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Root Graded Lie Superalgebras

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Abstract. We define root graded Lie superalgebras and study their connection with centerless cores of extended affine Lie superalgebras; our definition generalizes the known notions of root graded Lie superalgebras.
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0. Introduction

Motivated by a construction appearing in the classification of finite dimensional simple Lie algebras containing nonzero toral subalgebras [23], S. Berman and R. Moody [12] introduced the notion of a Lie algebra graded by an irreducible reduced finite root system. This notion was generalized to Lie algebras graded by a locally finite root system and well studied through a variety of papers; recognition theorems for root graded Lie algebras are found in [12], [7], [22], [4], [8], [25] and their central extensions have been studied in [3], [4] and [26]. Roughly speaking, a root graded Lie algebra is a Lie algebra which is graded by the root lattice of an irreducible locally finite root system R and contains a locally finite split simple Lie algebra whose root system is a full subsystem of R. One of the important phenomena in the study of root graded Lie algebras is their interaction with other classes of Lie algebras such as invariant affine reflection algebras [21](see also [1], [6] and [19]); more precisely, the main ingredient in constructing an invariant affine reflection algebra is a root graded Lie algebra [21, §6].

There have been two different approaches to define root graded Lie superalgebras. One is working with Lie superalgebras which are graded by the root lattice of a locally finite root system and satisfy modified properties of a root graded Lie algebra [22]; the other one is working with a Lie superalgebra $\mathcal{L}$ containing a basic classical Lie superalgebra with a Cartan subalgebra $\mathcal{H}$ with respect to which $\mathcal{L}$ has a weight space decomposition satisfying certain properties [9]. In fact in the latter case, $\mathcal{L}$ is graded by the root lattice of the root system of a basic classical Lie super-

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algebra. Root systems of basic classical Lie superalgebras are exactly generalized root systems introduced by V. Serganova in 1996 [24]. Generalized root systems are called finite root supersystems in [27] where the author introduces locally finite root supersystems and gives their classification. Locally finite root supersystems which are extended by abelian groups appear as the root systems of specific Lie superalgebras named extended affine Lie superalgebras [28]. The so called core of an extended affine Lie superalgebra is a Lie superalgebra satisfying certain properties which are in fact a super version of the features defining a root graded Lie algebra. This motivates us to define root graded Lie superalgebras in a general setting. Our definition is a generalization of both mentioned notions of root graded Lie superalgebras. In a series of papers, G. Benkart and A. Elduque studied Lie superalgebras graded by finite root supersystems $C(n), D(m,n), D(2,1;\alpha), F(4), G(3),\ A(m,n)$ and $B(m,n)$; see [9], [10] and [11]. We give a recognition theorem for Lie superalgebras graded by the locally finite root supersystem of type $BC(I,J)$.

This paper has been organized as follows. In the first section, we gather some preliminaries. In Section 2, we recall extended affine Lie superalgebras and their root systems. Root graded Lie superalgebras are defined in Section 3. We then realize extended affine Lie superalgebras using root graded Lie superalgebra; this can be considered as a first step of constructing extended affine Lie superalgebras. We begin Section 4 with a subsection devoted to some information regarding the locally finite Lie superalgebra $\mathfrak{osp}_{\mathbb{F}}(I,J)$. This will be followed by a subsection devoted to the study of finite dimensional $\mathfrak{osp}_{\mathbb{F}}(m,n)$-modules; the material of this subsection is used to prove our recognition theorem for $BC(I,J)$-graded Lie superalgebras. In the last subsection of Section 4, we study $BC(I,J)$-graded Lie superalgebras.

1. Preliminaries

Throughout this work, we take $\mathbb{Z}_2=\{0,1\}$ to be the unique abelian group of order 2 and $\mathbb{F}$ is a field of characteristic zero. Unless otherwise mentioned, all vector spaces are considered over $\mathbb{F}$. We denote the dual space of a vector space V by V^*; and for a nonempty subset X of V, by $\mathrm{span}_{\mathbb{F}}X$, we mean the subspace of V spanned by X. We denote the degree of a homogenous element v of a superspace by $|v|$ and make a convention that if in an expression, we use $|v|$ for an element v of a superspace, by default, we have assumed v is homogeneous. For a superspace V, by $\mathrm{End}_{\mathbb{F}}(V)$, we mean the superspace of linear endomorphisms of V. If A is an abelian group, we denote the group of automorphisms of A by $Aut(A)$ and for a subset X of A, by $\langle X\rangle$, we mean the subgroup of A generated by X. Also we denote the cardinality of a set S by $|S|$; and for two symbols i,j, by $\delta_{i,j}$, we mean the Kronecker delta. We use $\uplus$ to indicate the disjoint union and $\simeq$ to indicate isomorphism. For a map $f:A\longrightarrow B$ and $C\subseteq A$, by $f\mid_C$, we mean the restriction of f to C. Finally, we denote the center of a Lie superalgebra $\mathcal{G}$ by $Z(\mathcal{G})$.

Definition 1.1. Suppose that $\mathcal{G}$ is a Lie superalgebra and Λ and Γ are two additive abelian group.

(i) The Lie superalgebra $\mathcal{G}=\mathcal{G}_0\oplus\mathcal{G}_1$ is called a Λ-*graded* Lie superalgebra if there is a family $\{\mathcal{G}^\lambda \mid \lambda\in\Lambda\}$ of subsuperspaces of $\mathcal{G}$ such that $\mathcal{G}=\bigoplus_{\lambda\in\Lambda}\mathcal{G}^\lambda$ with $[\mathcal{G}^\lambda,\mathcal{G}^\mu]\subseteq\mathcal{G}^{\lambda+\mu}$ for $\lambda,\mu\in\Lambda$. In this case, for $i\in\mathbb{Z}_2$, we have $\mathcal{G}_i=\bigoplus_{\lambda\in\Lambda}\mathcal{G}_i^\lambda$ in which $\mathcal{G}_i^\lambda:=\mathcal{G}_i\cap\mathcal{G}^\lambda$ $(\lambda\in\Lambda)$. The subset $\mathrm{supp}_\Lambda\mathcal{G}:=\{\lambda\in\Lambda \mid \mathcal{G}^\lambda\neq\{0\}\}$ of Λ is called the *support* of $\mathcal{G}$ with respect to the Λ-grading.

(ii) We say a Λ-grading $\{\mathcal{G}^\lambda \mid \lambda\in\Lambda\}$ and a Γ-grading $\{{}^\gamma\mathcal{G} \mid \gamma\in\Gamma\}$ on $\mathcal{G}$ are *compatible* if for each $\gamma\in\Gamma$, ${}^\gamma\mathcal{G}=\oplus_{\lambda\in\Lambda}{}^\gamma\mathcal{G}^\lambda$ in which for $\lambda\in\Lambda$, ${}^\gamma\mathcal{G}^\lambda:={}^\gamma\mathcal{G}\cap\mathcal{G}^\lambda$.

Lemma 1.2. *Suppose that Λ is an additive abelian group and $\mathcal{G}=\oplus_{\lambda\in\Lambda}\mathcal{G}^\lambda$ a Λ-graded Lie superalgebra. Then $Z(\mathcal{G})$ is a Λ-graded subsuperalgebra.*

Proof. Suppose that $x\in Z(\mathcal{G})$, then $x=\sum_{\lambda\in\Lambda}x^\lambda$ with $x^\lambda\in\mathcal{G}^\lambda$ for $\lambda\in\Lambda$. Now for each $\mu\in\Lambda$ and $y\in\mathcal{G}^\mu$, $0=[x,y]=\sum_{\lambda\in\Lambda}[x^\lambda,y]$. This implies that for each $\lambda\in\Lambda$, $[x^\lambda,y]=0$. But μ and y are arbitrary, so we get that $x^\lambda\in Z(\mathcal{G})$ for each $\lambda\in\Lambda$. Next suppose $x\in\mathcal{G}^\lambda\cap Z(\mathcal{G})$ for some $\lambda\in\Lambda$. Then $x=x_0+x_1$ with $x_0\in\mathcal{G}_0$ and $x_1\in\mathcal{G}_1$. So as before, we get that $[x_1,y]=[x_2,y]=0$ for each $y\in\mathcal{G}_i$ $(i\in\mathbb{Z}_2)$. This in turn implies that x_0 and $x_1\in Z(\mathcal{G})$ and so we are done. ∎

Definition 1.3. Suppose that $\mathcal{G}$ is a Lie superalgebra. We say a superspace M together with a bilinear map $\cdot:\mathcal{G}\times M\longrightarrow M$ is a $\mathcal{G}$-module if

$$\begin{array}{l}\mathcal{G}_i\cdot M_j\subseteq M_{i+j}\\ \;[x,y]\cdot a=x\cdot(y\cdot a)-(-1)^{|x||y|}y\cdot(x\cdot a)\end{array}$$

for $x,y\in\mathcal{G}$, $a\in M$ and $i,j\in\{0,1\}$. Submodules are defined in a usual manner. Following [13, Def. 1.1], by a $\mathcal{G}$-module homomorphism from a $\mathcal{G}$-module V to a $\mathcal{G}$-module W, we mean a linear map $\varphi:V\longrightarrow W$ satisfying

$$\varphi(xv)=x\varphi(v);\quad x\in\mathcal{G},\ v\in V.$$

Monomorphisms, epimorphisms and isomorphisms are defined in the usual sense.

The following lemma is easily verified:

Lemma 1.4. *Suppose that $\mathcal{G}$ is a Lie superalgebra and V is a $\mathcal{G}$-module. Then we have the following:*

(i) The superspace $U:=U_0\oplus U_1$ where $U_0:=V_1$ and $U_1:=V_0$ is a $\mathcal{G}$-module isomorphic to V.

(ii) If V is irreducible and as $\mathcal{G}_0$-modules, $V_0\not\simeq V_1$, then each isomorphism from V to a $\mathcal{G}$-module W is homogeneous as a linear map.

2. Extended affine Lie superalgebras and their root systems

In this section, we recall the notions of extended affine Lie superalgebras and extended affine root supersystems from [28]. We prove Lemma 2.12 which is essential for the study of root graded Lie superalgebras. In the sequel, by a *symmetric form* on an additive abelian group A, we mean a map $(\cdot,\cdot) : A\times A \longrightarrow \mathbb{F}$ satisfying

- $(a,b) = (b,a)$ for all $a,b \in A$,
- $(a+b,c) = (a,c) + (b,c)$ and $(a,b+c) = (a,b) + (a,c)$ for all $a,b,c \in A$.

In this case, we set $A^0 := \{a \in A \mid (a,A) = \{0\}\}$ and call it the *radical* of the form $(\cdot,\cdot)$. The form is called *nondegenerate* if $A^0 = \{0\}$. We note that if the form is nondegenerate, A is torsion free and we can identify A as a subset of $\mathbb{Q}\otimes_{\mathbb{Z}} A$. In the following, if an abelian group A is equipped with a nondegenerate symmetric form, we consider A as a subset of $\mathbb{Q}\otimes_{\mathbb{Z}} A$ without further explanation. Also if A is a vector space over $\mathbb{F}$, bilinear forms are used in the usual sense.

We recall that by a *supersymmetric* bilinear form on a superspace V, we mean a bilinear map $(\cdot,\cdot) : V\times V \longrightarrow \mathbb{F}$ satisfying $(v,w) = (-1)^{|v||w|}(w,v)$ for $v,w \in V$. A bilinear form $(\cdot,\cdot)$ on a superspace V is called *even* if $(V_i,V_j) = \{0\}$ for $i,j \in \mathbb{Z}_2$ with $i+j = 1$. We also recall that a bilinear form on a Lie superalgebra $(\mathcal{L},[\cdot,\cdot])$ is called *invariant* if $([x,y],z) = (x,[y,z])$ for $x,y,z \in \mathcal{L}$.

We call a triple $(\mathcal{L},\mathcal{H},(\cdot,\cdot))$ a *super-toral triple* if

- $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ is a nonzero Lie superalgebra, $\mathcal{H}$ is a nontrivial subalgebra of $\mathcal{L}_0$ and $(\cdot,\cdot)$ is an invariant nondegenerate even supersymmetric bilinear form $(\cdot,\cdot)$ on $\mathcal{L}$,
- $\mathcal{L}$ has a weight space decomposition $\mathcal{L} = \oplus_{\alpha\in\mathcal{H}^*}\mathcal{L}^\alpha$ with respect to $\mathcal{H}$ via the adjoint representation; we note that as $\mathcal{L}_0$ as well as $\mathcal{L}_1$ are $\mathcal{H}$-submodules of $\mathcal{L}$, we have $\mathcal{L}_0 = \oplus_{\alpha\in\mathcal{H}^*}(\mathcal{L}_0)^\alpha$ and $\mathcal{L}_1 = \oplus_{\alpha\in\mathcal{H}^*}(\mathcal{L}_1)^\alpha$ with $(\mathcal{L}_i)^\alpha := \mathcal{L}_i \cap \mathcal{L}^\alpha$, $i = 0,1$,
- the restriction of the form $(\cdot,\cdot)$ to $\mathcal{H}\times\mathcal{H}$ is nondegenerate.

We call $R := \{\alpha \in \mathcal{H}^* \mid \mathcal{L}^\alpha \neq \{0\}\}$, the *root system* of $\mathcal{L}$ (with respect to $\mathcal{H}$). Each element of R is called a *root*. We refer to elements of $R_0 := \{\alpha \in \mathcal{H}^* \mid (\mathcal{L}_0)^\alpha \neq \{0\}\}$ (resp. $R_1 := \{\alpha \in \mathcal{H}^* \mid (\mathcal{L}_1)^\alpha \neq \{0\}\}$) as *even roots* (resp. *odd roots*). We note that $R = R_0 \cup R_1$. Suppose that $(\mathcal{L},\mathcal{H},(\cdot,\cdot))$ is a super-toral triple with corresponding root system R and take $\mathfrak{p} : \mathcal{H} \longrightarrow \mathcal{H}^*$ to be the function mapping $h \in \mathcal{H}$ to $(h,\cdot)$. Since the form is nondegenerate on $\mathcal{H}$, the map $\mathfrak{p}$ is one to one. So for each element α of the image $\mathcal{H}^{\mathfrak{p}}$ of $\mathcal{H}$ under the map $\mathfrak{p}$, there is a unique $t_\alpha \in \mathcal{H}$ representing α through the form $(\cdot,\cdot)$. Now we can transfer the form on $\mathcal{H}$ to a form on $\mathcal{H}^{\mathfrak{p}}$, denoted again by $(\cdot,\cdot)$, and defined by

$$(\alpha,\beta) := (t_\alpha,t_\beta) \quad (\alpha,\beta \in \mathcal{H}^{\mathfrak{p}}). \tag{1}$$

It is proved that if for $\alpha \in R_i \setminus \{0\}$ $(i \in \{0,1\})$, there are $x_\alpha \in (\mathcal{L}_i)^\alpha$ and $x_{-\alpha} \in (\mathcal{L}_i)^{-\alpha}$ such that $0 \neq [x_\alpha,x_{-\alpha}] \in \mathcal{H}$, then α is an element of $\mathcal{H}^{\mathfrak{p}}$ [28, Lem. 3.1].

Definition 2.1. A super-toral triple $(\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1,\mathcal{H},(\cdot,\cdot))$ (or $\mathcal{L}$ if there is no confusion), with root system $R = R_0 \cup R_1$, is called an *extended affine Lie superalgebra* if

- (1) for each $\alpha \in R_i \setminus \{0\}$ $(i \in \{0,1\})$, there are $x_\alpha \in (\mathcal{L}_i)^\alpha$ and $x_{-\alpha} \in (\mathcal{L}_i)^{-\alpha}$ such that $0 \neq [x_\alpha, x_{-\alpha}] \in \mathcal{H}$,

- (2) for each $\alpha \in R$ with $(\alpha,\alpha) \neq 0$ and $x \in \mathcal{L}^\alpha$, $adx : \mathcal{L} \longrightarrow \mathcal{L}$ is a locally nilpotent linear transformation.

An extended affine Lie superalgebra $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is called an *invariant affine reflection algebra* [21] if $\mathcal{L}_1 = \{0\}$ and it is called a *locally extended affine Lie algebra* [19] if $\mathcal{L}_1 = \{0\}$ and $\mathcal{L}^0 = \mathcal{H}$. Finally a locally extended affine Lie algebra $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is called an *extended affine Lie algebra* [1] if $\mathcal{L}^0 = \mathcal{H}$ is a finite dimensional subalgebra of $\mathcal{L}$.

Suppose that $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is an extended affine Lie superalgebra with root system R. It is proved that for $\alpha \in R_i$ $(i = 0, 1)$ with $(\alpha,\alpha) \neq 0$, there are $e_\alpha \in (\mathcal{L}_i)^\alpha$, $f_\alpha \in (\mathcal{L}_i)^{-\alpha}$ such that $(e_\alpha, f_\alpha, h_\alpha := \frac{2t_\alpha}{(\alpha,\alpha)})$ is an $\mathfrak{sl}_2$-super-triple in the sense that

$$[e_\alpha, f_\alpha] = h_\alpha, [h_\alpha, e_\alpha] = 2e_\alpha, [h_\alpha, f_\alpha] = -2f_\alpha.$$

Moreover, the subsuperalgebra $\mathcal{G}(\alpha)$ of $\mathcal{G}$ generated by $\{e_\alpha, f_\alpha, h_\alpha\}$ is either isomorphic to $\mathfrak{sl}_2$ or to $\mathfrak{osp}_{\mathbb{F}}(1,2)$ depending on $i = 0$ or $i = 1$; see [28, Lem.'s 2.2 & 3.6].

Definition 2.2. Suppose that $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is an extended affine Lie superalgebra with root system R. The subsuperalgebra $\mathcal{L}_c$ of $\mathcal{L}$ generated by $\mathcal{L}^\alpha$ for $\alpha \in \{\beta \in R \mid (\beta, R) \neq \{0\}\}$ is called the *core* of $\mathcal{L}$. The quotient Lie superalgebra $\mathcal{L}_c/Z(\mathcal{L}_c)$ is called the *centerless core* of $\mathcal{L}$.

Example 2.3. A finite dimensional basic classical simple Lie superalgebra $\mathcal{L}$ is an extended affine Lie superalgebra with $\mathcal{L} = \mathcal{L}_c$.

By [28, Cor. 3.9], the root system of an extended affine Lie superalgebra is an extended affine root supersystem in the following sense.

Definition 2.4. Suppose that A is a nontrivial additive abelian group, R is a subset of A and $(\cdot,\cdot) : A \times A \longrightarrow \mathbb{F}$ is a symmetric form. Set

$$\begin{aligned}
&R^0 := R \cap A^0,\\
&R^\times := R \setminus R^0,\\
&R_{re}^\times := \{\alpha \in R \mid (\alpha,\alpha) \neq 0\}, \quad R_{re} := R_{re}^\times \cup \{0\},\\
&R_{ns}^\times := \{\alpha \in R \setminus R^0 \mid (\alpha,\alpha) = 0\}, \quad R_{ns} := R_{ns}^\times \cup \{0\}.
\end{aligned}$$

We say $(A,(\cdot,\cdot),R)$ is an *extended affine root supersystem* if the following hold:

$(S1)$ $0\in R$, and $\langle R\rangle = A$,

$(S2)$ $R=-R$,

$(S3)$ for $\alpha\in R_{re}^{\times}$ and $\beta\in R$, $2(\alpha,\beta)/(\alpha,\alpha)\in\mathbb{Z}$,

$(S4)$ (*root string property*) for $\alpha\in R_{re}^{\times}$ and $\beta\in R$, there are nonnegative integers p,q with $2(\beta,\alpha)/(\alpha,\alpha)=p-q$ such that
$$\{\beta+k\alpha\mid k\in\mathbb{Z}\}\cap R=\{\beta-p\alpha,\dots,\beta+q\alpha\},$$

$(S5)$ for $\alpha\in R_{ns}$ and $\beta\in R$ with $(\alpha,\beta)\neq 0$, $\{\beta-\alpha,\beta+\alpha\}\cap R\neq\emptyset$.

If there is no confusion, for the sake of simplicity, we say R *is an extended affine root supersystem in* A. Each element of R is called a *root*. Elements of R_{re} (resp. R_{ns}) are called *real* (resp. *nonsingular*) roots. An extended affine root supersystem R is called *irreducible* if $R^{\times}$ cannot be written as a disjoint union of two nonempty orthogonal subsets. An extended affine root supersystem $(A,(\cdot,\cdot),R)$ is called a *locally finite root supersystem* if the form $(\cdot,\cdot)$ is nondegenerate; see [27, Lem. 3.10].

Example 2.5. Extended affine root systems [1] and invariant affine reflection systems [21] are examples of extended affine root supersystems. Also a generalized root system [24] is a locally finite root supersystem.

Definition 2.6. Suppose that $(A,(\cdot,\cdot),R)$ is a locally finite root supersystem.

- A subset S of R is called a *sub-supersystem* if the restriction of the form to $\langle S\rangle$ is nondegenerate, $0\in S$, for $\alpha\in S\cap R_{re}^{\times},\beta\in S$ and $\gamma\in S\cap R_{ns}$ with $(\beta,\gamma)\neq 0$, $r_\alpha(\beta)\in S$ and $\{\gamma-\beta,\gamma+\beta\}\cap S\neq\emptyset$.
- A sub-supersystem S of R is called *full* if $\mathrm{span}_{\mathbb{Q}}S=\mathbb{Q}\otimes_{\mathbb{Z}}A$.
- If $(A,(\cdot,\cdot),R)$ is irreducible, R is said to be of *type 1* if $\mathrm{span}_{\mathbb{Q}}R_{re}=\mathbb{Q}\otimes_{\mathbb{Z}}A$; otherwise, we say it is of *type 2*.
- If $\{R^i\mid i\in I\}$ is a class of sub-supersystems of R which are mutually orthogonal with respect the form $(\cdot,\cdot)$ and $R\setminus\{0\}=\uplus_{i\in I}(R^i\setminus\{0\})$, we say R is *the direct sum* of R^i's and write $R=\oplus_{i\in I}R^i$.
- The locally finite root supersystem $(A,(\cdot,\cdot),R)$ is called a *locally finite root system* if $R_{ns}=\{0\}$; see [16].

We have the following straightforward lemma; see [27, Lem. 3.20]:

Lemma 2.7. *If $\{(X_i,(\cdot,\cdot)_i,S_i)\mid i\in I\}$ is a class of locally finite root supersystems, then for $X:=\oplus_{i\in I}X_i$ and $(\cdot,\cdot):=\oplus_{i\in I}(\cdot,\cdot)_i$, $(X,(\cdot,\cdot),S:=\cup_{i\in I}S_i)$ is a locally finite root supersystem. Also each locally finite root supersystem is a direct sum of irreducible sub-supersystems.*

Definition 2.8. (i) Two irreducible extended affine root supersystems $(A, (\cdot,\cdot)_1, R)$ and $(B, (\cdot,\cdot)_2, S)$ are called *isomorphic* if there is a group isomorphism $\varphi : A \longrightarrow B$ and a nonzero scalar $r \in \mathbb{F}$ such that $\varphi(R) = S$ and $(a_1, a_2)_1 = r(\varphi(a_1), \varphi(a_2))_2$ for all $a_1, a_2 \in A$.

(ii) Suppose that $(A, (\cdot,\cdot), R)$ is an extended affine root supersystem. The subgroup $\mathcal{W}$ of $Aut(A)$ generated by r_α, $\alpha \in R^\times_{re}$, is called the *Weyl group* of R; we note that for $\alpha \in R^\times_{re}$ and $a \in A$, (S1) and (S3) imply that $2(a,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ and so $r_\alpha : A \longrightarrow A$ mapping $a \in A$ to $a - \frac{2(a,\alpha)}{(\alpha,\alpha)}\alpha$ is a group automorphism.

Theorem 2.9 ([16, §4.14, §8] and [27, Lem. 3.21])**.** *Suppose that T is a nonempty index set and $\mathcal{U} := \oplus_{i\in T}\mathbb{Z}\epsilon_i$ is the free $\mathbb{Z}$-module over the set T. Define the symmetric form*

$$(\cdot,\cdot) : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{F};\ \ (\epsilon_i, \epsilon_j) = \delta_{i,j},\ \ for\ i, j \in T$$

and set

$$\begin{array}{ll} \dot{A}_T := \{\epsilon_i - \epsilon_j \mid i, j \in T\} \ if\ |T| > 1, \\ D_T := \{\pm(\epsilon_i \pm \epsilon_j) \mid i, j \in T,\ i \neq j\} \ if\ |T| > 2, \\ B_T := \{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid i, j \in T,\ i \neq j\}, \\ C_T := \{\pm 2\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid i, j \in T,\ i \neq j\}, \\ BC_T := B_T \cup C_T. \end{array} \tag{2}$$

These are irreducible locally finite root systems in their $\mathbb{Z}$-spans. Moreover, each irreducible locally finite root system is either an irreducible finite root system or an infinite locally finite root system isomorphic to one of these locally finite root systems.

We refer to locally finite root systems listed in (2) as *type* A, D, B, C and BC respectively. We note that if R is an irreducible locally finite root system as above, then $(\alpha, \alpha) \in \mathbb{Z}^{>0}$ for all $\alpha \in R \setminus \{0\}$. This allows us to define

$$\begin{array}{l} R_{sh} := \{\alpha \in R^\times \mid (\alpha,\alpha) \leq (\beta,\beta);\ \ \text{for all } \beta \in R\}, \\ R_{ex} := R \cap 2R_{sh} \quad \text{and} \quad R_{lg} := R^\times \setminus (R_{sh} \cup R_{ex}) \\ R_{red} := \{0\} \cup R_{sh} \cup R_{lg}. \end{array}$$

The elements of R_{sh} (resp. R_{lg}, R_{ex} and R_{red}) are called *short roots* (resp. *long roots, extra-long roots* and *reduced roots*) of R. We point out that following the usual notation in the literature, the locally finite root system of type A is denoted by $\dot{A}$ instead of A, as all locally finite root systems listed above are spanning sets for $\mathbb{F} \otimes_{\mathbb{Z}} \mathcal{U}$ other than the one of type A which spans a subspace of codimension 1.

We make a convention that if a locally finite root system R is the direct sum of subsystems R_i, where i runs over a nonempty index set I, for $* \in \{sh, lg, ex, red\}$, by R_*, we mean $\cup_{i\in I}(R_i)_*$.

Theorem 2.10 ([27, Thm. 4.28])**.** *Suppose that T, T' are index sets with $|T|, |T'| > 1$ such that $|T| \neq |T'|$ if T, T' are both finite. Fix a symbol α^* and pick $t_0 \in T$ and $p_0 \in T'$. Consider the free $\mathbb{Z}$-module $X := \mathbb{Z}\alpha^* \oplus \oplus_{t\in T}\mathbb{Z}\epsilon_t \oplus \oplus_{p\in T'}\mathbb{Z}\delta_p$*

and define the symmetric form

$$(\cdot,\cdot) : X \times X \longrightarrow \mathbb{F}$$

by

$$\begin{array}{ll} (\alpha^*,\alpha^*) := 0, (\alpha^*,\epsilon_{t_0}) := 1, (\alpha^*,\delta_{p_0}) := 1 & \\ (\alpha^*,\epsilon_t) := 0, (\alpha^*,\delta_q) := 0 & t \in T\setminus\{t_0\}, q \in T'\setminus\{p_0\} \\ (\epsilon_t,\delta_p) := 0, (\epsilon_t,\epsilon_s) := \delta_{t,s}, (\delta_p,\delta_q) := -\delta_{p,q} & t,s \in T, p,q \in T'. \end{array}$$

Take R to be $R_{re} \cup R_{ns}^\times$ as in the following table:

type	R_{re}	$R_{ns}^\times$
$\dot{A}(0,T)$	$\{\epsilon_t - \epsilon_s \mid t,s \in T\}$	$\pm\mathcal{W}\alpha^*$
$\dot{C}(0,T)$	$\{\pm(\epsilon_t \pm \epsilon_s) \mid t,s \in T\}$	$\pm\mathcal{W}\alpha^*$
$\dot{A}(T,T')$	$\{\epsilon_t - \epsilon_s, \delta_p - \delta_q \mid t,s \in T, p,q \in T'\}$	$\pm\mathcal{W}\alpha^*$

in which $\mathcal{W}$ is the subgroup of $Aut(X)$ generated by the reflections r_α ($\alpha \in R_{re}\setminus\{0\}$) mapping $\beta \in X$ to $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$, then $(A := \langle R\rangle, (\cdot,\cdot)\mid_{A\times A}, R)$ is an irreducible locally finite root supersystem of type 2 and conversely, each irreducible locally finite root supersystem of type 2 is isomorphic to one and only one of these root supersystems.

We recall that for an irreducible finite root system R in an additive abelian group A with a root base $\Delta := \{\alpha_1,\ldots,\alpha_\ell\}$, setting $\mathcal{V} := \mathbb{Q}\otimes_{\mathbb{Z}} A$, each element of the basis $\{\omega_1,\ldots,\omega_\ell\}$ of $\mathcal{V}$ satisfying $2(\omega_j,\alpha_i)/(\alpha_i,\alpha_i) = \delta_{i,j}$, $1 \le i,j \le \ell$, is called a *fundamental weight* of R (with respect to Δ).

Theorem 2.11 ([27, Thm. 4.37]). *Consider the following:*

- *$n \in \{2,3\}$ and $(X_1,(\cdot,\cdot)_1,S_1)$, $\ldots,(X_n,(\cdot,\cdot)_n,S_n)$ are irreducible locally finite root systems.*
- *$X := X_1 \oplus\cdots\oplus X_n$ and $(\cdot,\cdot) := (\cdot,\cdot)_1 \oplus\cdots\oplus(\cdot,\cdot)_n$.*
- *$\mathcal{W}$ is the Weyl group of the locally finite root system $(X,(\cdot,\cdot),S := S_1 \oplus \cdots \oplus S_n)$.*
- *For $1 \le i \le n$, identify X_i with a subset of $\mathbb{Q}\otimes_{\mathbb{Z}} X_i$ in the usual manner and define ω_j^i's as following:*
  - *(i) If S_i is one of infinite locally finite root systems B_T, C_T, D_T or BC_T as in (2), by ω_1^i, we mean ϵ_1, where 1 is a distinguished element of T.*
  - *(ii) If S_i is one of the finite root systems $\{0,\pm\alpha\}$ of type A_1 or $\{0,\pm\alpha,\pm2\alpha\}$ of type BC_1, we set $\omega_1^i := \frac{1}{2}\alpha$.*

(iii) *If S_i is a finite root system of rank $\ell \geq 2$, we take $\{\omega_1^i, \ldots, \omega_\ell^i\} \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} X_i$ to be a set of fundamental weights for S_i (see [27, Prop. 2.7]).*

- *Consider δ^* and $R := R_{re} \cup R_{ns}^{\times}$ as in the following table:*

n	S_i $(1 \leq i \leq n)$	R_{re}	δ^*	$R_{ns}^{\times}$	*type*
2	$S_1 = A_\ell,\ S_2 = A_\ell$ $(\ell \in \mathbb{Z}^{\geq 1})$	$S_1 \oplus S_2$	$\omega_1^1 + \omega_1^2$	$\pm\mathcal{W}\delta^*$	$A(\ell, \ell)$
2	$S_1 = B_T,\ S_2 = BC_{T'}$ $(\lvert T\rvert, \lvert T'\rvert \geq 2)$	$S_1 \oplus S_2$	$\omega_1^1 + \omega_1^2$	$\mathcal{W}\delta^*$	$B(T, T')$
2	$S_1 = BC_T,\ S_2 = BC_{T'}$ $(\lvert T\rvert, \lvert T'\rvert > 1)$	$S_1 \oplus S_2$	$\omega_1^1 + \omega_1^2$	$\mathcal{W}\delta^*$	$BC(T, T')$
2	$S_1 = BC_T,\ S_2 = BC_{T'}$ $(\lvert T\rvert = 1, \lvert T'\rvert = 1)$	$S_1 \oplus S_2$	$2\omega_1^1 + 2\omega_1^2$	$\mathcal{W}\delta^*$	$BC(T, T')$
2	$S_1 = BC_T,\ S_2 = BC_{T'}$ $(\lvert T\rvert = 1, \lvert T'\rvert > 1)$	$S_1 \oplus S_2$	$2\omega_1^1 + \omega_1^2$	$\mathcal{W}\delta^*$	$BC(T, T')$
2	$S_1 = D_T,\ S_2 = C_{T'}$ $(\lvert T\rvert \geq 3, \lvert T'\rvert \geq 2)$	$S_1 \oplus S_2$	$\omega_1^1 + \omega_1^2$	$\mathcal{W}\delta^*$	$D(T, T')$
2	$S_1 = C_T,\ S_2 = C_{T'}$ $(\lvert T\rvert, \lvert T'\rvert \geq 2)$	$S_1 \oplus S_2$	$\omega_1^1 + \omega_1^2$	$\mathcal{W}\delta^*$	$C(T, T')$
2	$S_1 = A_1,\ S_2 = BC_T$ $(\lvert T\rvert = 1)$	$S_1 \oplus S_2$	$2\omega_1^1 + 2\omega_1^2$	$\mathcal{W}\delta^*$	$B(1, T)$
2	$S_1 = A_1,\ S_2 = BC_T$ $(\lvert T\rvert \geq 2)$	$S_1 \oplus S_2$	$2\omega_1^1 + \omega_1^2$	$\mathcal{W}\delta^*$	$B(1, T)$
2	$S_1 = A_1,\ S_2 = C_T$ $(\lvert T\rvert \geq 2)$	$S_1 \oplus S_2$	$\omega_1^1 + \omega_1^2$	$\mathcal{W}\delta^*$	$C(1, T)$
2	$S_1 = A_1,\ S_2 = B_3$	$S_1 \oplus S_2$	$\omega_1^1 + \omega_3^2$	$\mathcal{W}\delta^*$	$AB(1, 3)$
2	$S_1 = A_1,\ S_2 = D_T$ $(\lvert T\rvert \geq 3)$	$S_1 \oplus S_2$	$\omega_1^1 + \omega_1^2$	$\mathcal{W}\delta^*$	$D(1, T)$
2	$S_1 = BC_1,\ S_2 = B_T$ $(\lvert T\rvert \geq 2)$	$S_1 \oplus S_2$	$2\omega_1^1 + \omega_1^2$	$\mathcal{W}\delta^*$	$B(T, 1)$
2	$S_1 = BC_1,\ S_2 = G_2$	$S_1 \oplus S_2$	$2\omega_1^1 + \omega_1^2$	$\mathcal{W}\delta^*$	$G(1, 2)$
3	$S_1 = A_1,\ S_2 = A_1,\ S_3 = A_1$	$S_1 \oplus S_2 \oplus S_3$	$\omega_1^1 + \omega_1^2 + \omega_1^3$	$\mathcal{W}\delta^*$	$D(2, 1, \lambda)(\lambda \neq 0, -1)$
3	$S_1 = A_1,\ S_2 = A_1,\ S_3 := C_T$ $(\lvert T\rvert \geq 2)$	$S_1 \oplus S_2 \oplus S_3$	$\omega_1^1 + \omega_1^2 + \omega_1^3$	$\mathcal{W}\delta^*$	$D(2, T)$

- *For $1 \leq i \leq n$, normalize the form $(\cdot, \cdot)_i$ on X_i such that*

  (i) $(\delta^*, \delta^*) = 0$,

  (ii) *for type $D(2, T)$, $(\omega_1^1, \omega_1^1)_1 = (\omega_1^2, \omega_1^2)_2$.*

Then $(\langle R\rangle, (\cdot, \cdot) \mid_{\langle R\rangle \times \langle R\rangle}, R)$ is an irreducible locally finite root supersystem of type 1. Conversely, if $(X, (\cdot, \cdot), R)$ is an irreducible locally finite root supersystem of type 1, it is either an irreducible locally finite root system or isomorphic to one of the locally finite root supersystems listed in the above table.

Moreover, locally finite root supersystems in the above table are mutually non-isomorphic except for the ones of type $D(2, 1, \lambda)$. More precisely, For $\lambda, \mu \in \mathbb{F} \setminus \{0, -1\}$, $D(2, 1, \lambda)$ is isomorphic to $D(2, 1, \mu)$ if and only if λ, μ are in the same orbit under the action of the group of permutations on $\mathbb{F} \setminus \{0, -1\}$ generated by $\alpha \mapsto \alpha^{-1}$ and $\alpha \mapsto -1 - \alpha$.

Lemma 2.12. *Suppose that $(\mathcal{L}, \mathcal{H}, (\cdot, \cdot))$ is an extended affine Lie superalgebra with irreducible root system R. Set $\mathcal{V} := \mathrm{span}_{\mathbb{F}} R$ and denote the induced form on $\mathcal{V}$ again by $(\cdot, \cdot)$; see (1). Take $\mathcal{V}^0$ to be the radical of the form. Suppose that $\bar{\ }: \mathcal{V} \longrightarrow \bar{\mathcal{V}} := \mathcal{V}/\mathcal{V}^0$ is the canonical projection map and take $\bar{R}$ to be the image of R under the projection map " $\bar{\ }$ ". Denote by $\overline{(\cdot, \cdot)}$, the induced form on $\bar{\mathcal{V}}$, then we have the following:*

(i) *$(\bar{A} := \langle \bar{R}\rangle, \overline{(\cdot, \cdot)}|_{\bar{A} \times \bar{A}}, \bar{R})$ is an irreducible locally finite root supersystem.*

(ii) *There is a triple $(\dot{\mathcal{V}}, \dot{R}, \{S_{\dot\alpha}\}_{\dot\alpha \in \dot{R}})$ in which*

- *$\dot{\mathcal{V}}$ is a subspace of $\mathcal{V}$ with $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$,*
- *$\dot{R} \subseteq \dot{\mathcal{V}}$ and $\dot{R}$ is a locally finite root supersystem (in its $\mathbb{Z}$-span) isomorphic to $\bar{R}$; in particular, $\dot{R}_{re}$ is a locally finite root system,*
- *for each $\dot\alpha \in \dot{R}$, $S_{\dot\alpha}$ is a nonempty subset of $\mathcal{V}^0$ such that*

– $R = \cup_{\dot\alpha\in\dot R}(\dot\alpha + S_{\dot\alpha})$,

– $0 \in S_{\dot\alpha}$ *for* $\dot\alpha \in \begin{cases} (\dot R_{re})_{red} & \text{if } \dot R \text{ is of type 1,} \\ \dot R & \text{if } \dot R \text{ is of type 2,} \end{cases}$

– *if* $\dot R_{ns} \neq \{0\}$ *and* $\dot R$ *is of type* $X \neq A(\ell,\ell), C(T,T'), C(1,T)$, *then for all* $\dot\alpha, \dot\beta \in (\dot R_{re})_{sh}$, $S_{\dot\alpha} = S_{\dot\beta}$; *also for all* $\dot\alpha, \dot\beta \in (\dot R_{re})_{lg} \cup \dot R^{\times}_{ns}$, $S_{\dot\alpha} = S_{\dot\beta}$,

– *if* $\dot R_{ns} \neq \{0\}$ *and* $\dot R$ *is of type* $X \neq A(\ell,\ell), C(T,T'), C(1,T)$, *setting* $S := S_{\dot\alpha}$ *for some* $\dot\alpha \in (\dot R_{re})_{sh}$ *and* $F := S_{\dot\beta}$ *for some* $\dot\beta \in \dot R_{ns}$, *we get that* F *is a subgroup of* $\mathcal{V}^0$ *and*

$$S - 2S \subseteq S, \quad S + F \subseteq S \quad \text{and} \quad 2S + F \subseteq F.$$

Proof. Using the same argument as in [27, Lem. 3.10], one can see that $\bar R_{re}$ is locally finite in its $\mathbb{F}$-span in the sense that it intersects each finite dimensional subspace of $\text{span}_{\mathbb{F}}\bar R_{re}$ in a finite set. So using Lemmas 3.10, 3.12 and 3.21 of [27], we get that $\bar R$ is an irreducible locally finite root supersystem in its $\mathbb{Z}$-span. Also using [27, Lem. 3.5]; we get that $\bar R_{re}$ is a locally finite root system and the restriction of the form $\bar{(\cdot,\cdot)}$ to $\bar{\mathcal{V}}_{re} := \text{span}_{\mathbb{F}}\bar R_{re}$ is nondegenerate. Therefore we have

$$\text{the restriction of the form } \bar{(\cdot,\cdot)} \text{ to } \bar{\mathcal{V}}_{\mathbb{Q}} := \text{span}_{\mathbb{Q}}\bar R_{re} \text{ is nondegenerate.} \tag{3}$$

Since $\bar R_{re}$ is a locally finite root system, by [17, Lem. 5.1], it contains a $\mathbb{Z}$-linearly independent subset T such that

$$\mathcal{W}_T T = (\bar R_{re})^{\times}_{red} = \bar R_{re} \setminus \{2\bar\alpha \mid \alpha \in R_{re}\}, \tag{4}$$

in which by $\mathcal{W}_T$, we mean the subgroup of the Weyl group of $\bar R_{re}$ generated by $r_{\bar\alpha}$ for all $\bar\alpha \in T$. On the other hand, we know that there is a subset Π of R such that $\bar\Pi$ is a $\mathbb{Z}$-basis for $\text{span}_{\mathbb{Z}}\bar R$; see [29, Lem. 1.17]. This allows us to define the linear isomorphism

$$\varphi : \text{span}_{\mathbb{Q}}\bar R \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \text{span}_{\mathbb{Z}}\bar R$$

mapping $\bar\alpha$ to $1 \otimes \bar\alpha$ for all $\alpha \in \Pi$. Now suppose that $\bar R$ is of type 1, then

$$\varphi(\text{span}_{\mathbb{Q}}\bar R_{re}) = \text{span}_{\mathbb{Q}}(1 \otimes \bar R_{re}) = \mathbb{Q} \otimes \text{span}_{\mathbb{Z}}\bar R = \varphi(\text{span}_{\mathbb{Q}}\bar R)$$

which in turn implies that $\text{span}_{\mathbb{Q}}\bar R = \text{span}_{\mathbb{Q}}\bar R_{re}$. Therefore, $\text{span}_{\mathbb{Q}}\bar R = \text{span}_{\mathbb{Q}}T$ and so $\text{span}_{\mathbb{F}}\bar R = \text{span}_{\mathbb{F}}T$. But T is $\mathbb{Z}$-linearly independent and so it is $\mathbb{Q}$-linearly independent. We now prove that T is $\mathbb{F}$-linearly independent. Suppose that $\{\bar\alpha_1, \ldots, \bar\alpha_n\} \subseteq T$ and $\{r_1, \ldots, r_n\} \subseteq \mathbb{F}$ with $\sum_{i=1}^n r_i\bar\alpha_i = 0$. Take $\{a_j \mid j \in J\}$ to be a basis for $\mathbb{Q}$-vector space $\mathbb{F}$. For each $1 \leq i \leq n$, suppose $\{r_i^j \mid j \in J\} \subseteq \mathbb{Q}$ is such that $r_i = \sum_{j\in J} r_i^j a_j$. Then for each $\bar\alpha \in T$, we have

$$0 = \sum_{i=1}^n r_i \frac{2\bar{(\bar\alpha_i,\bar\alpha)}}{\bar{(\bar\alpha,\bar\alpha)}} = \sum_{i=1}^n \sum_{j\in J} r_i^j a_j \frac{2\bar{(\bar\alpha_i,\bar\alpha)}}{\bar{(\bar\alpha,\bar\alpha)}} = \sum_{j\in J}\sum_{i=1}^n r_i^j \frac{2\bar{(\bar\alpha_i,\bar\alpha)}}{\bar{(\bar\alpha,\bar\alpha)}} a_j.$$

Since $\frac{2\bar{(\bar\alpha_i,\bar\alpha)}}{\bar{(\bar\alpha,\bar\alpha)}} \in \mathbb{Z}$, we get that for each $j \in J$ and $\bar\alpha \in T$,

$$\bar{(\sum_{i=1}^n r_i^j \bar\alpha_i, \bar\alpha)} = \sum_{i=1}^n r_i^j \bar{(\bar\alpha_i,\bar\alpha)} = 0.$$

So by (3), $\sum_{i=1}^{n} r_i^j \bar{\alpha}_i = 0$ for all $j \in J$. But T is $\mathbb{Q}$-linearly independent and so $r_i^j = 0$ for all $1 \leq i \leq n$ and $j \in J$. This means that

$$T \text{ is } \mathbb{F}\text{-linearly independent.} \tag{5}$$

Next suppose that $\bar{R}$ is of type 2 and fix $\alpha^* \in R_{ns}^{\times}$. Using a modified version of the above argument together with [27, Lem 3.14] (see also [27, Lem. 3.21]), we get that

$$T \cup \{\bar{\alpha^*}\} \text{ is } \mathbb{F}\text{-linearly independent.} \tag{6}$$

For each element $\alpha \in T$, we fix a preimage $\dot{\alpha} \in R$ of α under $^-$ and set

$$K := \begin{cases} \{\dot{\alpha} \mid \alpha \in T\} & \text{if } \bar{R} \text{ is of type 1,} \\ \{\dot{\alpha} \mid \alpha \in T\} \cup \{\alpha^*\} & \text{if } \bar{R} \text{ is of type 2.} \end{cases}$$

We have using [27, Prop. 3.14] together with (4) that $\bar{\mathcal{V}} = \text{span}_{\mathbb{F}} \bar{K}$. Therefore setting $\dot{\mathcal{V}} := \text{span}_{\mathbb{F}} K$ and using (5) and (6), we get that $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$. We set $\dot{R} := \{\dot{\alpha} \in \dot{\mathcal{V}} \mid \exists \sigma \in \mathcal{V}^0, \dot{\alpha} + \sigma \in R\}$, then $\dot{R}$ is a locally finite root supersystem in its $\mathbb{Z}$-span isomorphic to $\bar{R}$. Also since $K \subseteq R \cap \dot{R}$, $-K \subseteq R \cap \dot{R}$. So taking $\mathcal{W}_K$ to be the subgroup of the Weyl group of R generated by the reflections based on real roots of K, we have

$$\mathcal{W}_K(\pm K) \subseteq R \cap \dot{R} \quad \text{and} \quad \pm \mathcal{W}_K K = \begin{cases} (\dot{R}_{re})_{red}^{\times} & \text{if } \bar{R} \text{ is of type 1,} \\ \dot{R}^{\times} & \text{if } \bar{R} \text{ is of type 2.} \end{cases}$$

We finally set $S_{\dot{\alpha}} := \{\sigma \in \mathcal{V}^0 \mid \dot{\alpha} + \sigma \in R\}$ for $\dot{\alpha} \in \dot{R}$. Then $R = \cup_{\dot{\alpha} \in \dot{R}} (\dot{\alpha} + S_{\dot{\alpha}})$ and

$$0 \in S_{\dot{\alpha}} \text{ for } \dot{\alpha} \in \begin{cases} (\dot{R}_{re})_{red} & \dot{R} \text{ is of type 1,} \\ \dot{R} & \dot{R} \text{ is of type 2.} \end{cases}$$

Other assertions in the statement follow from the same argument as in Claims 3,4 of the proof of Theorem 2.2 of [29]. ∎

3. Root graded Lie superalgebras

Definition 3.1. For a locally finite root supersystem R of type X. Set

$$R_0 := \begin{cases} \{\alpha \in R_{re} \mid 2\alpha \notin R\} \cup \{0\} & \text{if } X \neq BC(T, T') \\ \\ R_{re} \setminus (R_{re}^2)_{sh} & \text{if } X = BC(T, T') \text{ and } R_{re} = R_{re}^1 \oplus R_{re}^2 \end{cases}$$

and

$$R_1 := R \setminus R_0.$$

We call elements of R_0 (resp. R_1) *even* (resp. *odd*) roots.

We note that for a locally finite root supersystem R, R_0 is a locally finite root system.

Definition 3.2. Suppose that $(A, (\cdot, \cdot), R)$ is a locally finite root supersystem and Λ is an additive abelian group. A Lie superalgebra $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ is called an (R, Λ)*-graded* Lie superalgebra if

- the Lie superalgebra $\mathcal{L}$ is equipped with a $\langle R\rangle$-grading $\mathcal{L}=\oplus_{\alpha\in\langle R\rangle}\mathcal{L}^\alpha$,
- the support of $\mathcal{L}$ with respect to the $\langle R\rangle$-grading is a subset of R,
- $\mathcal{L}^0=\sum_{\alpha\in R\setminus\{0\}}[\mathcal{L}^\alpha,\mathcal{L}^{-\alpha}]$,
- the Lie superalgebra $\mathcal{L}$ is equipped with a Λ-grading $\mathcal{L}=\oplus_{\lambda\in\Lambda}{}^\lambda\mathcal{L}$ which is compatible with the $\langle R\rangle$-grading on $\mathcal{L}$,
- there is a full subsystem Φ of R such that for $0\neq\alpha\in\Phi$, there are $0\neq e\in{}^0\mathcal{L}\cap\mathcal{L}^\alpha$ and $0\neq f\in{}^0\mathcal{L}\cap\mathcal{L}^{-\alpha}$ such that $k_\alpha:=[e,f]\in\mathcal{L}_0\setminus\{0\}$ and for $\beta\in R$ and $x\in\mathcal{L}^\beta$, $[k_\alpha,x]=(\beta,\alpha)x$. We call $\{k_\alpha\mid\alpha\in\Phi\setminus\{0\}\}$ a *set of toral elements* and refer to Φ as a *grading subsystem.*

An (R,Λ)-graded Lie superalgebra $\mathcal{L}$ is called *fine* if for $i=0,1$, the support $\mathcal{L}_i$ with respect to the $\langle R\rangle$-grading is a subset of R_i; also it is called *predivision* if for $\alpha\in R\setminus\{0\}$ and $\lambda\in\Lambda$ with ${}^\lambda\mathcal{L}^\alpha:={}^\lambda\mathcal{L}\cap\mathcal{L}^\alpha\neq\{0\}$, there are $e\in{}^\lambda\mathcal{L}^\alpha$ and $f\in{}^{-\lambda}\mathcal{L}^{-\alpha}$ such that $k:=[e,f]\in\mathcal{L}_0\setminus\{0\}$ and for $\beta\in R$ and $x\in\mathcal{L}^\beta$, $[k,x]=(\beta,\alpha)x$. An $(R,\{0\})$-graded Lie superalgebra is called an *R-graded Lie superalgebra.*

Lemma 3.3. *Suppose that $(A,(\cdot,\cdot),R)$ is a locally finite root supersystem and Λ an additive abelian group. If $\mathcal{G}=\oplus_{\alpha\in R}\oplus_{\sigma\in\Lambda}{}^\sigma\mathcal{G}^\alpha$ is an (R,Λ)-graded Lie superalgebra with a grading subsystem Φ, then so is $\mathcal{G}/Z(\mathcal{G})$. Moreover, if $\mathcal{G}$ is predivision, then $\mathcal{G}/Z(\mathcal{G})$ is also predivision.*

Proof. Since the center $Z(\mathcal{G})$ of G inherits the gradings on $\mathcal{G}$ (Lemma 1.2), for $\alpha\in R$ and $\sigma\in\Lambda$, we have

$$\frac{{}^\sigma\mathcal{G}+Z(\mathcal{G})}{Z(\mathcal{G})}\cap\frac{\mathcal{G}^\alpha+Z(\mathcal{G})}{Z(\mathcal{G})}=\frac{{}^\sigma\mathcal{G}^\alpha}{Z(\mathcal{G})}$$

and that

$$\frac{\mathcal{G}}{Z(\mathcal{G})}=\frac{\mathcal{G}_0+Z(\mathcal{G})}{Z(\mathcal{G})}\oplus\frac{\mathcal{G}_1+Z(\mathcal{G})}{Z(\mathcal{G})}=\bigoplus_{\alpha\in R,\sigma\in\Lambda}\frac{{}^\sigma\mathcal{G}^\alpha+Z(\mathcal{G})}{Z(\mathcal{G})}.$$

More precisely, $\mathcal{G}/Z(\mathcal{G})$ is equipped with compatible $\langle R\rangle$ and Λ-gradings. Now we prove that $Z(\mathcal{G})\subseteq\mathcal{G}^0$. For this, we suppose $\alpha\in R\setminus\{0\}$ and show that $\mathcal{G}^\alpha\cap Z(\mathcal{G})=\{0\}$. If $\mathcal{G}^\alpha=\{0\}$, there is nothing to prove, so suppose $\mathcal{G}^\alpha\neq\{0\}$. Since $\mathrm{span}_{\mathbb{Q}}\Phi=\mathbb{Q}\otimes_{\mathbb{Z}}R$, for each $\beta\in R$, there is a nonzero integer n with $n\beta\in\mathrm{span}_{\mathbb{Z}}\Phi$. This together with the fact that the form $(\cdot,\cdot)$ is nondegenerate on $A=\mathrm{span}_{\mathbb{Z}}R$, guarantees the existence of an element $\gamma\in\Phi$ with $(\alpha,\gamma)\neq 0$. Suppose that k_γ is a toral element of $\mathcal{G}$ corresponding to γ. For each $0\neq x\in\mathcal{G}^\alpha$, we have $[k_\gamma,x]=(\alpha,\gamma)x\neq 0$, so $x\notin Z(\mathcal{G})$. This shows that $\mathcal{G}^\alpha\cap Z(\mathcal{G})=\{0\}$. To complete the proof, it is enough to show if $e\in{}^\lambda\mathcal{G}^\alpha$ and $f\in{}^{-\lambda}\mathcal{G}^{-\alpha}$ for some $\alpha\in R\setminus\{0\}$ and $\lambda\in\Lambda$ with $k:=[e,f]\in\mathcal{G}_0\setminus\{0\}$ such that $[k,x]=(\beta,\alpha)x$ for $\beta\in R,x\in\mathcal{G}^\beta$, then $k\notin Z(\mathcal{G})$. So consider α,λ,e,f and k as above. Since $\alpha\neq 0$, as before, there is $\beta\in\Phi$ with $(\alpha,\beta)\neq 0$. Now for $0\neq y\in{}^0\mathcal{G}^\beta$, we have $[k,y]=(\beta,\alpha)y\neq 0$. This shows that $k\notin Z(\mathcal{G})$ and so we are done. ■

Lemma 3.4. *Suppose that $(\mathcal{L},\mathcal{H},(\cdot,\cdot))$ is an extended affine Lie superalgebra with irreducible root system R. Keep the same notation as in Lemma 2.12 and set $\Lambda := \langle \cup_{\dot\alpha\in\dot R} S_{\dot\alpha}\rangle$. Then the core $\mathcal{L}_c$ of $\mathcal{L}$ is a predivision $(\dot R,\Lambda)$-graded Lie superalgebra. Moreover, if $R^0\subseteq R_0$, then for $i=0,1$, the support $(\mathcal{L}_c)_i$ with respect to the $\langle\dot R\rangle$-grading is $\dot R_i$.*

Proof. We note that for each root $\alpha\in R$, $\mathcal{L}^\alpha$ is a $\mathbb{Z}_2$-graded subspace, so $\mathcal{L}_c$ is a $\mathbb{Z}_2$-graded subalgebra of $\mathcal{L}$. Moreover, we have

$$\mathcal{L}_c=\sum_{\dot\alpha\in\dot R^\times,\sigma\in S_{\dot\alpha}}\mathcal{L}^{\dot\alpha+\sigma}+\sum_{\dot\alpha\in\dot R^\times,\sigma\in S_{\dot\alpha},\tau\in S_{-\dot\alpha}}[\mathcal{L}^{\dot\alpha+\sigma},\mathcal{L}^{-\dot\alpha+\tau}].$$

Therefore, we have

$$\mathcal{L}_c=\sum_{\dot\alpha\in\dot R}(\mathcal{L}_c)^{\dot\alpha}=(\mathcal{L}_c)_0\oplus(\mathcal{L}_c)_1=\sum_{\sigma\in\Lambda}{}^\sigma(\mathcal{L}_c)$$

where

$$\begin{aligned}
&(\mathcal{L}_c)^{\dot\alpha}=\textstyle\sum_{\sigma\in S_{\dot\alpha}}\mathcal{L}^{\dot\alpha+\sigma}\qquad(\dot\alpha\in\dot R^\times),\\
&(\mathcal{L}_c)^0=\textstyle\sum_{\dot\alpha\in\dot R^\times}\sum_{\sigma\in S_{\dot\alpha}}\sum_{\tau\in S_{-\dot\alpha}}[\mathcal{L}^{\dot\alpha+\sigma},\mathcal{L}^{-\dot\alpha+\tau}],\\
&(\mathcal{L}_c)_0=\mathcal{L}_0\cap\mathcal{L}_c\quad\text{and}\quad(\mathcal{L}_c)_1=\mathcal{L}_1\cap\mathcal{L}_c,\\
&{}^\lambda(\mathcal{L}_c)=\textstyle\sum_{\dot\beta\in\dot R^\times}\mathcal{L}^{\dot\beta+\lambda}+\sum_{\dot\beta\in\dot R^\times}\sum_{\sigma\in S_{\dot\beta}}[\mathcal{L}^{\dot\beta+\sigma},\mathcal{L}^{-\dot\beta+\lambda-\sigma}]\qquad(\lambda\in\Lambda).
\end{aligned}$$

These define compatible $\langle\dot R\rangle$ and Λ-gradings on $\mathcal{L}_c$. Now set

$$\dot\Phi:=\begin{cases}(\dot R_{re})_{red} & \text{if } \dot R \text{ is of type 1}\\ \dot R & \text{if } \dot R \text{ is of type 2.}\end{cases}$$

We know from Lemma 2.12 that $\dot\Phi\subseteq R$. Now for $\dot\alpha\in\dot\Phi\setminus\{0\}$, since $\mathcal{L}$ is an extended affine Lie superalgebra, by [28, Lem. 3.1], there are $e\in\mathcal{L}^{\dot\alpha}={}^0(\mathcal{L}_c)^{\dot\alpha}$ and $f\in\mathcal{L}^{-\dot\alpha}={}^0(\mathcal{L}_c)^{-\dot\alpha}$ such that $[e,f]=t_{\dot\alpha}$ (we recall $t_{\dot\alpha}$ from Section 2). Therefore, for $x\in{}^\lambda(\mathcal{L}_c)^{\dot\beta}\subseteq\mathcal{L}^{\dot\beta+\lambda}$ ($\dot\beta\in\dot R$, $\lambda\in\Lambda$), we have

$$[t_{\dot\alpha},x]=(\dot\beta+\lambda)(t_{\dot\alpha})x=(t_{\dot\beta+\lambda},t_{\dot\alpha})x=(\dot\beta+\lambda,\dot\alpha)x=(\dot\beta,\dot\alpha)x.$$

Now assume $R^0\subseteq R_0$, then using the same argument as in [28, Prop. 3.10], one gets that

$$\begin{aligned}
&\bullet\ \text{if } \dot\alpha\in\dot R_{re} \text{ and } 2\dot\alpha\notin\dot R, \text{ then } \dot\alpha+S_{\dot\alpha}\subseteq R_0,\\
&\bullet\ \text{if } \dot\alpha\in\dot R_{re}^\times \text{ and } 2\dot\alpha\in\dot R, \text{ then } 2\dot\alpha+S_{2\dot\alpha}\subseteq R_0,\\
&\bullet\ \text{if } \dot\alpha\in\dot R_{ns}^\times, \text{ then } \dot\alpha+S_{\dot\alpha}\subseteq R_1,\\
&\bullet\ \text{if } \dot\alpha\in\dot R_{re}^\times \text{ and } \dot\alpha+\sigma\in R_0 \text{ for some } \sigma\in S_{\dot\alpha}, \text{ then } \dot\alpha+\tau\notin R_1 \text{ for all } \tau\in S_{\dot\alpha}.
\end{aligned}\tag{7}$$

Now the result easily follows if $\dot R$ is not of type $BC(T,T'),B(T,T'),B(1,T),B(T,1)$ and $G(1,2)$. So we just consider these mentioned types. From the classification table of Theorem 2.11, we know that for types $BC(T,T'),B(T,T'),B(1,T),B(T,1)$ and $G(1,2)$, $\dot R_{re}$ has two irreducible components $\dot R_{re}^1$ and $\dot R_{re}^2$ and that $\dot R_{ns}^\times=$

$(\dot{R}^1_{re})_{sh}+(\dot{R}^2_{re})_{sh}$. We also recall from Lemma 2.12 that $S=S_{\dot\alpha}$, for all $\dot\alpha\in(\dot R_{re})_{sh}$, and $F=S_{\dot\beta}$, for all $\dot\beta\in\dot R_{lg}\cup\dot R_{ns}$, satisfy $S+F=S$. Considering (7), to complete the proof, we just need to show that if $\{i,j\}=\{1,2\}$, $\{r,s\}=\{0,1\}$ and $\dot\alpha+\sigma\in R_r$ for some $\dot\alpha\in(\dot R^i_{re})_{sh}$ and $\sigma\in S$, then

$$(\dot R^i_{re})_{sh}+S\subseteq R_r \quad\text{and}\quad (\dot R^j_{re})_{sh}+S\subseteq R_s.$$

So suppose that $\dot\alpha+\sigma\in R_r$ for some $\dot\alpha\in(\dot R^i_{re})_{sh}$ and $\sigma\in S$. By (7), $\dot\alpha+\tau\in R_r$ for all $\tau\in S$. Fix $\dot\beta\in(\dot R^j_{re})_{sh}$ and $\tau\in S$. Set $\alpha:=\dot\alpha+\tau\in R_r$ and pick $e_\alpha\in\mathcal{L}^\alpha_r$ and $e_{-\alpha}\in\mathcal{L}^{-\alpha}_r$ such that $[e_\alpha,e_{-\alpha}]=t_\alpha$. We know that for each $\zeta\in F$, $\gamma:=\dot\beta-\dot\alpha+\zeta\in R_1$ and that $[e_{-\alpha},\mathcal{L}^\gamma]\subseteq\mathcal{L}^{\dot\beta-2\dot\alpha-\tau+\zeta}_{r+1}=\{0\}$, so we get

$$\begin{aligned}\{0\}\neq(\gamma,\alpha)\mathcal{L}^\gamma &= [t_\alpha,\mathcal{L}^\gamma]\\ &= [[e_\alpha,e_{-\alpha}],\mathcal{L}^\gamma]\\ &= [e_\alpha,[e_{-\alpha},\mathcal{L}^\gamma]]-(-1)^r[e_{-\alpha},[e_\alpha,\mathcal{L}^\gamma]]\\ &= -(-1)^r[e_{-\alpha},[e_\alpha,\mathcal{L}^\gamma]]\end{aligned}$$

which implies that $\{0\}\neq[e_\alpha,\mathcal{L}^\gamma]\subseteq\mathcal{L}^{\dot\beta+\tau+\zeta}_{r+1}$. Therefore,

$$\dot\beta+\tau+\zeta\in R_s \qquad (\tau\in S,\zeta\in F).$$

But $S+F=S$, so we have $\dot\beta+\eta\in R_s$ for $\eta\in S$. This means that

$$(\dot R^j_{re})_{sh}+S\subseteq R_s.$$

Finally using the same argument as above, we get that $(\dot R^i_{re})_{sh}+S\subseteq R_r$. This completes the proof. ■

Lemma 3.5. *Suppose that*

- $(\dot A,(\cdot,\cdot)\dot{},\dot R)$ *is a locally finite root supersystem with the decomposition* $\dot R=\oplus_{j\in J}\dot R^{(j)}$ *into irreducible sub-supersystems,*
- Λ *is a torsion free additive abelian group,*
- $\mathcal{G}=\mathcal{G}_0\oplus\mathcal{G}_1=\oplus_{\lambda\in\Lambda}\oplus_{\dot\alpha\in\dot R}{}^\lambda\mathcal{G}^{\dot\alpha}$ *is an* $(\dot R,\Lambda)$*-graded Lie superalgebra with a grading subsystem* $\dot\Phi$,
- $T:=\{k_{\dot\alpha}\mid\dot\alpha\in\dot\Phi\setminus\{0\}\}$ *is a set of toral elements,*
- $(\cdot,\cdot)$ *is an invariant nondegenerate even supersymmetric bilinear form on* $\mathcal{G}$.

For $\dot\alpha\in\dot\Phi\setminus\{0\}$, *fix* $e_{\dot\alpha}\in{}^0\mathcal{G}^{\dot\alpha}$ *and* $f_{\dot\alpha}\in{}^0\mathcal{G}^{-\dot\alpha}$ *such that* $k_{\dot\alpha}=[e_{\dot\alpha},f_{\dot\alpha}]$. *Then we have the following:*

(i) Take $\{\dot\alpha_i\mid i\in I\}\subseteq\dot\Phi$ *to be such that* $\{k_{\dot\alpha_i}\mid i\in I\}$ *is a basis for* $span_{\mathbb F}T$. *If* $\dot\gamma\in\dot\Phi^{(j)}:=\dot R^{(j)}\cap\dot\Phi$, *for some* $j\in J$, *then* $k_{\dot\gamma}\in span_{\mathbb F}\{k_{\dot\alpha_i}\mid i\in I,\dot\alpha_i\in\dot\Phi^{(j)}\}$. *Moreover, if* $\{r_i\mid i\in I\}\subseteq\mathbb F$ *with* $k_{\dot\gamma}=\sum_{i\in I}r_ik_{\dot\alpha_i}$, *we have* $\dot\gamma=\sum_{i\in I}r_i\dot\alpha_i\in\mathbb F\otimes_{\mathbb Z}\dot A$ *(here we identify* $\dot A$ *as a subset of* $\mathbb F\otimes_{\mathbb Z}\dot A$*); in particular, for* $\dot\beta\in\dot R$,

$$\begin{aligned}&\tilde{\dot\beta}:T\longrightarrow\mathbb F\\ &k_{\dot\alpha}\mapsto(\dot\alpha,\dot\beta)\dot{}\quad(\dot\alpha\in\dot\Phi\setminus\{0\})\end{aligned}$$

is a well-defined linear function.

(ii) $\mathcal{G}$ *has a weight space decomposition with respect to* T *with the set of weights contained in* $\{\tilde{\dot\beta} \mid \dot\beta \in \dot R\}$.

(iii) Suppose that $\mathcal{G}$ *is centerless. Assume that* $\dot\gamma \in \dot R \setminus \{0\}$ *and that there are* $e \in \mathcal{G}^{\dot\gamma}$ *and* $f \in \mathcal{G}^{-\dot\gamma}$ *such that* $0 \neq k := [e, f]$ *satisfies*

$$[k, x] = (\dot\beta, \dot\alpha)^{\cdot} x \ \ (x \in \mathcal{G}^{\dot\beta}).$$

If $\{r, r_{\dot\alpha} \mid \dot\alpha \in \dot\Phi\setminus\{0\}\} \subseteq \mathbb{Z}\setminus\{0\}$ *and* $r\dot\gamma = \sum_{\dot\alpha\in\dot\Phi\setminus\{0\}} r_{\dot\alpha}\dot\alpha$, *then* $rk = \sum_{\dot\alpha\in\dot\Phi\setminus\{0\}} r_{\dot\alpha} k_{\dot\alpha}$; *in particular,* $k \in T$ *and* $(e, f) \neq 0$.

(iv) Suppose that $\mathcal{G}$ *is centerless. Assume that* $\dot\gamma \in \dot R \setminus \{0\}$ *and that there are* $e, x \in \mathcal{G}^{\dot\gamma}$ *and* $f, y \in \mathcal{G}^{-\dot\gamma}$ *such that* $t := [x, y]$ *and* $k := [e, f]$ *satisfy*

$$[t, x] = (\dot\beta, \dot\alpha)^{\cdot} x \quad \text{and} \quad [k, x] = (\dot\beta, \dot\alpha)^{\cdot} x \ \ (x \in \mathcal{G}^{\dot\beta}).$$

Then $t = k$ *and* $(x, y) = (e, f)$.

Proof. *(i)* The form $(\cdot,\cdot)^{\cdot}$ induces the $\mathbb{F}$-bilinear form

$(\cdot,\cdot)_{\mathbb{F}} : (\mathbb{F}\otimes_{\mathbb{Z}} \dot A) \times (\mathbb{F}\otimes_{\mathbb{Z}} \dot A) \to \mathbb{F}$ defined by $(r\otimes a, s\otimes b)_{\mathbb{F}} := rs(a, b)^{\cdot}$; $r, s \in \mathbb{F}$, $a, b \in \dot A$.

This is a nondegenerate symmetric bilinear form satisfying

$$(\mathrm{span}_{\mathbb{Z}}\dot R^{(i)}, \mathrm{span}_{\mathbb{Z}}\dot R^{(j)})_{\mathbb{F}} = \{0\} \text{ for } i, j \in J \text{ with } i \neq j$$

(see [27, Lem. 3.21]). Since $\mathrm{span}_{\mathbb{Q}}\dot\Phi = \mathbb{Q}\otimes_{\mathbb{Z}} \dot R$ and $\mathrm{span}_{\mathbb{Z}}\dot R = \oplus_{j\in J}\mathrm{span}_{\mathbb{Z}}\dot R^{(j)}$, we get that

$$\mathrm{span}_{\mathbb{F}}\dot\Phi^{(j)} = \mathrm{span}_{\mathbb{F}}\dot R^{(j)}.$$

Suppose that $j \in J$ and $\dot\gamma \in \dot\Phi^{(j)}$. Let $i_1, \ldots, i_n,\ j_1, \ldots, j_m \in I$ are such that $\dot\alpha_{i_1}, \ldots, \dot\alpha_{i_n} \in \dot\Phi^{(j)}$, $\dot\alpha_{j_1}, \ldots, \dot\alpha_{j_m} \notin \dot\Phi^{(j)}$ and $k_{\dot\gamma} = r_1 k_{\dot\alpha_{i_1}} + \cdots + r_n k_{\dot\alpha_{i_n}} + s_1 k_{\dot\alpha_{j_1}} + \cdots + s_m k_{\dot\alpha_{j_m}}$ for some $r_1, \ldots, r_n,\ s_1, \ldots, s_m \in \mathbb{F}$. For $\dot\beta \in \dot\Phi^{(j)} \setminus \{0\}$, we have $\mathcal{G}^{\dot\beta} \neq \{0\}$ and for $0 \neq x \in \mathcal{G}^{\dot\beta}$, we have

$$\begin{aligned}
(\dot\gamma, \dot\beta)_{\mathbb{F}} x &= (\dot\gamma, \dot\beta)^{\cdot} x \\
&= [k_{\dot\gamma}, x] \\
&= [r_1 k_{\dot\alpha_{i_1}} + \cdots + r_n k_{\dot\alpha_{i_n}} + s_1 k_{\dot\alpha_{j_1}} + \cdots + s_m k_{\dot\alpha_{j_m}}, x] \\
&= r_1[k_{\dot\alpha_{i_1}}, x] + \cdots + r_n[k_{\dot\alpha_{i_n}}, x] + s_1[k_{\dot\alpha_{j_1}}, x] + \cdots + s_m[k_{\dot\alpha_{j_m}}, x] \\
&= r_1(\dot\alpha_{i_1}, \dot\beta)^{\cdot} x + \cdots + r_n(\dot\alpha_{i_n}, \dot\beta)^{\cdot} x + s_1(\dot\alpha_{j_1}, \dot\beta)^{\cdot} x + \cdots + s_m(\dot\alpha_{j_m}, \dot\beta)^{\cdot} x \\
&= r_1(\dot\alpha_{i_1}, \dot\beta)^{\cdot} x + \cdots + r_n(\dot\alpha_{i_n}, \dot\beta)^{\cdot} x \\
&= (r_1\dot\alpha_{i_1} + \cdots + r_n\dot\alpha_{i_n}, \dot\beta)_{\mathbb{F}} x
\end{aligned}$$

This implies that $\dot\gamma = r_1\dot\alpha_{i_1} + \cdots + r_n\dot\alpha_{i_n}$ as the form $(\cdot,\cdot)_{\mathbb{F}}$ on $\mathrm{span}_{\mathbb{F}}\dot R^{(j)} = \mathrm{span}_{\mathbb{F}}\dot\Phi^{(j)}$ is nondegenerate. Now we claim that $k_{\dot\gamma} = r_1 k_{\dot\alpha_{i_1}} + \cdots + r_n k_{\dot\alpha_{i_n}}$. To show this, it is enough to prove that $(k_{\dot\gamma} - (r_1 k_{\dot\alpha_{i_1}} + \cdots + r_n k_{\dot\alpha_{i_n}}), k_{\dot\beta}) = 0$ for all $\dot\beta \in \dot\Phi \setminus \{0\}$ as the form is nondegenerate on T. Assume $\dot\beta \in \dot\Phi \setminus \{0\}$, then

$$\begin{aligned}
(k_{\dot\gamma} - (r_1 k_{\dot\alpha_{i_1}} + \cdots + r_n k_{\dot\alpha_{i_n}}), k_{\dot\beta}) &= (k_{\dot\gamma} - (r_1 k_{\dot\alpha_{i_1}} + \cdots + r_n k_{\dot\alpha_{i_n}}), [e_{\dot\beta}, f_{\dot\beta}]) \\
&= ([k_{\dot\gamma} - (r_1 k_{\dot\alpha_{i_1}} + \cdots + r_n k_{\dot\alpha_{i_n}}), e_{\dot\beta}], f_{\dot\beta}) \\
&= (\dot\gamma - r_1\dot\alpha_{i_1} + \cdots + r_n\dot\alpha_{i_n}, \dot\beta)_{\mathbb{F}}(e_{\dot\beta}, f_{\dot\beta}) \\
&= 0(e_{\dot\beta}, f_{\dot\beta}) = 0.
\end{aligned}$$

This completes the proof.

(*ii*) We know that $\mathcal{G} = \oplus_{\dot\beta\in\dot R}\mathcal{G}^{\dot\beta}$. If $x \in \mathcal{G}^{\dot\beta}$ $(\dot\beta \in \dot R)$, we have $[k_{\dot\alpha}, x] = (\dot\beta, \dot\alpha)\dot{} x = \tilde{\dot\beta}(k_{\dot\alpha})x$ for all $\dot\alpha \in \dot\Phi \setminus \{0\}$ and so we have $[t, x] = \tilde{\dot\beta}(t)x$ for all $t \in T$. So $\mathcal{G}$ has a weight space decomposition $\mathcal{G} = \oplus_{\dot\beta\in\dot R}\mathcal{G}^{(\tilde{\dot\beta})}$ with respect to T in which for $\dot\beta \in \dot R$, $\mathcal{G}^{(\tilde{\dot\beta})} = \mathcal{G}^{\dot\beta}$.

(*iii*) We know $\mathcal{G} = \sum_{\dot\beta\in\dot R}\mathcal{G}^{\dot\beta}$ and that for all $a \in \mathcal{G}^{\dot\beta}$ $(\dot\beta \in \dot R)$,

$$\begin{aligned}[][rk - \sum_{\dot\alpha\in\dot\Phi\setminus\{0\}} r_{\dot\alpha}k_{\dot\alpha}, a] &= r(\dot\beta, \dot\gamma)\dot{}a - \sum_{\dot\alpha\in\dot\Phi\setminus\{0\}} r_{\dot\alpha}(\dot\beta, \dot\alpha)\dot{}a \\ &= (\dot\beta, r\dot\gamma - \sum_{\dot\alpha\in\dot\Phi\setminus\{0\}} r_{\dot\alpha}\dot\alpha)\dot{}a = 0.\end{aligned}$$

This means that $rk - \sum_{\dot\alpha\in\dot\Phi\setminus\{0\}} r_{\dot\alpha}k_{\dot\alpha}$ is an element of the center of $\mathcal{G}$ and so it is zero, i.e. $rk = \sum_{\dot\alpha\in\dot\Phi\setminus\{0\}} r_{\dot\alpha}k_{\dot\alpha}$; in particular, $k \in T$. Now to the contrary, assume $(e, f) = 0$, then for each $\dot\alpha \in \dot\Phi \setminus \{0\}$,

$$(k, k_{\dot\alpha}) = ([e, f], k_{\dot\alpha}) = (e, [f, k_{\dot\alpha}]) = (\dot\alpha, \dot\gamma)\dot{}(e, f) = 0.$$

This contradicts the fact that the form on T is nondegenerate.

(*iv*) As $\mathcal{G}$ is centerless, it is immediate that $t = k$. We shall show that $(e, f) = (x, y)$. Since $\dot\gamma \neq 0$, there is $\dot\alpha \in \dot\Phi$ with $(\dot\alpha, \dot\gamma)\dot{} \neq 0$. Now we have

$$\begin{aligned}(e, f)(\dot\gamma, \dot\alpha)\dot{} = ((\dot\gamma, \dot\alpha)\dot{}e, f) = ([k_{\dot\alpha}, e], f) = (k_{\dot\alpha}, [e, f]) = (k_{\dot\alpha}, k) &= (k, k_{\dot\alpha}) \\ &= (k, [e_{\dot\alpha}, f_{\dot\alpha}]) \\ &= ([k, e_{\dot\alpha}], f_{\dot\alpha}) \\ &= (\dot\gamma, \dot\alpha)\dot{}(e_{\dot\alpha}, f_{\dot\alpha}).\end{aligned}$$

This implies that $(e, f) = (e_{\dot\alpha}, f_{\dot\alpha})$. Similarly $(x, y) = (e_{\dot\alpha}, f_{\dot\alpha})$. This completes the proof. ∎

One knows from [2] that affine Lie algebras are extended affine Lie algebras of nullity one. Affinization is a process to get an affine Lie algebra starting with a finite dimensional simple Lie algebra over $\mathbb{C}$. To get an untwisted affine Lie algebra $\hat{\mathcal{G}}$, one forms the tensor product of a finite dimensional simple Lie algebra $\mathcal{G}$ with the Laurent polynomials in one variable and then adds some central elements as well as some derivations. The centerless core of $\hat{\mathcal{G}}$ is the Lie algebra $\mathcal{G} \otimes \mathbb{C}[t^{\pm1}]$ which is in fact a $\mathbb{Z}$-graded Lie algebra. In general, the centerless core of an extended affine Lie algebra is an $(R, \mathbb{Z}^n)$-graded Lie algebra satisfying certain properties for some finite root system R and a positive integer n. Conversely, if $\mathfrak{K}$ is an $(R, \mathbb{Z}^n)$-graded $\mathbb{C}$-Lie algebra satisfying these certain properties, adding two n-dimensional vector spaces C and D to $\mathfrak{K}$, one can impose a nondegenerate bilinear form on $\mathfrak{K} \oplus C \oplus D$ such that C and D have bases which are dual with respect to this form. Then $\mathfrak{K} \oplus C \oplus D$ will be an extended affine Lie algebra with centerless core $\mathfrak{K}$; see [1, Ch. III]. But one can identify C with $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}^n$ and D with C^*. This motivates us to extend affinization process to get a so-called extended affinization. More precisely, in [5, §7], we consider the tensor product of an invariant affine reflection algebra $\mathcal{G}$ and a so-called predivision unital commutative associative Λ-graded algebra $\mathcal{A}$, then we add $\mathcal{V} := \mathbb{F} \otimes \Lambda$ and a certain dual of $\mathcal{V}$ to $\mathcal{G} \otimes \mathcal{A}$ to get another Lie algebra. In the following theorem, we show that for a locally finite root supersystem R and an abelian group Λ, an (R, Λ)-graded Lie superalgebra $\mathcal{L}$ satisfying specific conditions is the centerless core of an extended affine Lie superalgebra; in fact, we add $\mathcal{V} := \mathbb{F} \otimes_{\mathbb{Z}} \Lambda$ and a certain dual $\mathcal{V}^\dagger$ of $\mathcal{V}$ to $\mathcal{L}$ and then equip $\mathcal{L} \oplus \mathcal{V} \oplus \mathcal{V}^\dagger$ with a Lie bracket and show that this is an extended affine Lie superalgebra with centerless core $\mathcal{L}$.

Theorem 3.6. *Suppose that $(\dot{A}, \dot{(\cdot,\cdot)}, \dot{R})$ is a locally finite root supersystem and Λ is a torsion free additive abelian group. Suppose that $\mathcal{G} = \oplus_{\lambda \in \Lambda} \oplus_{\dot\alpha \in \dot R} {}^{\lambda}\mathcal{G}^{\dot\alpha}$, together with the Lie bracket $[\cdot,\cdot]_{\mathcal{G}}$, is a centerless $(\dot R, \Lambda)$-graded Lie superalgebra, with a grading subsystem $\dot\Phi$, equipped with an invariant nondegenerate even supersymmetric bilinear form $(\cdot,\cdot)$. Suppose that*

- *for $\lambda, \mu \in \Lambda$ with $\lambda + \mu \neq 0$, $({}^{\lambda}\mathcal{G}, {}^{\mu}\mathcal{G}) = \{0\}$,*
- *the form is nondegenerate on the span of a set of toral elements of $\mathcal{G}$,*
- *for $\lambda \in \Lambda$ with ${}^{\lambda}\mathcal{G}_i^0 := \mathcal{G}_i \cap {}^{\lambda}\mathcal{G} \cap \mathcal{G}^0 \neq \{0\}$ $(i = 0, 1)$, there are $e \in {}^{\lambda}\mathcal{G}_i^0$ and $f \in {}^{-\lambda}\mathcal{G}_i^0$ such that $[e, f]_{\mathcal{G}} = 0$ and $(e, f) \neq 0$,*
- *for $\dot\alpha \in \dot R \setminus \{0\}$ and $\lambda \in \Lambda$ with ${}^{\lambda}\mathcal{G}_i^{\dot\alpha} := \mathcal{G}_i \cap {}^{\lambda}\mathcal{G} \cap \mathcal{G}^{\dot\alpha} \neq \{0\}$ $(i = 0, 1)$, there are $e \in {}^{\lambda}\mathcal{G}_i^{\dot\alpha}$ and $f \in {}^{-\lambda}\mathcal{G}_i^{-\dot\alpha}$ such that $k := [e, f]_{\mathcal{G}} \in \mathcal{G}_0 \setminus \{0\}$ and $[k, x]_{\mathcal{G}} = \dot{(\dot\beta, \dot\alpha)} x$, for $\dot\beta \in \dot R$ and $x \in \mathcal{G}^{\dot\beta}$,*

then $\mathcal{G}$ is isomorphic to the centerless core of an extended affine Lie superalgebra.

Proof. Set $\mathcal{V} := \mathbb{F} \otimes_{\mathbb{Z}} \Lambda$. Identify Λ with a subset of $\mathcal{V}$ and fix a basis $\{\lambda_i \mid i \in I\} \subseteq \Lambda$ of $\mathcal{V}$. Suppose that $\mathcal{V}^\dagger$ is the restricted dual (see [18, Example 1.3.1]) of $\mathcal{V}$ with respect to this basis. We suppose $\{d_i \mid i \in I\}$ is the corresponding basis for $\mathcal{V}^\dagger$. Consider d_i $(i \in I)$ as a derivation of $\mathcal{G}$ mapping $x \in {}^{\lambda}\mathcal{G}$ to $d_i(\lambda)x$ for all $\lambda \in \Lambda$. Set

$$\mathcal{L} := \mathcal{G} \oplus \mathcal{V} \oplus \mathcal{V}^\dagger$$

and for $x, y \in \mathcal{G}$, $d \in \mathcal{V}^\dagger$ and $v \in \mathcal{V} \oplus \mathcal{V}^\dagger$, define

$$\begin{array}{ll} deg(v) := 0, \quad [d, x] = -[x, d] := d(\lambda)x, & \\ {[\mathcal{L}, \mathcal{V}] = [\mathcal{V}, \mathcal{L}] = [\mathcal{V}^\dagger, \mathcal{V}^\dagger] := \{0\}}, \quad [x, y] := [x, y]_{\mathcal{G}} + \sum_{i \in I} (d_i(x), y)\lambda_i. & \end{array} \tag{8}$$

We next extend the form on $\mathcal{G}$ to a bilinear form on $\mathcal{L}$ by

$$\begin{array}{ll} (\mathcal{V}, \mathcal{V}) = (\mathcal{V}^\dagger, \mathcal{V}^\dagger) = (\mathcal{V}, \mathcal{G}) = (\mathcal{V}^\dagger, \mathcal{G}) := \{0\}, & \\ (v, d) = (d, v) := d(v), \qquad d \in \mathcal{V}^\dagger, v \in \mathcal{V}. & \end{array} \tag{9}$$

Then $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$, where $\mathcal{L}_0 := \mathcal{G}_0 \oplus \mathcal{V} \oplus \mathcal{V}^\dagger$ and $\mathcal{L}_1 := \mathcal{G}_1$, together with the Lie bracket $[\cdot,\cdot]$ is a Lie superalgebra and $(\cdot,\cdot)$ is an invariant nondegenerate even supersymmetric bilinear form.

For $\dot\alpha \in \dot\Phi \setminus \{0\}$, we fix $e_{\dot\alpha} \in {}^{0}\mathcal{G}^{\dot\alpha}$ and $f_{\dot\alpha} \in {}^{0}\mathcal{G}^{-\dot\alpha}$ such that $k_{\dot\alpha} = [e_{\dot\alpha}, f_{\dot\alpha}]$ and that the form on

$$T := \mathrm{span}_{\mathbb{F}}\{k_{\dot\alpha} \mid \dot\alpha \in \dot\Phi \setminus \{0\}\}$$

is nondegenerate. We next set

$$\mathcal{H} := T \oplus \mathcal{V} \oplus \mathcal{V}^\dagger$$

and note that the form restricted to $\mathcal{H}$ is nondegenerate. We identify $\mathcal{H}^*$ with $T^* \oplus \mathcal{V}^* \oplus (\mathcal{V}^\dagger)^*$ in the usual manner. We also consider $\lambda \in \mathcal{V}$ as an element of $\mathcal{H}^*$ by $\lambda(t+v+d) = d(\lambda)$ for $t \in T,\ v \in \mathcal{V}$ and $d \in \mathcal{V}^\dagger$. We know that

$$\mathcal{L}_0 = \sum_{\lambda\in\Lambda}\sum_{\dot\alpha\in\dot R} {}^\lambda\mathcal{G}_0^{\dot\alpha} \oplus \mathcal{V} \oplus \mathcal{V}^\dagger \quad \text{and} \quad \mathcal{L}_1 = \sum_{\lambda\in\Lambda}\sum_{\dot\alpha\in\dot R} {}^\lambda\mathcal{G}_1^{\dot\alpha}.$$

For $i \in \{0,1\},\ t \in T,\ v \in \mathcal{V},\ d \in \mathcal{V}^\dagger,\ \dot\beta \in \dot R,\ \lambda \in \Lambda$ and $x \in {}^\lambda\mathcal{G}_i^{\dot\beta}$, we have using Lemma 3.5(ii) that

$$\begin{aligned}&[t+v+d, x] = [t,x]_{\mathcal{G}} + [d,x] = (\tilde{\dot\beta} + \lambda)(t+v+d)x,\\ &[t+v+d, \mathcal{V} \oplus \mathcal{V}^\dagger] = \{0\},\end{aligned}$$

so for $i = 0,1,\ \mathcal{L}_i$ has a weight space decomposition with respect to $\mathcal{H}$ with the set of weights $\{\tilde{\dot\beta} + \lambda \mid \dot\beta \in \dot R, \lambda \in \Lambda, {}^\lambda\mathcal{G}_i^{\dot\beta} \neq \{0\}\}$. Now suppose $i \in \{0,1\}$, $\dot\beta \in \dot R,\ \lambda \in \Lambda$ with $\tilde{\dot\beta} + \lambda \neq 0$ and $\mathcal{L}_i^{\tilde{\dot\beta}+\lambda} \neq \{0\}$. If $\dot\beta \neq 0$, there are $e \in {}^\lambda\mathcal{G}_i^{\dot\beta}$ and $f \in {}^{-\lambda}\mathcal{G}_i^{-\dot\beta}$ such that $k := [e,f]_{\mathcal{G}} \in \mathcal{G}_0 \setminus \{0\}$ and for $x \in \mathcal{G}^{\dot\gamma}$ $(\dot\gamma \in \dot R)$, $[k,x]_{\mathcal{G}} = (\dot\beta, \dot\gamma)x$. But since $\text{span}_{\mathbb{Q}}\dot\Phi = \mathbb{Q} \otimes_{\mathbb{Z}} \text{span}_{\mathbb{Z}}\dot R$, there is a nonzero integer $r \in \mathbb{Z}$ such that $r\dot\beta = \sum_{\dot\alpha\in\dot\Phi\setminus\{0\}} r_{\dot\alpha}\dot\alpha$. So $k = \frac{1}{r}\sum_{\dot\alpha\in\dot\Phi\setminus\{0\}} r_{\dot\alpha}k_{\dot\alpha} \in T$ by Lemma 3.5. This implies that $[e,f] \in \mathcal{H} \setminus \{0\}$. Also if $\dot\beta = 0$, take $e \in {}^\lambda\mathcal{G}_i^0$ and $f \in {}^{-\lambda}\mathcal{G}_i^0$ such that $[e,f]_{\mathcal{G}} = 0$ and $(e,f) \neq 0$, then $[e,f] = (e,f)\lambda \in \mathcal{H} \setminus \{0\}$. Therefore there are $e \in \mathcal{L}_i^{\tilde{\dot\beta}+\lambda} = {}^\lambda\mathcal{G}_i^{\dot\beta}$ and $f \in \mathcal{L}_i^{-\tilde{\dot\beta}-\lambda} = {}^{-\lambda}\mathcal{G}_i^{-\dot\beta}$ with $0 \neq [e,f] \in \mathcal{H}$.

Now take R to be the root system of $\mathcal{L}$ with respect to $\mathcal{H}$ and suppose $\dot\alpha \in \dot R$ and $\lambda \in \Lambda$ with $\tilde{\dot\alpha} + \lambda \in R$. If $\dot\alpha = 0$, then it is easy to see that $t_{\tilde{\dot\alpha}+\lambda} = \lambda$ in which $t_{\tilde{\dot\alpha}+\lambda}$ as before is the unique element of $\mathcal{H}$ representing $\tilde{\dot\alpha} + \lambda$ through the form $(\cdot,\cdot)$. Also if $\dot\alpha \neq 0$, we fix $e \in {}^\lambda\mathcal{G}^{\dot\alpha}$ and $f \in {}^{-\lambda}\mathcal{G}^{-\dot\alpha}$ such that for $k := [e,f] \in \mathcal{G}_0 \setminus \{0\}$ and for all $x \in \dot{\mathcal{G}}^{\dot\beta}$ $(\dot\beta \in \dot R)$ $[k,x] = (\dot\alpha, \dot\beta)x$. Then considering Lemma 3.5, it is easily verified that $t_{\tilde{\dot\alpha}+\lambda} = (e,f)^{-1}k + \lambda$. Now it follows that $R^0 = R \cap \Lambda$ and $R^\times = \{\tilde{\dot\beta} + \lambda \mid \dot\beta \in \dot R^\times, \lambda \in \Lambda, {}^\lambda\mathcal{G}^{\dot\beta} \neq \{0\}\}$. We next show that adx is locally nilpotent for $x \in {}^\lambda\mathcal{G}^{\dot\alpha} = \mathcal{L}^{\tilde{\dot\alpha}+\lambda}$ $(\dot\alpha \in \dot R^\times$ and $\lambda \in \Lambda$ with $\tilde{\dot\alpha} + \lambda \in R)$. Let $v \in \mathcal{V}$ and $d \in \mathcal{V}^\dagger$, then $adx(v) = 0$ and if $\lambda = 0$, $adx(d) = 0$. Next suppose that $\lambda \neq 0$, then we have

$$\begin{aligned}(adx)^3(d) = -\lambda(d)(adx)^2(x) &= -\lambda(d)[x,[x,x]]\\ &= -\lambda(d)[x,[x,x]_{\mathcal{G}}]\\ &= -\lambda(d)([x,[x,x]_{\mathcal{G}}]_{\mathcal{G}} + \sum_{i\in I}(d_i(x),[x,x]_{\mathcal{G}})\lambda_i\\ &= -\lambda(d)[x,[x,x]_{\mathcal{G}}]_{\mathcal{G}} \in \mathcal{G}^{3\dot\alpha} = \{0\}.\end{aligned}$$

Also for $\dot\beta \in \dot R,\ \mu \in \Lambda$ and $y \in {}^\mu\mathcal{G}^{\dot\beta}$, since $\dot R$ is a locally finite root supersystem, $\{k\dot\alpha + \dot\beta \mid k \in \mathbb{Z}\} \cap \dot R$ is a finite set. Fix a positive integer N such that for $m \geq N$,

$m\dot{\alpha}+\dot{\beta}\notin\dot{R}$. If $\lambda=0$, we have $(adx)^N(y)=(ad_{\mathcal{G}}x)^N(y)\in\mathcal{G}^{N\dot{\alpha}+\dot{\beta}}=\{0\}$ in which $ad_{\mathcal{G}}$ denotes the adjoint representation of $\mathcal{G}$. If $\lambda\neq 0$, we choose a positive integer $n>N$ such that $n\lambda+\mu\neq 0$, then

$$\begin{aligned}(adx)^n(y) &= (ad_{\mathcal{G}}x)^n(y)+\sum_{i\in I}(d_i(x),(ad_{\mathcal{G}}x)^{n-1}(y))\lambda_i\\ &= (ad_{\mathcal{G}}x)^n(y)\in\mathcal{G}^{n\dot{\alpha}+\dot{\beta}}=\{0\}.\end{aligned}$$

Therefore adx is locally nilpotent. Thus $(\mathcal{L},(\cdot,\cdot),\mathcal{H})$ is an extended affine Lie superalgebra with root system $R=\{\tilde{\dot{\beta}}+\lambda\mid\dot{\beta}\in\dot{R},\lambda\in\Lambda,{}^{\lambda}\mathcal{G}^{\dot{\beta}}\neq\{0\}\}$. We now show that $\mathcal{L}_c/Z(\mathcal{L}_c)$ is a Lie superalgebra isomorphic to $\mathcal{G}$. We know that

$$\mathcal{L}_c=\sum_{\dot{\alpha}\in\dot{R}^{\times},\lambda\in\Lambda}{}^{\lambda}\mathcal{G}^{\dot{\alpha}}+\sum_{\dot{\alpha}\in\dot{R}^{\times}}\sum_{\lambda,\mu\in\Lambda}[{}^{\lambda}\mathcal{G}^{\dot{\alpha}},{}^{\mu}\mathcal{G}^{-\dot{\alpha}}]\subseteq\mathcal{G}\oplus\mathcal{V}.$$

Take Π to be the restriction of the canonical projection map $\mathcal{L}\longrightarrow\mathcal{G}$ to $\mathcal{L}_c$ with respect to the decomposition $\mathcal{L}=\mathcal{G}\oplus\mathcal{V}\oplus\mathcal{V}^{\dagger}$. Since $\mathcal{G}^{\dot{\alpha}}\subseteq\mathcal{L}_c$ for all $\dot{\alpha}\in\dot{R}\setminus\{0\}$ and $\mathcal{G}^0=\sum_{\dot{\alpha}\in\dot{R}\setminus\{0\}}[\mathcal{G}^{\dot{\alpha}},\dot{\mathcal{G}}^{-\dot{\alpha}}]_{\mathcal{G}}$, Π is surjective. Also if $x\in\mathcal{G}$ and $v\in\mathcal{V}$ are such that $x+v\in Z(\mathcal{L}_c)$, then $[x+v,\mathcal{G}^{\dot{\alpha}}]=\{0\}$ for all $\dot{\alpha}\in\dot{R}\setminus\{0\}$. So $[x,\mathcal{G}^{\dot{\alpha}}]_{\mathcal{G}}=\{0\}$ for all $\dot{\alpha}\in\dot{R}\setminus\{0\}$. Then it follows that $x\in Z(\mathcal{G})=\{0\}$. Therefore $Z(\mathcal{L}_c)=\mathcal{L}_c\cap\mathcal{V}=ker\Pi$. This implies that $\mathcal{L}_c$ is isomorphic to $\mathcal{G}$. ■

4. $BC(I,J)$-graded Lie superalgebras

4.1. The locally finite Lie superalgebra $\mathfrak{osp}_{\mathbb{F}}(I,J)$. For two disjoint nonempty index sets I,J, suppose that $\{0,\imath,\bar{\imath},\mid\imath\in I\cup J\}$ is a superset with $|0|=|\imath|=|\bar{\imath}|=0$ for $\imath\in I$ and $|\jmath|=|\bar{\jmath}|=1$ for $\jmath\in J$. Take $\mathfrak{u}$ to be a vector superspace with a basis $\{v_\imath\mid\imath\in I\cup\bar{I}\cup J\cup\bar{J}\cup\{0\}\}$ and

$$|v_\imath|:=|\imath|;\quad \imath\in I\cup\bar{I}\cup J\cup\bar{J}\cup\{0\}$$

in which by $\bar{I}$ (resp. $\bar{J}$), we mean $\{\bar{\imath}\mid\imath\in I\}$ (resp. $\{\bar{\jmath}\mid\jmath\in J\}$). Take $(\cdot,\cdot)$ to be the supersymmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{u}$ defined by

$$(v_0,v_0):=1,(v_\imath,v_\jmath):=0,(v_{\bar{\imath}},v_{\bar{\jmath}}):=0,(v_\imath,v_{\bar{\jmath}})=(-1)^{|\imath||\jmath|}(v_{\bar{\jmath}},v_\imath):=\delta_{\imath,\jmath}\tag{10}$$

for $\imath,\jmath\in I\cup J$. Now for $\jmath,k\in\{0\}\cup I\cup\bar{I}\cup J\cup\bar{J}$, define

$$e_{\jmath,k}:\mathfrak{u}\longrightarrow\mathfrak{u};\quad v_\imath\mapsto\delta_{k,\imath}v_\jmath,\quad(\imath\in\{0\}\cup I\cup\bar{I}\cup J\cup\bar{J}).\tag{11}$$

Then

$$\mathfrak{gl}_{\mathbb{F}}(I,J):=\mathrm{span}_{\mathbb{F}}\{e_{\jmath,k}\mid\jmath,k\in\{0\}\cup I\cup\bar{I}\cup J\cup\bar{J}\}\tag{12}$$

is a Lie subsuperalgebra of $\mathrm{End}_{\mathbb{F}}(\mathfrak{u})$. For $A=\sum_{\imath,\jmath\in I\uplus J}a_{\imath,\jmath}e_{\imath,\jmath}\in\mathfrak{gl}_{\mathbb{F}}(I,J)$, define the *supertrace* of A to be $str(A):=\sum_{\imath\in\{0\}\cup I\cup\bar{I}\cup J\cup\bar{J}}(-1)^{|\imath|}a_{\imath,\imath}$; we note that it is well-defined as $a_{\imath,\imath}$'s are nonzero for at most finitely many index $\imath$. Suppose that $\gamma\in\{1,-1\}$. For $i=0,1$, take

$$(\mathcal{A}_\gamma)_i:=\{s\in\mathfrak{gl}_{\mathbb{F}}(I,J)_i\mid str(s)=0\quad\text{and}\quad(su,v)=\gamma(-1)^{|s||u|}(u,sv),\ \forall u,v\in\mathfrak{u}\}$$

and set
$$\mathcal{A}_\gamma := (\mathcal{A}_\gamma)_0 \oplus (\mathcal{A}_\gamma)_1.$$
We next put
$$\mathfrak{g} := \mathcal{A}_{-1} \quad \text{and} \quad \mathfrak{s} := \mathcal{A}_1. \tag{13}$$
One knows that $\mathfrak{g}$ is a Lie subsuperalgebra of $\mathrm{End}_{\mathbb{F}}(\mathfrak{u})$. Set
$$\mathfrak{h} := \mathrm{span}_{\mathbb{F}}\{h_t, d_k \mid t \in I,\ k \in J\} \tag{14}$$
in which for $t \in I$ and $k \in J$,
$$h_t := e_{t,t} - e_{\bar{t},\bar{t}} \quad \text{and} \quad d_k := e_{k,k} - e_{\bar{k},\bar{k}}$$
and for $\imath \in I$ and $\jmath \in J$, define
$$\begin{array}{cc} \epsilon_\imath : \mathfrak{h} \longrightarrow \mathbb{F} & \delta_\jmath : \mathfrak{h} \longrightarrow \mathbb{F} \\ h_t \mapsto \delta_{\imath,t}, \quad d_k \mapsto 0, & h_t \mapsto 0, \quad d_k \mapsto \delta_{\jmath,k}, \end{array}$$
in which $t \in I$ and $k \in J$. Then $\mathfrak{u}$ is a $\mathfrak{g}$-module equipped with a weight space decomposition $\mathfrak{u} = \oplus_{\alpha \in \Delta_{\mathfrak{u}}} \mathfrak{u}^\alpha$ with respect to $\mathfrak{h}$, where
$$\Delta_{\mathfrak{u}} = \{0, \pm\epsilon_\imath, \pm\delta_\jmath \mid \imath \in I, \jmath \in J\} \tag{15}$$
with
$$\mathfrak{u}^0 = \mathbb{F}v_0,\ \ \mathfrak{u}^{\epsilon_\imath} = \mathbb{F}v_\imath,,\ \ \mathfrak{u}^{-\epsilon_\imath} = \mathbb{F}v_{\bar{\imath}},\ \mathfrak{u}^{\delta_\jmath} = \mathbb{F}v_\jmath,,\ \ \mathfrak{u}^{-\delta_\jmath} = \mathbb{F}v_{\bar{\jmath}}$$
for $\imath \in I$ and $\jmath \in J$. Also for $\gamma \in \{1, -1\}$, $\mathcal{A}_\gamma$ is a $\mathfrak{g}$-module having a weight space decomposition with respect to $\mathfrak{h}$. Taking R (resp. $\Delta_{\mathfrak{s}}$) to be the set of weights of $\mathcal{A}_{-1}$ (resp. $\mathcal{A}_1$) with respect to $\mathfrak{h}$, we have
$$R = \{\pm\epsilon_r, \pm(\epsilon_r \pm \epsilon_s), \pm\delta_p, \pm(\delta_p \pm \delta_q), \pm(\epsilon_r \pm \delta_p) \mid r, s \in I,\ p, q \in J,\ r \neq s\},$$
$$\Delta_{\mathfrak{s}} = \{\pm\epsilon_r, \pm(\epsilon_r \pm \epsilon_s), \pm\delta_p, \pm(\delta_p \pm \delta_q), \pm(\epsilon_r \pm \delta_p) \mid r, s \in I,\ p, q \in J,\ p \neq q\}. \tag{16}$$
Moreover, for $r, s \in I,\ p, q \in J,\ r \neq s$ and $p \neq q$, we have
$$\begin{array}{ll}
(\mathcal{A}_\gamma)^{\epsilon_r} = \mathrm{span}_{\mathbb{F}}(e_{r,0} + \gamma e_{0,\bar{r}}), & (\mathcal{A}_\gamma)^{-\epsilon_r} = \mathrm{span}_{\mathbb{F}}(e_{\bar{r},0} + \gamma e_{0,r}), \\
(\mathcal{A}_\gamma)^{\epsilon_r+\epsilon_s} = \mathrm{span}_{\mathbb{F}}(e_{r,\bar{s}} + \gamma e_{s,\bar{r}}), & (\mathcal{A}_\gamma)^{-\epsilon_r-\epsilon_s} = \mathrm{span}_{\mathbb{F}}(e_{\bar{r},s} + \gamma e_{\bar{s},r}), \\
(\mathcal{A}_\gamma)^{\epsilon_r-\epsilon_s} = \mathrm{span}_{\mathbb{F}}(e_{r,s} + \gamma e_{\bar{s},\bar{r})}, & (\mathcal{A}_\gamma)^{2\epsilon_r} = \mathrm{span}_{\mathbb{F}}\delta_{\gamma,1}e_{r,\bar{r}}, \\
(\mathcal{A}_\gamma)^{-2\epsilon_r} = \mathrm{span}_{\mathbb{F}}\delta_{\gamma,1}e_{\bar{r},r}, & (\mathcal{A}_\gamma)^{2\delta_p} = \mathrm{span}_{\mathbb{F}}\delta_{\gamma,-1}e_{p,\bar{p}}, \\
(\mathcal{A}_\gamma)^{-2\delta_p} = \mathrm{span}_{\mathbb{F}}\delta_{\gamma,-1}e_{\bar{p},p}, & (\mathcal{A}_\gamma)^{\delta_p+\delta_q} = \mathrm{span}_{\mathbb{F}}(e_{p,\bar{q}} - \gamma e_{p,\bar{q}}), \\
(\mathcal{A}_\gamma)^{-\delta_p-\delta_q} = \mathrm{span}_{\mathbb{F}}(e_{\bar{p},q} - \gamma e_{\bar{q},p}), & (\mathcal{A}_\gamma)^{\delta_p-\delta_q} = \mathrm{span}_{\mathbb{F}}(e_{p,q} + \gamma e_{\bar{q},\bar{p}}), \\
(\mathcal{A}_\gamma)^{\delta_p} = \mathrm{span}_{\mathbb{F}}(e_{0,\bar{p}} - \gamma e_{p,0}), & (\mathcal{A}_\gamma)^{-\delta_p} = \mathrm{span}_{\mathbb{F}}(e_{0,p} + \gamma e_{\bar{p},0}), \\
(\mathcal{A}_\gamma)^{\epsilon_r+\delta_p} = \mathrm{span}_{\mathbb{F}}(e_{r,\bar{p}} + \gamma e_{p,\bar{r}}), & (\mathcal{A}_\gamma)^{-\epsilon_r-\delta_p} = \mathrm{span}_{\mathbb{F}}(e_{\bar{r},p} - \gamma e_{\bar{p},r}), \\
(\mathcal{A}_\gamma)^{\epsilon_r-\delta_p} = \mathrm{span}_{\mathbb{F}}(e_{r,p} - \gamma e_{\bar{p},\bar{r}}), & (\mathcal{A}_\gamma)^{-\epsilon_r+\delta_p} = \mathrm{span}_{\mathbb{F}}(e_{\bar{r},\bar{p}} + \gamma e_{p,r}).
\end{array}$$

In the literature, $\mathfrak{g}$ is denoted by $\mathfrak{osp}_{\mathbb{F}}(I,J)$ (or $\mathfrak{osp}_{\mathbb{F}}(m,n)$ if I,J are finite sets with cardinalities m,n respectively) and referred to as an *orthosymplectic* Lie superalgebra. We also refer to $\mathfrak{g}$ as the *split locally finite Lie superalgebra of type* $B(I,J)$ (or $B(m,n)$ if $|I|=m$ and $|J|=n$) and say $\mathfrak{h}$ is the standard *splitting Cartan subalgebra* of $\mathfrak{g}$. We also refer to the $\mathfrak{g}$-module $\mathfrak{u}$ as the *natural module* of $\mathfrak{g}$ and to the $\mathfrak{g}$-module $\mathfrak{s}$ as the *second natural module* of $\mathfrak{g}$. We take

$$\mathfrak{g}_{_B} := \mathfrak{g}\cap \mathrm{span}_{\mathbb{F}}\{e_{\imath,\jmath}\mid \imath,\jmath\in I\cup\bar{I}\cup\{0\}\}\quad\text{and}\quad \mathfrak{g}_{_C} := \mathfrak{g}\cap \mathrm{span}_{\mathbb{F}}\{e_{\imath,\jmath}\mid \imath,\jmath\in J\cup\bar{J}\}.$$

Then $\mathfrak{g}_{_B}$ (resp. $\mathfrak{g}_{_C}$) is a locally finite split simple Lie algebra of type B_I (resp. C_J) with splitting Cartan subalgebra $\mathrm{span}_{\mathbb{F}}\{h_\imath\mid \imath\in I\}$ (resp. $\mathrm{span}_{\mathbb{F}}\{d_\jmath\mid \jmath\in J\}$) and corresponding root system $\{0,\pm\epsilon_\imath,\pm(\epsilon_\imath\pm\epsilon_\jmath)\mid \imath,\jmath\in I,\ \imath\neq\jmath\}$ (resp. $\{0,\pm2\delta_p,\pm(\delta_p\pm\delta_q)\mid p,q\in J,\ p\neq q\}$); see [20]. Moreover,

$$\mathfrak{s}_{_B} := \{x\in\mathfrak{s}\cap\mathrm{span}_{\mathbb{F}}\{e_{\imath,\jmath}\mid \imath,\jmath\in I\cup\bar{I}\cup\{0\}\}\mid tr(x)=0\}$$

is a $\mathfrak{g}_{_B}$-module and

$$\mathfrak{s}_{_C} := \{x\in\mathfrak{s}\cap\mathrm{span}_{\mathbb{F}}\{e_{\imath,\jmath}\mid \imath,\jmath\in J\cup\bar{J}\}\mid tr(x)=0\}$$

is a $\mathfrak{g}_{_C}$-module. We mention that $tr(x)$ in these expressions is well-defined as x is a linear combination of finitely many $e_{\imath,\jmath}$'s. We finally note that if

$$|I|=m,\ |J|=n\quad\text{and}\quad \mathfrak{I} := \frac{1}{2m+1}\sum_{\imath\in\{0\}\cup I\cup\bar{I}} e_{\imath\imath}+\frac{1}{2n}\sum_{\imath\in J\cup\bar{J}} e_{\imath\imath},$$

then we have

$$\mathfrak{s}_0=\mathfrak{s}_{_B}\oplus\mathfrak{s}_{_C}\oplus\mathbb{F}\mathfrak{I}. \tag{17}$$

Example 4.1. Suppose that $\mathbb{F}=\mathbb{C}$. For $\jmath,k\in\{0\}\cup I\cup\bar{I}\cup J\cup\bar{J}$, we recall $e_{\jmath,k}$ and $\mathfrak{gl}_{\mathbb{F}}(I,J)$ from (11) and (12) and for $T=\sum_{\jmath,k} r_{\jmath,k}e_{\jmath,k}\in\mathfrak{gl}_{\mathbb{F}}(I,J)$, set $\bar{T} := \sum_{\jmath,k}\overline{r_{\jmath,k}}e_{\jmath,k}$, where " $^-$ " denotes the complex conjugate. Now set

$$\begin{aligned}&\mathcal{L}_i := \{X\in\mathfrak{gl}_{\mathbb{F}}(I,J)_i\mid (Xv,w)=-(-1)^{|X||v|}(v,\bar{X}w);\forall v,w\in\mathfrak{u}\};\ i=0,1,\\ &\mathcal{L}=\mathcal{L}_0\oplus\mathcal{L}_1,\\ &\mathcal{H} := \mathrm{span}_{\mathbb{R}}\{h_t := e_{t,t}-e_{\bar{t},\bar{t}}, d_k := e_{k,k}-e_{\bar{k},\bar{k}}\mid t\in I,\ k\in J\}.\end{aligned}$$

We note that $\mathcal{L}\cap\mathrm{span}_{\mathbb{R}}\{e_{\jmath,k}\mid \jmath,k\in\{0\}\cup I\cup\bar{I}\cup J\cup\bar{J}\}\simeq\mathfrak{osp}_{\mathbb{R}}(I,J)$. For $\imath\in I$ and $\jmath\in J$, define

$$\begin{array}{ll}\dot\epsilon_\imath:\mathcal{H}\longrightarrow\mathbb{R} & \dot\delta_\jmath:\mathcal{H}\longrightarrow\mathbb{R}\\ h_t\mapsto\delta_{\imath,t},\ \ d_k\mapsto 0, & h_t\mapsto 0,\ \ d_k\mapsto\delta_{\jmath,k},\end{array}\quad (t\in I,k\in J).$$

One can see that with respect to $\mathcal{H}$, $\mathcal{L}$ has a weight space decomposition $\mathcal{L}=\oplus_{\alpha\in\dot{R}}\mathcal{L}^\alpha$ with the set of weights

$$\dot{R} := \{\pm\dot\epsilon_r,\pm(\dot\epsilon_r\pm\dot\epsilon_s),\pm\dot\delta_p,\pm(\dot\delta_p\pm\dot\delta_q),\pm(\dot\epsilon_r\pm\dot\delta_p)\mid r,s\in I,\ p,q\in J\}$$

and for $r,s\in I$ and $p,q\in J$ with $r\neq s$ and $p\neq q$,

$$
\begin{aligned}
\mathcal{L}^{\dot\epsilon_r} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{r,0}-(-1)^\alpha e_{0,\bar r}) \mid \alpha=0,1\},\\
\mathcal{L}^{-\dot\epsilon_r} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{\bar r,0}-(-1)^\alpha e_{0,r}) \mid \alpha=0,1\},\\
\mathcal{L}^{2\dot\epsilon_r} &= \mathrm{span}_{\mathbb{R}} i e_{r,\bar r},\\
\mathcal{L}^{-2\dot\epsilon_r} &= \mathrm{span}_{\mathbb{R}} i e_{\bar r,r},\\
\mathcal{L}^{2\dot\delta_p} &= \mathrm{span}_{\mathbb{R}} e_{p,\bar p},\\
\mathcal{L}^{-2\dot\delta_p} &= \mathrm{span}_{\mathbb{R}} e_{\bar p,p},\\
\mathcal{L}^{\dot\epsilon_r+\dot\epsilon_s} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{r,\bar s}-(-1)^\alpha e_{s,\bar r}) \mid \alpha=0,1\},\\
\mathcal{L}^{-\dot\epsilon_r-\dot\epsilon_s} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{\bar r,s}-(-1)^\alpha e_{\bar s,r}) \mid \alpha=0,1\},\\
\mathcal{L}^{\dot\epsilon_r-\dot\epsilon_s} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{r,s}-(-1)^\alpha e_{\bar s,\bar r}) \mid \alpha=0,1\},\\
\mathcal{L}^{\dot\delta_p+\dot\delta_q} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{p,\bar q}-(-1)^{\alpha+1} e_{q,\bar p}) \mid \alpha=0,1\},\\
\mathcal{L}^{-\dot\delta_p-\dot\delta_q} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{\bar p,q}-(-1)^{\alpha+1} e_{\bar q,p}) \mid \alpha=0,1\},\\
\mathcal{L}^{\dot\delta_p-\dot\delta_q} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{p,q}+(-1)^{\alpha+1} e_{\bar q,\bar p}) \mid \alpha=0,1\},\\
\mathcal{L}^{\dot\delta_p} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{0,\bar p}+(-1)^{\alpha+1} e_{p,0}) \mid \alpha=0,1\},\\
\mathcal{L}^{-\dot\delta_p} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{0,p}-(-1)^{\alpha+1} e_{\bar p,0}) \mid \alpha=0,1\},\\
\mathcal{L}^{\dot\epsilon_r+\dot\delta_p} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{r,\bar p}+(-1)^{\alpha+1} e_{p,\bar r}) \mid \alpha=0,1\},\\
\mathcal{L}^{-\dot\epsilon_r-\dot\delta_p} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{\bar r,p}-(-1)^{\alpha+1} e_{\bar p,r}) \mid \alpha=0,1\},\\
\mathcal{L}^{\dot\epsilon_r-\dot\delta_p} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{r,p}-(-1)^{\alpha+1} e_{\bar p,\bar r}) \mid \alpha=0,1\},\\
\mathcal{L}^{-\dot\epsilon_r+\dot\delta_p} &= \mathrm{span}_{\mathbb{R}}\{i^\alpha(e_{\bar r,\bar p}+(-1)^{\alpha+1} e_{p,r}) \mid \alpha=0,1\}.
\end{aligned}
$$

It is easy to see that the Lie superalgebra $\mathfrak{L} := \sum_{\alpha\in R^\times}\mathcal{L}^\alpha + \sum_{\alpha\in R^\times}[\mathcal{L}^\alpha,\mathcal{L}^{-\alpha}]$ is a $BC(I,J)$-graded Lie superalgebra with grading subsystem $\dot R\setminus\{\pm 2\dot\epsilon_\imath \mid \imath\in I\}$.

To get the structure of a $BC(I,J)$-graded Lie superalgebra $\mathcal{L}$, we shall write $\mathcal{L}$ as a direct union of subsuperalgebras each of which is graded by a finite root supersystem of type $BC(m,n)$, for some positive integers m,n. The components of the mentioned direct union are compatible in some sense. To study these compatibilities, the following straightforward proposition plays an important role.

Proposition 4.2. *Use the same notation as in §3 and suppose that $I'\subseteq I$, $J'\subseteq J$. Consider (13) and (16) and take*

$$R' := R\cap \mathit{span}_{\mathbb{Z}}\{\epsilon_\imath,\delta_p \mid \imath\in I', \jmath\in J'\} \text{ and } S := \Delta_{\mathfrak{s}}\cap \mathit{span}_{\mathbb{Z}}\{\epsilon_\imath,\delta_p \mid \imath\in I', \jmath\in J'\}$$

as well as

$$\mathcal{G} := \bigoplus_{\alpha\in R\setminus\{0\}}\mathfrak{g}^\alpha \oplus \sum_{\alpha\in R\setminus\{0\}}[\mathfrak{g}^\alpha,\mathfrak{g}^{-\alpha}] \text{ and } \mathcal{S} := \bigoplus_{\alpha\in S\setminus\{0\}}\mathfrak{s}^\alpha \oplus \sum_{\alpha\in S\setminus\{0\}}\mathfrak{g}^\alpha\cdot\mathfrak{s}^{-\alpha}.$$

Then we have the following:

(i) $\mathcal{G}$ is a Lie subsuperalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{osp}_{\mathbb{F}}(I',J')$.

(ii) Consider $\mathfrak{s}$ as a $\mathcal{G}$-module, then $\mathcal{S}$ is a $\mathcal{G}$-submodule of $\mathfrak{s}$ isomorphic to the second natural module of $\mathcal{G}$.

(iii) Suppose that V is a $\mathfrak{g}$-module isomorphic to $\mathfrak{g}$ and set

$$W := \bigoplus_{\alpha\in R'\setminus\{0\}} V^{\alpha} \oplus \sum_{\alpha\in R'\setminus\{0\}} \mathfrak{g}^{\alpha}\cdot V^{-\alpha},$$

then W is a $\mathcal{G}$-module isomorphic to $\mathcal{G}$.

(iv) If K is a $\mathfrak{g}$-module isomorphic to $\mathfrak{s}$, then

$$T := \bigoplus_{\alpha\in S\setminus\{0\}} K^{\alpha} \oplus \sum_{\alpha\in S\setminus\{0\}} \mathfrak{g}^{\alpha}\cdot K^{-\alpha}$$

is a $\mathcal{G}$-module isomorphic to the second natural module of $\mathcal{S}$.

(v) Set $\Gamma_1 := \{0, \pm\epsilon_\imath, \pm\delta_\jmath \mid \imath\in I', \jmath\in J'\}$. If U is a $\mathfrak{g}$-module isomorphic to $\mathfrak{u}$, then

$$M := \bigoplus_{\alpha\in\Gamma_1\setminus\{0\}} U^{\alpha} \oplus \sum_{\alpha\in\Gamma_1\setminus\{0\}} \mathfrak{g}^{\alpha}\cdot U^{-\alpha}$$

is a $\mathcal{G}$-module isomorphic to the natural module of $\mathfrak{osp}_{\mathbb{F}}(I', J')$.

4.2. Finite dimensional $\mathfrak{osp}_{\mathbb{F}}(m,n)$-modules.. In this subsection, we suppose the field $\mathbb{F}$ is algebraically closed and gather some facts regarding finite dimensional orthosympletic Lie superalgebras. We keep the same notation as in the previous subsection and suppose $I = \{1,\dots,m\}$ and $J = \{1,\dots,n\}$. We denote the set of $\mathfrak{g}_0$-module homomorphisms from a $\mathfrak{g}_0$-module X to a $\mathfrak{g}_0$-module Y by $\hom_{\mathfrak{g}_0}(X,Y)$.

Proposition 4.3. *Recall (17) and suppose that $\frac{2n}{2m+1}\notin\mathbb{Z}$, then*

$$\begin{array}{ll}
\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_0,\mathfrak{s}_0)=\{0\}, & \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_1,\mathfrak{s}_0)=\{0\},\\
\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_0,\mathfrak{s}_1)=\{0\}, & \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_1,\mathfrak{s}_1)=\{0\},\\
\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_0,\mathfrak{u}_0)=\{0\}, & \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_1,\mathfrak{u}_0)=\{0\},\\
\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_0,\mathfrak{u}_1)=\{0\}, & \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_1,\mathfrak{u}_1)=\{0\},\\
\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_0,\mathfrak{g}_0)=\{0\}, & \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_1,\mathfrak{g}_0)=\{0\},\\
\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_0,\mathfrak{g}_1)=\{0\}, & \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_1,\mathfrak{g}_1)=\{0\},\\
\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{g}_0,\mathfrak{u}_0)=\{0\}, & \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{g}_1,\mathfrak{u}_0)=\{0\},\\
\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{g}_0,\mathfrak{u}_1)=\{0\}, & \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{g}_1,\mathfrak{u}_1)=\{0\}.
\end{array}$$

Proof. We first note that $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module isomorphic to $\mathfrak{u}_0\otimes\mathfrak{u}_1$ and fix the base

$$\{\epsilon_1-\epsilon_2,\dots,\epsilon_{m-1}-\epsilon_m,\epsilon_m,\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,2\delta_n\}$$

for the root system of $\mathfrak{g}_0$. With respect to this base, we denote the finite dimensional irreducible $\mathfrak{g}_0$-module of highest weight λ by $V(\lambda)$ and recall that

for two finite dimensional irreducible highest weight modules $V(\lambda)$ and $V(\mu)$, $V(\lambda)\otimes V(\mu)$ is decomposed into finite dimensional irreducible highest weight modules of highest weights of the form $\mu+\lambda'$ for some λ' in the set of weights of $V(\lambda)$; (18)

see [14, Exercise 24.12]. We also recall that if V is an irreducible $\mathfrak{g}_{B}$-module and W is an irreducible $\mathfrak{g}_{C}$-module, then $V \otimes W$ is an irreducible $\mathfrak{g}_0$-module.

$\underline{\hom_{\mathfrak{g}_0}(\mathfrak{g}_1 \otimes \mathfrak{u}_0, \mathfrak{s}_0) = \{0\}}$: Suppose that $\mathfrak{u}_0 \otimes \mathfrak{u}_0 = \oplus_{i=1}^{r} V_i$ is the decomposition of the $\mathfrak{g}_0$-module $\mathfrak{u}_0 \otimes \mathfrak{u}_0$ into finite dimensional irreducible highest weight $\mathfrak{g}_0$-modules. Now we have $\hom_{\mathfrak{g}_0}(\mathfrak{g}_1 \otimes \mathfrak{u}_0, \mathfrak{s}_0) \simeq \hom_{\mathfrak{g}_0}((\mathfrak{u}_0 \otimes \mathfrak{u}_0) \otimes \mathfrak{u}_1, \mathfrak{s}_0)$

$$\begin{aligned} &\simeq \hom_{\mathfrak{g}_0}(\oplus_{i=1}^{r} V_i \otimes \mathfrak{u}_1, \mathfrak{s}_0) \\ &\simeq \oplus_{i=1}^{r} \hom_{\mathfrak{g}_0}(V_i \otimes \mathfrak{u}_1, \mathfrak{s}_0) \\ &\simeq \oplus_{i=1}^{r} \hom_{\mathfrak{g}_0}(V_i \otimes \mathfrak{u}_1, \mathfrak{s}_{C}) \oplus \oplus_{i=1}^{r} \hom_{\mathfrak{g}_0}(V_i \otimes \mathfrak{u}_1, \mathfrak{s}_{B}) \oplus \oplus_{i=1}^{r} \hom_{\mathfrak{g}_0}(V_i \otimes \mathfrak{u}_1, \mathbb{F}\mathfrak{I}). \end{aligned}$$

If $\hom_{\mathfrak{g}_0}(V_i \otimes \mathfrak{u}_1, \mathfrak{s}_{C}) \neq \{0\}$ for some i, then there is a nonzero $\mathfrak{g}_0$-module homomorphism $\varphi \in \hom_{\mathfrak{g}_0}(V_i \otimes \mathfrak{u}_1, \mathfrak{s}_{C}) \neq \{0\}$. But $V_i \otimes \mathfrak{u}_1$ and $\mathfrak{s}_{C}$ are irreducible, so φ is an isomorphism. We note that the set of weights of $\mathfrak{s}_{C}$ as a $\mathfrak{g}_0$-module is $\{0, \pm(\delta_p \pm \delta_q) \mid 1 \leq p \neq q \leq n\}$ while the set of weights of $V_i \otimes \mathfrak{u}_1$ is a subset of $\{\pm\epsilon_\imath \pm \delta_p, \pm\epsilon_\imath \pm \epsilon_\jmath \pm \delta_p \mid 1 \leq \imath, \jmath \leq m, 1 \leq p \leq n\}$ which is a contradiction. Using the same argument as above, we get that $\hom_{\mathfrak{g}_0}(V_i \otimes \mathfrak{u}_1, \mathfrak{s}_{B}) = \{0\}$ for all $1 \leq i \leq r$. Also as $\dim(V_i \otimes \mathfrak{u}_1) > 1$, there is no isomorphism from $V_i \otimes \mathfrak{u}_1$ to $\mathbb{F}\mathfrak{I}$ and so $\hom_{\mathfrak{g}_0}(V_i \otimes \mathfrak{u}_1, \mathbb{F}\mathfrak{I}) = \{0\}$.

$\underline{\hom_{\mathfrak{g}_0}(\mathfrak{g}_1 \otimes \mathfrak{u}_1, \mathfrak{s}_0) = \{0\}}$: Consider the decomposition $\mathfrak{u}_1 \otimes \mathfrak{u}_1 = \oplus_{i=1}^{s} V_i$ of the $\mathfrak{g}_{C}$-module $\mathfrak{u}_1 \otimes \mathfrak{u}_1$ into irreducible submodules. We have $\hom_{\mathfrak{g}_0}(\mathfrak{g}_1 \otimes \mathfrak{u}_1, \mathfrak{s}_0) \simeq \hom_{\mathfrak{g}_0}(\mathfrak{u}_0 \otimes (\mathfrak{u}_1 \otimes \mathfrak{u}_1), \mathfrak{s}_0)$

$$\begin{aligned} &\simeq \hom_{\mathfrak{g}_0}(\oplus_{i=1}^{s} \mathfrak{u}_0 \otimes V_i, \mathfrak{s}_0) \\ &\simeq \oplus_{i=1}^{s} \hom_{\mathfrak{g}_0}(\mathfrak{u}_0 \otimes V_i, \mathfrak{s}_0) \\ &\simeq \oplus_{i=1}^{s} \hom_{\mathfrak{g}_0}(\mathfrak{u}_0 \otimes V_i, \mathfrak{s}_{B}) \oplus \oplus_{i=1}^{s} \hom_{\mathfrak{g}_0}(\mathfrak{u}_0 \otimes V_i, \mathfrak{s}_{C}) \oplus \oplus_{i=1}^{s} \hom_{\mathfrak{g}_0}(\mathfrak{u}_0 \otimes V_i, \mathbb{F}\mathfrak{I}). \end{aligned}$$

As before, since $\dim(\mathfrak{u}_0 \otimes V_i) > 1$, there is no $\mathfrak{g}_0$-module isomorphism from $\mathfrak{u}_0 \otimes V_i$ to $\mathbb{F}\mathfrak{I}$, so $\hom_{\mathfrak{g}_0}(\mathfrak{u}_0 \otimes V_i, \mathbb{F}\mathfrak{I}) = \{0\}$. Also the set of weights of $\mathfrak{u}_0 \otimes V_i$ nontrivially intersects $\{\pm\delta_p \pm \delta_q \pm \epsilon_j \mid 1 \leq p, q \leq m, 1 \leq j \leq n\}$ while the set of weights of $\mathfrak{g}_0$-module $\mathfrak{s}_{B}$ and the set of weights of $\mathfrak{g}_0$-module $\mathfrak{s}_{C}$ are $\{\pm\epsilon_\imath \pm \epsilon_\jmath \mid 1 \leq \imath, \jmath \leq m\}$ and $\{0, \pm(\delta_p \pm \delta_q) \mid 1 \leq p < q \leq n\}$ respectively. Therefore $\hom_{\mathfrak{g}_0}(\mathfrak{u}_0 \otimes V_i, \mathfrak{s}_{B}) = \{0\}$ and $\hom_{\mathfrak{g}_0}(\mathfrak{u}_0 \otimes V_i, \mathfrak{s}_{C}) = \{0\}$, for all $1 \leq i \leq s$.

$\underline{\hom_{\mathfrak{g}_0}(\mathfrak{g}_1 \otimes \mathfrak{s}_0, \mathfrak{u}_0) = \{0\}}$: Suppose that $\mathfrak{u}_1 \otimes \mathfrak{s}_{C} = \oplus_{i=1}^{t} V(\eta_i)$ is the decomposition of the $\mathfrak{g}_{C}$-module $\mathfrak{u}_1 \otimes \mathfrak{s}_{C}$ into irreducible submodules and note that by (18), $\{\eta_i \mid 1 \leq i \leq t\} \subseteq \{(\delta_1 + \delta_2) \pm \delta_p \mid 1 \leq p \leq n\}$, so for $1 \leq i \leq t$, $\eta_i \neq 0$. Therefore, $\dim(V(\eta_i)) \neq 1$ which in turn implies that $\dim(\mathfrak{u}_0 \otimes V(\eta_i)) \neq \dim(\mathfrak{u}_0)$. In particular, since $\mathfrak{u}_0 \otimes V(\eta_i)$ and $\mathfrak{u}_0$ are irreducible $\mathfrak{g}_0$-modules, we have

$$\hom_{\mathfrak{g}_0}(\mathfrak{u}_0 \otimes V(\eta_i), \mathfrak{u}_0) = \{0\};\ 1 \leq i \leq t. \tag{19}$$

Next suppose that $\mathfrak{u}_0 \otimes \mathfrak{s}_{B} = \oplus_{i=1}^{s} V(\theta_i)$ is the decomposition of the $\mathfrak{g}_{B}$-module $\mathfrak{u}_0 \otimes \mathfrak{s}_{B}$ into finite dimensional irreducible highest weight submodules, then by (18), $\{\theta_i \mid 1 \leq i \leq s\} \subseteq \{2\epsilon_1, 2\epsilon_1 \pm \epsilon_\jmath \mid 1 \leq \jmath \leq m\}$. Therefore, for each $1 \leq i \leq s$, the set of weights of $V(\theta_i) \otimes \mathfrak{u}_1$ nontrivially intersects

$$\{2\epsilon_1 \pm \delta_p, 2\epsilon_1 \pm \epsilon_\jmath \pm \delta_p \mid 1 \leq \jmath \leq m, 1 \leq p \leq n\}.$$

So $V(\theta_i)\otimes\mathfrak{u}_1$ is not isomorphic to $\mathfrak{u}_0$ or $\mathfrak{u}_1$ as the set of weights of $\mathfrak{u}_0$ is $\{0,\pm\epsilon_\jmath \mid 1\leq \jmath\leq m\}$ and the set of weights of $\mathfrak{u}_1$ is $\{\pm\delta_p \mid 1\leq p\leq n\}$; in particular, since $V(\theta_i)\otimes\mathfrak{u}_1,\mathfrak{u}_0$ and $\mathfrak{u}_1$ are irreducible $\mathfrak{g}_0$-module, we have

$$\hom_{\mathfrak{g}_0}(V(\theta_i)\otimes\mathfrak{u}_1,\mathfrak{u}_0)=\{0\}\quad\text{and}\quad \hom_{\mathfrak{g}_0}(V(\theta_i)\otimes\mathfrak{u}_1,\mathfrak{u}_1)=\{0\}. \tag{20}$$

We also note that $\mathfrak{g}_1$ and $\mathfrak{u}_0$ as well as $\mathfrak{g}_1$ and $\mathfrak{u}_1$ are non-isomorphic irreducible $\mathfrak{g}_0$-modules, so we get that

$$\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathbb{F}\mathfrak{I},\mathfrak{u}_0)=\{0\}\quad\text{and}\quad \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathbb{F}\mathfrak{I},\mathfrak{u}_1)=\{0\}. \tag{21}$$

Now using (19)-(21), we have

$$\begin{aligned}\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_0,\mathfrak{u}_0)&\simeq\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_B,\mathfrak{u}_0)\oplus\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_C,\mathfrak{u}_0)\oplus\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathbb{F}\mathfrak{I},\mathfrak{u}_0)\\ &\simeq\hom_{\mathfrak{g}_0}((\mathfrak{u}_0\otimes\mathfrak{s}_B)\otimes\mathfrak{u}_1,\mathfrak{u}_0)\oplus\hom_{\mathfrak{g}_0}(\mathfrak{u}_0\otimes(\mathfrak{u}_1\otimes\mathfrak{s}_C),\mathfrak{u}_0)\\ &\simeq\hom_{\mathfrak{g}_0}(\oplus_{i=1}^{s}V(\theta_i)\otimes\mathfrak{u}_1,\mathfrak{u}_0)\oplus\hom_{\mathfrak{g}_0}(\mathfrak{u}_0\otimes\oplus_{i=1}^{t}V(\eta_i),\mathfrak{u}_0)\\ &\simeq\oplus_{i=1}^{s}\hom_{\mathfrak{g}_0}(V(\theta_i)\otimes\mathfrak{u}_1,\mathfrak{u}_0)\oplus\oplus_{i=1}^{t}\hom_{\mathfrak{g}_0}(\mathfrak{u}_0\otimes V(\eta_i),\mathfrak{u}_0)\\ &=\ \{0\}.\end{aligned}$$

$\underline{\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_0,\mathfrak{u}_1)=\{0\}}$: For this, we first note that if

$$0\neq\varphi\in\hom_{\mathfrak{g}_0}(\mathfrak{u}_0\otimes V(\eta_i),\mathfrak{u}_1),$$

then φ is an isomorphism and so $\dim(V(\eta_i))=2n/(2m+1)\notin\mathbb{Z}$, a contradiction. This together with (20) and (21) implies that

$$\begin{aligned}\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_0,\mathfrak{u}_1)&\simeq\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_B,\mathfrak{u}_1)\oplus\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{s}_C,\mathfrak{u}_1)\oplus\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathbb{F}\mathfrak{I},\mathfrak{u}_1)\\ &\simeq\hom_{\mathfrak{g}_0}((\mathfrak{u}_0\otimes\mathfrak{s}_B)\otimes\mathfrak{u}_1,\mathfrak{u}_1)\oplus\hom_{\mathfrak{g}_0}(\mathfrak{u}_0\otimes(\mathfrak{u}_1\otimes\mathfrak{s}_C),\mathfrak{u}_1)\\ &\simeq\hom_{\mathfrak{g}_0}(\oplus_{i=1}^{s}V(\theta_i)\otimes\mathfrak{u}_1,\mathfrak{u}_1)\oplus\hom_{\mathfrak{g}_0}(\mathfrak{u}_0\otimes\oplus_{i=1}^{t}V(\eta_i),\mathfrak{u}_1)\\ &\simeq\oplus_{i=1}^{s}\hom_{\mathfrak{g}_0}(V(\theta_i)\otimes\mathfrak{u}_1,\mathfrak{u}_1)\oplus\oplus_{i=1}^{t}\hom_{\mathfrak{g}_0}(\mathfrak{u}_0\otimes V(\eta_i),\mathfrak{u}_1)\\ &\simeq\oplus_{i=1}^{t}\hom_{\mathfrak{g}_0}(\mathfrak{u}_0\otimes V(\eta_i),\mathfrak{u}_1)=\{0\}.\end{aligned}$$

So considering the material of Section 3 of [11], we are done using the fact that $\mathfrak{s}_1$ is a $\mathfrak{g}_0$-module isomorphic to the $\mathfrak{g}_0$-module $\mathfrak{g}_1$. ∎

Recall (16) and suppose that $|I|=m,|J|=n$. One knows that

$$\Pi:=\{\delta_1-\delta_2,\delta_2-\delta_3,\ldots,\delta_{n-1}-\delta_n,\delta_n-\epsilon_1,\epsilon_1-\epsilon_2,\ldots,\epsilon_{m-1}-\epsilon_m,\epsilon_m\}$$

is a fundamental system for the root system R of the finite dimensional basic classical simple Lie superalgebra $\mathfrak{g}$ with respect to the positive system

$$\{\delta_p\pm\delta_q,2\delta_p,\delta_p,\delta_p\pm\epsilon_\imath,\epsilon_\imath\pm\epsilon_\jmath,\epsilon_\imath \mid 1\leq\imath<\jmath\leq m,\ 1\leq p<q\leq n\}.$$

Set $\rho:=(1/2)\sum_{\alpha\in R_0^+}\alpha-(1/2)\sum_{\alpha\in R_1^+}\alpha$, where R_0^+ (resp. R_1^+) is the set of positive even (resp. odd) roots and recall from [15] that the *Casimir element* Γ of $\mathfrak{g}$ is $\sum_{i=1}^{k}(-1)^{|x_i|}x_ix^i$ where $\{x_i\mid 1\leq i\leq k\}$ and $\{x^i\mid 1\leq i\leq k\}$ are dual bases of $\mathfrak{g}$ with respect to the Killing form on $\mathfrak{g}$. We know from [15, (2.2)] that

the Casimir element Γ of $\mathfrak{g}$ acts on the highest weight $\mathfrak{g}$-module of highest weight λ as $(\lambda,\lambda+2\rho)\mathrm{id}$ (22)

where " id " indicates the identity map. Moreover, up to a scalar product, we have

$$(\lambda,\lambda+2\rho)=\begin{cases} m-n & \text{if } \lambda=\delta_1\\ -1-2(n-m) & \text{if } \lambda=2\delta_1\\ 1-2(n-m) & \text{if } \lambda=\delta_1+\delta_2.\end{cases} \tag{23}$$

Using [13, Thm. 2.11] (see also [15, Thm. 8]), we get that the only nonzero elements of

$$\Psi := R\cup\{\pm 2\epsilon_\imath \mid 1\leq \imath\leq m\} \tag{24}$$

which can be the highest weight for a finite dimensional irreducible $\mathfrak{g}$-module are

$$\begin{array}{ll} 2\delta_1,\;\; \delta_1+\delta_2,\;\; \delta_1 & \text{if } n\geq 2,\\ 2\delta_1,\;\; \delta_1+\epsilon_1,\;\; \delta_1 & \text{if } n=1.\end{array}$$

One knows that up to isomorphism, the only finite dimensional irreducible $\mathfrak{g}$-module whose highest weight is $2\delta_1$ (resp. δ_1) is $\mathfrak{g}$ (resp. $\mathfrak{u}$). Also up to isomorphism, $\mathfrak{s}$ is the only finite dimensional irreducible $\mathfrak{g}$-module whose highest weight is $\delta_1+\delta_2$ if $n\neq 1$ and $\epsilon_1+\delta_1$ if $n=1$. The following lemma and its corollary are a slight generalization of a result of [11, §3].

Lemma 4.4. *Let $n\neq 1$ and consider (24). Suppose that X is a finite dimensional $\mathfrak{g}$-module equipped with a weight space decomposition with respect to $\mathfrak{h}$ whose set of weights is contained in Ψ. Suppose that Y is an irreducible $\mathfrak{g}$-submodule of X isomorphic to one of the $\mathfrak{g}$-modules $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{s}$ or the trivial module such that X/Y is also an irreducible $\mathfrak{g}$-module isomorphic to one of the above $\mathfrak{g}$-modules, then X is completely reducible.*

Proof. For $x\in X$, we denote the image of x in X/Y under the canonical epimorphism $^{-}: X\longrightarrow X/Y$ by $\bar{x}$. Since Y and X/Y are finite dimensional irreducible $\mathfrak{g}$-modules, they are highest weight modules. Suppose that λ and μ are the highest weights of Y and X/Y respectively. We first suppose that $(\lambda,\lambda+2\rho)\neq(\mu,\mu+2\rho)$. If r is an eigenvalue of the action of the Casimir element Γ on X, then there is a nonzero $x\in X$ with $\Gamma x=rx$, so $\Gamma\bar{x}=r\bar{x}$. This means that either $\bar{x}=0$ or $r=(\mu,\mu+2\rho)$ by (22). In the former case, $x\in Y$ and so $r=(\lambda,\lambda+2\rho)$. Therefore, the only eigenvalues for the action of Γ on X are $(\lambda,\lambda+2\rho)$ and $(\mu,\mu+2\rho)$; in particular $X=X_\lambda\oplus X_\mu$ in which X_λ and X_μ are the generalized eigenspaces corresponding to $(\lambda,\lambda+2\rho)$ and $(\mu,\mu+2\rho)$ respectively. Since Γ is a $\mathfrak{g}$-module homomorphism, X_λ and X_μ are $\mathfrak{g}$-submodules of X with $Y\subseteq X_\lambda$, therefore, we have $\frac{X}{Y}=\frac{X_\lambda}{Y}\oplus\frac{X_\mu+Y}{Y}$. But the only eigenvalue for the action of Γ on X/Y is $(\mu,\mu+2\rho)$, so $X_\lambda/Y=\{0\}$; i.e., $X_\lambda=Y$ is an irreducible $\mathfrak{g}$-module. This also implies that $X_\mu\simeq X/Y$ is an irreducible $\mathfrak{g}$-module. Therefore, $X=X_\lambda\oplus X_\mu$ is completely reducible. This completes the proof in this case. So from now till the end of the proof, we assume $(\lambda,\lambda+2\rho)=(\mu,\mu+2\rho)$. By (23), one of the following cases can happen:

- Y is isomorphic to X/Y,
- one of Y and X/Y is the trivial module and the other one is isomorphic to $\mathfrak{u}$,

- one of Y and X/Y is isomorphic to $\mathfrak{g}$ and the other one is isomorphic to $\mathfrak{u}$,
- one of Y and X/Y is isomorphic to $\mathfrak{s}$ and the other one is isomorphic to $\mathfrak{u}$.

Using the same argument as in [11, §3] together with Proposition 4.3, we get that in the first case, X is completely reducible and that the last three cases result in a contradiction but for the convenience of readers, we carry out the proof for one case. Suppose that Y is isomorphic to $\mathfrak{s}$ and X/Y is isomorphic to $\mathfrak{u}$, then by (23), $m-n=1-2(n-m)$ and so $\frac{2n}{2m+1}\notin\mathbb{Z}$. Consider X as a $\mathfrak{g}_0$-module, then X_0 as well as X_1 are completely reducible $\mathcal{G}_0$-modules and so for $i=0,1$, there is a $\mathfrak{g}_0$-submodule Z_i of X_i with $X_i=Y_i\oplus Z_i$. Set $Z:=Z_0\oplus Z_1$ which is a $\mathbb{Z}_2$-graded subspace of X. Since $\mathfrak{g}$-module X/Y is isomorphic to $\mathfrak{u}$, Z as a $\mathfrak{g}_0$-module is isomorphic to $\mathfrak{u}$. So $X=Y_0\oplus Y_1\oplus Z_0\oplus Z_1$ is a decomposition of X into $\mathfrak{g}_0$-modules with either $Z_0\simeq\mathfrak{u}_0$ and $Z_1\simeq\mathfrak{u}_1$ or $Z_0\simeq\mathfrak{u}_1$ and $Z_1\simeq\mathfrak{u}_0$. Since by Proposition 4.3,

$$\begin{aligned}&\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_0,\mathfrak{s}_0)=\{0\},\quad \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_1,\mathfrak{s}_0)=\{0\},\\ &\hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_0,\mathfrak{s}_1)=\{0\},\quad \hom_{\mathfrak{g}_0}(\mathfrak{g}_1\otimes\mathfrak{u}_1,\mathfrak{s}_1)=\{0\},\end{aligned}$$

it follows that for $i=0,1$, $\mathfrak{g}_1Z_i\subseteq Z$, and so $\mathfrak{g}Z\subseteq Z$. This together with the fact that Z is a $\mathbb{Z}_2$-graded subspace of X implies that Z is a $\mathfrak{g}$-submodule of X. Also as $\mathfrak{g}$-module X/Y is isomorphic to $\mathfrak{u}$, Z as a $\mathfrak{g}$-module is isomorphic to $\mathfrak{u}$. Therefore, X is completely reducible. ■

Corollary 4.5. *Suppose that X is a finite dimensional $\mathfrak{g}$-module equipped with a weight space decomposition with respect to $\mathfrak{h}$. If the set of weights of X is a subset of Ψ, then X is completely reducible such that its irreducible constituents are isomorphic to one of $\mathfrak{g}$-modules $\mathfrak{g}$, $\mathfrak{s}$, $\mathfrak{u}$ or the trivial $\mathfrak{g}$-module.*

Proof. One knows that X has a composition series, say $\{0\}=X_0\subseteq X_1\subseteq X_2\subseteq\cdots\subseteq X_t=X$. For each $1\leq i\leq t$, X_i is an $\mathfrak{h}$-submodule of X and so it inherits the weight space decomposition of X with respect to $\mathfrak{h}$. This implies that the set of weights of X_i is contained in Ψ and so the set of weights of the irreducible $\mathfrak{g}$-module X_i/X_{i-1} is contained in Ψ. Therefore, X_i/X_{i-1} is a finite dimensional irreducible $\mathfrak{g}$-module whose highest weight is an element of Ψ; in particular, it either is isomorphic to one of $\mathfrak{g}$-modules $\mathfrak{g}$, $\mathfrak{s}$, $\mathfrak{u}$ or is the trivial $\mathfrak{g}$-module. Now the result follows using Lemma 4.4. ■

4.3. Structure of $BC(I,J)$-graded Lie superalgebras.. Throughout this subsection, we suppose $\mathbb{F}$ is an algebraically closed field of characteristic zero and use the same notation as in Subsection 3; in particular, we recall $\mathfrak{g}$ as well as $\mathfrak{s}$ from (13), $\mathfrak{h}$ from (14), $\Delta_{\mathfrak{u}}$ from (15) and $R,\Delta_{\mathfrak{s}}$ from (16). We set

$$\Psi:=\{\pm\epsilon_\imath\pm\epsilon_\jmath,\pm\epsilon_\imath,\pm\delta_p\pm\delta_q,\pm\epsilon_\imath,\pm\delta_p\mid\imath\in I,\jmath\in J\}=R\cup\{\pm2\epsilon_\imath\mid\imath\in I\};$$

in fact Ψ is a locally finite root supersystem of type $BC(I,J)$. Suppose that $\mathfrak{L}$ is a Lie superalgebra such that

- $\mathfrak{L}$ contains $\mathfrak{g}$ as a subsuperalgebra,
- $\mathfrak{L}$ is equipped with a weight space decomposition $\mathfrak{L}=\oplus_{\alpha\in\Psi}\mathfrak{L}^\alpha$, with respect to $\mathfrak{h}$,
- $\mathfrak{L}^0=\sum_{\alpha\in\Psi\setminus\{0\}}[\mathfrak{L}^\alpha,\mathfrak{L}^{-\alpha}]$.

(25)

It is easy to see that $\mathfrak{L}$ is a Ψ-graded Lie superalgebra with R as its grading subsystem. One knows that (25) is just a generalization of the notion of a root graded Lie superalgebra in the sense of [9] by switching from finite root supersystems to locally finite root supersystems.

We study the structure of a Lie superalgebra $\mathfrak{L}$ satisfying (25). We make a convention that for a map f defined on a set X, by x^f, for $x \in X$, we mean the image of x under f.

4.0.1. Some Conventions

Suppose that

- $\mathfrak{a}$ is an associative superalgebra and η is a *superinvolution* of $\mathfrak{a}$, i.e., η is an even linear map with $\eta^2 = id$ and $\eta(ab) = (-1)^{|a||b|}\eta(b)\eta(a)$ for all $a, b \in \mathfrak{a}$,
- $\mathcal{C}$ is a module for the associative superalgebra $\mathfrak{a}$; i.e., $\mathcal{C}$ is a superspace equipped with an action $\mathfrak{a} \times \mathcal{C} \longrightarrow \mathcal{C}$ mapping $(x, c) \in \mathfrak{a} \times \mathcal{C}$ to xc such that $\mathfrak{a}_i\mathcal{C}_j \subseteq \mathcal{C}_{i+j}$ for $i, j \in \mathbb{Z}_2$ and that $x(yc) = (xy)c$ for $x, y \in \mathfrak{a}$ and $c \in \mathcal{C}$,
- $\chi : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is a *superhermitian* $\mathfrak{a}$-form of $\mathcal{C}$, in the sense that χ is an even bilinear form satisfying

$$\chi(x, y)^\eta = (-1)^{|x||y|}\chi(y, x) \quad \text{and} \quad \chi(ax, y) = a\chi(x, y)$$

for all $x, y \in \mathcal{C}$ and $a \in \mathfrak{a}$.

Then $\mathfrak{b} := \mathfrak{b}(\mathfrak{a}, \mathcal{C}) := \mathfrak{a} \oplus \mathcal{C}$ together with

$$(\alpha + c)(\alpha' + c') = (\alpha \cdot \alpha' + 2\chi(c, c')) + (\alpha \cdot c' + (-1)^{|\alpha'||c|}(\alpha')^\eta \cdot c)$$

is a superalgebra. We set

$$\mathcal{A} := \{\alpha \in \mathfrak{a} \mid \alpha^\eta = \alpha\} \quad \text{and} \quad \mathcal{B} := \{\alpha \in \mathfrak{a} \mid \alpha^\eta = -\alpha\}$$

and note that

$$\mathfrak{a} = \mathcal{A} \oplus \mathcal{B}.$$

We next define

$$\diamond : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{A} \quad (c, c') \mapsto \tfrac{1}{2}(\chi(c, c') + (-1)^{|c||c'|}\chi(c', c))$$

$$\heartsuit : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{B} \quad (c, c') \mapsto \tfrac{1}{2}(\chi(c, c') - (-1)^{|c||c'|}\chi(c', c)).$$

Finally for $\beta_1 = a_1 + b_1 + c_1, \beta_2 = a_2 + b_2 + c_3 \in \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$, we set

$$\beta^*_{\beta_1,\beta_2} := [a_1, a_2] + [b_1, b_2] + 2c_1 \heartsuit c_2,\ \beta^*_1 := c_1 \quad \text{and} \quad \beta^*_2 := c_2. \tag{26}$$

4.0.2. Structure Theorem

We suppose $|I|, |J| > 4$ and fix a subset I_0 of I and a subset J_0 of J with $m := |I_0| > 3$ and $n := |J_0| > 3$. We set

$$\begin{array}{l}\Phi := \{\pm\epsilon_\imath \pm \epsilon_\jmath, \pm\epsilon_\imath, \pm\delta_p \pm \delta_q, \pm\delta_p, \pm\epsilon_\imath \pm \delta_p \mid \imath \in I_0, \jmath \in J_0\},\\ R^{m,n} := \Phi \setminus \{\pm 2\epsilon_\imath \mid \imath \in I_0\}.\end{array} \tag{27}$$

We next define the linear endomorphism $\mathrm{id}_{m,n}$ on $\mathfrak{u}$ by

$$\begin{array}{c}\mathrm{id}_{m,n} : \mathfrak{u} \longrightarrow \mathfrak{u}\\ v_0 \mapsto v_0,\ v_\imath \mapsto v_\imath,\ v_{\bar\imath} \mapsto v_{\bar\imath},\ v_\jmath \mapsto 0,\ v_{\bar\jmath} \mapsto 0,\\ (\imath \in I_0 \cup J_0,\ \jmath \in I \cup J \setminus (I_0 \cup J_0))\end{array}$$

and for $u, v \in \mathfrak{u}$ and $x, y \in \mathfrak{g} \cup \mathfrak{s}$, define

$$\begin{array}{l}[u, v] : \mathfrak{u} \longrightarrow \mathfrak{u};\ w \mapsto (v, w)u + (-1)^{|u||v|}(u, w)v - \frac{2(u,v)}{2m+1-2n}\mathrm{id}_{m,n}(w);\quad w \in \mathfrak{u},\\ u \circ v : \mathfrak{u} \longrightarrow \mathfrak{u};\ w \mapsto (v, w)u - (-1)^{|u||v|}(u, w)v;\quad w \in \mathfrak{u},\\ x \circ y := xy + (-1)^{|x||y|}yx - \frac{2str(xy)}{2m+1-2n}\mathrm{id}_{m,n}.\end{array}$$

Theorem 4.6. *Suppose that $\mathfrak{L}$ is a Lie superalgebra satisfying the following:*

- *$\mathfrak{L}$ contains $\mathfrak{g}$ as a subsuperalgebra,*
- *$\mathfrak{L}$ is equipped with a weight space decomposition $\mathfrak{L} = \oplus_{\alpha\in\Psi}\mathfrak{L}^\alpha$, with respect to $\mathfrak{h}$,*
- *$\mathfrak{L}^0 = \sum_{\alpha\in\Psi\setminus\{0\}}[\mathfrak{L}^\alpha, \mathfrak{L}^{-\alpha}]$.*

Then there are a subsuperalgebra $\mathcal{D}$ of $\mathfrak{L}$, superspaces $\mathcal{A}, \mathcal{B}, \mathcal{C}$, even bilinear maps

$$\cdot : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a} \quad \cdot : \mathfrak{a} \times \mathcal{C} \longrightarrow \mathcal{C},\quad \chi : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a},\quad \langle\cdot,\cdot\rangle : \mathfrak{b} \times \mathfrak{b} \longrightarrow \mathcal{D}$$

in which $\mathfrak{b} := \mathfrak{a} \oplus \mathcal{C}$, and linear maps

$$\eta : \mathfrak{a} \longrightarrow \mathfrak{a} \quad \text{and} \quad \phi : \mathcal{D} \longrightarrow End(\mathfrak{b})$$

such that $(\mathfrak{a}, \cdot)$ is an associative superalgebra, $(\mathcal{C}, \cdot)$ is an associative $\mathfrak{a}$-module, η is a superinvolution and χ is a superhermitian $\mathfrak{a}$-form such that the following properties hold:

- *$\langle\cdot,\cdot\rangle$ is supersymmetric and if $\mathcal{A}$ and $\mathcal{B}$ are respectively the fixed and skew-fixed points of $\mathfrak{a}$ under the map η, $\langle\cdot,\cdot\rangle$ satisfies $\langle\mathcal{A}, \mathcal{B}\rangle = \langle\mathcal{A}, \mathcal{C}\rangle = \langle\mathcal{B}, \mathcal{C}\rangle = \{0\}$ and $\langle\mathfrak{b}, \mathfrak{b}\rangle = \mathcal{D}$,*

- *considering the superalgebraic structure on $\mathfrak{b}$ as constructed in Subsection 4.0.1, for each $d \in \mathcal{D}$, we have $\phi(d)$ is a superderivation of $\mathfrak{b}$; i.e., $\mathcal{D}$ acts on $\mathfrak{b}$ as superderivations,*

- *$d\mathcal{A} \subseteq \mathcal{A}$, $d\mathcal{B} \subseteq \mathcal{B}$ and $d\mathcal{C} \subseteq \mathcal{C}$, for all $d \in \mathcal{D}$,*

- $[d, \langle \beta, \beta' \rangle] = \langle d\beta, \beta' \rangle + (-1)^{|d||b'|}\langle \beta, d\beta' \rangle$,
- $\sum_{\circlearrowleft}(-1)^{|\beta_1||\beta_3|}\langle \beta_1, \beta_2\beta_3 \rangle = 0$, *in which " $\circlearrowleft$ " indicates the cyclic permutation,*
- $\langle \alpha, \alpha' \rangle \alpha'' = \frac{1}{2(2m+1-2n)}[[\alpha, \alpha'] - [\alpha, \alpha']^\eta, \alpha'']$,
- $\langle \alpha, \alpha' \rangle c = \frac{1}{2(2m+1-2n)}([\alpha, \alpha'] - [\alpha, \alpha']^\eta)c$,
- $\langle c, c' \rangle \alpha = \frac{1}{2m+1-2n}[\chi(c, c') - \chi(c, c')^\eta, \alpha]$,
- $\langle c, c' \rangle c'' = \frac{1}{2m+1-2n}(\chi(c, c') - \chi(c, c')^\eta) \cdot c'' + (-1)^{|c|(|c'|+|c''|)}\chi(c', c'')^\eta \cdot c - (-1)^{|c'||c''|}\chi(c, c'')^\eta \cdot c'$.

Moreover, we have the following:

(i) There are subsuperspaces $\mathfrak{L}^1, \mathfrak{L}^2$ *and* $\mathfrak{L}^3$ *of* $\mathfrak{L}$ *isomorphic to* $\mathfrak{g} \otimes \mathcal{A}, \mathfrak{s} \otimes \mathcal{B}$ *and* $\mathfrak{u} \otimes \mathcal{C}$ *respectively such that*

$$\mathfrak{L} = (\mathfrak{L}^1 \oplus \mathfrak{L}^2 \oplus \mathfrak{L}^3) + \mathcal{D}.$$

Furthermore, if either $|I| = m$ *and* $|J| = n$ *or* $I \cup J$ *is an infinite set, we have*

$$\mathfrak{L} = \mathfrak{L}^1 \oplus \mathfrak{L}^2 \oplus \mathfrak{L}^3 \oplus \mathcal{D},$$

more precisely, in these cases $\mathfrak{L}$ *can be identified with*

$$(\mathfrak{g} \otimes \mathcal{A}) \oplus (\mathfrak{s} \otimes \mathcal{B}) \oplus (\mathfrak{u} \otimes \mathcal{C}) \oplus \mathcal{D}.$$

(ii) Identify $\mathfrak{L}^1, \mathfrak{L}^2$ *and* $\mathfrak{L}^3$ *with* $\mathfrak{g} \otimes \mathcal{A}, \mathfrak{s} \otimes \mathcal{B}$ *and* $\mathfrak{u} \otimes \mathcal{C}$ *respectively. The Lie bracket on* $\mathfrak{L}$ *is given by the following:*

$$\begin{aligned}
&[x \otimes a, y \otimes a'] &&= (-1)^{|a||y|}([x, y] \otimes \tfrac{1}{2}(a \circ a') + (x \circ y) \otimes \tfrac{1}{2}[a, a'] + str(xy)\langle a, a' \rangle),\\
&[x \otimes a, e \otimes b] &&= (-1)^{|a||e|}((x \circ e) \otimes \tfrac{1}{2}[a, b] + [x, e] \otimes \tfrac{1}{2}(a \circ b)),\\
&[e \otimes b, f \otimes b'] &&= (-1)^{|b||f|}([e, f] \otimes \tfrac{1}{2}(b \circ b') + (e \circ f) \otimes \tfrac{1}{2}[b, b'] + str(ef)\langle b, b' \rangle),\\
&[x \otimes a, u \otimes c] &&= (-1)^{|a||u|}xu \otimes a \cdot c,\\
&[e \otimes b, u \otimes c] &&= (-1)^{|b||u|}eu \otimes b \cdot c,\\
&[u \otimes c, v \otimes c'] &&= (-1)^{|c||v|}((u \circ v) \otimes (c \diamond c') + [u, v] \otimes (c \heartsuit c') + (u, v)\langle c, c' \rangle)\\
&[\langle \beta_1, \beta_2 \rangle, \langle \beta'_1, \beta'_2 \rangle] &&= \langle \langle \beta_1, \beta_2 \rangle \beta'_1, \beta'_2 \rangle + (-1)^{(|\beta_1|+|\beta_2|)|\beta'_1|}\langle \beta'_1, \langle \beta_1, \beta_2 \rangle \beta'_2 \rangle,\\
&[\langle \beta_1, \beta_2 \rangle, x \otimes a] &&= \tfrac{(-1)^{|\beta_1||x|+|\beta_2||x|}}{2(2m+1-2n)}([id_{m,n}, x] \otimes (\beta^*_{\beta_1,\beta_2} \circ a) + (id_{m,n} \circ x) \otimes [\beta^*_{\beta_1,\beta_2}, a])\\
&[\langle \beta_1, \beta_2 \rangle, e \otimes b] &&= \tfrac{(-1)^{|\beta_1||x|+|\beta_2||x|}}{2(2m+1-2n)}([id_{m,n}, e] \otimes (\beta^*_{\beta_1,\beta_2} \circ b) + (id_{m,n} \circ e) \otimes [\beta^*_{\beta_1,\beta_2}, b])\\
& &&\quad - \tfrac{1}{2m+1-2n}str(id_{m,n}e)\langle [b_1, b_2], b \rangle\\
&[\langle \beta_1, \beta_2 \rangle, u \otimes c] &&= \tfrac{(-1)^{|\beta_1||u|+|\beta_2||u|}}{2m+1-2n}(id_{m,n}u \otimes \beta^*_{\beta_1,\beta_2}c) + (-1)^{|\beta_1||u|+|\beta_2||u|}u \otimes\\
& &&\quad ((-1)^{|\beta^*_1||\beta^*_2|+|\beta^*_1||c|}\chi(\beta^*_2, c)^\eta \beta^*_1 - (-1)^{|\beta^*_2||c|}\chi(\beta^*_1, c)^\eta \beta^*_2).
\end{aligned} \tag{28}$$

Remark 4.7. We mention that if $|I| = m$ and $|J| = n$, then the last three Lie brackets in the above display will be converted to the following ones:

$$\begin{aligned} [\langle\beta_1,\beta_2\rangle, x\otimes a] &= (-1)^{|\beta_1||x|+|\beta_2||x|} x\otimes\langle\beta_1,\beta_2\rangle a \\ [\langle\beta_1,\beta_2\rangle, e\otimes b] &= (-1)^{|\beta_1||e|+|\beta_2||e|} e\otimes\langle\beta_1,\beta_2\rangle b \\ [\langle\beta_1,\beta_2\rangle, u\otimes c] &= (-1)^{|\beta_1||u|+|\beta_2||u|} u\otimes\langle\beta_1,\beta_2\rangle c. \end{aligned} \tag{29}$$

To prove Theorem 4.6, we first carry out the proof for the case that $|I|, |J| < \infty$; at the first step, we suppose $I = I_0$ and $J = J_0$.

• *Finite Case-The First Step.* In this part, we assume $I = I_0$, $J = J_0$ and that $\mathfrak{L}$ is a Lie superalgebra satisfying (25). Consider $\mathfrak{L}$ as a $\mathfrak{g}$-module via the adjoint representation, then $\mathfrak{L}$ is a locally finite $\mathfrak{g}$-module, i.e., any finite subset of $\mathfrak{L}$ generates a finite dimensional $\mathfrak{g}$-submodule (see [11, Lem. 2.2]). Therefore, it is a sum of finite dimensional $\mathfrak{g}$-submodules. Using Corollary 4.5, $\mathfrak{L}$ is completely reducible such that each of its irreducible components is either isomorphic to one of $\mathfrak{g}$-modules $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{s}$ or it is a trivial $\mathfrak{g}$-module. Now collecting the isomorphic $\mathfrak{g}$-submodules of the same parity, we may assume that as a vector space, $\mathfrak{L}$ is isomorphic to

$$(\mathfrak{g}\otimes\mathcal{A}_0)\oplus(\mathfrak{g}\otimes\mathcal{A}_1)\oplus(\mathfrak{s}\otimes\mathcal{B}_0)\oplus(\mathfrak{s}\otimes\mathcal{B}_1)\oplus(\mathfrak{u}\otimes\mathcal{C}_0)\oplus(\mathfrak{u}\otimes\mathcal{C}_1)\oplus\mathcal{D};$$

in which $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{C}_0$ and $\mathcal{C}_1$ are vector spaces and $\mathcal{D}$ is the centralizer of $\mathcal{G}$ in $\mathfrak{L}$; in particular, $\mathcal{D}$ is a subsuperalgebra of $\mathcal{L}$. Setting

$$\mathcal{A} := \mathcal{A}_0\oplus\mathcal{A}_1,\ \mathcal{B} := \mathcal{B}_0\oplus\mathcal{B}_1,\ \mathcal{C} := \mathcal{C}_0\oplus\mathcal{C}_1,\ \mathcal{D} := \mathcal{D}_0\oplus\mathcal{D}_1,$$

we can consider

$$\mathfrak{L} = (\mathfrak{g}\otimes\mathcal{A})\oplus(\mathfrak{s}\otimes\mathcal{B})\oplus(\mathfrak{u}\otimes\mathcal{C})\oplus\mathcal{D}.$$

Now using the same argument as in [11, § 5], one can see that the Lie superalgebraic structure on $\mathcal{L}$ induces a superalgebraic structure on $\mathcal{A}\oplus\mathcal{B}\oplus\mathcal{C}$ and that the stated properties in Theorem 4.6 are fulfilled.

• *Finite Case-Compatibility of Subsuperalgebras.* Throughout this part, we assume that $m' > m$, $n' > n$, $I = \{1,\ldots,m'\}, J = \{1,\ldots,n'\}$ and $I_0 = \{1,\ldots,m\}, J_0 = \{1,\ldots,n\}$. We also assume

$$\mathfrak{L} := \sum_{\alpha\in\Psi\setminus\{0\}}\mathfrak{L}^\alpha\oplus\sum_{\alpha\in\Psi\setminus\{0\}}[\mathfrak{L}^\alpha,\mathfrak{L}^{-\alpha}]$$

is a Lie superalgebra satisfying (25). Consider $\mathfrak{L}$ as a $\mathfrak{g}$-module. As in the previous subsection, it follows from Corollary 4.5 that $\mathfrak{L}$ is decomposed into irreducible submodules, more precisely, there are index sets T_1, T_2 and T_3 such that

$$\mathfrak{L} = \sum_{i\in T_1}\mathfrak{g}^{(i)}\oplus\sum_{j\in T_2}\mathcal{V}^{(j)}\oplus\sum_{t\in T_3}\mathfrak{s}^{(t)}\oplus E \tag{30}$$

in which $\mathfrak{g}^{(i)}$ is isomorphic to $\mathfrak{g}$, $\mathcal{V}^{(j)}$ is isomorphic to $\mathfrak{u}$, $\mathfrak{s}^{(t)}$ is isomorphic to $\mathfrak{s}$ for all $i \in T_1, j \in T_2, t \in T_3$ and E is a trivial $\mathfrak{g}$-module. As before, collecting the isomorphic $\mathfrak{g}$-submodules, we may assume

$$\mathfrak{L} = (\mathfrak{g} \otimes \mathcal{A}) \oplus (\mathfrak{s} \otimes \mathcal{B}) \oplus (\mathfrak{u} \otimes \mathcal{C}) \oplus E, \tag{31}$$

in which $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are vector superspaces. We recall (27) and use Proposition 4.2 to get that

$$\mathcal{G} := \sum_{\alpha \in R^{m,n} \setminus \{0\}} \mathfrak{g}^{\alpha} + \sum_{\alpha \in R^{m,n} \setminus \{0\}} [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}]$$

is a subsuperalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{osp}_{\mathbb{F}}(m,n)$ with Cartan subalgebra

$$\mathcal{H} = \sum_{\alpha \in R^{m,n} \setminus \{0\}} [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}]$$

and root system $R^{m,n}$. Consider (27) and set

$$\mathcal{L} := \sum_{\alpha \in \Phi \setminus \{0\}} \mathfrak{L}^{\alpha} \oplus \sum_{\alpha \in \Phi \setminus \{0\}} [\mathfrak{L}^{\alpha}, \mathfrak{L}^{-\alpha}].$$

It is easy to see that $\mathcal{L}$ has a weight space decomposition $\mathcal{L} = \sum_{\alpha \in \Phi} \mathcal{L}^{\alpha}$ with respect to $\mathcal{H}$ with

$$\mathcal{L}^{\alpha} := \begin{cases} \mathfrak{L}^{\alpha} & \alpha \in \Phi \setminus \{0\} \\ \sum_{\alpha \in \Phi \setminus \{0\}} [\mathfrak{L}^{\alpha}, \mathfrak{L}^{-\alpha}] & \alpha = 0. \end{cases}$$

This in particular implies that

$$\mathcal{L}^{\alpha} = \sum_{i \in T_1} (\mathfrak{g}^{(i)})^{\alpha} \oplus \sum_{j \in T_2} (\mathcal{V}^{(j)})^{\alpha} \oplus \sum_{t \in T_3} (\mathfrak{s}^{(t)})^{\alpha} \quad (\alpha \in \Phi \setminus \{0\}). \tag{32}$$

Moreover, setting

$$\Delta_1 := \mathrm{span}_{\mathbb{Z}} R^{m,n} \cap \Delta_{\mathfrak{u}} \quad \text{and} \quad \Delta_2 := \mathrm{span}_{\mathbb{Z}} R^{m,n} \cap \Delta_{\mathfrak{s}},$$

and using Proposition 4.2, for $i \in T_1,\ j \in T_2$ and $t \in T_3$, we have the following $\mathcal{G}$-modules

$$\mathcal{G}^{(i)} := \textstyle\sum_{\alpha \in R^{m,n} \setminus \{0\}} (\mathfrak{g}^{(i)})^{\alpha} + \sum_{\alpha \in R^{m,n} \setminus \{0\}} [\mathfrak{g}^{\alpha}, (\mathfrak{g}^{(i)})^{-\alpha}],$$

$$\mathcal{U}^{(j)} := \textstyle\sum_{\alpha \in \Delta_1 \setminus \{0\}} (\mathcal{V}^{(j)})^{\alpha} + \sum_{\alpha \in \Delta_1 \setminus \{0\}} [\mathfrak{g}^{\alpha}, (\mathcal{V}^{(j)})^{-\alpha}],$$

$$\mathcal{S}^{(t)} := \textstyle\sum_{\alpha \in \Delta_2 \setminus \{0\}} (\mathfrak{s}^{(t)})^{\alpha} + \sum_{\alpha \in \Delta_2 \setminus \{0\}} [\mathfrak{g}^{\alpha}, (\mathfrak{s}^{(t)})^{-\alpha}]$$

which are respectively isomorphic to $\mathcal{G}$, to the natural module $\mathcal{U}$ of $\mathcal{G}$ and to the second natural module $\mathcal{S}$ of $\mathcal{G}$. Also it is immediate that

1) $\mathcal{L}$ contains $\mathcal{G}$ as a subalgebra,
2) $\mathcal{L}$ is equipped with a weight space decomposition $\mathcal{L} = \bigoplus_{\alpha \in \Phi} \mathcal{L}^{\alpha}$, with respect to $\mathcal{H}$,
3) $\mathcal{L}^0 = \sum_{\alpha \in \Phi \setminus \{0\}} [\mathcal{L}^{\alpha}, \mathcal{L}^{-\alpha}]$

and so as above $\mathcal{L}$ is completely reducible with irreducible constituents isomorphic to $\mathcal{G}$, $\mathcal{U}$, $\mathcal{S}$ or to the trivial module. Since $\sum_{i\in T_1}\mathcal{G}^{(i)}\oplus\sum_{j\in T_2}\mathcal{U}^{(j)}\oplus\sum_{t\in T_3}\mathcal{S}^{(t)}$ is a $\mathcal{G}$-submodule of $\mathcal{L}$, there is a submodule $\mathcal{D}$ of $\mathcal{L}$ such that

$$\mathcal{L}=\sum_{i\in T_1}\mathcal{G}^{(i)}\oplus\sum_{j\in T_2}\mathcal{U}^{(j)}\oplus\sum_{t\in T_3}\mathcal{S}^{(t)}\oplus\mathcal{D}.$$

But for each nonzero $\alpha\in\Phi\setminus\{0\}$,

$$\mathcal{L}^\alpha=\mathfrak{L}^\alpha=\sum_{i\in T_1}(\mathfrak{g}^{(i)})^\alpha\oplus\sum_{j\in T_2}(\mathcal{V}^{(j)})^\alpha\oplus\sum_{t\in T_3}(\mathfrak{s}^{(t)})^\alpha\subseteq\sum_{i\in T_1}\mathcal{G}^{(i)}\oplus\sum_{j\in T_2}\mathcal{U}^{(j)}\oplus\sum_{t\in T_3}\mathcal{S}^{(t)}.$$

This means that $\mathcal{D}$ is a trivial $\mathcal{G}$-module. Next we note that vector spaces are flat and so $(\mathcal{G}\otimes\mathcal{A})$, $(\mathcal{S}\otimes\mathcal{B})$ and $(\mathcal{U}\otimes\mathcal{C})$ can be embedded in $(\mathfrak{g}\otimes\mathcal{A})$, $(\mathfrak{s}\otimes\mathcal{B})$ and $(\mathfrak{u}\otimes\mathcal{C})$ respectively; so identifying $\mathcal{G}\otimes\mathcal{A}$, $\mathcal{S}\otimes\mathcal{B}$ and $\mathcal{U}\otimes\mathcal{C}$ with subspaces of $\mathfrak{g}\otimes\mathcal{A}$, $\mathfrak{s}\otimes\mathcal{B}$ and $\mathfrak{u}\otimes\mathcal{C}$ respectively, so we may assume

$$\mathcal{L}=(\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus(\mathcal{U}\otimes\mathcal{C})\oplus\mathcal{D}.$$

Now using the same argument as in [25, Lem. 3.6 and Prop. 3.10], $\mathcal{A}\oplus\mathcal{B}\oplus\mathcal{C}$ is equipped with a superalgebraic structure derived from the Lie superalgebraic structures on $\mathcal{L}$ and $\mathfrak{L}$,

$$\mathfrak{L}=((\mathfrak{g}\otimes\mathcal{A})\oplus(\mathfrak{u}\otimes\mathcal{B})\oplus(\mathfrak{s}\otimes\mathcal{C}))+\mathcal{D}$$

and the stated properties in Theorem 4.6 hold.

• *Proof of Theorem 4.6.* We recall that $I_0\subseteq I$ and $J_0\subseteq J$ are finite subsets with $|I_0|=m$ and $|J_0|=n$. Take Λ and Γ to be index sets with a symbol " 0 " belonging to $\Lambda\cap\Gamma$ such that $\{I_\lambda\mid\lambda\in\Lambda\}$ (resp. $\{J_\gamma\mid\gamma\in\Gamma\}$) is the set of finite subsets of I (resp. J) containing I_0 (resp. J_0). For $(\lambda,\gamma)\in\Lambda\times\Gamma$, set

$$\begin{aligned}\Psi^{\lambda,\gamma}&:=\Psi\cap\mathrm{span}_{\mathbb{Z}}\{\epsilon_\imath,\delta_p\mid\imath\in I_\lambda,p\in J_\gamma\}, & \Delta_1^{\lambda,\gamma}&:=\Delta_{\mathfrak{u}}\cap\mathrm{span}_{\mathbb{Z}}\{\epsilon_\imath,\delta_p\mid\imath\in I_\lambda,p\in J_\gamma\},\\ R^{\lambda,\gamma}&:=R\cap\mathrm{span}_{\mathbb{Z}}\{\epsilon_\imath,\delta_p\mid\imath\in I_\lambda,p\in J_\gamma\}, & \Delta_2^{\lambda,\gamma}&:=\Delta_{\mathfrak{s}}\cap\mathrm{span}_{\mathbb{Z}}\{\epsilon_\imath,\delta_p\mid\imath\in I_\lambda,p\in J_\gamma\}.\end{aligned}$$

and take

$$\begin{aligned}\mathfrak{L}^{\lambda,\gamma}&:=\textstyle\sum_{\alpha\in\Psi^{\lambda,\gamma}}\mathfrak{L}^\alpha+\sum_{\alpha\in\Psi^{\lambda,\gamma}}[\mathfrak{L}^\alpha,\mathfrak{L}^{-\alpha}], & \mathfrak{u}^{\lambda,\gamma}&:=\textstyle\sum_{\alpha\in\Delta_1^{\lambda,\gamma}}\mathfrak{u}^\alpha+\sum_{\alpha\in\Delta_1^{\lambda,\gamma}}\mathfrak{g}^\alpha\mathfrak{u}^{-\alpha},\\ \mathfrak{g}^{\lambda,\gamma}&:=\textstyle\sum_{\alpha\in R^{\lambda,\gamma}}\mathfrak{g}^\alpha+\sum_{\alpha\in R^{\lambda,\gamma}}[\mathfrak{g}^\alpha,\mathfrak{g}^{-\alpha}], & \mathfrak{s}^{\lambda,\gamma}&:=\textstyle\sum_{\alpha\in\Delta_2^{\lambda,\gamma}}\mathfrak{s}^\alpha+\sum_{\alpha\in\Delta_2^{\lambda,\gamma}}\mathfrak{g}^\alpha\mathfrak{s}^{-\alpha}.\end{aligned}$$

Using the result of Subsection 4.0.2, we find a subsuperalgebra $\mathcal{D}$ of $\mathfrak{L}^{0,0}$ and superspaces $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ such that the properties stated in Theorem 4.6 are satisfied and

$$\mathfrak{L}^{0,0}=(\mathfrak{g}^{0,0}\otimes\mathcal{A})\oplus(\mathfrak{s}^{0,0}\otimes\mathcal{B})\oplus(\mathfrak{u}^{0,0}\otimes\mathcal{C})\oplus\mathcal{D},$$

moreover, for $\lambda\in\Lambda$ and $\gamma\in\Gamma$,

$$\mathfrak{L}^{\lambda,\gamma}=((\mathfrak{g}^{\lambda,\gamma}\otimes\mathcal{A})\oplus(\mathfrak{s}^{\lambda,\gamma}\otimes\mathcal{B})\oplus(\mathfrak{u}^{\lambda,\gamma}\otimes\mathcal{C}))+\mathcal{D}.$$

Now the result follows using the same argument as in [25, Thm. 4.1]. ■

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