Journal of Lie Theory
Volume **26** (2016) 767–775

Classification of Multiplicative Simple Hom-Lie Algebras

Xue Chen*and Wei Han

Communicated by M. Schlichenmaier

Abstract. In this paper, we firstly study the dimensions problem of simple Hom-Lie algebras. Then we classify finite dimensional multiplicative simple Hom-Lie algebras by investigating the corresponding semisimple Lie algebras.
Mathematics Subject Classification 2010: 17B40, 17B60.
Key Words and Phrases: Hom-Lie algebra, simple, classification.

1. Introduction

The Hom-Lie algebras were initially introduced by Hartwig, Larson and Silvestrov in [3] motivated by examples of the deformed Witt and Virasoro algebras coming from twisted discretizations of vector fields. A Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ can be considered as a deformation of a Lie algebra, in which the Jacobi identity is twisted by the linear map α, called the Hom-Jacobi identity. Hom-Lie algebras have close connections with various aspects of mathematics and physics, for instance, discrete and deformed vector fields, differential calculus, the Yang-Baxter equation and so on. For this reason, recently Hom-Lie algebras have been more and more considered[1, 5, 8, 9, 10, 7].

In [6], Hom-associative, Hom-Leibniz, and Hom-Lie admissible algebraic structures were introduced which generalize the well known associative, Leibniz and Lie admissible algebras. In [11], D. Yau introduced the twisting principle, constructed classes of G-Hom-associative algebras and Chevalley-Eilenberg type homology for Hom-Lie algebras. In [4], it was proved that the Hom-Lie algebra structures on finite-dimensional simple Lie algebras are trivial and the authors found when a finite dimensional semi-simple Lie algebra admits non-trivial Hom-Lie algebra structures and the isomorphic classes of non-trivial Hom-Lie algebras were determined.

It is well known that simple Lie algebras play an important role in Lie theory. Similarly, it is very necessary to study simple Hom-Lie algebras in Hom-Lie theory. In particular, [1] proved some key results of simple Hom-Lie algebras, and the main results of Section 4 in this paper are based on [1]. The paper is organized

* Supported in part by China NSF grant 11426191 and the Scientific Research Foundation of Xiamen University of Technology (YKJ13028R).

ISSN 0949–5932 / $2.50

as follows. In Section 2, we recall some basic definitions about Hom-Lie algebras. In Section 3, we prove that there exist n dimensional simple Hom-Lie algebras for any positive integer n larger than two. In Section 4, we give the necessary and sufficient conditions for two finite dimensional multiplicative simple Hom-Lie algebras to be isomorphic by studying the corresponding semisimple Lie algebras, and then classify finite dimensional multiplicative simple Hom-Lie algebras.

Throughout this paper, all algebras are finite dimensional and defined on the algebraically closed field $\mathbb{C}$ of characteristic 0 unless otherwise specified.

2. Preliminaries

Definition 2.1. ([3]) A Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ consisting of a linear space $\mathfrak{g}$, bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and a linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ satisfying

(L1) $[x, y] = -[y, x]$ (skew-symmetry)

(L2) $[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0$ (Hom-Jacobi identity)

for all $x, y, z \in \mathfrak{g}$.

A Hom-Lie algebra is called a multiplicative Hom-Lie algebra if α is an algebraic morphism, i.e. for any $x, y \in \mathfrak{g}$, we have $\alpha([x, y]) = [\alpha(x), \alpha(y)]$.

Definition 2.2. Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a Hom-Lie subalgebra of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ if $\alpha(\mathfrak{h}) \subseteq \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. In particular, a Hom-Lie subalgebra $\mathfrak{h}$ is said to be an ideal of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. $\mathfrak{g}$ is called an abelian Hom-Lie algebra if $[x, y] = 0$ for any $x, y \in \mathfrak{g}$.

Definition 2.3.

$$C(\mathfrak{g}) = \{x \in \mathfrak{g} |\ [x, y] = 0,\ [\alpha(x), y] = 0,\ \forall\ y \in \mathfrak{g}\}$$

is called the center of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$.

Proposition 2.4. *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie algebra, then $(Ker(\alpha), [\cdot,\cdot], \alpha)$ is an ideal.*

Proof. Obviously $\alpha(x) = 0 \in Ker(\alpha)$ for any $x \in Ker(\alpha)$. Since $\alpha([x, y]) = [\alpha(x), \alpha(y)] = [0, y] = 0$ for any $x \in Ker(\alpha)$ and $y \in \mathfrak{g}$, we get $[x, y] \in Ker(\alpha)$. Therefore $(Ker(\alpha), [\cdot,\cdot], \alpha)$ is an ideal of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$. ∎

Definition 2.5. ([4]) Let $(\mathfrak{g}_1, [\cdot,\cdot]_1, \alpha)$ and $(\mathfrak{g}_2, [\cdot,\cdot]_2, \beta)$ be two Hom-Lie algebras. A linear map $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a homomorphism of Hom-Lie algebras if

$$\varphi([x, y]_1) = [\varphi(x), \varphi(y)]_2,\ \ \beta \circ \varphi(x) = \varphi \circ \alpha(x),\ \forall\ x, y \in \mathfrak{g}.$$

In particular, Hom-Lie algebras $(\mathfrak{g}_1, [\cdot,\cdot]_1, \alpha)$ and $(\mathfrak{g}_2, [\cdot,\cdot]_2, \beta)$ are isomorphic if φ is a bijective linear map.

Definition 2.6. Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)(\alpha \neq 0)$ be a Hom-Lie algebra. $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ is called a simple Hom-Lie algebra if $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ has no proper ideals and is not abelian. $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ is called a semisimple Hom-Lie algebra if $\mathfrak{g}$ is a direct sum of certain simple ideals.

3. The construction of simple Hom-Lie algebras

In this section, we shall study the dimensions problem of simple Hom-Lie algebras. It is easy to check that there do not exist one and two dimensional simple Hom-Lie algebras. For a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$, when the linear map $\alpha = id$, the Hom-Lie algebra definition degenerates to the the definition of a Lie algebra. Hence the simple Lie algebra of type A_1 is indeed a three dimensional simple Hom-Lie algebra ($\alpha = id$). Do there exist any finite dimensional simple Hom-Lie algebras when dimensions are larger than three? We will answer this question in the following.

When $n \geq 4$, we shall construct n dimensional simple Hom-Lie algebra. Let $\{a_{\bar{i}} \mid i \in \mathbb{Z}_n\}$ be a basis of n dimensional vector space $\mathfrak{g}$ over $\mathbb{C}$, then define a bilinear skew-symmetric binary operation:

$$[a_{\bar{i}}, a_{\overline{i+1}}] = a_{\overline{i+2}}, \quad \text{otherwise, } [\cdot,\cdot] = 0. \tag{1}$$

The next task is to find a nonzero α satisfying the Hom-Jacobi identity. Since the bracket product is bilinear skew-symmetric, the Hom-Jacobi identity holds for arbitrary three elements of $\mathfrak{g}$ if and only if it holds for arbitrary three elements of the basis $\{a_{\bar{i}} \mid i \in \mathbb{Z}_n\}$. In addition, by the definition of the bracket product, if the Hom-Jacobi identity holds for the three different basis vectors $\{a_{\bar{i}}, a_{\overline{i+1}}, a_{\bar{j}}\}$, then $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra.((L2) holds for any α if there are two same vectors in $\{x, y, z\}$)

When $j = i+2$, applying the Hom-Jacobi identity, we have

$$[\alpha(a_{\bar{i}}), [a_{\overline{i+1}}, a_{\overline{i+2}}]] + [\alpha(a_{\overline{i+1}}), [a_{\overline{i+2}}, a_{\bar{i}}]] + [\alpha(a_{\overline{i+2}}), [a_{\bar{i}}, a_{\overline{i+1}}]] = 0.$$

Using (1), the above equation can be simplified as

$$[\alpha(a_{\bar{i}}), a_{\overline{i+3}}] + [\alpha(a_{\overline{i+2}}), a_{\overline{i+2}}] = 0 . \tag{2}$$

When $\bar{j} \notin \{\overline{i-1}, \overline{i+2}\}$, the Hom-Jacobi identity is

$$[\alpha(a_{\bar{i}}), [a_{\overline{i+1}}, a_{\bar{j}}]] + [\alpha(a_{\overline{i+1}}), [a_{\bar{j}}, a_{\bar{i}}]] + [\alpha(a_{\bar{j}}), [a_{\bar{i}}, a_{\overline{i+1}}]] = 0.$$

Then it follows from (1) and the above equation that

$$[\alpha(a_{\bar{j}}), a_{\overline{i+2}}] = 0. \tag{3}$$

Let $\alpha(a_{\bar{0}}, a_{\bar{1}}, \ldots, a_{\overline{n-1}}) = (a_{\bar{0}}, a_{\bar{1}}, \ldots, a_{\overline{n-1}})(e_{ij})_{n\times n}$, then by (2) we get

$$e_{i+2,i+1} = e_{i+3,i+3}, \quad e_{i+4,i+1} = 0, \quad e_{i+1,i+3} = 0. \tag{4}$$

Similarly, by (3) we have

$$e_{i+1,j+1} = 0, \quad e_{i+3,j+1} = 0. \tag{5}$$

Therefore, if α satisfies (4) and (5), then $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra.(When the two under indexes of $e_{i,j}$ are larger than n, simply subtract n.)

When $n = 4$, it is sufficient that α satisfies (4). We obtain that

$$(e_{ij}) = \begin{pmatrix} b & 0 & 0 & c \\ d & c & 0 & 0 \\ 0 & a & d & 0 \\ 0 & 0 & b & a \end{pmatrix},$$

where $a, b, c, d \in \mathbb{C}$.

When $n > 4$, it follows from (4) and (5) that

$$(e_{ij}) = \begin{pmatrix} b_1 & 0 & 0 & \cdots & \cdots & 0 & 0 & b_2 \\ b_3 & b_2 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & b_4 & b_3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & b_5 & b_4 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & b_6 & b_5 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_n & b_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_n \end{pmatrix},$$

where $b_i (i = 1, 2, \cdots, n) \in \mathbb{C}$.

In the following we shall prove that this Hom-Lie algebra is simple.

If elements of the (e_{ij}) are not all zero, then α is nonzero. By (1), clearly we have $[\mathfrak{g}, \mathfrak{g}] \neq 0$. Let $(\mathfrak{h}, [\cdot,\cdot], \alpha)$ be a nonzero ideal of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$, then there exists a nonzero element $x = \sum_{i=0}^{n-1} x_i a_{\bar{i}} \in \mathfrak{h}$. Suppose $x_t \neq 0$. Since $[a_{\bar{t}}, [a_{\overline{t-1}}, x]] = x_t a_{\overline{t+2}} \in \mathfrak{h}$, we get $a_{\overline{t+2}} \in \mathfrak{h}$. By (1) we deduce that $\mathfrak{h}$ contains all $a_{\bar{i}} (i \in \mathbb{Z}_n)$. Therefore $\mathfrak{h} = \mathfrak{g}$ and $\mathfrak{g}$ is a simple Hom-Lie algebra.

From the above discussion, we conclude:

Theorem 3.1. *There exist n-dimensional simple Hom-Lie algebras for any positive integer n larger than two.*

4. Classification of multiplicative simple Hom-Lie algebras

Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a multiplicative simple Hom-Lie algebra. By Proposition 2.4, α must be a monomorphism, thus α is an automorphism of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$.

Definition 4.1. Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra. The Lie algebra $(\mathfrak{g}, [\cdot,\cdot]')$ is called the induced Lie algebra of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ if $[x, y] = \alpha([x, y]') = [\alpha(x), \alpha(y)]'$, $\forall\, x, y \in \mathfrak{g}$.

Proposition 4.2. ([2]) *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a multiplicative simple Hom-Lie algebra, define $[x, y]' = \alpha^{-1}([x, y])$, $\forall\, x, y \in \mathfrak{g}$, then $(\mathfrak{g}, [\cdot,\cdot]')$ is a Lie algebra and α is also a Lie algebra automorphism.*

Proof. Clearly, the bracket product $[\cdot,\cdot]'$ is bilinear skew-symmetric, we only need to check the Jacobi identity.

$$\begin{aligned} & [x,[y,z]']' + [y,[z,x]']' + [z,[x,y]']' \\ = & [x,\alpha^{-1}([y,z])]' + [y,\alpha^{-1}([z,x])]' + [z,\alpha^{-1}([x,y])]' \\ = & \alpha^{-2}([\alpha(x),[y,z]] + [\alpha(y),[z,x]] + [\alpha(z),[x,y]]) \\ = & 0, \end{aligned}$$

where $x, y, z \in \mathfrak{g}$. Moreover, we have

$$\alpha([x,y]') = \alpha\alpha^{-1}([x,y]) = \alpha^{-1}([\alpha(x),\alpha(y)]) = [\alpha(x),\alpha(y)]'.$$

Thus $(\mathfrak{g}, [\cdot,\cdot]')$ is indeed a Lie algebra and α is an automorphism of $(\mathfrak{g}, [\cdot,\cdot]')$. ■

By the above proposition, there exists an induced Lie algebra for any multiplicative simple Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ and α is an automorphism of the induced Lie algebra, in addition their bracket products are mutually determined. Therefore problems about multiplicative simple Hom-Lie algebras perhaps can be solved by investigating the corresponding induced Lie algebras. It is based on the idea that we classify all multiplicative simple Hom-Lie algebras in this section.

Theorem 4.3. *The two multiplicative simple Hom-Lie algebras* $(\mathfrak{g}_1, [\cdot,\cdot]_1, \alpha)$ *and* $(\mathfrak{g}_2, [\cdot,\cdot]_2, \beta)$ *are isomorphic if and only if there exists a Lie algebra isomorphism* φ *(between their induced Lie algebras):* $\mathfrak{g}_1 \to \mathfrak{g}_2$ *satisfying* $\varphi \circ \alpha = \beta \circ \varphi$. *In other words, the two Lie algebra automorphisms* α *and* β *are conjugate.*

Proof. Let $(\mathfrak{g}_1, [\cdot,\cdot]_1')$ and $(\mathfrak{g}_2, [\cdot,\cdot]_2')$ be induced Lie algebras of $(\mathfrak{g}_1, [\cdot,\cdot]_1, \alpha)$ and $(\mathfrak{g}_2, [\cdot,\cdot]_2, \beta)$, respectively. Suppose $\varphi : (\mathfrak{g}_1, [\cdot,\cdot]_1, \alpha) \to (\mathfrak{g}_2, [\cdot,\cdot]_2, \beta)$ is an isomorphism of Hom-Lie algebras, then $\varphi \circ \alpha = \beta \circ \varphi$, thus $\varphi \circ \alpha^{-1} = \beta^{-1} \circ \varphi$. Moreover,

$$\begin{aligned} \varphi([x,y]_1') = \varphi \circ \alpha^{-1} \circ \alpha([x,y]_1') &= \varphi \circ \alpha^{-1}([x,y]_1) = \beta^{-1} \circ \varphi([x,y]_1) \\ &= \beta^{-1}([\varphi(x),\varphi(y)]_2) = [\varphi(x),\varphi(y)]_2'. \end{aligned}$$

So φ is an isomorphism between the two induced Lie algebras.

On the other hand, if there exists an isomorphism φ from the induced Lie algebras $(\mathfrak{g}_1, [\cdot,\cdot]_1')$ to $(\mathfrak{g}_2, [\cdot,\cdot]_2')$ such that $\varphi \circ \alpha = \beta \circ \varphi$, then

$$\varphi([x,y]_1) = \varphi \circ \alpha([x,y]_1') = \beta \circ \varphi([x,y]_1') = \beta([\varphi(x),\varphi(y)]_2') = [\varphi(x),\varphi(y)]_2.$$

Hence φ is an isomorphism between the two Hom-Lie algebras. ■

The following theorem comes from [5].

Theorem 4.4. ([5]) *The induced Lie algebra of the multiplicative simple Hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ *is semisimple and can be decomposed into direct sum of isomorphic simple ideals, in addition,* α *acts simply transitively on simple ideals of the induced Lie algebra.*

Proof. Let $(\mathfrak{g}, [\cdot,\cdot]')$ be an induced Lie algebra of the multiplicative simple Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$. Clearly, α is both an automorphism of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ and $(\mathfrak{g}, [\cdot,\cdot]')$.

Suppose that $\mathfrak{g}_1 \neq 0$ is the maximal solvable ideal of $(\mathfrak{g}, [\cdot,\cdot]')$. Because $\alpha(\mathfrak{g}_1)$ is also a solvable ideal of $(\mathfrak{g}, [\cdot,\cdot]')$, then $\alpha(\mathfrak{g}_1) \subseteq \mathfrak{g}_1$. Moreover,

$$[\mathfrak{g}_1, \mathfrak{g}] = \alpha([\mathfrak{g}_1, \mathfrak{g}]') \subseteq \alpha(\mathfrak{g}_1) \subseteq \mathfrak{g}_1,$$

so $\mathfrak{g}_1$ is an ideal of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$. Then $\mathfrak{g}_1 = \mathfrak{g}$, and we have

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1] = \alpha([\mathfrak{g}_1, \mathfrak{g}_1]') \subsetneqq \alpha(\mathfrak{g}_1) \subseteq \mathfrak{g}_1 = \mathfrak{g}.$$

Furthermore, since $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ is a multiplicative simple Hom-Lie algebra, clearly we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. It's a contradiction. Hence $\mathfrak{g}_1 = 0$.

According to Lie theory, $(\mathfrak{g}, [\cdot,\cdot]')$ is a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s (s \geq 1)$, where $(\mathfrak{g}_i, [\cdot,\cdot]')(i = 1, 2, \cdots, s))$ are simple ideas of $(\mathfrak{g}, [\cdot,\cdot]')$. Because there may be isomorphic Lie algebras in $\mathfrak{g}_1, \mathfrak{g}_2, \cdots \mathfrak{g}_s$, we rearrange the order as the following

$$\mathfrak{g} = \mathfrak{g}_{11} \oplus \mathfrak{g}_{12} \oplus \cdots \oplus \mathfrak{g}_{1m_1} \oplus \mathfrak{g}_{21} \oplus \mathfrak{g}_{22} \oplus \cdots \oplus \mathfrak{g}_{2m_2} \oplus \cdots \oplus \mathfrak{g}_{t1} \oplus \cdots \oplus \mathfrak{g}_{tm_t},$$

where

$$(\mathfrak{g}_{ij}, [\cdot,\cdot]') \cong (\mathfrak{g}_{ik}, [\cdot,\cdot]'), \quad 1 \leq j, k \leq m_i,\ i = 1, 2, \cdots, t.$$

Since

$$\alpha(\mathfrak{g}_{i1} \oplus \mathfrak{g}_{i2} \oplus \cdots \oplus \mathfrak{g}_{im_i}) = \mathfrak{g}_{i1} \oplus \mathfrak{g}_{i2} \oplus \cdots \oplus \mathfrak{g}_{im_i},\ (1 \leq i \leq t)$$
$$[\mathfrak{g}, \mathfrak{g}_{i1} \oplus \mathfrak{g}_{i2} \oplus \cdots \oplus \mathfrak{g}_{im_i}] = \mathfrak{g}_{i1} \oplus \mathfrak{g}_{i2} \oplus \cdots \oplus \mathfrak{g}_{im_i},$$

then $\mathfrak{g}_{i1} \oplus \mathfrak{g}_{i2} \oplus \cdots \oplus \mathfrak{g}_{im_i}$ is an ideal of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$. Because of the simpleness of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$, we have $\mathfrak{g}_{i1} \oplus \mathfrak{g}_{i2} \oplus \cdots \oplus \mathfrak{g}_{im_i} = 0$ or $\mathfrak{g}$. So all but one $\mathfrak{g}_{i1} \oplus \mathfrak{g}_{i2} \oplus \cdots \oplus \mathfrak{g}_{im_i}$ must be 0. Without loss of generality, we can assume

$$\mathfrak{g} = \mathfrak{g}_{11} \oplus \mathfrak{g}_{12} \oplus \cdots \oplus \mathfrak{g}_{1m_1}.$$

When $m_1 = 1$, $(\mathfrak{g}, [\cdot,\cdot]')$ is a simple Lie algebra. When $m_1 > 1$, if

$$\alpha(\mathfrak{g}_{1p}) = \mathfrak{g}_{1p} (1 \leq p \leq m_1),$$

then $\mathfrak{g}_{1p}$ is a nontrivial ideal of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$, which contradicts with the fact that $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ is simple. Hence

$$\alpha(\mathfrak{g}_{1p}) = \mathfrak{g}_{1l} (1 \leq l \neq p \leq m_1).$$

In addition, it is easy to check that $\mathfrak{g}_{11} \oplus \alpha(\mathfrak{g}_{11}) \oplus \alpha^2(\mathfrak{g}_{11}) \oplus \cdots \oplus \alpha^{m_1-1}(\mathfrak{g}_{11})$ is an ideal of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$. Therefore

$$\mathfrak{g} = \mathfrak{g}_{11} \oplus \alpha(\mathfrak{g}_{11}) \oplus \alpha^2(\mathfrak{g}_{11}) \oplus \cdots \oplus \alpha^{m_1-1}(\mathfrak{g}_{11}).$$

That is α acts simply transitively on simple ideals of the induced Lie algebra. ∎

According to the above theorem, the dimension of a multiplicative simple Hom-Lie algebra can only be an integer multiple of the dimension of a simple Lie algebra.

By the Theorem 4.3 and 4.4, in order to classify multiplicative simple Hom-Lie algebras, we just classify automorphisms on their induced Lie algebras, in particular, automorphisms on semisimple Lie algebras which are direct sum of finite isomorphic simple ideals.

Theorem 4.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra and its n simple ideals are isomorphic mutually, moreover $\mathfrak{g}$ can be generated by its automorphism α (or β) and any simple ideal. Taking $\alpha_n = \alpha^n$, $\beta_n = \beta^n$, then α_n (or β_n) leaves each simple ideal of $\mathfrak{g}$ invariant. There exists an automorphism φ on $\mathfrak{g}$ satisfying $\varphi \circ \alpha = \beta \circ \varphi$ if and only if there exists an automorphism ϕ on the simple ideal of $\mathfrak{g}$ satisfying $\phi \circ \alpha^n = \beta^n \circ \phi$.*

Proof. Let $\mathfrak{g}_1$ be a simple ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ can be generated by $\mathfrak{g}_1$ and α(or β), then $\alpha^n(\mathfrak{g}_1) = \mathfrak{g}_1$(or $\beta^n(\mathfrak{g}_1) = \mathfrak{g}_1$) and we have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \oplus \alpha^2(\mathfrak{g}_1) \oplus \cdots \oplus \alpha^{n-1}(\mathfrak{g}_1)$$

$$(or\ \mathfrak{g} = \mathfrak{g}_1 \oplus \beta(\mathfrak{g}_1) \oplus \beta^2(\mathfrak{g}_1) \oplus \cdots \oplus \beta^{n-1}(\mathfrak{g}_1)).$$

Choose a basis $x = (x_1, x_2, \ldots, x_m)$ of $\mathfrak{g}_1$, then

$$x' = (x, \alpha(x), \alpha^2(x), \ldots, \alpha^{n-1}(x)), \quad x'' = (x, \beta(x), \beta^2(x), \ldots, \beta^{n-1}(x))$$

are both bases of $\mathfrak{g}$. Let $\alpha(x') = (x')A$, $\beta(x'') = (x'')B$, then

$$A = \begin{pmatrix} 0 & & & & A_1 \\ I & 0 & & & \\ & I & 0 & & \\ & & \ddots & \ddots & \\ & & & I & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & & & & B_1 \\ I & 0 & & & \\ & I & 0 & & \\ & & \ddots & \ddots & \\ & & & I & 0 \end{pmatrix},$$

where $\alpha_n(x) = (x)A_1$, $\beta_n(x) = (x)B_1$.

If there exists an automorphism ϕ on $\mathfrak{g}_1$ such that $\phi \circ \alpha_n = \beta_n \circ \phi$, let $\phi(x) = (x)M$, then $MA_1 = B_1M$. Defining $\varphi(x') = (x'')diag(\underbrace{M, \cdots, M}_{n})$, then φ is an automorphism on $\mathfrak{g}$ and $\varphi \circ \alpha = \beta \circ \varphi$.

Now suppose there exists an automorphism φ on $\mathfrak{g}$ satisfying $\varphi \circ \alpha = \beta \circ \varphi$. Let $\varphi(\mathfrak{g}_1) = \beta^i(\mathfrak{g}_1)$, then $\varphi \circ \alpha^j(\mathfrak{g}_1) = \beta^{i+j}(\mathfrak{g}_1)(0 \leq i, j \leq n-1)$. Let $\varphi(x) = \beta^i(x)M_1$, then $\varphi(x') = (x'')M$, where

$$M = \begin{pmatrix} M_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & M_1 \end{pmatrix} \ (i = 0),$$

$$M = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ i+1 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & \cdots & n-i & n-i+1 & \cdots & n \end{array} \\ \left(\begin{array}{cccccc} & & & B_1M_1 & & \\ & & & & B_1M_1 & \\ & & & & & \ddots \\ & & & & & B_1M_1 \\ M_1 & & & & & \\ & \ddots & & & & \\ & & M_1 & & & \end{array} \right) \end{array} (1 \leq i \leq n-1).$$

Using $\varphi \circ \alpha = \beta \circ \varphi$, we can get $MA = BM$, thus $M_1 A_1 = B_1 M_1$. Defining $\phi(x) = (x)M_1$, then ϕ is an automorphism on $\mathfrak{g}_1$ and $\phi \circ \alpha_n = \beta_n \circ \phi$. ■

By the above theorem, it is obvious that the two multiplicative simple Hom-Lie algebras $(\mathfrak{g}_1, [\cdot,\cdot]_1, \alpha)$ and $(\mathfrak{g}_2, [\cdot,\cdot]_2, \beta)$ are isomorphic if and only if the automorphisms α^n and β^n on two simple ideals (as simple Lie algebras) of the corresponding induced Lie algebras are conjugate.

Combing Theorems 4.3-4.5, we get the following theorem.

Theorem 4.6. *All finite dimensional multiplicative simple Hom-Lie algebras can be denoted as* (X, n, Γ_α)*, where* X *represents the type of the simple ideal (as the simple Lie algebra) of the corresponding induced Lie algebra,* n *represents numbers of simple ideals,* Γ_α *represents the sets of conjugate classes of the automorphism* α^n *of the simple Lie algebra* X*, i.e.* $\Gamma_\alpha = \{\phi \circ \alpha^n \circ \phi^{-1} \mid \phi \in Aut(X)\}$.

References

[1] Benayadi, S., and A. Makhlouf, *Hom-Lie Algebras with Symmetric Invariant Nondegenerate Bilinear Forms*, J. of Geometry and Physics **76** (2014), 38–60.

[2] Gohr, A., *On Hom-algebras with surjective twisting*, J. Algebra **324** (2010), 1483–1491.

[3] Hartwig, J. T., D. Larsson, and S. D. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra **295** (2006), 314–361.

[4] Jin, Q. Q, and X.C. Li, *Hom-Lie algebra structures on semi-simple Lie algebras*, J. Algebra **319** (2008), 1398–1408.

[5] Li, X. X., *Structures of multiplicative Hom-Lie algebras*, Advances in Mathematics(China) **43** (2014), 817–823.

[6] Makhlouf, A., and S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2008), 51–64.

[7] —, *Notes on Formal Deformations of Hom-associative and Hom- Lie Algebras*, Forum Math. **22** (2010), 715–739.

[8] Sheng, Y. H. *Representations of hom-Lie algebras*, Alg. Represent. Theory **15** (2012), 1081–1098.

[9] Yau, D., *Enveloping algebras of Hom-Lie algebras*, J. Gen. Lie Theory Appl. **2** (2008), 95–108.

[10] Yau, D., *Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras*, J. Phys. A: Math. Theor. **42** (2009), 165202.

[11] —, *Hom-algebras and homology*, J. Lie Theory **19** (2009), 409–421.

Xue Chen
School of Applied Mathematics
Xiamen University of Technology
No.600 Ligong Road
Xiamen, 361024, China
chenxue@sjtu.edu.cn

Wei Han
Department of Mathematics
Shanghai Jiaotong University
No.800 Dongchuan Road
Shanghai, 200240, China
hw163@hotmail.com

Received June 26, 2015
and in final form November 10, 2015