O})$, which were constructed by Shalika [16], using a different approach.

We split our construction according to $n = 2r+1$ or $n = 2r$. In the first case, we can directly apply Clifford theory to provide a part of primitive representations of K. In the case $n = 2r$, we use the ideas of Hill [6], adapting and checking that his methods and results are still valid in our case.

It is worth to emphasize that our work is explicit, detailed and general in the sense that we construct the primitive representations for $\mathrm{U}(1,1)(\mathcal{O})$ and $\mathrm{SL}_*^{-1}(2, \mathcal{O})$ (where $*$ is the identity) simultaneously.

Main Results for $\mathrm{U}(1,1)(\mathcal{O})$. We briefly present our main results for $K = \mathrm{U}(1,1)(\mathcal{O})$: the construction of primitive representations; their dimensions and the number of them (see sections 6, 7). Our results are summarized in the following three theorems.

Let E be a ramified quadratic extension of a non-archimedean local field F, $\mathcal{O}$ the ring of integers of E, $*$ an element of the Galois group $\mathrm{Gal}(E/F)$ of E/F and $n = 2r$.

Theorem 1.1. *The primitive representations ρ of K of level $n+1$ are of the following kind:*

1. *If r is odd, then*
$$\rho = \mathrm{ind}_{T(\psi_b)}^{K} \eta\psi_b,$$
where η is a linear character of $\mathrm{C}_K(b_0)$ such that $\eta = \psi_b$ on $\mathrm{C}_K(b_0) \cap K_{r+1}$.

2. *If r is even and the characteristic polynomial of b_0 has its roots in $\mathcal{O}/\mathcal{P}$, then*
$$\rho = \mathrm{ind}_{T(\psi_b)}^{K} (\omega\zeta_b),$$
where $\zeta_b = \mathrm{ind}_{H_b \mathrm{C}_K(b_0)}^{T(\psi_b)} (\widetilde{\psi_b})$, $\widetilde{\psi_a}$ is an extension of ψ_a to $H_b \mathrm{C}_K(b_0)$ with $H_b = (B \cap K_r)K_{r+1}$ and ω is a linear character of $T(\psi_b)$ determined by a linear character of $T(\psi_b)/K_r$.

3. *If r is even and the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$, then*
$$\rho = \mathrm{ind}_{T(\psi_b)}^{K} \widetilde{\rho}_{\widetilde{\psi_b}},$$
where $\rho_{\widetilde{\psi_b}}$ is the unique irreducible representation of $C_1 K_r$ containing $\widetilde{\psi_b}$ and $\widetilde{\rho}_{\widetilde{\psi_b}}$ is an extension of $\widetilde{\psi_b}$ to $T(\psi_b)$.

We put $b_0 = \varpi^n b$ and denote by $\lfloor m \rfloor$ the integer part of m.

Theorem 1.2. *The dimensions of the primitive representations ρ_b of K of level $n+1$ with $n = 2r$ are given as follows:*

- *If the characteristic polynomial of b_0 has different roots in $\mathcal{O}/\mathcal{P}$, then*

$$\dim \rho_b = (q+1)q^r.$$

- *If the characteristic polynomial of b_0 over $\mathcal{O}/\mathcal{P}$ is x^2, then*

$$\dim \rho_b = \frac{q^2-1}{2}\, q^{r-1}.$$

- *If the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$, then*

$$\dim \rho_b = (q-1)q^{2\lfloor \frac{r+2}{2} \rfloor - 1}.$$

Theorem 1.3. *The number of inequivalent primitive representations ρ_b of K of level $n+1$ is given as follows:*

- *If the characteristic polynomial of b_0 has different roots in $\mathcal{O}/\mathcal{P}$, then*

$$\text{Number of } \rho_b = \frac{(q-1)^2}{2} q^{n-1}.$$

- *If the characteristic polynomial of b_0 over $\mathcal{O}/\mathcal{P}$ is x^2, then*

$$\text{Number of } \rho_b = 4q^{r+\lfloor \frac{r}{2} \rfloor}.$$

- *If the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$, then*

$$\text{Number of } \rho_b = \frac{q^2-1}{2} q^{r+\lfloor \frac{r}{2} \rfloor - 1}.$$

This paper is organized as follows. In section 2, we give the basic definitions of $\mathrm{SL}_*^{\varepsilon}(2, A)$ for a unitary ring A endowed with an involution $*$. In section 3, we study the structure of the groups $\mathrm{SL}_*^{-1}(2, A_n)$, where $A_n = \mathcal{O}/\mathcal{P}^n$ is endowed with the involution determined by $*$. We also construct all the characters ψ_b of K_{r+1} which are trivial on K_{n+1}, but not on K_n, (here, $1 \le r < n$ and $2(r+1) \ge n+1$). In section 4, we characterize the primitive representations of K of level $n+1$. Section 5 studies the structure of the normalizer group $T(\psi_b)$ of ψ_b. In section 6, we give the construction of all primitive representations of K of level $n+1$ according to the structure of $T(\psi_b)$ and the corresponding type of involution: in subsection 6.1, we construct the primitive representations for the cases $T(\psi_b) = \mathrm{C}_K(b_0)K_{r+1}$ while the subsection 6.2 shows the case $T(\psi_b) = \mathrm{C}_K(b_0)K_r$. Finally, in section 7, we explicitly calculate the number and the dimensions of the primitive representations so constructed, computing all stabilizers and indices of the related groups.

2. The group $\mathrm{SL}_*^\varepsilon(2, A)$

Let A be a unitary ring with an involution $a \mapsto a^*$, *i.e.*, an antiautomorphism of order two of the ring A. Let $Z(A)$ be the center of A and denote by $A^\times$ the group of invertible elements of A. We write A^s for the set of all elements $a \in A$ such that $a^* = a$, *i.e.*, the set of symmetric elements with respect to $*$.

Let $(A, *)$ be an involute ring. One induces an involution on the ring of matrices $\mathrm{M}(2, A)$ (denoted also by $*$) as follows: for a matrix g with entries in A, the matrix g^* is defined by $(g^*)_{ij} = (g_{ji})^*$. We now consider $\varepsilon = 1$ or $\varepsilon = -1$ and we set $ML_*^\varepsilon(2, A)$ be the set of matrices g in $\mathrm{M}(2, A)$ such that $g^* J g J^{-1} = \lambda_g I$, where $\lambda_g \in Z(A)$, $J = \left(\begin{smallmatrix} 0 & 1 \\ \varepsilon 1 & 0 \end{smallmatrix}\right) \in \mathrm{M}(2, A)$ and I is the identity matrix of $\mathrm{M}(2, A)$. $\mathrm{GL}_*^\varepsilon(2, A)$ is the set of invertible elements in $ML_*^\varepsilon(2, A)$ and $\mathrm{SL}_*^\varepsilon(2, A)$ denotes the subset of all matrices g in $\mathrm{GL}_*^\varepsilon(2, A)$ such that $g^* J g J^{-1} = I$. Also, a $*$-determinant function on $ML_*^\varepsilon(2, A)$ is defined by $\det_*(g) = ad^* + \varepsilon bc^*$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in ML_*^\varepsilon(2, A)$.

Pantoja and Soto-Andrade proved the following result:

Proposition 2.1. $\mathrm{GL}_*^\varepsilon(2, A)$ *is a group under multiplication and* $\det_*$ *is an epimorphism of* $\mathrm{GL}_*^\varepsilon(2, A)$ *onto the group of all central symmetric invertible elements of* A*, such that* $\ker \det_*$ *is* $\mathrm{SL}_*^\varepsilon(2, A)$*.*

Proof. See [14], Lemma 1.5. ■

Remark 2.1. Also in [14], the authors proved that $\mathrm{SL}_*^\varepsilon(2, A)$ is the group of matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with entries in A that satisfy the following equalities; $a^*c = -\varepsilon c^* a$, $ab^* = -\varepsilon ba^*$, $b^*d = -\varepsilon d^* b$, $cd^* = -\varepsilon dc^*$ and $ad^* + \varepsilon bc^* = a^*d + \varepsilon c^* b = 1$.

3. Construction of representations of $\mathrm{U}(1, 1)(\mathcal{O})$

Let E be a ramified quadratic extension of a non-archimedean local field F of odd residual characteristic p and let τ be an element of the Galois group $\mathrm{Gal}(E/F)$ of the extension E/F. We denote by $\mathcal{O}$ (resp. $\mathcal{O}_F$) the ring of integers of E (resp. F) and by $\mathcal{P}$ (resp. $\mathcal{P}_F$) its maximal ideal. We write q to indicate the cardinality of the residue field $\mathcal{O}/\mathcal{P}$. If τ is nontrivial, let ϖ be a prime element in E such that $E = F(\varpi)$ and $\tau(\varpi) = -\varpi$.

Since $\tau(\mathcal{O}) \subset \mathcal{O}$, the restriction of τ to $\mathcal{O}$ is an involution of $\mathcal{O}$. We will denote by $*$ the involution τ of E and its restriction to $\mathcal{O}$. We then write $a^* = \tau(a)$ for all $a \in E$. In addition, let us set

$$K = \mathrm{SL}_*^{-1}(2, \mathcal{O}).$$

We observe that if $*$ is nontrivial, K is the quasi-split unitary group $\mathrm{U}(1, 1)(\mathcal{O})$ defined over the ring of integers $\mathcal{O}$, that is, the group of matrices g with entries in $\mathcal{O}$ that preserves the (-1)-hermitian form h, with respect to $*$, from $E^2 \times E^2$ to E given by $h((x, y), (z, w)) = x^* w - y^* z$.

We now define a map on the ring of matrices $\mathrm{M}(2,E)$ setting $\sigma(y) = J^{-1}y^*J$, where $(y^*)_{ij} = (y_{ji})^*$ and $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The map σ clearly satisfies $\sigma(x+y) = \sigma(x) + \sigma(y)$; $\sigma(yx) = \sigma(x)\sigma(y)$, and $\sigma^2 = \mathrm{id}$ for any x, y in $\mathrm{M}(2,E)$. Using this map, the group $K = \mathrm{SL}_*^{-1}(2,\mathcal{O})$ can be described as the subgroup of $\mathrm{GL}(2,E)$ of the invertible matrices k with entries in $\mathcal{O}$ such that $\sigma(k) = k^{-1}$.

For each $n \in \mathbb{Z}$, we denote by $\mathcal{G}_n$ the additive subgroup of $\mathrm{M}(2,E)$ consisting of the matrices x with entries in $\mathcal{P}^n$ such that $\sigma(x) = -x$. Then we have:

Lemma 3.1. *The group K acts on $\mathcal{G}_n$ by conjugation.*

Proof. Let $k \in K$ and $y \in \mathcal{G}_n$. By using the description of K and the properties of σ, we have that $\sigma(k^{-1}yk) = \sigma(k)\sigma(y)\sigma(k^{-1}) = k^{-1}(-y)k$ and hence our lemma follows. ∎

Notice that $*$ induces naturally an involution on the ring $A_n = \mathcal{O}/\mathcal{P}^n$ for every positive integer n, which is also denoted by $*$. We set A_n^s for the set of symmetric elements in A_n.

We consider the following matrices in $\mathrm{SL}_*^{-1}(2,A_n)$:

$$h_t = \begin{pmatrix} t & 0 \\ 0 & t^{*^{-1}} \end{pmatrix}, \quad t \in A_n^\times; \quad u_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in A_n^s \quad \text{and} \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We will prove that the matrices h_t $(t \in A_n^\times)$, u_b $(b \in A_n^s)$, w and its commutation relations provide a presentation of $\mathrm{SL}_*^{-1}(2,A_n)$. To do this, we prove the following result:

Lemma 3.2. *Let a, c be two elements in A_n such that a or c is invertible and $ac^* = ca^*$. Then there is a symmetric element s in A_n such that $a + sc$ is an invertible element in A_n.*

Proof. If a is invertible, then $s = 0$ satisfies the lemma. On the other hand, if c is invertible, we consider $s = 1$ to verify the result. ∎

Proposition 3.3. *The group $\mathrm{SL}_*^{-1}(2,A_n)$ is generated by the set of matrices h_t with $t \in A_n^\times$, u_b with $b \in A_n^s$, and w.*

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $\mathrm{SL}_*^{-1}(2,A_n)$. If $c = 0$, we see that $g = h_a u_{a^{-1}b}$. Now if $c \in A_n^\times$, we verify that $g = h_{-c^{*-1}} u_{c^*a} w u_{c^{-1}d}$. Finally, if c is not in $A_n^\times \cup \{0\}$, the elements a, c satisfy the conditions of Lemma 3.2. Thus we can consider a symmetric element $s \in A_n$ such that $a + sc$ is an invertible element. Then we see that $g = u_{-s} h_{-a-sc} w u_{(-a^*-sc^*)c} w u_{(a+sc)^{-1}(b+sd)}$. ∎

Also, we need to prove:

Lemma 3.4. *Let a, b be two non-invertible symmetric elements in A_n. Then there is an invertible symmetric element x in A_n such that $a - x^{-1}$ and $b + x$ are invertible symmetric elements.*

Proof. It suffices to consider $x = 1$. ■

Finally, we get:

Theorem 3.5. *The set of matrices h_t, u_b, and w in $\mathrm{SL}_*^{-1}(2, A_n)$ together with the commutation relations $h_{t_1}h_{t_2} = h_{t_1t_2}$, $u_{b_1}u_{b_2} = u_{b_1+b_2}$, $h_tu_b = u_{tbt^*}h_t$, $w^2 = h_{-1}$, $wh_t = h_{t^{*-1}}w$, $u_twu_{t^{-1}}wu_t = wh_{-t^{-1}}$ provide a presentation of the group $\mathrm{SL}_*^{-1}(2, A_n)$.*

Proof. By Proposition 3.3, we already have that $\mathrm{SL}_*^{-1}(2, A_n)$ is generated by the matrices h_t, u_b, and w. The Lemma 3.4 allows us to argue as in [13] Theorem 15, to prove the result. ■

Filtrations and characters

The group K contains a natural (subgroup) filtration as follows. For each $n \in \mathbb{N}$, we denote by K_n the kernel of the canonical map (reduction $\mod \mathcal{P}^n$) from K to $\mathrm{SL}_*^{-1}(2, A_n)$. Explicitly, K_n is the subgroup of K consisting of all matrices of the form

$$\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix},$$

where a, b, c and d are in $\mathcal{P}^n$. By the theorem above, we see that the natural map from K to $\mathrm{SL}_*^{-1}(2, A_n)$ is surjective.

The filtrations K_n and $\mathcal{G}_n$ are related by:

Proposition 3.6. *Let (n, r) be a pair of integers such that $1 \le r < n$ and $2r \ge n$. If $\overline{a}$ denotes the equivalence class of a, the map*

$$\overline{1+x} \longmapsto \overline{x - \sigma(x)}$$

from K_r/K_n to $\mathcal{G}_r/\mathcal{G}_n$ is an isomorphism.

Proof. We first check that it is well defined. To do that, let $\overline{1+x}$ in K_r/K_n. For $1 + z \in K_n$ we have that $z - \sigma(z)$ and $xz - \sigma(xz)$ are in $\mathcal{G}_n$. Hence

$$\begin{aligned} x + z + xz - \sigma(x + z + xz) &= (x - \sigma(x)) - (z - \sigma(z)) + (xz - \sigma(xz)) \\ &\equiv x - \sigma(x) \mod (\mathcal{G}_n), \end{aligned}$$

and then the map is well defined.

To prove that the map above is a homomorphism, let $\overline{1+x}$, $\overline{1+y}$ be two elements in K_r/K_n. Since $xy - \sigma(xy) \in \mathcal{G}_n$ we observe that

$$\overline{1+x}\cdot\overline{1+y} = \overline{1+x+y+xy} \mapsto \overline{x+y+xy-\sigma(x+y+xy)} = \overline{x-\sigma(x)} + \overline{y-\sigma(y)},$$

and from this, $\overline{1+x} \mapsto \overline{x-\sigma(x)}$ is a homomorphism from K_r/K_n to $\mathcal{G}_r/\mathcal{G}_n$. Now, we suppose that $\overline{1+x} \in K_r/K_n$ such that $\overline{1+x} \mapsto \overline{x-\sigma(x)} = \overline{0}$. We will prove that $\overline{1+x} = \overline{1}$. Since $1 + x \in K_r \subset K$, we get $x + \sigma(x) + x\sigma(x) = 0$, so $x + \sigma(x) = -x\sigma(x) \in \mathrm{M}(2, \mathcal{P}^n)$, and $x - \sigma(x)$ also belongs to $\mathcal{G}_n \subset \mathrm{M}(2, \mathcal{P}^n)$ by assumption. Thus $x \in \mathrm{M}(2, \mathcal{P}^n)$, which implies $\overline{1+x} = \overline{1}$, *i.e.*, $\overline{1+x} \mapsto \overline{x - \sigma(x)}$ is injective.

Finally, in order to prove the map is surjective, let $\overline{x}$ be an element in $\mathcal{G}_r/\mathcal{G}_n$. Setting $y = 4^{-1}x$, we get that $\sigma(y) = -y$. If we write $z = 1+y$ and $s = (1-y^2)^{-1}$, we verify $\sigma(z)zs = 1$. Now $v = (1-y)^{-1}$ implies $s = \sigma(v)v$ and from here $1 = \sigma(z)z\sigma(v)v = \sigma(zv)zv$. Hence $zv \in K_r$ and $\overline{zv - \sigma(zv)} = \overline{x}$. Therefore the map $\overline{1+x} \mapsto \overline{x - \sigma(x)}$ is an isomorphism and our proposition follows. ■

We also have.

Lemma 3.7. *The groups $\mathcal{G}_{-n}/\mathcal{G}_{-r}$ and $\mathcal{G}_{r+1}/\mathcal{G}_{n+1}$ have the same cardinality.*

Proof. If $* = \mathrm{id}$, we verify that the map $b \mapsto \varpi^{n+r+1}b$ gives a bijection from $\mathcal{G}_{-n}/\mathcal{G}_{-r}$ to $\mathcal{G}_{r+1}/\mathcal{G}_{n+1}$. On the other hand, if $*$ is nontrivial, the map above works if $n = 2r$, with r odd. Now, if $n = 2r$ where r is even, the map

$$\begin{pmatrix} a & c \\ d & -a^* \end{pmatrix} \mapsto \begin{pmatrix} a\varpi^{n+r+1} & c\varpi^{n+r+2} \\ d\varpi^{n+r+2} & -a^*\varpi^{n+r+1} \end{pmatrix}$$

is a bijection from $\mathcal{G}_{-n}/\mathcal{G}_{-r}$ to $\mathcal{G}_{r+1}/\mathcal{G}_{n+1}$ and the lemma follows. ■

We denote by tr the usual trace on the ring of matrices $\mathrm{M}(2, E)$.

We fix a character ψ of F^+ of conductor $\mathcal{P}_F$ (that is, $\mathcal{P}_F$ is the largest fractional ideal of F contained in the kernel $\ker(\psi)$). We also consider a pair of integers (n, r) such that $1 \leq r < n$ and $2(r+1) \geq n+1$.

Definition 3.8. *Let b be an element in $\mathcal{G}_{-n}$. We define the map ψ_b from K_{r+1} to $\mathbb{C}^\times$ by*

$$\psi_b(1+x) = \psi\left(\mathrm{tr}\left(b\left(x - \sigma(x)\right)\right)\right),$$

where $1 + x \in K_{r+1}$.

Notice that if $b \in \mathcal{G}_{-n}$, we get $\mathrm{tr}(b(x - \sigma(x)))$ is a symmetric element in E for each $x \in K_{r+1}$. In other words, $\mathrm{tr}(b(x-\sigma(x))) \in F$, and so ψ_b is well defined.

From now on, we will keep the above notations.

Proposition 3.9. *For each $b \in \mathcal{G}_{-n}$, the function ψ_b (defined above) is a character of the group K_{r+1}.*

Proof. It follows from the definition. ■

The group of linear characters of K_{r+1}/K_{n+1} is denoted by $\widehat{K_{r+1}/K_{n+1}}$.

For the next proposition, we assume $n = 2r$ if $* \neq \mathrm{id}$.

Proposition 3.10. *The map $b \mapsto \psi_b$ from $\mathcal{G}_{-n}$ to $\widehat{K_{r+1}}$ induces an isomorphism from $\mathcal{G}_{-n}/\mathcal{G}_{-r}$ to $\widehat{K_{r+1}/K_{n+1}}$.*

Proof. The character ψ_b, $b \in \mathcal{G}_{-n}$, determines naturally a character $\widetilde{\psi}_b$ of K_{r+1}/K_{n+1}. By using Proposition 3.6 and Lemma 3.7 the groups K_{r+1}/K_{n+1} and $\mathcal{G}_{-n}/\mathcal{G}_{-r}$ have the same cardinality. Hence, the fact that K_{r+1}/K_{n+1} is abelian implies that $\widehat{K_{r+1}/K_{n+1}}$ and $\mathcal{G}_{-n}/\mathcal{G}_{-r}$ have also the same cardinality. So, the proposition will follow if we prove that the induced map $\bar{b} \mapsto \widetilde{\psi}_b$ is injective ($\bar{b}$ is the equivalence class of b). To prove this, let $b = (b_{ij}) \in \mathcal{G}_{-n} - \mathcal{G}_{-r}$. Since the conductor of ψ is $\mathcal{P}_F$, we see that ψ is nontrivial on $\mathcal{O}^s$. Let us consider then $\alpha \in \mathcal{O}^s$ such that $\psi(\alpha) \neq 1$. If $b_{11} \notin \mathcal{P}^{-r}$, setting

$$1 + x = \begin{pmatrix} 1 + 4^{-1} b_{11}^{-1} \alpha & 0 \\ 0 & (1 + 4^{-1} b_{11}^{-1} \alpha)^{*-1} \end{pmatrix},$$

we verify that

$$b(x - \sigma(x)) = \begin{pmatrix} 2^{-1}\alpha & 2^{-1}\alpha b_{11}^{*^{-1}} b_{12} \\ 2^{-1}\alpha b_{11}^{-1} b_{21} & 2^{-1}\alpha \end{pmatrix},$$

so $\widetilde{\psi}_b(\overline{1+x}) = \psi(\mathrm{tr}\,(b(x - \sigma(x))) = \psi(\alpha) \neq 1$. On the other hand, if $b_{11} \in \mathcal{P}^{-r}$, then b_{12} or b_{21} is not in $\mathcal{P}^{-r}$. Supposing, for instance, that $b_{21} \notin \mathcal{P}^{-r}$, we set

$$1 + x = \begin{pmatrix} 1 & 2^{-1} b_{21}^{-1} \alpha \\ 0 & 1 \end{pmatrix}$$

and we check again that $\widetilde{\psi}_b(\overline{1+x}) = \psi(\mathrm{tr}\,(b(x - \sigma(x))) = \psi(\alpha) \neq 1$. Similarly if $b_{12} \notin \mathcal{P}^{-r}$. Then $\bar{b} \mapsto \widetilde{\psi}_b$ is injective and therefore our result follows. ∎

4. Primitive representations

An irreducible representation ρ of K is called smooth if there exists $n \in \mathbb{N}$ such that $K_n \subset \ker \rho$. We will say that a smooth representation ρ of K has level $n+1$ if $K_{n+1} \subset \ker \rho$ and $K_n \not\subset \ker \rho$. Now a smooth representation ρ of K of level $n+1$ is a primitive representation of K of level $n+1$ if $\rho \otimes \chi \circ \det$ has level greater than or equal to $n+1$ for any character χ of $\mathcal{O}^\times$.

Remark 4.1. Immediately from the definition, the primitive representations of $K = \mathrm{SL}(2, \mathcal{O})$ of level $n+1$ are only the irreducible representation of K of level $n+1$.

Concerning $*$ nontrivial, we get the following two propositions.

Proposition 4.1. *Let E be a ramified quadratic extension of a non-archimedean local field F, $\mathcal{O}$ is the ring of integers of E and $*$ is an element of the Galois group* $\mathrm{Gal}(E/F)$ *of E/F. If ρ is a smooth representation of K of level $n+1$, where n is odd, then ρ is not a primitive representation of K of level $n+1$.*

Proof. Let ψ_b be an irreducible component of ρ restricted to K_n, where b belongs to $\mathcal{G}_{-n}/\mathcal{G}_{-r}$. Since n is odd, we can assume that b has the form $\left(\begin{smallmatrix} u\varpi^{-n} & 0 \\ 0 & u\varpi^{-n} \end{smallmatrix}\right)$ modulo $\mathcal{G}_{1-n}/\mathcal{G}_{-r}$, with u a unit in $\mathcal{O}$. Writing

$$1 + z = \begin{pmatrix} 1 + a & b \\ c & 1 + d \end{pmatrix} \in K_n$$

it follows that $\psi_b(1+z) = \psi(u\varpi^{-n}[(a-a^*)+(d-d^*)])$. Since n is odd, we note that $a-a^* = 2a \mod \mathcal{P}^{1-n}$ and $d-d^* = 2d \mod \mathcal{P}^{1-n}$. We now consider the character χ of $\mathcal{O}^\times$ such that the restriction of χ to $\mathcal{U}_F^n$ is $\psi_{2u\varpi^{-n}}$, this is, $\chi(1+t) = \psi(2u\varpi^{-n}t)$. Then

$$\psi_b(1+z)\cdot\chi^{-1}\circ\det(1+z) = 1,$$

for any $1+z\in K_n$. Therefore the representation $\rho\otimes\chi^{-1}\circ\det$ contains the trivial representation of K_n and hence $K_n\subset\ker(\rho\otimes\chi^{-1}\circ\det)$. In other words, the level of $(\rho\otimes\chi^{-1}\circ\det)$ is less than or equal to n and therefore ρ is not primitive of level $n+1$. ■

Proposition 4.2. *Let E be a ramified quadratic extension of a non-archimedean local field F, $\mathcal{O}$ is the ring of integers of E and $*$ is an element of the Galois group* $\mathrm{Gal}(E/F)$ *of E/F. If ρ is a smooth representation of K of level $n+1$, where n is even, then ρ is primitive of level $n+1$.*

Proof. It suffices to show that $\rho\otimes\chi\circ\det$ is nontrivial on K_n for any character χ of $\mathcal{O}^\times$. To this end, let χ be a character of $\mathcal{O}^\times$ and m its conductor. We have three cases. If $m<n$, then $(\chi\circ\det)$ is trivial on K_n. Since ρ is nontrivial on K_n, we get that $\rho\otimes\chi\circ\det$ is nontrivial on K_n. On the other hand, if $m>n$, by using the trivial action of ρ on K_{n+1}, we see that $\rho\otimes\chi\circ\det$ is nontrivial on K_{n+1}. Finally, if $m=n$, consider $e\in\mathcal{P}^{-n}$ such that χ restricted to $\mathcal{U}_F^n$ is $\chi=\psi_e$. Let

$$1+z = \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \in K_n.$$

Then $(\chi\circ\det)(1+z) = \chi(1+a+d+ad-bc) = \psi(e(a+d))$. Given that $1+z\in K_n$, we observe $d = -a^* - d^*a + c^*b$, which implies $(\chi\circ\det)(1+z) = \psi(e(a-a^*))$. Since n is even, we see that $a-a^*\in\mathcal{P}^{n+1}$. Verifying $(\chi\circ\det)(1+z)=1$ for any $1+z\in K_n$, it follows that $(\rho\otimes\chi\circ\det)$ is nontrivial on K_n. This completes the proof. ■

From the propositions above, if $*\neq 1$, our goal will be the construction of smooth representations of K of level $n+1$, where n is even.

5. The normalizer group $T(\psi_b)$

Let ψ_b be a character of K_{r+1} (as in 3), where $b\in\mathcal{G}_{-n}$. For each $k\in K$, we put $\psi_b^k(1+z) = \psi_b(k^{-1}(1+z)k)$ for any $1+z\in K_{r+1}$. Let us denote by $T(\psi_b)$ the normalizer group of ψ_b in K, that consists of the elements $k\in K$ such that $\psi_b^k = \psi_b$.

In order to find the structure of the group $T(\psi_b)$, let us introduce some notation. Let $G=\mathrm{GL}(2,\mathcal{O})$ be the general linear group over $\mathcal{O}$. We write G_n, $n\in\mathbb{N}$, for the subgroup of G of matrices of the form $1+x$, where 1 is the identity matrix in G and $x\in\mathrm{M}(2,\mathcal{P}^n)$. Let ψ be a character of E of conductor $\mathcal{P}$, and we fix a pair of integers (n,r) such that $2(r+1)\geq n+1$. For $b\in\mathrm{M}(2,\mathcal{P}^{-n})$, we set

$$\psi_b(1+z) = \psi\left(\mathrm{tr}\,(bz)\right), \quad 1+z\in G_n.$$

The structure of the normalizer group $T_G(\psi_b)$ of ψ_b in G, consisting of the elements $g \in G$ such that $\psi_b(g^{-1}(1+z)g) = \psi_b(1+z)$, for all $1+z \in G_n$ is given by the following proposition:

Proposition 5.1. *Set $b_0 = \varpi^n b$. If $n = 2r$ or $n = 2r+1$, then*

$$T_G(\psi_b) = \mathrm{C}_G(b_0)G_{n-r},$$

where $\mathrm{C}_G(b_0)$ is the centralizer of b_0 in G.

Proof. Notice that

$$\begin{aligned} T_G(\psi_b) &= \{g \in G : \psi_{gbg^{-1}} = \psi_b\} \\ &= \left\{g \in G : gbg^{-1} \equiv b \mod \mathrm{M}(2, \mathcal{P}^{-r})\right\} \\ &= \left\{g \in G : gb_0g^{-1} \equiv b_0 \mod \mathrm{M}(2, \mathcal{P}^{n-r})\right\} \\ &= \left\{g \in G : gb_0 - b_0 g \in \mathrm{M}(2, \mathcal{P}^{n-r})\right\}, \end{aligned}$$

from which $\mathrm{C}_G(b_0)G_{n-r} \subset T_G(\psi_b)$. On the other hand, we first define the $\mathcal{O}$-linear map from $\mathrm{M}(2, \mathcal{P}^{n-r})$ to $\mathrm{M}(2, \mathcal{P}^{n-r})$ given by $\varphi(z) = zb_0 - b_0 z$. We claim that $\operatorname{Im}\varphi = (\mathrm{M}(2, \mathcal{P}^{n-r}) \cap \mathrm{C}(b_0))^{\perp}$ (with respect to the non-degenerate pairing $\langle\ ,\ \rangle$ of $\mathrm{M}(2, P^{n-r})$ with $\mathrm{M}(2, P^{n-r})$ defined by $\langle x, y\rangle = \mathrm{tr}(xy)$), where $\mathrm{C}(b_0)$ is the set of elements $z \in \mathrm{M}(2, \mathcal{O})$ such that $zb_0 - b_0 z = 0$. In fact, considering $zb_0 - b_0 z$ an element of $\operatorname{Im}\varphi$ and $u \in \mathrm{M}(2, \mathcal{P}^{n-r}) \cap \mathrm{C}(b_0)$, we have that

$$\langle u, zb_0 - b_0 z\rangle = \mathrm{tr}(u(zb_0 - b_0 z)) = \mathrm{tr}(uzb_0 - b_0 uz) = 0.$$

Thus $\operatorname{Im}\varphi \subset (\mathrm{M}(2, \mathcal{P}^{n-r}) \cap \mathrm{C}(b_0))^{\perp}$. Since $\ker\varphi$ is $\mathrm{M}(2, \mathcal{P}^{n-r}) \cap \mathrm{C}(b_0)$ and $\operatorname{Im}\varphi$ is isomorphic to $\mathrm{M}(2, \mathcal{P}^{n-r})/\ker\varphi$, it follows that

$$\dim_{\mathcal{O}} \operatorname{Im}\varphi = \dim_{\mathcal{O}}(\mathrm{M}(2, \mathcal{P}^{n-r})) - \dim_{\mathcal{O}}(\ker\varphi) = \dim_{\mathcal{O}}(\mathrm{M}(2, \mathcal{P}^{n-r}) \cap \mathrm{C}(b_0))^{\perp}$$

and our claim follows. Finally, letting $x \in T_G(\psi_b)$, the element $xb_0 - b_0 x$ belongs to $\operatorname{Im}\varphi = (\mathrm{M}(2, \mathcal{P}^{n-r}) \cap \mathrm{C}(b_0))^{\perp}$. Then there exists $z \in \mathrm{M}(2, \mathcal{P}^{n-r})$ such that $xb_0 - b_0 x = zb_0 - b_0 z$, which implies that $(x-z)b_0 - b_0(x-z) = 0$. If we set $\beta = x - z \in \mathrm{C}_G(b_0)$, we can write $\beta = x(1 - x^{-1}z)$. Then β is invertible, *i.e.*, $\beta \in \mathrm{C}_G(b_0)$ and $x = \beta(1 - x^{-1}z)^{-1} \in \mathrm{C}_G(b_0)G_{n-r}$. This completes the proof. ∎

For the lemmas below, we recall the map σ on $\mathrm{M}(2, E)$ defined by $\sigma(y) = J^{-1}y^* J$, where $(y^*)_{ij} = (y_{ji})^*$ and $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Lemma 5.2. *Let $b_0 \in \mathrm{M}(2, \mathcal{O})$. If $\overline{s} \in (\mathrm{C}_G(b_0) \cap G_r) / (\mathrm{C}_G(b_0) \cap G_{r+1})$ and $\underline{\sigma(s)} = s$, then there exists $\overline{v} \in (\mathrm{C}_G(b_0) \cap G_r) / (\mathrm{C}_G(b_0) \cap G_{r+1})$ such that $\overline{s} = \overline{\sigma(v)v}$.*

Proof. Let $\overline{s} \in (\mathrm{C}_G(b_0) \cap G_r) / (\mathrm{C}_G(b_0) \cap G_{r+1})$ such that $\sigma(s) = s$. We then write $s = 1 + x = \lambda + \mu b_0$ for some matrix $x \in \mathrm{M}(2, \mathcal{P}^r)$ and scalars λ, μ in $\mathcal{O}$. Hence, the matrix $v = 1 + 2^{-1}x = 2^{-1}(\lambda+1) + 2^{-1}\mu b_0$ satisfies $\overline{v} \in \mathrm{C}_G(b_0) \cap G_r / \mathrm{C}_G(b_0) \cap G_{r+1}$ and $\overline{s} = \overline{\sigma(v)v}$. ∎

Lemma 5.3. *If $s \in \mathrm{C}_G(b_0) \cap G_r$ and $\sigma(s) = s$, then there is $v \in \mathrm{C}_G(b_0) \cap G_r$ such that $s = \sigma(v)v$.*

Proof. Let $s \in \mathrm{C}_G(b_0) \cap G_r$ and $\sigma(s) = s$. By Lemma 5.2, there exist elements $v_1 \in \mathrm{C}_G(b_0) \cap G_r$ and $s_1 \in \mathrm{C}_G(b_0) \cap G_{r+1}$ such that $s = \sigma(v_1)v_1s_1$. Since $\sigma(s) = s$, we get that $\sigma(s_1) = s_1$. So applying the procedure to s_{i-1} $(i > 1)$, there are $v_i \in \mathrm{C}_G(b_0) \cap G_{r+i-1}$ and $s_i \in \mathrm{C}_G(b_0) \cap G_{r+i}$, such that $s_{i-1} = \sigma(v_i)v_is_i$ and $s_i = \sigma(s_i)$. After n-steps we get

$$s = \sigma(v_1)v_1\sigma(v_2)v_2 \cdots \sigma(v_n)v_ns_n = \sigma(v_1v_2 \cdots v_n)v_1v_2 \cdots v_ns_n.$$

From here, the sequence $(v_1, v_1v_2, v_1v_2v_3, \cdots, v_1v_2v_3 \cdots v_n, \cdots)$ converges to an element v in $\mathrm{C}_G(b_0) \cap G_r$. The continuity of σ implies that $\sigma(v)$ is the limit of the sequence $(\sigma(v_1), \sigma(v_1v_2), \sigma(v_1v_2v_3), \cdots, \sigma(v_1v_2v_3 \cdots v_n), \cdots)$. So, we observe that the sequence

$$(\sigma(v_1)v_1, \sigma(v_1v_2)v_1v_2, \sigma(v_1v_2v_3)v_1v_2v_3, \sigma(v_1v_2v_3v_4)v_1v_2v_3v_4, \cdots)$$

converges to $\sigma(v)v$. Since this sequence also converges to s, we conclude that $s = \sigma(v)v$ and the result follows. ∎

Finally, we can prove:

Proposition 5.4. *Let E/F be a ramified quadratic extension of local fields and $*$ be an element of the Galois group $\mathrm{Gal}(E/F)$ and set $b_0 = \varpi^n b$. The structure of the normalizer group $T(\psi_b)$ of ψ_b in K is given as follows:*

(i) If $n = 2r$, then

$$T(\psi_b) = \mathrm{C}_K(b_0)K_r.$$

(ii) If $n = 2r + 1$ and $*$ is the trivial involution on E, then

$$T(\psi_b) = \mathrm{C}_K(b_0)K_{r+1}.$$

Proof. We will only prove the case when $*$ is nontrivial and $n = 2r$. The other cases can be proved similarly.

We note that

$$T(\psi_b) = \{k \in K : \psi_{kbk^{-1}} = \psi_b\} = \left\{k \in K : kbk^{-1} \equiv b \mod \mathcal{G}_{-r}\right\}.$$

Since the map from $\mathcal{G}_{-n}/\mathcal{G}_{-r}$ to $\mathcal{G}_0/\mathcal{G}_r$ given by $b \mapsto \varpi^n b$ is an isomorphism, we observe that

$$\begin{aligned} T(\psi_b) &= \left\{k \in K : kbk^{-1} \equiv b \mod \mathcal{G}_{-r}\right\} \\ &= \left\{k \in K : kb_0k^{-1} \equiv b_0 \mod \mathcal{G}_r\right\} \\ &= \left\{k \in K : kb_0 - b_0k \in \mathrm{M}(2, \mathcal{P}^r)\right\}. \end{aligned}$$

It follows from above that $\mathrm{C}_K(b_0)K_r \subset T(\psi_b)$. Conversely, given x in $T(\psi_b)$, we will prove that $x = tk$, for some t in $\mathrm{C}_K(b_0)$ and k in K_r. Clearly $T(\psi_b) \subset$

$T_G\left(\widetilde{\psi}_b\right)$, where $\widetilde{\psi}$ is a character of E of conductor $\mathcal{P}$ extending ψ. So by the description of $T_G(\widetilde{\psi}_b)$ given in Proposition 5.1, there exist $l \in \mathrm{C}_G(b_0)$ and $g \in G_r$ such that $x = lg$. Since $x \in K$, we have that $\sigma(x) = x^{-1}$, *i.e.*, $\sigma(g)\sigma(l) = g^{-1}l^{-1}$, we can see that $\sigma(l)l = \sigma(g^{-1})g^{-1}$. Now the inclusions $\sigma\left(\mathrm{C}_G(b_0)\right) \subset \mathrm{C}_G(b_0)$ and $\sigma\left(G_r\right) \subset G_r$ imply that $s = \sigma(l)l \in \mathrm{C}_G(b_0) \cap G_r$ satisfies Lemma 5.3. Then there is an element $v \in \mathrm{C}_G(b_0) \cap G_r$ such that $\sigma(l)l = \sigma(v)v$ or equivalently $\sigma(lv^{-1}) = (lv^{-1})^{-1}$, which implies $lv^{-1} \in K \cap \mathrm{C}_G(b_0) = \mathrm{C}_K(b_0)$. So writing $x = g = lv^{-1}vg$, we observe that $vg \in K_r$. This completes the proof. ∎

6. Construction of primitive representation of level $n+1$

In this section, our aim is the construction of the primitive representations of K of level $n+1$, which will be separated into cases. To this end, we recall that if the involution is trivial, we have to construct every irreducible representation of K of level $n+1$. On the other hand, if the involution is nontrivial, we need to construct the smooth representation of K of level $n+1$ for n even (Propositions 4.2 and 4.3).

The construction will be achieved according to the structure of the normalizer group $T\left(\psi_b\right)$ of ψ_b. From Proposition 5.4, it will be separated into:

- First kind: $* = \mathrm{id}$ and $n = 2r+1$, or $* \neq \mathrm{id}$ and $n = 2r$, where r is odd.
- Second kind: $*$ is trivial and $n = 2r$, or $*$ and $n = 2r$, where r is even.

6.1. Construction for the first kind. Suppose that $* = \mathrm{id}$ and $n = 2r+1$, or $* \neq \mathrm{id}$ and $n = 2r$, where r is odd.

Notice that if $*$ is trivial and $n = 2r+1$, by using Proposition 5.4, we have that

$$T\left(\psi_b\right) = \mathrm{C}_K(b_0)K_{r+1}.$$

On the other hand, if $*$ is nontrivial and $n = 2r$, where r is odd, we get that K_r/K_{r+1} is isomorphic to the group of diagonal matrices of the form $\left(\begin{smallmatrix} 1+a & 0 \\ 0 & 1+a \end{smallmatrix}\right)$, with $1+a \in 1+\mathcal{P}^r$ such that $(1+a)(1+a^*) = 1$. Since the group $\mathrm{C}_K(b_0)K_{r+1}$ contains these matrices, we get that

$$T\left(\psi_b\right) = \mathrm{C}_K(b_0)K_r = \mathrm{C}_K(b_0)K_{r+1}.$$

Therefore, we have that

$$T\left(\psi_b\right) = \mathrm{C}_K(b_0)K_{r+1},$$

for both cases. Thus the character ψ_b can be extended to $T\left(\psi_b\right)$ by considering a character η of $\mathrm{C}_K(b_0)$ such that $\eta = \psi_b$ on $\mathrm{C}_K(b_0) \cap K_{r+1}$ and taking the character $\eta\psi_b$ of $T\left(\psi_b\right)$.

Therefore, we can prove:

Theorem 6.1. *Let E be a ramified quadratic extension of a non-archimedean local field F, $\mathcal{O}$ is the ring of integers of E and $*$ is an element of the Galois group* $\mathrm{Gal}(E/F)$ *of E/F. Then* *(i) The primitive representations of K of level $n+1$ are exactly those of the form*

$$\rho_{\eta,\psi_b} = \mathrm{ind}_{T(\psi_b)}^{K} \eta\psi_b .$$

(ii) If the representations ρ_{η,ψ_b} and $\rho_{\eta',\psi_{b'}}$ are equivalent, then $\psi_b^k = \psi_{b'}$ for some $k \in K$. In this case, the representations $\mathrm{ind}_{K_{r+1}}^{K} \psi_b$ and $\mathrm{ind}_{K_{r+1}}^{K} \psi_{b'}$ are equivalent.

(iii) The representations ρ_{η,ψ_b} and ρ_{η',ψ_b} are equivalent if and only if $\eta = \eta'$.

Proof. Notice that ψ_b determines a character of K_{r+1}/K_{n+1} (recall $K_{n+1} \subset \ker \psi_b$). Since the one-dimensional representation $\psi_b\eta$ of $T(\psi_b)$ contains ψ_b, then we have

$$\begin{array}{ccc}
K/K_{n+1} & & \mathrm{ind}_{T(\psi_b)/K_{n+1}}^{K/K_n+1} \widetilde{\psi_b}\widetilde{\eta} \\
| & & | \\
T(\psi_b)/K_{n+1} & & \widetilde{\psi_b}\widetilde{\eta} \\
| & & | \\
K_{r+1}/K_{n+1} & & \widetilde{\psi_b}
\end{array}$$

where $\widetilde{\psi_b}$ is a character of K_{r+1}/K_{n+1} determined by ψ_b and $\widetilde{\eta}$ is a character of $\mathrm{C}_K(b_0)K_{n+1}/K_{n+1}$ such that $\widetilde{\psi_b} = \widetilde{\eta}$ on $(\mathrm{C}_K(b_0 \cap K_{r+1})\, K_{n+1}/K_{n+1}$. By using Clifford theory, we get that the induced representation $\mathrm{ind}_{T(\psi_b)/K_{n+1}}^{K/K_n+1} \widetilde{\psi_b}\widetilde{\eta}$ is irreducible. It follows that $\mathrm{ind}_{T(\psi_b)}^{K} \eta\psi_b$ is an irreducible representation of K, and also it is primitive because its restriction to K_{r+1} contains ψ_b. On the other hand, let ρ be a primitive representation of level $n+1$ of K. Then ρ restricted to $T(\psi_b)$ contains a representation $\psi_b\eta$, for some character η of $C_K(b_0)$ with $\eta = \psi_b$ on $\mathrm{C}_K(b_0) \cap K_{r+1}$. By Frobenius reciprocity, ρ and $\mathrm{ind}_{T(\psi_b)}^{K} \eta\psi_b$ intertwine, and since they are irreducible, the representations ρ and $\mathrm{ind}_{T(\psi_b)}^{K} \eta\psi_b$must be isomorphic and (i) follows.

To prove (ii), suppose that ρ_{η,ψ_b}, $\rho_{\eta',\psi_{b'}}$ are equivalent. Notice first that

$$\mathrm{ind}_{K_{r+1}}^{K} \psi_b = \mathrm{ind}_{T(\psi_b)}^{K} \mathrm{ind}_{K_{r+1}}^{T(\psi_b)} \psi_b = \mathrm{ind}_{T(\psi_b)}^{K} \bigoplus_{\eta} \eta\psi_b = \bigoplus_{\eta} \mathrm{ind}_{T(\psi_b)}^{K} \eta\psi_b,$$

where η, in the above sums, runs over all irreducible representations η of $\mathrm{C}_K(b_0)$ such that $\eta = \psi_b$ on $\mathrm{C}_K(b_0) \cap K_{r+1}$. Using the decomposition of $\mathrm{ind}_{K_{r+1}}^{K} \psi_b$ and Frobenius reciprocity, the representations $\mathrm{ind}_{K_{r+1}}^{K} \psi_b$ and $\mathrm{ind}_{K_{r+1}}^{K} \psi_{b'}$ intertwine. So, there exists $k \in K$ such that $\psi_b = \psi_b^k$ and therefore $\mathrm{ind}_{K_{r+1}}^{K} \psi_b$ and $\mathrm{ind}_{K_{r+1}}^{K} \psi_{b'}$ are isomorphic.

(iii) Finally, we suppose ρ_{η,ψ_b} and ρ_{η',ψ_b} are equivalent. Then $\eta\psi_b$ and $\eta'\psi_b$ are equivalent. Since the representations $\eta\psi_b$ and $\eta'\psi_b$ are one-dimensional, we get that $\eta\psi_b = \eta'\psi_b$, and from here $\eta = \eta'$. ■

6.2. Construction for the second kind. Assume that $*$ is trivial and $n = 2r$, or $*$ is nontrivial and $n = 2r$, where r is even. This second kind, in turn, will be split into cases according to whether the characteristic polynomial of b_0 has roots in $\mathcal{O}/\mathcal{P}$ or not.

The characteristic polynomial of b_0 has roots in $\mathcal{O}/\mathcal{P}$.

For the cases; $* = \mathrm{id}$ and $n = 2r$ or $* \neq 1$ and $n = 2r$, where r is even, we get $T(\psi_b) = \mathrm{C}_K(b_0)K_r$. It is not so evident how to extend the character ψ_b to $T(\psi_b)$ as in the previous cases. Therefore, we introduce the following notations: for any integer $r \geq 1$, let $G(r)$ be the general linear group over the ring $\mathcal{O}_r = \mathcal{O}/\mathcal{P}^r$. If $1 \leq i \leq r-1$, let G_i be the kernel of the natural map $\eta_i : G(r) \to G(i)$.

Consider an odd integer $r \geq 2$ and set $l = (r+1)/2$ and $l' = (r-1)/2$. For an irreducible character ζ of G, $\Omega_l(\zeta)$ denotes the regular orbit.

Theorem 6.2 ([6], Theorem 4.6). *Let $a \in M_n(\mathcal{O}_{l'})$ be split regular. Choose $\hat{a} \in \mathcal{O}_r$ with $\eta_{l'}(\hat{a}) = a$. For every irreducible character ζ such that $a \in \Omega_l(\zeta)$ there exists a subgroup H_a and an extension $\widetilde{\psi_a}$ of ψ_a to $H_aC_G(\hat{a})$ such that*

$$\zeta = \mathrm{ind}_{H_aC_G(\hat{a})}^{G(r)} \widetilde{\psi_a}$$

The main steps to prove this theorem are:

- To prove that the subgroup $H_a = (B \cap K_{l'})K_l$ (B is the subgroup of upper triangular matrices of G) is a normal subgroup of $\mathrm{Stab}(\psi_a)$. To do this, to show that H_b/K_l is a maximal isotropic subspace of $K_{l'}/K_l$ with respect to the pairing
$$\langle xK_l, yK_l\rangle_a = \mathrm{tr}(\overline{a}(\overline{nm} - \overline{mn})),$$
where $x = 1 + m\pi^{l'}$ and $x = 1 + n\pi^{l'}$ (π a prime element).
- There exists q^n extensions $\widetilde{\psi_b}$ (one dimensional) of ψ_b to $H_a\,\mathrm{C}_G(a)$. Set
$$\rho_b = \mathrm{ind}_{H_b}^{K_r}(\widetilde{\psi_b}|_{H_b}) \qquad \text{and} \qquad \zeta_b = \mathrm{ind}_{H_b\,\mathrm{C}_K(b_0)}^{K_r\,\mathrm{C}_K(b_0)}(\widetilde{\psi_b}),$$
and to show that
$$\mathrm{ind}_{K_l'}^{\mathrm{Stab}(\psi_a)} \rho_a = \sum_{\omega} \omega\zeta_a,$$
where ω are the linear characters of $\mathrm{Stab}(\psi_a)/K_r$.
- To verify that every irreducible constituent of $\mathrm{ind}_{K_l}^{G} \psi_a$ has the form $\omega\zeta$ and that each one of these produces an irreducible character of G

Returning to the primitive representations of K of level $n+1$, we are assuming that $*$ is trivial and $n = 2r$ or $*$ is nontrivial and $n = 2r$, where r is even.

Considering the alternating bilinear form $\langle\ ,\ \rangle_b$ from $K_r/K_{r+1} \times K_r/K_{r+1}$ to $\mathcal{O}/\mathcal{P}$ given by:

$$(\overline{1+x}, \overline{1+y}) \mapsto \langle\overline{1+x}, \overline{1+y}\rangle_b = \overline{\mathrm{tr}(b((xy - yx) + (y\sigma(x) - x\sigma(y))))},$$

we observe that the construction due to Hill works, step by step, for the group K, the filtration K_n and the characters ψ_b.

In fact, if $H_b = (B \cap K_r)K_{r+1}$, the alternating form helps to extend ψ_b to a character $\widetilde{\psi}_b$ of $H_b \, \mathrm{C}_K(b_0)$. From here, one proves:

Theorem 6.3. *The primitive representations of K are of the form*

$$\mathrm{ind}_{\mathrm{C}_K(b_0)K_r}^{K}(\omega\zeta_b),$$

where $\zeta_b = \mathrm{ind}_{H_b \, \mathrm{C}_K(b_0)}^{K_r \, \mathrm{C}_K(b_0)K_r}(\widetilde{\psi}_b)$ and ω is a linear character of $T(\psi_b)$ determined by a linear character of $T(\psi_b)/K_r$, that is,

$$\begin{array}{ccc}
T(\psi_b) & \qquad & \zeta_b = \mathrm{ind}_{\mathrm{C}_K(b)H_b}^{T(\psi_b)} \widetilde{\psi}_b \\
| & & | \\
\mathrm{C}_K(b_0)H_b & & \widetilde{\psi}_b \\
| & & | \\
K_{r+1} & & \psi_b
\end{array}$$

The characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$

Following the same strategy as in the other cases, we need to extend the character ψ_b to $T(\psi_b)$. To do this, we will follow the method used by Stasinski [19] for the general linear group $\mathrm{GL}_2(\mathcal{O})$. We first recall a theorem due to Bushnell-Fröhlich [2].

Theorem 6.4. *Let G be a finite group and N a normal subgroup such that G/N is an elementary abelian p-group (so G/N has a structure of $\mathbb{F}_p$-vector space). Let ψ be a one dimensional representation of N stabilized by G. If the alternating bilinear form h_ψ from $G/N \times G/N$ to $\mathbb{C}^\times$ given by $h_\psi(\overline{g_1}, \overline{g_2}) = \psi(g_1 g_2 g_1^{-1} g_2^{-1})$ is non-degenerate, then there exists a unique (up to isomorphism) irreducible representation ρ_ψ of G so that ρ_ψ restricted to N contains ψ.*

To apply the theorem above, we set $C_1 = \mathrm{C}_K(b_0) \cap K_1$, $G = C_1 K_r$ and define the subgroup $N = C_1 K_{r+1}$ of G. Since N/K_{r+1} is abelian there exists a character $\widetilde{\psi}_b$ of N extending ψ_b. Hence we can define an alternating form $h_{\widetilde{\psi}_b}$ from $G/N \times G/N$ to $\mathbb{C}^\times$ by

$$(\overline{g_1}, \overline{g_2}) \mapsto h_{\widetilde{\psi}_b}(\overline{g_1}, \overline{g_2}) = \widetilde{\psi}_b(g_1 g_2 g_1^{-1} g_2^{-1}).$$

We follow Stasinski [19] to verify that the group $G = C_1 K_r$, the subgroup $N = C_1 K_{r+1}$ of G, the one-dimensional representation $\widetilde{\psi}_b$ and the alternating form $h_{\widetilde{\psi}_b}$ from $G/N \times G/N$ to $\mathbb{C}^\times$ satisfy the conditions of Bushnell-Fröhlich theorem [2].

We then denote by $\rho_{\widetilde{\psi_b}}$ the unique irreducible representation of C_1K_r containing $\widetilde{\psi_b}$. Given that $T(\psi_b)/C_1K_r$ is abelian, $\rho_{\widetilde{\psi_b}}$ has an extension $\widetilde{\rho}_{\widetilde{\psi_b}}$ to $T(\psi_b)$. As in [19], we can prove:

Theorem 6.5. *The induced representation* $\mathrm{ind}_{T(\psi_b)}^{K}\widetilde{\rho}_{\widetilde{\psi_b}}$ *is primitive of level* $n+1$*, for any extension* $\widetilde{\rho}_{\widetilde{\psi_b}}$ *of* $\widetilde{\psi_b}$ *to* $T(\psi_b)$*.*

Proof. We consider an extension $\widetilde{\rho}_{\widetilde{\psi_b}}$ of $\rho_{\widetilde{\psi_b}}$ to $T(\psi_b)$. We now prove that $T(\psi_b)$ normalizes the character $\rho_{\widetilde{\psi_b}}$. To see this, let $z \in T(\psi_b)$. We observe that $(\rho_{\widetilde{\psi_b}})^z$ is another irreducible representation of C_1K_r, whose restriction to C_1K_{r+1} contains the representation $\widetilde{\psi_b}^{\,z} = \widetilde{\psi_b}$. So, by uniqueness of $\rho_{\widetilde{\psi_b}}$ (Theorem 6.4), we get that $\rho_{\widetilde{\psi_b}}$ is isomorphic to $(\rho_{\widetilde{\psi_b}})^z$. By Clifford's theorem the induced representation

$$\mathrm{ind}_{T(\psi_b)}^{K}\widetilde{\rho}_{\widetilde{\psi_b}}$$

is irreducible. This proves the theorem. ∎

Remark 6.1. Given that every irreducible constituent of $\mathrm{ind}_{K_{r+1}}^{T(\psi_b)}\psi_b$ has the form $\widetilde{\rho}_{\widetilde{\psi_b}}$ and induces an irreducible representation of K, we have constructed the primitive representation of K of level $n+1$.

Theorem 1.1 summarizes the constructions of the primitive representations from Theorems 6.1, 6.3 and 6.5 for $K = \mathrm{U}(1,1)(\mathcal{O})$. Recall that these representations only appear in level $n+1$, when n is even (Propositions 4.1 and 4.2).

7. Dimensions and numbers of primitive representations

Recall that we are setting $A_n = \mathcal{O}/\mathcal{P}^n$ for $n \in \mathbb{N}$. The involution $*$ on $\mathcal{O}$ induces naturally involutions on the rings A_n, which are also denoted by $*$. We set A_n^s for the set of symmetric elements in A_n. If m is a real number, we write $\lfloor m \rfloor$ to indicate the integer part of m. If H is a subgroup of a group G, we denote by $(G:H)$ the index of H in G.

Proposition 7.1. *The cardinality* $|K/K_n|$ *of the quotient group* K/K_n *is given by*

$$\left|K/K_n\right| = \begin{cases} (q-1)q^{3n-2}(q+1), & \textit{if } * = \mathrm{id}, \\ (q-1)q^{n+2\lfloor \frac{n+1}{2} \rfloor -2}(q+1), & \textit{if } * \neq \mathrm{id}\,. \end{cases}$$

In order to prove the proposition above, we first show:

Lemma 7.2. *Let* $(A_n, *)$ *be the involutive ring described above. Then*

(i) The cardinality of the set of symmetric elements A_n^s *of* A_n *is* q^n *if the involution is trivial, and* $q^{\lfloor \frac{n+1}{2} \rfloor}$ *otherwise.*

(ii) The cardinality of the set of all invertible symmetric elements $A_n^\times \cap (A_n)^s$ *of*

A_n is $(q-1)q^{n-1}$ if the involution is trivial, and $(q-1)q^{\lfloor\frac{n-1}{2}\rfloor}$ for the other case.

Proof. (i) Since $\mathcal{O}=\mathcal{O}_F[\varpi]$, $\mathcal{P}=\mathcal{P}_F+\mathcal{P}_F[\varpi]$ and $*$ belongs to the Galois group of the extension E/F, we get $\mathcal{O}^s=\mathcal{O}_F$. So, for any even integer i with $0\le i\le n-1$, we get that the set of symmetric elements in $\mathcal{P}^i/\mathcal{P}^{i+1}$ is $\mathcal{P}_F^i/\mathcal{P}_F^{i+1}$. On the other hand, for an odd integer i with $0\le i\le n-1$, we observe that the set of symmetric elements in $\mathcal{P}^i/\mathcal{P}^{i+1}$ is $\mathcal{P}_F^i/\mathcal{P}_F^{i+1}$ if $*=\mathrm{id}$, and $\{0\}$ otherwise. From here, the cardinality of symmetric elements in $\mathcal{O}/\mathcal{P}^n$ is q^n if the involution is trivial and $q^{\lfloor\frac{n+1}{2}\rfloor}$ for the other case.

(ii) follows from (i) using that A_n is a local ring. ■

Proof of Proposition 7.1. Since the natural map from K to the group $\mathrm{SL}_*^{-1}(2,A_n)$ is surjective and its kernel is K_n, we only need to compute the cardinality $\left|\mathrm{SL}_*^{-1}(2,A_n)\right|$ of $\mathrm{SL}_*^{-1}(2,A_n)$. To do this, we consider the group $\mathrm{SL}_*^{-1}(2,A_n)$ acting on $\mathrm{M}_{2\times1}(A_n)$ by left multiplication. In order to compute the cardinality of the orbit of $\begin{pmatrix}1\\0\end{pmatrix}$, we observe that every matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ of $\mathrm{SL}_*^{-1}(2,A_n)$ has either a or c invertible. Setting

$$O_1=\left\{\begin{pmatrix}a\\ua\end{pmatrix}\in\mathrm{M}(2,A_n):a\in A_n^\times,u\in A_n^s\right\},$$
$$O_2=\left\{\begin{pmatrix}uc\\c\end{pmatrix}\in\mathrm{M}(2,A_n):c\in A_n^\times,u\in A_n^s-A_n^\times\right\},$$

we claim that the orbit $Orb_{\mathrm{SL}_*^{-1}(2,A_n)}\begin{pmatrix}1\\0\end{pmatrix}$ is the (disjoint) union of O_1 and O_2. In fact, the first column $\begin{pmatrix}a\\c\end{pmatrix}$ of a matrix in $\mathrm{SL}_*^{-1}(2,A_n)$ satisfies $a^*c=c^*a$. So, either ca^{-1} is symmetric if a is invertible or ac^{-1} is symmetric. Thus we can write $c=ua$ or $a=uc$, where u is symmetric, which implies that $Orb_{\mathrm{SL}_*^{-1}(2,A_n)}\begin{pmatrix}1\\0\end{pmatrix}$ is contained in the union $O_1\cup O_2$. On the other hand, if $\begin{pmatrix}a\\ua\end{pmatrix}\in O_1$, where a is invertible and u is symmetric, we can observe that

$$\begin{pmatrix}a\\ua\end{pmatrix}=\begin{pmatrix}a&0\\ua&a^{*^{-1}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}a&0\\ua&a^{*^{-1}}\end{pmatrix}\in\mathrm{SL}_*^{-1}(2,A_n).$$

Then O_1 is contained in $Orb_{\mathrm{SL}_*^{-1}(2,A_n)}\begin{pmatrix}1\\0\end{pmatrix}$. Similarly, $Orb_{\mathrm{SL}_*^{-1}(2,A_n)}\begin{pmatrix}1\\0\end{pmatrix}$ contains O_2 and our claim follows. By using Lemma 7.2, we compute the cardinalities of O_1 and O_2 to get that the cardinality of $Orb_{\mathrm{SL}_*^{-1}(2,A_n)}\begin{pmatrix}1\\0\end{pmatrix}$ is $(q-1)q^{2(n-1)}(q+1)$ or $(q-1)q^{n+\lfloor\frac{n+1}{2}\rfloor-2}(q+1)$ according to whether $*$ is trivial or not. Finally, we see that the isotropy group has cardinality q^n if $*=\mathrm{id}$ and $q^{\lfloor\frac{n+1}{2}\rfloor}$ otherwise. From here, our proposition follows. ■

For the theorems below, we will denote by ρ_b the primitive representations of K of level $n+1$ independent of how they were constructed. We keep considering $n = 2r+1$ or $n = 2r$.

Theorem 7.3. *Let E be a ramified quadratic extension of a non-archimedean local field F, $\mathcal{O}$ is the ring of integers of E and $*$ is an element of the Galois group $\mathrm{Gal}(E/F)$ of E/F. We set $b_0 = \varpi^n b$. Then*

(i) If $K = \mathrm{SL}(2,\mathcal{O})$, i.e., $*$ is trivial, the dimensions of the primitive representations ρ_b of K of level $n+1$ are given as follows:

- *If the characteristic polynomial of b_0 has different roots in $\mathcal{O}/\mathcal{P}$, then*

$$\dim \rho_b = (q+1)q^n.$$

- *If the characteristic polynomial of b_0 over $\mathcal{O}/\mathcal{P}$ is x^2, then*

$$\dim \rho_b = \frac{q^2-1}{2} q^{n-1}.$$

- *If the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$, then*

$$\dim \rho_b = (q-1)q^n.$$

(ii) If $K = \mathrm{U}(1,1)(\mathcal{O})$, i.e., $*$ is nontrivial, the dimensions of the primitive representations ρ_b of K of level $n+1$ with $n = 2r$ are given as follows:

- *If the characteristic polynomial of b_0 has different roots in $\mathcal{O}/\mathcal{P}$, then*

$$\dim \rho_b = (q+1)q^r.$$

- *If the characteristic polynomial of b_0 over $\mathcal{O}/\mathcal{P}$ is x^2, then*

$$\dim \rho_b = \frac{q^2-1}{2} q^{r-1}.$$

- *If the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$, then*

$$\dim \rho_b = (q-1)q^{2\lfloor \frac{r+2}{2} \rfloor - 1}.$$

Proof. The proof will be achieved by cases according to the division made in the construction of the primitive representations.

- Firstly, suppose that $*$ is trivial and $n = 2r+1$, or $*$ is nontrivial and $n = 2r$, where r is odd. In these cases we have that

$$\rho_b = \mathrm{ind}_{T(\psi_b)}^K \eta\psi_b.$$

  Then the dimension $\dim \mathrm{ind}_{T(\psi_b)}^K \eta\psi_b$ of ρ_b is the index $(K : \mathrm{C}_K(b_0)K_{r+1})$, which will be computed by cases:

1. If the characteristic polynomial of b_0 has two different roots, we see that $\mathrm{C}_K(b_0)$ consists of matrices in K of the form:

$$\begin{pmatrix} c & 0 \\ 0 & c^{*^{-1}} \end{pmatrix}.$$

Then $(\mathrm{C}_K(b_0)K_{r+1} : K_{r+1}) = (q-1)q^r$ for both involutions. So, by using Proposition 7.1, it follows that

$$\dim \operatorname{ind}_{T(\psi_b)}^K \eta\psi_b = \frac{(K : K_{r+1})}{(\mathrm{C}_K(b_0)K_{r+1} : K_{r+1})} = \begin{cases} q^{2r+1}(q+1), & * = \mathrm{id}, \\ q^r(q+1), & * \neq \mathrm{id}\,. \end{cases}$$

2. If the characteristic polynomial of b_0 is x^2, $\mathrm{C}_K(b_0)$ is the group of matrices in K of the form:

$$\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}.$$

Hence if $* = \mathrm{id}$, we get $(\mathrm{C}_K(b_0)K_{r+1} : K_{r+1}) = 2q^{r+1}$. On the other hand, if $* \neq \mathrm{id}$, we see that $(\mathrm{C}_K(b_0)K_{r+1} : K_{r+1})$ is equal to $2q^{\lfloor \frac{r+1}{2} \rfloor}q^{\lfloor \frac{r+2}{2} \rfloor} = 2q^{r+1}$. Hence (from Proposition 7.1) we get that

$$\dim \operatorname{ind}_{T(\psi_b)}^K \eta\psi_b = \begin{cases} \frac{q^2-1}{2}q^{n-1}, & * = \mathrm{id} \\ \frac{q^2-1}{2}q^{r-1}, & * \neq \mathrm{id}\,. \end{cases}$$

3. Finally, if the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$, then $\mathrm{C}_K(b_0)$ is given by

$$\mathrm{C}_K(b_0) = \left\{ \begin{pmatrix} c & d \\ d\epsilon & c \end{pmatrix} \in \mathrm{M}(2, \mathcal{O}) : cc^* - dd^*\epsilon^* = 1 \right\},$$

where $\epsilon \in \mathcal{O}_E$ is such that $\overline{\epsilon}$ is a nonsquare in $\mathcal{O}_E/\mathcal{P}_E$. We need, as above, to compute $(\mathrm{C}_K(b_0)K_{r+1} : K_{r+1}) = (\mathrm{C}_K(b_0) : \mathrm{C}_K(b_0) \cap K_{r+1})$. To do this, let δ be a square root of ϵ in some extension and we set $L = E(\delta)$. We define the automorphism $\dagger$ of L given by $(c+d\delta)^\dagger = c^* - d^*\delta$ and an epimorphism $\widetilde{N}$ from L to E by $\widetilde{N}(c+d\delta) = (c+d\delta)(c+d\delta)^\dagger$. It follows from definition that $\widetilde{N}(\alpha\beta) = \widetilde{N}(\alpha)\widetilde{N}(\beta)$ for any α, β in L. Setting $\widetilde{\mathcal{O}} = \mathcal{O} + \mathcal{O}\delta$ and $\widetilde{\mathcal{P}} = \mathcal{P} + \mathcal{P}\delta$, we observe that $\mathrm{C}_K(b_0)$ is isomorphic to the group M_δ of elements $\alpha \in \widetilde{\mathcal{O}}$ such that $\widetilde{N}(\alpha) = 1$ via

$$\begin{pmatrix} c & d \\ d\epsilon & c \end{pmatrix} \mapsto c + d\delta.$$

Then we have $(\mathrm{C}_K(b_0) : \mathrm{C}_K(b_0) \cap K_1) = (M_\delta : M_\delta \cap (1+\mathcal{P}))$. If we put $h = \mathcal{O}/\mathcal{P}$, we note that $\widetilde{\mathcal{O}}/\widetilde{\mathcal{P}} = h(\overline{\delta})$, and hence $M_\delta/M_\delta \cap (1+\widetilde{\mathcal{P}})$ is isomorphic to the group of elements α in $\widetilde{\mathcal{O}}$ such that $\widetilde{N}(\alpha) = N_{h(\overline{\delta})|h}(\overline{\alpha}) = 1$. Then

$$(\mathrm{C}_K(b_0) : \mathrm{C}_K(b_0) \cap K_1) = (M_\delta : M_\delta \cap (1+\widetilde{\mathcal{P}})) = q+1.$$

We now compute $(\mathrm{C}_K(b_0) \cap K_i : \mathrm{C}_K(b_0) \cap K_{i+1})$ for a positive integer $i \geq 1$. Let α be an element in $M_\delta \cap (1 + \widetilde{\mathcal{P}}^i)$, then $\alpha = 1 + c + d\delta$. Since $\widetilde{N}(\alpha) = 1$ we have

$$1 = \widetilde{N}(\alpha) = 1 + (c + d\delta) + (c^* - d^*\delta) \mod \widetilde{\mathcal{P}}^{i+1}.$$

If $* = \mathrm{id}$, we get that $c = 0$ and d is arbitrary. On the other hand, if $*$ is nontrivial and i is even, $c = 0$ and d is arbitrary. Finally, if i is odd, we see that c is arbitrary and $d = 0$. Then

$$(M_\delta \cap (1 + \widetilde{\mathcal{P}}^i) : M_\delta \cap (1 + \widetilde{\mathcal{P}}^{i+1})) = q,$$

and $(\mathrm{C}_K(b_0) : \mathrm{C}_K(b_0) \cap K_{r+1}) = (q+1)q^r$. Therefore, we have that

$$\dim \operatorname{ind}_{T(\psi_b)}^K \eta\psi_b = \begin{cases} (q-1)q^{2r+1}, & * = \mathrm{id} \\ (q-1)q^r, & * \neq \mathrm{id}\,. \end{cases}$$

- Suppose that $* = \mathrm{id}$ and $n = 2r$, or $* \neq \mathrm{id}$ and $n = 2r$ with r even.

1. If the characteristic polynomial of b_0 has roots in $\mathcal{O}/\mathcal{P}$, then the primitive representations are of the form $\operatorname{ind}_{\mathrm{C}_K(b_0)K_r}^K(\omega\zeta_b)$, where ω is a linear character of $T(\psi_b)$ determined by a linear character of $T(\psi_b)/K_r$ and $\zeta_b = \operatorname{ind}_{H_b\,\mathrm{C}_K(b_0)}^{K_r\,\mathrm{C}_K(b_0)K_r}(\widetilde{\psi}_b)$. So, we get

$$\dim \operatorname{ind}_{\mathrm{C}_K(b_0)K_r}^K(\omega\zeta_b) = (K : \mathrm{C}_K(b_0)K_r)(K_r : H_b).$$

We compute that $\dim \zeta_b = (K_r : H_b) = q$ for both involutions. Since the characteristic polynomial of b_0 has different roots, we observe in both cases that $(\mathrm{C}_K(b_0)K_r : K_r) = (q-1)q^{r-1}$. Then, as above, we have that $(K : \mathrm{C}_K(b_0)K_r)$ is $(q+1)q^{n-1}$ if $* = \mathrm{id}$ and $(q+1)q^{r-1}$ otherwise. Hence

$$\dim \operatorname{ind}_{\mathrm{C}_K(b_0)K_r}^K(\omega\zeta_b) = \begin{cases} (q+1)q^n, & \text{if } * = \mathrm{id}, \\ (q+1)q^r, & \text{if } * \neq \mathrm{id}. \end{cases}$$

2. Assume now that the characteristic polynomial of b_0 is x^2. If $* = \mathrm{id}$, the index $(\mathrm{C}_K(b_0)K_r : K_r)$ is $2q^r$. On the other hand, if $* \neq \mathrm{id}$, then $(\mathrm{C}_K(b_0)K_r : K_r) = 2q^{r-1}$. So

$$(K : \mathrm{C}_K(b_0)K_r) = \frac{(K : K_r)}{(\mathrm{C}_K(b_0)K_r : K_r)} = \begin{cases} \frac{(q^2-1)}{2}q^{n-2}, & \text{if } * = \mathrm{id}, \\ \frac{(q^2-1)}{2}q^{r-2}, & \text{if } * \neq \mathrm{id}. \end{cases}$$

Therefore we see that

$$\dim \operatorname{ind}_{\mathrm{C}_K(b_0)K_r}^K(\omega\zeta_b) = \begin{cases} \frac{(q^2-1)}{2}q^{n-1}, & \text{if } * = \mathrm{id}, \\ \frac{(q^2-1)}{2}q^{r-1}, & \text{if } * \neq \mathrm{id}. \end{cases}$$

3. Finally, if the characteristic polynomial of b has no roots in $\mathcal{O}/\mathcal{P}$. Then the primitive representations of level $n+1$ are of the form $\operatorname{ind}_{T(\psi_b)}^{K} \widetilde{\rho}_{\widetilde{\psi_b}}$. So the dimension $\dim \operatorname{ind}_{T(\psi_b)}^{K} \widetilde{\rho}_{\widetilde{\psi_b}}$ is the index $(K : T(\psi_b))$, which was computed in (i).

   Thus the proof follows. ■

We now prove:

Theorem 7.4. *Let E be a ramified quadratic extension of a non-archimedean local field F, $\mathcal{O}$ is the ring of integers of E and $*$ is an element of the Galois group $\mathrm{Gal}(E/F)$ of E/F. We set $b_0 = \varpi^n b$. Then*

(i) If $K = \mathrm{SL}(2, \mathcal{O})$, i.e., $*$ is trivial, the number of inequivalent primitive representations ρ_b of K of level $n+1$ is given as follows:

- *If the characteristic polynomial of b_0 has different roots in $\mathcal{O}/\mathcal{P}$, then*

$$\textit{Number of } \rho_b = \frac{(q-1)^2}{2} q^{n-1}.$$

- *If the characteristic polynomial of b_0 over $\mathcal{O}/\mathcal{P}$ is x^2, then*

$$\textit{Number of } \rho_b = 4q^n.$$

- *If the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$, then*

$$\textit{Number of } \rho_b = \frac{q^2-1}{2} q^{n-1}.$$

(ii) If $K = \mathrm{U}(1,1)(\mathcal{O})$, i.e., $*$ is nontrivial, the number of inequivalent primitive representations ρ_b of K of level $n+1$ is given as follows:

- *If the characteristic polynomial of b_0 has different roots in $\mathcal{O}/\mathcal{P}$, then*

$$\textit{Number of } \rho_b = \frac{(q-1)^2}{2} q^{n-1}.$$

- *If the characteristic polynomial of b_0 over $\mathcal{O}/\mathcal{P}$ is x^2, then*

$$\textit{Number of } \rho_b = 4q^{r+\lfloor \frac{r}{2} \rfloor}.$$

- *If the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$, then*

$$\textit{Number of } \rho_b = \frac{q^2-1}{2} q^{r+\lfloor \frac{r}{2} \rfloor - 1}.$$

Proof. We will prove the theorem by cases according to the division made in the construction of the primitive representations.

- We suppose that $*$ is trivial and $n = 2r + 1$, or $*$ is nontrivial and $n = 2r$, where r is odd. In these cases (see Theorem 6.1), the primitive representations of K of level $n + 1$ are of the form:

$$\rho_b = \operatorname{ind}_{T(\psi_b)}^{K} \eta\psi_b.$$

Every irreducible representation of $T(\psi_b)$ containing ψ_b has the form $\eta\psi_b$, where η is a character of $\mathrm{C}_K(b_0)$ such that $\eta{=}\psi_b$ on the group $\mathrm{C}_K(b_0){\cap}K_{r+1}$. Then, we only have to compute the number of different characters ψ_b of K_{r+1} and the number of characters η of $\mathrm{C}_K(b_0)$ such that $\eta = \psi_b$ on $\mathrm{C}_K(b_0){\cap}K_{r+1}$. So:

1. Suppose that the characteristic polynomial of b_0 has different roots. The number of different characters ψ_b of K_{r+1} is the number of conjugacy classes of b in $\mathcal{G}_{-n}/\mathcal{G}_{-r}$ such that $\varpi^n b$ has characteristic polynomial with different roots. So: if $* = \mathrm{id}$, the number different characters ψ_b is $\frac{q-1}{2}q^r$. On the other hand, if $* \neq \mathrm{id}$, this number of is $\frac{q-1}{2}q^{r-1}$. Now the number of characters η of $\mathrm{C}_K(b_0)$ such that $\eta = \psi_b$ on $\mathrm{C}_K(b_0) \cap K_{r+1}$ is:

$$(\mathrm{C}_K(b_0) : \mathrm{C}_K(b_0) \cap K_{r+1}) = (\mathrm{C}_K(b_0)K_{r+1} : K_{r+1}) = (q-1)q^r$$

for both involutions. Therefore, the number of primitive representations of K of level $n+1$ of this kind is $\dfrac{(q-1)^2}{2}q^{n-1}$.

2. We assume that the characteristic polynomial of b_0 is x^2. The number of conjugacy classes of b such that $\varpi^n b$ has characteristic polynomial x^2 is $2q^r$ for $* = \mathrm{id}$, and $2q^{\lfloor \frac{r}{2} \rfloor}$ otherwise. We also compute that

$$(\mathrm{C}_K(b_0) : \mathrm{C}_K(b_0) \cap K_{r+1}) = \begin{cases} 2q^{r+1} & \text{for } * = \mathrm{id}, \\ 2q^r, & \text{if } * \neq \mathrm{id}. \end{cases}$$

Hence, the number of primitive representations of K of level $n+1$ in this case is $4q^n$ or $4q^{r+\lfloor \frac{r}{2} \rfloor}$ according to whether $*$ is trivial or not.

3. Finally, suppose that the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$. The number of conjugacy classes is $\frac{q-1}{2}q^r$ if $* = \mathrm{id}$ and $\frac{q-1}{2}q^{\lfloor \frac{r}{2} \rfloor -1}$ in the other case. On the other hand, by proof of Theorem 7.3, we get that $(\mathrm{C}_K(b_0) : \mathrm{C}_K(b_0) \cap K_{r+1}) = (q+1)q^r$ for both involutions. Then we see that

$$\text{the number of } \rho_b = \begin{cases} \frac{q^2-1}{2}q^{n-1} & \text{if } * = \mathrm{id}, \\ \frac{q^2-1}{2}q^{r+\lfloor \frac{r}{2} \rfloor -1} & \text{if } * \neq \mathrm{id}. \end{cases}$$

- We assume that $* = \mathrm{id}$ and $n = 2r$, or $* \neq \mathrm{id}$ and $n = 2r$ with r even. As above, we will compute by cases:

1. Suppose that the characteristic polynomial of b_0 has two roots in $\mathcal{O}/\mathcal{P}$. By Theorem 6.3 each primitive representation of K of level $n+1$ has

the form $\zeta = \operatorname{ind}_{T(\psi_b)}^{K} \omega\zeta_b$, where $\zeta_b = \operatorname{ind}_{\mathrm{C}_K(b_0)H_b}^{T(\psi_b)} \widetilde{\psi}_b$ and ω is a linear character of $T(\psi_b)$ determined by a character of $T(\psi_b)/K_r$, *i.e.*,

$$\begin{array}{ccc} T(\psi_b) & \qquad & \zeta_b = \operatorname{ind}_{\mathrm{C}_K(b)H_b}^{T(\psi_b)} \widetilde{\psi}_b \\ \Big| & & \Big| \\ \mathrm{C}_K(b_0)H_b & & \widetilde{\psi}_b \\ \Big| & & \Big| \\ K_{r+1} & & \psi_b \end{array}$$

As before, the number of characters ψ_b is the number of conjugacy classes of $b \in \mathcal{G}_{-n}/\mathcal{G}_{-r}$ such that the characteristic polynomial of b_0 has two roots in $\mathcal{O}/\mathcal{P}$. This number is $\frac{q-1}{2}q^{r-1}$ for both cases. Now, each character ψ_b determines q extensions ζ_b.

On the other hand, the number of linear characters ω of $T(\psi_b)$ determined by the characters of $T(\psi_b)/K_r$ is the index $(\mathrm{C}_K(b)K_r : K_r) = (q-1)q^{r-1}$ for both involutions (as in (i)). Then the number of primitive representations of level $n+1$ is $\frac{(q-1)^2}{2}q^{n-1}$ for both cases.

2. We now suppose that the characteristic polynomial of b_0 is x^2. The number of conjugacy classes of $b \in \mathcal{G}_{-n}/\mathcal{G}_{-r}$ with x^2 as characteristic polynomial is $2q^r$ for $* = \mathrm{id}$ and $2q^{\lfloor \frac{r}{2} \rfloor - 1}$ for the other case. Now we observe that each character ψ_b has q extensions ζ_b. On the other hand, the number of linear characters ω determined by characters of $T(\psi_b)$ is, as above, the index $(\mathrm{C}_K(b)K_r : K_r) = 2q^r$ for both involutions. So the number of primitive representations of level $n+1$ of this type is $4q^n$ or $4q^{r+\lfloor \frac{r}{2} \rfloor}$, according to whether $*$ is trivial or not respectively.
3. Finally, when the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$, the representations $\rho_b = \operatorname{ind}_{T(\psi_b)}^{K} \widetilde{\psi}_b$ were constructed in Theorem 6.5 considering an extension $\widetilde{\psi}_b$ of ψ_b to $\mathrm{C}_1 K_{r+1} = (\mathrm{C}_K(b_0) \cap K_1)K_{r+1}$, and

$$\begin{array}{ccc} K & \qquad & \rho_b = \operatorname{ind}_{T(\psi_b)}^{K} \widetilde{\rho}_{\widetilde{\psi}_b} \\ \Big| & & \Big| \\ T(\psi_b) & & \widetilde{\rho}_{\widetilde{\psi}_b} \\ \Big| & & \Big| \\ \mathrm{C}_1 K_r & & \rho_{\widetilde{\psi}_b} \\ \Big| & & \Big| \\ \mathrm{C}_1 K_{r+1} & & \widetilde{\psi}_b \end{array}$$

The number of characters ψ_b, as before, is the number of conjugacy classes of $b \in \mathcal{G}_{-n}/\mathcal{G}_{-r}$ such that the characteristic polynomial of b_0 has no roots in $\mathcal{O}/\mathcal{P}$. This number is $\frac{q-1}{2}q^{r-1}$ for $* = \mathrm{id}$ and $\frac{q-1}{2}q^{\lfloor \frac{r}{2}\rfloor -1}$ in the other case.

For each character ψ_b, there are $(\mathrm{C}_1\, K_{r+1} : K_{r+1}) = q^r$ extensions $\widetilde{\psi_b}$ of ψ_b for both involutions. By Bushnell-Fröhlich Theorem 6.4 and the construction above, for each character $\widetilde{\psi_b}$, there exists a unique irreducible representation $\rho_{\widetilde{\psi_b}}$ of $\mathrm{C}_1\, K_r$ containing $\widetilde{\psi_b}$ and for each one of them, the number of extensions $\widetilde{\rho}_{\widetilde{\psi_b}}$ to $T(\psi_b)$ is the index $(T(\psi_b) : \mathrm{C}_1\, K_r) = q+1$. Since one induces the irreducible representation $\widetilde{\rho}_{\widetilde{\psi_b}}$ from $T(\psi_b)$ to K, we see that

$$\text{The number of } \rho_b = \begin{cases} \frac{q^2-1}{2}q^{n-1} & \text{for } * = \mathrm{id}, \\ \frac{q^2-1}{2}q^{r+\lfloor \frac{r}{2}\rfloor -1} & \text{if } * \neq \mathrm{id}\,. \end{cases}$$

This finally completes the proof. ∎

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References

[1] Aubert, A-M., U. Onn, A. Prasad, and A. Stasinski, *On cuspidal representations of general linear groups over discrete valuation rings*, Israel J. Math. **175** (2010), 391–420.

[2] Bushnell C. J., and A. Fröhlich, *Non-abelian congruence Gauss sums and p-adic simple algebras*, Proc. London Math. Soc. **50** (1985), 207–264.

[3] Bushnell C. J., and P. Kutzko, "The admissible dual of GL(N) via compact open subgroups," Princeton University Press, 1993.

[4] Gutiérrez Frez, L., *A generalized Weil representation for* $\mathrm{SL}_*(2, A_m)$*, where* $A_m = \mathbb{F}_q[x]/\langle x^m\rangle$, J. Algebra **322** (2009), 42–53.

[5] Hill, G., *On the nilpotent representations of* $\mathrm{GL}_n(\mathcal{O})$, Manuscripta Math. **82** (1994), 293–311.

[6] —, *Regular elements and regular characters of* $\mathrm{GL}_n(\mathcal{O})$, J. Algebra **174** (1995), 610–635.

[7] —, *Semisimple and cuspidal characters of* $\mathrm{GL}_n(\mathcal{O})$, Comm. Algebra **23** (1995), 7–25.

[8] Howe, R., *Tamely ramified supercuspidal representations of* Gl_n, Pacific J. Math. **73** (1977), 437–460.

[9] Kloosterman, H., *The behaviour of general theta functions under the modular groups and the characters of binary modular congruence groups, I and II*, Ann. of Math. **47** (1946), 317–375, 376–447.

[10] Kutzko, P., *On the supercuspidal representations of* Gl_2, Amer. J. Math. **100** (1978), 43–60.

[11] Lansky, J., and A. Raghuram, *Conductors and newforms for* $\mathrm{SL}(2)$, Proc. Indian Acad. Sci. **114** (2004), 319–343.

[12] Nevins, M., *Branching rules for supercuspidal representations of* $\mathrm{SL}_2(k)$, J. Algebra **377** (2013), 204–231.

[13] Pantoja, J., *A presentation of the group* $\mathrm{SL}_*(2, A)$*, A a simple artinian ring with involution*, Manuscripta Math **121** (2006), 97–104.

[14] Pantoja, P., and J. Soto-Andrade, *A Bruhat decomposition of the group* $\mathrm{SL}_*(2, A)$, J. Algebra **262** (2003), 401-412.

[15] —, *Bruhat presentations for* $*$*-classical groups*, Comm. Algebra **37** (2009), 4170–4191.

[16] Shalika, J. A., *Representation of the two by two unimodular group over local fields*, 1–38. In: "Contributions to automorphic Forms, Geometry, and Number Theory," Johns Hopkins Univ. Press, 2004.

[17] Shintani, T., *On certain square-integrable irreducible unitary representations of some* $\mathfrak{p}$*-adic linear groups*, J. Math. Soc. Japan **20** (1968), 522–565.

[18] Singla, P., *On representations of general linear groups over principal ideal local rings of length two*, J. Algebra **324** (2010), 2543–2563.

[19] Stasinski, A., *The smooth representations of* $\mathrm{GL}_2(\mathcal{O})$, Comm. Algebra **37** (2009), 4416–4430.

[20] Tanaka, S., *Construction and classification of irreducible representations of special linear group of the second order over a finite field*, Osaka J. Math. **4** (1967), 65–84.

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