Riemannian Metrics on Infinite Dimensional Self-Adjoint Operator Groups

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Abstract.   The aim of this paper is the study of the geodesic distance in operator groups with several Riemannian metrics. More precisely we study the geodesic distance in self-adjoint operator groups with the left invariant Riemannian metric induced by the infinite trace and extend known results about the completeness of some classical Banach-Lie groups to this general class. We will focus on Banach-Lie subgroups of the group of all invertible operators which differ from the identity operator by a Hilbert-Schmidt operator.
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1.   Background and definitions

In [8] we studied Riemannian metrics on an infinite dimensional symplectic group. There we proved that if we endow the tangent spaces with the classical left invariant metric then the metric space corresponding to the geodesic distance is complete. In this paper we will extend this result to a more general class of Riemannian operator groups, the self-adjoint operator groups. Examples of these self-adjoint operator groups are the classical unitary group $U_2(\mathcal{H})$ and the symplectic group $Sp_2(\mathcal{H})$. We will follow the notations and definitions of [7] and [8], so we recall some of the facts stated there.

Let $\mathcal{B}(\mathcal{H})$ be the space of bounded operators acting on an infinite dimensional complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{B}_2(\mathcal{H})$ the Hilbert-Schmidt class

$$\mathcal{B}_2(\mathcal{H}) = \{a \in \mathcal{B}(\mathcal{H}) : Tr(a^*a) < \infty\}$$

where Tr is the usual trace in $\mathcal{B}(\mathcal{H})$. This space is a Hilbert space with the inner product

$$\langle a, b \rangle = Tr(b^*a).$$

The norm induced by this inner product is called the 2-norm and denoted by

$$\|a\|_2 = Tr(a^*a)^{1/2}.$$

The usual operator norm will be denoted by $\| \; \|$. We use the subscript h (resp. ah) to denote the sets of hermitian (resp. anti-hermitian) operators, e.g. $\mathcal{B}_2(\mathcal{H})_{ah} = \{x \in \mathcal{B}_2(\mathcal{H}) : x = -x^*\}$. In $\mathcal{B}(\mathcal{H})$ we have the classical general linear group

$$\mathrm{GL}_2(\mathcal{H}) = \{g \in \mathrm{GL}(\mathcal{H}) : g - 1 \in \mathcal{B}_2(\mathcal{H})\}.$$

This group has a differentiable structure when endowed with the metric $\|g_1 - g_2\|_2$ (note that $g_1 - g_2 \in \mathcal{B}_2(\mathcal{H})$); it is a Banach-Lie group with Banach-Lie algebra $\mathcal{B}_2(\mathcal{H})$. We recall the meaning of Banach-Lie subgroups.

Definition 1.1.   A subgroup G of a Banach-Lie group H is a *Banach-Lie subgroup* if:

1. G is a Banach-Lie group and its topology coincides with the topology it inherits from H.

2. The map $i : G \hookrightarrow H$ is an immersion with closed range.

3. There exists a closed subspace $\mathcal{F}$ such that $d_e i(\mathfrak{g}) \oplus \mathcal{F} = \mathfrak{h}$.

Because of condition 2 we always identify $\mathfrak{g}$ (the Banach-Lie algebra of G) with $Ran(d_e i)$, so that we think of $\mathfrak{g}$ as a closed subalgebra of the Banach-Lie algebra $\mathfrak{h}$. In this way $d_e i$ is just the inclusion map $\mathfrak{g} \hookrightarrow \mathfrak{h}$. Since we are working in a Hilbert-Lie group the third condition in the above definition is always satisfied.

The following statement supplies a very useful characterization of Banach-Lie subgroups. It can be found in the Chapter 4, Proposition 4.4 in the book [3].

Theorem 1.2.   *Assume that H is a Banach-Lie group with the Lie algebra $\mathfrak{h}$, G is a closed subgroup of H and denote*

$$\mathfrak{g} = \{x \in \mathfrak{h} : \exp_H(tx) \in G, \; \forall t \in \mathbb{R}\}$$

Then $\mathfrak{g}$ is a closed Lie subalgebra of $\mathfrak{h}$ and there exist on G an uniquely determined topology τ and a manifold structure making G into a Banach-Lie group such that $L(G) = \mathfrak{g}$, the inclusion map $G \hookrightarrow H$ is smooth and $d_e i : \mathfrak{g} \hookrightarrow \mathfrak{h}$ is an inclusion.

Corollary 1.3.   *In the setting of the above theorem, if we assume that there exist an open neighborhood V of $0 \in \mathfrak{h}$ and an open neighborhood U of $1 \in H$ such that $\exp_H$ induces a diffeomorphism of V onto U and $\exp_H(V \cap \mathfrak{g}) = U \cap G$. Then the topology τ coincides with the topology inherited by G from H.*

Let $\mathrm{U}_2(\mathcal{H}) \subset \mathrm{GL}_2(\mathcal{H})$ be the subgroup consisting of unitary operators. It is known that $\mathrm{U}_2(\mathcal{H})$ is a Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$ with Banach-Lie algebra $\mathcal{B}_2(\mathcal{H})_{ah}$.

The following definition is related to those of Sections 3 and 7 of Chapter IV in [6].

Definition 1.4.   Let G be a connected subgroup of $\mathrm{GL}_2(\mathcal{H})$. We say that G is a self-adjoint subgroup if $g^* \in G$ whenever $g \in G$ (for short, we write $G^* = G$). Moreover, if G is a Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$, then it is clear that G is self-adjoint if and only if $\mathfrak{g}^* = \mathfrak{g}$, where $\mathfrak{g}$ denotes the Banach-Lie algebra of G.

Throughout this paper, if M is any submanifold of $\mathrm{GL}_2(\mathcal{H})$, we will denote by $\mathfrak{b}(g, \cdot)$ the metric in each tangent space $T_g M$. The length of a smooth curve measured with the metric $\mathfrak{b}$ will be denoted by

$$L_{\mathfrak{b}}(\alpha) = \int_0^1 \mathfrak{b}(\alpha(t), \dot{\alpha}(t)) dt.$$

We define the geodesic distance between two points $p, q \in M \subseteq \mathrm{GL}_2(\mathcal{H})$ as the infimum of the length of all piecewise smooth curves in M joining p to q,

$$d_{\mathfrak{b}}(p, q) = \inf \left\{ L_{\mathfrak{b}}(\alpha) : \alpha \subset M, \alpha(0) = p, \ \alpha(1) = q \right\}.$$

We will denote with G a closed, connected self-adjoint Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$ and by $\mathfrak{g} \subset \mathcal{B}_2(\mathcal{H})$ its closed Banach-Lie algebra. Using the left action on itself, the tangent space at $g \in G$ is

$$T_g G = g.\mathfrak{g}.$$

We introduce the classical left invariant metric for $v \in T_g G$ by

$$\mathcal{I}(g, v) = \|g^{-1} v\|_2. \tag{1}$$

One of the purposes of this paper is to prove that for such G the metric space $(G, d_{\mathcal{I}})$ is complete. Using the polar decomposition we will define a mixed metric related to the unitary and positive part of the group. We also study the geodesic curves with these metrics and we will describe them in terms of exponentials of operators.

If $A \subset \mathrm{GL}_2(\mathcal{H})$ is a set, $\langle A \rangle$ will denote the abstract subgroup generated by A (the group whose elements are the inverses and the finite products of elements in A). Let $\mathrm{GL}_2^+(\mathcal{H}) := \{g > 0 : g \in \mathrm{GL}_2(\mathcal{H})\}$ be the subset of positive invertible operators. It is known that $\mathrm{GL}_2^+(\mathcal{H})$ is a submanifold of the open set $\Delta = \{\beta + X \in \mathbb{C} \oplus \mathcal{B}_2(\mathcal{H}) : \beta + X > 0\}$. Let $|x| = (x^* x)^{1/2} = \exp(\frac{1}{2} \ln(x^* x))$ be the modulus operator, then it is clear that $|x| \in \mathrm{GL}_2^+(\mathcal{H}) \subset \mathrm{GL}_2(\mathcal{H})$ if $x \in 1 + \mathcal{B}_2(\mathcal{H})$.

For $p \in \mathrm{GL}_2^+(\mathcal{H})$, we identify the tangent space $T_p \mathrm{GL}_2^+(\mathcal{H})$ with $\mathcal{B}_2(\mathcal{H})_h$ and endow this manifold with a complete Riemannian metric by means of the formula

$$\mathfrak{p}(p, x) = \|p^{-1/2} x p^{-1/2}\|_2 \tag{2}$$

for $p \in \mathrm{GL}_2^+(\mathcal{H})$ and $x \in T_p \mathrm{GL}_2^+(\mathcal{H})$. Euler's equation $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ for the covariant derivative introduced by the Riemannian connection reads $\ddot{\gamma} = \dot{\gamma} \gamma^{-1} \dot{\gamma}$, and it is not hard to see that the unique solution of this equation with $\gamma(0) = p$ and $\gamma(1) = q$ is given by the smooth curve

$$\gamma_{pq}(t) = p^{1/2}(p^{-1/2} q p^{-1/2})^t p^{1/2}.$$

The exponential map of $GL_2^+(\mathcal{H})$ is given by

$$Exp_p : T_p GL_2^+(\mathcal{H}) \to GL_2^+(\mathcal{H}), \ Exp_p(v) = p^{1/2} \exp(p^{-1/2} v p^{-1/2}) p^{1/2}.$$

In [7] G. Larotonda obtained general geometric results about $GL_2^+(\mathcal{H})$ with the above metric: Riemannian connection, geodesic, sectional curvature, convexity of geodesic distance and completeness.

2. A polar Riemannian structure on $GL_2(\mathcal{H})$

The polar decomposition of $g \in GL_2(\mathcal{H})$ induces a diffeomorphism

$$GL_2(\mathcal{H}) \xrightarrow{\varphi} U_2(\mathcal{H}) \times GL_2^+(\mathcal{H}) \tag{3}$$
$$g \longmapsto (u, |g|).$$

This fact was noted in Prop.14 (iv) on page 98 of the book [5]. If we put the left invariant metric on $U_2(\mathcal{H})$ and the positive metric (2) on $GL_2^+(\mathcal{H})$, then we can endow the product manifold $U_2(\mathcal{H}) \times GL_2^+(\mathcal{H})$ with the usual product metric, that is: if $v = (x, y) \in T_u U_2(\mathcal{H}) \times T_{|g|} GL_2^+(\mathcal{H})$ we put

$$\mathcal{P}\big((u, |g|), v\big) \ := \ \left(\mathcal{I}(u, x)^2 + \mathfrak{p}(|g|, y)^2 \right)^{1/2}$$

$$= \ \left(\|x\|_2^2 + \||g|^{-1/2} y |g|^{-1/2}\|_2^2 \right)^{1/2}. \tag{4}$$

This is the product metric in the Riemannian manifold $U_2(\mathcal{H}) \times GL_2^+(\mathcal{H})$. The map φ is in particular an immersion, from this we can define a new Riemannian metric in the group in the following way: if $v, w \in T_g GL_2(\mathcal{H})$ we put

$$\langle v, w \rangle_g := \langle d\varphi_g(v), d\varphi_g(w) \rangle_{(u, |g|)}.$$

It is clear that φ is an isometric map with the above metric and if α is any curve in the group $GL_2(\mathcal{H})$ we can measure its length as $L_{\mathcal{P}}(\varphi \circ \alpha)$.

Theorem 2.1. *Let $g \in GL_2(\mathcal{H})$ with polar decomposition $u|g|$ and suppose that $u = e^x$ with $x \in \mathcal{B}_2(\mathcal{H})_{ah}$ and $\|x\| \leq \pi$, then the curve $\alpha(t) = e^{tx} |g|^t \subset GL_2(\mathcal{H})$ has minimal length among all curves joining 1 to g, if we endow $GL_2(\mathcal{H})$ with the polar Riemannian metric (4).*

Proof. By the polar decomposition, $\varphi \circ \alpha(t) = (e^{tx}, |g|^t)$ and its length is

$$L_{\mathcal{P}}(\varphi \circ \alpha) = \int_0^1 \mathcal{P}\big((e^{tx}, |g|^t), (x e^{tx}, \ln|g| |g|^t)\big) dt = \big(\|x\|_2^2 + \| \ln|g| \|_2^2 \big)^{1/2}.$$

Let β be another curve that joins the same endpoints and suppose that $\beta = \beta_1 \beta_2$ is its polar decomposition where $\beta_1 \subset U_2(\mathcal{H})$ and $\beta_2 \subset GL_2^+(\mathcal{H})$, then

$$L_{\mathcal{P}}(\varphi \circ \beta) = \int_0^1 \mathcal{P}\big((\beta_1, \beta_2), (\dot{\beta}_1, \dot{\beta}_2)\big) dt = \int_0^1 \big(\mathcal{I}(\beta_1, \dot{\beta}_1)^2 + \mathfrak{p}(\beta_2, \dot{\beta}_2)^2 \big)^{1/2} dt.$$

Using the Minkowski inequality (see inequality 201 of [4]) we have,

$$\int_0^1 \left(\mathcal{I}(\beta_1, \dot{\beta}_1)^2 + \mathfrak{p}(\beta_2, \dot{\beta}_2)^2\right)^{1/2} dt \geq \left(\left\{\int_0^1 \mathcal{I}(\beta_1, \dot{\beta}_1)\right\}^2 + \left\{\int_0^1 \mathfrak{p}(\beta_2, \dot{\beta}_2)\right\}^2\right)^{1/2}$$

$$= \left(L_{\mathcal{I}}(\beta_1)^2 + L_{\mathfrak{p}}(\beta_2)^2\right)^{1/2}. \tag{5}$$

It is known that the geodesic curve e^{tx} has minimal length among all smooth curves in $U_2(\mathcal{H})$ joining the same endpoints (see [1]); using this fact and since the curve $|g|^t$ has minimal length with the positive metric $\mathfrak{p}$ (see [7]) we have,

$$L_{\mathcal{I}}(\beta_1) \geq L_{\mathcal{I}}(e^{tx}) = \|x\|_2 \quad \text{and} \quad L_{\mathfrak{p}}(\beta_2) \geq L_{\mathfrak{p}}(e^{t\ln(|g|)}) = \|\ln|g|\|_2$$

then it is clear that $L_{\mathcal{P}}(\varphi \circ \beta) \geq L_{\mathcal{P}}(\varphi \circ \alpha)$. ∎

Remark 2.2. Let $p, q \in GL_2(\mathcal{H})$, suppose that $u_p|p|$ and $u_q|q|$ are their polar decompositions. From the surjectivity of the exponential map we can choose $z \in \mathcal{B}_2(\mathcal{H})_{ah}$ such that $u_q = u_p e^z$ with $\|z\| \leq \pi$. Then the curve

$$\alpha_{p,q}(t) = u_p e^{tz} |p|^{1/2} (|p|^{-1/2} |q| |p|^{-1/2})^t |p|^{1/2} \subset GL_2(\mathcal{H})$$

has minimal length among all curves joining p to q.

The above fact shows that the curve $\alpha_{p,q}$ is a geodesic of the Levi-Civita connection of the polar metric. Its length is

$$\left(\|z\|_2^2 + \|\ln|p|^{-1/2}|q||p|^{-1/2}\|_2^2\right)^{1/2}.$$

From this, the geodesic distance is

$$d_{\mathcal{P}}(p, q) = \left(d_{\mathcal{I}}(u_p, u_q)^2 + d_{\mathfrak{p}}(|p|, |q|)^2\right)^{1/2}.$$

Proposition 2.3. *The metric space* $(GL_2(\mathcal{H}), d_{\mathcal{P}})$ *is complete.*

Proof. Let $(x_n) \subset GL_2(\mathcal{H})$ be a Cauchy sequence with $d_{\mathcal{P}}$, if $x_n = u_{x_n}|x_n|$ is its polar decomposition, we have that

$$d_{\mathcal{I}}(u_{x_n}, u_{x_m}) \leq d_{\mathcal{P}}(x_n, x_m) = \left(d_{\mathcal{I}}(u_{x_n}, u_{x_m})^2 + d_{\mathfrak{p}}(|x_n|, |x_m|)^2\right)^{1/2}$$

then the unitary part is a Cauchy sequence in $(U_2(\mathcal{H}), d_{\mathcal{I}})$ and by [1] it is $d_{\mathcal{I}}$ convergent to an element $u \in U_2(\mathcal{H})$. Analogously the positive part is a Cauchy sequence in $(GL_2^+(\mathcal{H}), d_{\mathfrak{p}})$ then it is convergent to an element $g \in GL_2^+(\mathcal{H})$ (see [7]). If we put $x := ug \in GL_2(\mathcal{H})$ then,

$$d_{\mathcal{P}}(x_n, x) = \left(d_{\mathcal{I}}(u_{x_n}, u)^2 + d_{\mathfrak{p}}(|x_n|, g)^2\right)^{1/2} \to 0.$$ ∎

In the next proposition we will compare the geodesic distance measured with the polar metric versus the left invariant metric.

Proposition 2.4. *Given $p, q \in GL_2(\mathcal{H})$, if we denote $v := |p|^{-1/2}|q||p|^{-1/2}$ we can estimate the geodesic distance $d_{\mathcal{I}}$ by the geodesic distance $d_{\mathcal{P}}$ as,*

$$d_{\mathcal{I}}(p, q) \leq c(p, q)d_{\mathcal{P}}(p, q)$$

where

$$c(p, q)^2 = 2 \max \left\{ e^{4\|\ln(v)\|} \left(\|p\|\|p^{-1}\| \right)^2, \|p\|\|p^{-1}\| \right\}.$$

Proof. The proof is similar to Proposition 6.4 in [8]. If we differentiate $\alpha_{p,q}$ we have,

$$\dot{\alpha}_{p,q} = u_p z e^{tz} |p|^{1/2} e^{t\ln(v)} |p|^{1/2} + u_p e^{tz} |p|^{1/2} \ln(v) e^{t\ln(v)} |p|^{1/2}$$

and the inverse of the curve $\alpha_{p,q}$ is

$$\alpha_{p,q}^{-1} = |p|^{-1/2} e^{-t\ln(v)} |p|^{-1/2} e^{-tz} u_p^{-1}.$$

After some simplifications we can write

$$\alpha_{p,q}^{-1}\dot{\alpha}_{p,q} = |p|^{-1/2} e^{-t\ln(v)} |p|^{-1/2} z |p|^{1/2} e^{t\ln(v)} |p|^{1/2} + |p|^{-1/2} \ln(v) |p|^{1/2}.$$

Let $x := |p|^{1/2} e^{t\ln(v)} |p|^{1/2}$, taking the norm and using the parallelogram rule we have,

$$
\begin{aligned}
\|\alpha_{p,q}^{-1}\dot{\alpha}_{p,q}\|_2^2 &= \|x^{-1}zx + |p|^{-1/2}\ln(v)|p|^{1/2}\|_2^2 \\
&\leq 2\left(\|x^{-1}zx\|_2^2 + \||p|^{-1/2}\ln(v)|p|^{1/2}\|_2^2 \right) \\
&\leq 2\left(\|x^{-1}\|^2 \|x\|^2 \|z\|_2^2 + \||p|^{-1/2}\|^2 \|\ln(v)\|_2^2 \||p|^{1/2}\|^2 \right). \quad (6)
\end{aligned}
$$

We can estimate $\|x\|^2$ and $\|x^{-1}\|^2$ by

$$\|x\|^2 \leq \||p|^{1/2}\|^4 \ e^{2\|\ln(v)\|} = \|p\|^2 e^{2\|\ln(v)\|}$$

and

$$\|x^{-1}\|^2 \leq \||p|^{-1/2}\|^4 \ e^{2\|\ln(v)\|} = \|p^{-1}\|^2 e^{2\|\ln(v)\|}.$$

If we define

$$c(p, q)^2 = 2 \max \left\{ e^{4\|\ln(v)\|} \left(\|p\|\|p^{-1}\| \right)^2, \|p\|\|p^{-1}\| \right\}.$$

from (6) and taking square roots we have,

$$\|\alpha_{p,q}^{-1}\dot{\alpha}_{p,q}\|_2 \leq c(p, q)\left(\|z\|_2^2 + \|\ln(v)\|_2^2 \right)^{1/2} = c(p, q)d_{\mathcal{P}}(p, q),$$

then

$$d_{\mathcal{I}}(p, q) \leq L_{\mathcal{I}}(\alpha_{p,q}) \leq c(p, q)d_{\mathcal{P}}(p, q). \qquad \blacksquare$$

3. Riemannian geometry of a self-adjoint subgroup

3.1. Riemannian geometry with the left invariant metric. It is known that the covariant derivative given by the left invariant Riemannian metric on the Banach-Lie group $GL_2(\mathcal{H})$ can be expressed as

$$\alpha^{-1}D_t\eta = \dot{\mu} + 1/2\{[\beta, \mu] + [\beta, \mu^*] + [\mu, \beta^*]\} \tag{7}$$

where $\alpha : (-\epsilon, \epsilon) \to GL_2(\mathcal{H})$ is any smooth curve, η is a tangent field along α and $\beta = \alpha^{-1}\dot{\alpha}$, $\mu = \alpha^{-1}\eta \subset \mathcal{B}_2(\mathcal{H})$ are the fields at the identity (see [8] for a proof).

Proposition 3.1. *Let G be a closed, connected self-adjoint Banach-Lie subgroup of $GL_2(\mathcal{H})$ with the left invariant metric (1), then G is totally geodesic. That is, if $\alpha \subset G$ is a curve and η is a field along α then*

$$D_t\eta \in (TG)_\alpha = \alpha.\mathfrak{g}.$$

Proof. Let $\beta = \alpha^{-1}\dot{\alpha}$ and $\mu = \alpha^{-1}\eta$ be the fields at $\mathfrak{g}$, we will show that $\alpha^{-1}D_t\eta \subset \mathfrak{g}$. Indeed, since $\mu \subset \mathfrak{g}$ then it is clear that $\dot{\mu}$ belongs to $\mathfrak{g}$, because it is a limit of operators that belong to the closed algebra $\mathfrak{g}$. Since G is self-adjoint, we have $\mathfrak{g}^* = \mathfrak{g}$, then μ^* and β^* belong to $\mathfrak{g}$ and the brackets $[\beta, \mu], [\beta, \mu^*], [\mu, \beta^*]$ are all in $\mathfrak{g}$ because it is a closed Banach-Lie algebra. Then using equation (7) we have $\alpha^{-1}D_t\eta \subset \mathfrak{g}$. ∎

This shows that the Riemannian connection given by the left invariant metric in the group G matches the one of $GL_2(\mathcal{H})$. In particular, the geodesics of G are the same than those of $GL_2(\mathcal{H})$; if $g_0 \in G$ and $g_0 v_0 \in g_0.\mathfrak{g}$ are the initial position and the initial velocity then

$$\alpha(t) = g_0 e^{tv_0^*} e^{t(v_0 - v_0^*)} \subset G$$

satisfies $D_t\dot{\alpha} = 0$ (see [2]). In this context the Riemannian exponential for $g \in G$ is

$$Exp_g(v) = g e^{v^*} e^{v - v^*}$$

with $v \in \mathfrak{g}$.

3.2. Riemannian geometry with the polar metric. The next theorem summarizes the most important properties of self-adjoint Banach-Lie groups. It was proved by G. Larotonda in [7].

Theorem 3.2. *Let $G = \langle \exp(\mathfrak{g}) \rangle$ be a connected, self-adjoint Banach-Lie group with Banach-Lie algebra $\mathfrak{g} \subset \mathcal{B}_2(\mathcal{H})$, where the inclusion is an immersion. Let P be the analytic map $g \mapsto g^*g$, $P : G \to G$. Let $\mathfrak{k} = \ker(d_1 P)$, $\mathfrak{m} = Ran(d_1 P)$. Let $M_G = \exp(\mathfrak{m})$ and $K = G \cap U_2(\mathcal{H}) = P^{-1}(1)$. Then*

1. The set $\mathfrak{m}$ is a closed Lie triple system. We have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. In particular $\mathfrak{k}$ is a Banach-Lie subalgebra of $\mathfrak{g}$.

2. $P(G) = M_G$ and M_G is a geodesically convex submanifold of $\mathrm{GL}_2^+(\mathcal{H})$.

3. For any $g = u_g|g|$ (polar decomposition), we have $|g| \in M_G$ and $u_g \in K$.

4. Let $g \in G$, $p \in M_G$, $I_g(p) = gpg^*$. Then $I_g \in I(M_G)$ (the group of isometries of M_G). If $g = p^{\frac{1}{2}}(p^{-\frac{1}{2}}qp^{-\frac{1}{2}})^{\frac{1}{2}}p^{\frac{1}{2}}$, then $I_g(p) = q$, namely G acts isometrically and transitively on M_G.

5. Let $u \in K$ and $x \in \mathfrak{m}$ (resp. $\mathfrak{m}^\perp$). Then $I_u(x) = uxu^* \in \mathfrak{m}$ (resp. $\mathfrak{m}^\perp$). If $p, q \in M_G$ then I_p maps $T_q M_G$ (resp. $T_q M_G^\perp$) isometrically onto $T_{I_p(q)} M_G$ (resp. $T_{I_p(q)} M_G^\perp$).

6. The group K is a Banach-Lie subgroup of G with Lie algebra $\mathfrak{k}$.

7. $G \simeq K \times M_G$ as Hilbert manifolds. In particular K is connected and $G/K \simeq M_G$.

Since $M_G = \exp(\mathfrak{m})$ is closed and a geodesically convex submanifold of $\mathrm{GL}_2^+(\mathcal{H})$, then for any $p = e^x \in M_G$,

$$T_p M_G = Exp_p^{-1}(M_G) = \{p^{1/2}\ln(p^{-1/2}qp^{-1/2})p^{1/2} : q \in M_G\}.$$

For this reason it is clear, as in the case of the full space $\mathrm{GL}_2^+(\mathcal{H})$, that given any $p, q \in M_G$ the curve

$$\gamma_{pq}(t) = p^{1/2}(p^{-1/2}qp^{-1/2})^t p^{1/2} \subset M_G$$

has minimal length among all curves in M_G that join p to q. Its length is measured with the metric (2) and it is $\|\ln(p^{-1/2}qp^{-1/2})\|_2$. Moreover the metric space $(M_G, d_{\mathfrak{p}})$ is complete.

The diffeomorphism given in point seven of the Theorem 3.2 is the restriction of the map (3) on G. So, we can endow G with the polar Riemannian metric (4) using the product manifold $K \times M_G$.

If we consider a curve $\alpha \subset K$ and η is a tangent field along α then the covariant derivative (7) given by the left invariant metric is reduced to

$$\alpha^{-1}D_t\eta = \dot{\mu} + 1/2[\beta, \mu] \in \mathfrak{k}.$$

Therefore, the geodesics of the Banach-Lie subgroup K with the left invariant metric are one-parameter groups.

Using the above facts, it is not difficult to see that given any initial velocity $v \in \mathfrak{g}$, if $v = x + y$ is the decomposition into $\mathfrak{k} \oplus \mathfrak{m}$, then the geodesics of the polar metric starting at the identity are

$$\alpha(t) = e^{tx}e^{ty}.$$

Proposition 3.3. *The geodesics of the left invariant metric coincide with the geodesics of the polar metric if the initial velocity $v \in \mathfrak{g}$ is normal.*

Proof. Let $v = x + y \in \mathfrak{k} \oplus \mathfrak{m}$ be the decomposition into its hermitian and anti-hermitian part, since v is normal a straightforward computation shows that x commutes with y, thus we have

$$e^{tv^*}e^{t(v-v^*)} = e^{tv} = e^{tx}e^{ty}.$$

This equation shows that the geodesics are one-parameter groups. ∎

4. Completeness of the geodesic distance

4.1. Completeness in finite dimension with p-norms. Let $\mathrm{GL}_n(\mathbb{C})$ be the general linear group in finite dimension. Let $\mathcal{M}_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. Since $\mathrm{GL}_n(\mathbb{C})$ is open in the space $\mathcal{M}_n(\mathbb{C})$, we can identify the tangent space of $\mathrm{GL}_n(\mathbb{C})$ at any point with $\mathcal{M}_n(\mathbb{C})$. In this algebra we consider the classical *p-norms*, if $x \in \mathcal{M}_n(\mathbb{C})$ and τ denote the trace we put,

$$\|x\|_p^p = \tau\big((x^*x)^{p/2}\big) \text{ for any } p \geq 1.$$

Since the dimension of $\mathrm{GL}_n(\mathbb{C})$ is finite, it is known that any closed subgroup G has a structure of Banach-Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$. In this context we denote the left invariant metric for any self-adjoint closed subgroup as $\mathcal{I}_p(g,v) = \|g^{-1}v\|_p$ for $g \in G$ and $v \in T_gG$.

Theorem 4.1. *Let G be a self-adjoint closed subgroup of $\mathrm{GL}_n(\mathbb{C})$, then the metric space $(G, d_{\mathcal{I}_p})$ is complete.*

Proof. If $p = 2$, by Hopf-Rinow's theorem, the space $(G, d_{\mathcal{I}_2})$ is complete since the manifold G is geodesically complete with the 2-norm (Proposition 3.1). Now, we claim that $d_{\mathcal{I}_p}$ is equivalent to $d_{\mathcal{I}_2}$ for any $p \geq 2$. Indeed, at each tangent space of G, the p-norm is equivalent with the 2-norm with constants which depend only on the dimension of $\mathcal{M}_n(\mathbb{C})$. Examining the length functionals, it follows that the metrics are equivalent, with the same constants. ∎

4.2. Completeness in the infinite dimensional case.

Since the map $g \mapsto g|g|^{-1} = u_g$ is continuous, it is clear that if $p, q \in G$ are close to each other its unitary parts (u_p, u_q) are close too. So, if $g \in G$ is close to the identity 1, then its unitary part u_g is close to 1 too and since the polar decomposition is in the group (Theorem 3.2) we can assume that u_g lies in $K \cap U$ where U is a neighbourhood of the identity in G. Since K is a Banach-Lie subgroup of G, if we reduce the neighbourhood U, we can assume that $u_g = \exp(z)$ where z belongs to a neighbourhood of 0 in the Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. Now, if $p, q \in G$ are close to each other and $u_p|p|, u_q|q|$ are their polar decompositions, then the unitary element $u_p^{-1}u_q \in K$ is close to 1 and we can choose an element $z \in \mathfrak{k} \subset \mathfrak{g}$ such that $u_p^{-1}u_q = e^z$. So, we can build the following smooth curve in G;

$$\alpha_{p,q}(t) = u_p e^{tz}|p|^{1/2}(|p|^{-1/2}|q||p|^{-1/2})^t|p|^{1/2} \subset G \tag{8}$$

that joins p to q. This curve will be the key of the completeness with both metrics.

Theorem 4.2.   *If G is a closed, connected self-adjoint Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$, then the metric space $(G, d_\mathcal{P})$ is complete.*

Proof.   Let (x_n) be a $d_\mathcal{P}$-Cauchy sequence, let $x_n = u_{x_n}|x_n|$ be its polar decomposition. First we will prove that $u_{x_n} \subset K$ and $|x_n| \subset M_G$ are $d_\mathcal{I}$ and $d_\mathfrak{p}$-Cauchy sequences respectively. Indeed, given $\varepsilon = 1/n$ there exist curves $\beta_n \subset G$ such that $\beta_n(0) = x_n$, $\beta_n(1) = x_m$ and $d_\mathcal{P}(x_n, x_m) + 1/n > L_\mathcal{P}(\beta_n)$. If $\beta_{1n} \subset K$ and $\beta_{2n} \subset M_G$ denote the unitary and positive part of β_n, then since $x_n = \beta_n(0) = \beta_{1n}(0)\beta_{2n}(0) = u_{x_n}|x_n|$, it is clear that β_{1n} joins u_{x_n} to u_{x_m} and β_{2n} joins $|x_n|$ to $|x_m|$. Using the inequality (5) we have,

$$d_\mathcal{P}(x_n, x_m) + 1/n > L_\mathcal{P}(\beta_n) \geq L_\mathcal{I}(\beta_{1n}) \geq d_\mathcal{I}(u_{x_n}, u_{x_m})$$

for all n, m, then it is clear that $d_\mathcal{I}(u_{x_n}, u_{x_m}) \to 0$ when $n, m \to \infty$. An analogous computation using β_{2n} shows that $d_\mathfrak{p}(|x_n|, |x_m|) \to 0$ when $n, m \to \infty$. Since (u_{x_n}) is an unitary sequence it is known that $\|u_{x_n} - u_{x_m}\|_2 \leq d_\mathcal{I}(u_{x_n}, u_{x_m})$ and therefore $(u_{x_n}) \subset K$ is a 2-norm Cauchy sequence. Since K is closed we can take $u \in K$ such that $u_{x_n} \xrightarrow{\|\cdot\|_2} u$. Then if n is large, we can suppose that $u^{-1}u_{x_n}$ is close to 1, therefore there exists a sequence $(z_n) \subseteq \mathfrak{k}$ such that $u^{-1}u_{x_n} = e^{z_n}$ and $z_n \xrightarrow{\|\cdot\|_2} 0$. On the other hand, there exists $g \in M_G$ such that $d_\mathfrak{p}(|x_n|, g) = \|\ln(g^{-1/2}|x_n|g^{-1/2})\|_2 \to 0$. It is clear that $ug \in G$, then if n is large we can consider the curve $\alpha_{ug,x_n}(t) = ue^{tz_n}g^{1/2}(g^{-1/2}|x_n|g^{-1/2})^t g^{1/2}$ (8) that joins x_n and ug, therefore we have

$$d_\mathcal{P}(x_n, ug) \leq L_\mathcal{P}(\alpha_{ug,x_n}) = \left(\|z_n\|_2^2 + \|\ln(g^{-1/2}|x_n|g^{-1/2})\|_2^2\right)^{1/2} \to 0. \qquad \blacksquare$$

The following proposition is a generalization of Proposition 2.4 on G.

Proposition 4.3.   *Suppose $p, q \in G$ are close to each other and let $v := |p|^{-1/2}|q||p|^{-1/2}$ then we can estimate the geodesic distance $d_\mathcal{I}$ by*

$$d_\mathcal{I}(p, q) \leq c(p, q)\left(\|z\|_2^2 + \|\ln(v)\|_2^2\right)^{1/2}$$

where

$$c(p, q)^2 = 2\max\left\{ e^{4\|\ln(v)\|}\left(\|p\|\|p^{-1}\|\right)^2, \|p\|\|p^{-1}\|\right\}.$$

Proof.   The proof is similar to Proposition 2.4. Since $p, q \in G$ are close to each other, then we can build the smooth curve $\alpha_{p,q} \subset G$ (8) that joins p to q; so we can repeat the argument that we gave in Proposition 2.4. Then in this case we have,

$$\|\alpha_{p,q}^{-1}\dot{\alpha}_{p,q}\|_2 \leq c(p, q)\left(\|z\|_2^2 + \|\ln(v)\|_2^2\right)^{1/2}.$$

and then $d_\mathcal{I}(p, q) \leq L_\mathcal{I}(\alpha_{p,q}) \leq c(p, q)\left(\|z\|_2^2 + \|\ln(v)\|_2^2\right)^{1/2}$. $\qquad \blacksquare$

Lemma 4.4.   *If $(x_n) \subset G$ is a Cauchy sequence in $(G, d_\mathcal{I})$ then it is a Cauchy sequence in $(G, \|.\|_2)$.*

Proof.   Since the geodesics of the Riemannian connection are of the form $\alpha(t) = Exp_g(tv) = ge^{tv^*}e^{t(v-v^*)}$, the proof of this lemma can be adapted easily from Lemma 7.1 in [8]. ∎

Now we are in position to obtain our final result.

Theorem 4.5.   *If G is a closed, connected self-adjoint Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$, then the metric space $(G, d_{\mathcal{I}})$ is complete.*

Proof.   Let $(x_n) \subset G$ be a $d_{\mathcal{I}}$-Cauchy sequence, by the above lemma it is $\|.\|_2$-Cauchy; then since G is closed there exists $x \in G$ such that $x_n \xrightarrow{\|.\|_2} x$. Now we will show that $x_n \xrightarrow{d_{\mathcal{I}}} x$; indeed from the continuity of the module we have that $|x_n|$ converges to $|x|$ in $\|.\|_2$ and its unitary part $u_{x_n} = x_n|x_n|^{-1}$ converges to $u_x = x|x|^{-1}$. The sequence $|x|^{-1/2}|x_n||x|^{-1/2}$ converges to 1 and then $\|\ln(|x|^{-1/2}|x_n||x|^{-1/2})\|_2 \to 0$. Since x_n converges to x, we can assume that x_n is close to x if $n \geq n_0$, therefore we can use Proposition 4.3 to estimate the geodesic distance. Then we have

$$d_{\mathcal{I}}(x, x_n) \leq c(x, x_n)\big(\|z_n\|_2^2 + \|\ln(|x|^{-1/2}|x_n||x|^{-1/2})\|_2^2\big)^{1/2}$$

where $z_n \in \mathfrak{k} \subset \mathfrak{g}$ is such that $u_x^{-1}u_{x_n} = e^{z_n}$ (since $u_x^{-1}u_{x_n}$ is close to 1 and K is a Banach-Lie subgroup of G). We also have $\|z_n\|_2 \to 0$. Now we will see that $c(x, x_n)$ is uniformly bounded. Indeed, since $\|\ln(|x|^{-1/2}|x_n||x|^{-1/2})\| \leq \|\ln(|x|^{-1/2}|x_n||x|^{-1/2})\|_2 \to 0$, then for n large we can assume that $\|\ln(v_n)\| \leq 1$ where $v_n = |x|^{-1/2}|x_n||x|^{-1/2}$ as we denoted in Proposition 4.3. Finally we have

$$c(x, x_n)^2 = 2\max\{\ e^{4\|\ln(v_n)\|}\big(\|x\|\|x^{-1}\|\big)^2,\ \|x\|\|x^{-1}\|\}$$

$$\leq 2\max\{\ e^4\big(\|x\|\|x^{-1}\|\big)^2,\ \|x\|\|x^{-1}\|\}$$

is clearly uniformly bounded and then it is clear that $d_{\mathcal{I}}(x, x_n) \to 0$. ∎

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