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Linear Commuting Maps and Biderivations on the Lie Algebras $\mathcal{W}(a,b)$

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Abstract. Let $\mathcal{W}(a,b)$ be the semidirect product Lie algebra of the centerless Virasoro algebra and its module of the intermediate series, where $a, b \in \mathbb{C}$. In this paper, we first determine all the skew-symmetric biderivations of $\mathcal{W}(a,b)$. Based on the result of biderivations, we give the explicit form of each linear commuting map on $\mathcal{W}(a,b)$. In particular, we find that there exist non-inner biderivations and non-standard linear commuting maps for certain $\mathcal{W}(a,b)$.
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1. Introduction

Let $\mathcal{A}$ be an associative algebra (or ring). A map $\psi : \mathcal{A} \to \mathcal{A}$ is called *commuting* if

$$\psi(x)x = x\psi(x) \quad \text{for all} \quad x \in \mathcal{A}. \tag{1.1}$$

Note that (1.1) can be also written as $[\psi(x), x] = 0$ if we use bracket $[x, y]$ to denote $xy - yx$ for $x, y \in \mathcal{A}$. So, it is natural to give the definition of commuting maps on Lie algebras. Let $\mathcal{L}$ be a Lie algebra with Lie brackets $[\cdot, \cdot]$. A map $\psi : \mathcal{L} \to \mathcal{L}$ is called *commuting* if

$$[\psi(x), x] = 0 \quad \text{for all} \quad x \in \mathcal{L}. \tag{1.2}$$

The theory of commuting maps on associative algebras (or rings) has a long and rich history. The initial work can be traced back to the well-known Posner's 1957 theorem stating that there are no nonzero commuting derivations on noncommutative prime rings [15]. We refer the reader to the survey paper [6] by Brešar for a detailed discussion on this theory and an extensive bibliography. However, different from the extensive research on associative algebras, commuting maps on

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Lie algebras were studied only when their structure is determined by their associative enveloping algebras. Until recently, motivated by this, the authors in [17] investigated the commuting maps on the Schrödinger-Virasoro Lie algebra. In this paper, we shall consider this problem on another interesting Lie algebra $\mathcal{W}(a,b)$, where a, b are complex parameters.

Let us first recall the definition of the Lie algebra $\mathcal{W}(a,b)$. For any fixed complex numbers a, b, the Lie algebra $\mathcal{W}(a,b)$ has basis $\{L_\alpha, W_\beta \mid \alpha, \beta \in \mathbb{Z}\}$ and relations

$$[L_\alpha, L_\beta] = (\beta - \alpha)L_{\alpha+\beta}, \tag{1.3}$$

$$[L_\alpha, W_\beta] = (a + \beta + b\alpha)W_{\alpha+\beta}, \tag{1.4}$$

$$[W_\alpha, W_\beta] = 0, \tag{1.5}$$

for $\alpha, \beta \in \mathbb{Z}$. Note that $\mathcal{W}(a,b)$ contains a subalgebra Vir $=\{ L_\alpha \mid \alpha \in \mathbb{Z}\}$ isomorphic to the well-known centerless Virasoro algebra [14]. The Lie algebra $\mathcal{W}(a,b)$ is in fact the semidirect product of Vir by a module of the intermediate series of Vir. This Lie algebra is interesting in another aspect that it contains some meaningful Lie algebras as special cases. For example,

- $\mathcal{W}(0,-1)$ is exactly the W-algebra $W(2,2)$, which was first introduced in [20] in order to give a classification of vertex operator algebras satisfying certain conditions;
- $\mathcal{W}(0,0)$ is exactly the well-known twisted Heisenberg-Virasoro Lie algebra, which was first introduced in [1], where a connection with the second cohomology of certain moduli spaces of curves is established.

One-dimensional (Leibniz) central extensions, the derivation algebra, and the automorphism group of $\mathcal{W}(a,b)$ were determined in [12]. Recently, indecomposable modules of the intermediate series over $\mathcal{W}(a,b)$ were classified in [16].

A bilinear map $\varphi: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a *biderivation* of $\mathcal{A}$ if it is a derivation with respect to both components, namely,

$$\varphi(xy, z) = x\varphi(y,z) + \varphi(x,z)y \quad \text{and} \quad \varphi(x, yz) = \varphi(x,y)z + y\varphi(x,z)$$

for all $x, y, z \in \mathcal{A}$. As Brešar concluded in [6], biderivation is an effective tool in characterizing commuting maps on associative algebras (or rings) [5, 8]. Of course, biderivations and their generalizations are interesting in their own right and have been exclusively studied by many authors (e.g., [3, 11, 13, 19]). To characterize commuting maps on Lie algebras, it is natural to transfer the concept of biderivation from associative algebras to Lie algebras [17]. A bilinear map $\varphi: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is called a *biderivation* of $\mathcal{L}$ if it is a derivation with respect to both components, namely,

$$\varphi([x,y], z) = [x, \varphi(y,z)] + [\varphi(x,z), y] \quad \text{and} \quad \varphi(x, [y,z]) = [\varphi(x,y), z] + [y, \varphi(x,z)]$$

for all $x, y, z \in \mathcal{L}$. Clearly, the following map

$$\mathcal{L} \times \mathcal{L} \to \mathcal{L}, \qquad (x,y) \mapsto [x,y]$$

is a biderivation of $\mathcal{L}$. Any scalar multiple of such a map is called an *inner biderivation*. An inner biderivation of an associative algebra (or ring) has a similar form. Brešar pointed in [6] that it happens quite often that all biderivations of a noncommutative ring $\mathcal{A}$ are inner, and in this case each linear commuting map ψ has a standard form: $\psi(x) = \lambda x + \mu(x)$, where λ is a central element of $\mathcal{A}$, and μ is a linear map from $\mathcal{A}$ to its center. In this paper, we surprisingly find that there exist non-inner biderivations for certain $\mathcal{W}(a,b)$ (c.f. (3.1)), which renders the corresponding commuting maps non-standard (c.f. (4.1)).

We also would like to mention that commuting maps in the associative algebra (ring) setting are applicable to another fertile theory, the theory of linear preservers, especially the following two topics: commutativity preservers and normal preservers (e.g., [2, 7, 9]), see [6] for detailed explanation. So, one may expect that commuting maps in the Lie algebra setting could be useful when considering preservers problems on Lie algebras. This is also our motivation to present this paper.

This paper is organized as follows. In Section 2, we recall some general results on biderivations of Lie algebras. Then, in Section 3, we determine all the skew-symmetric biderivations of $\mathcal{W}(a,b)$ (Theorem 3.1). Finally, in Section 4, we give the explicit form of each linear commuting map on $\mathcal{W}(a,b)$ (Theorem 4.1), and obtain an interesting classification of $\mathcal{W}(a,b)$ as a byproduct (Remark 4.2).

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers.

2. General results on biderivations

Let $\mathcal{L}$ be a Lie algebra. Let $C_{\mathcal{L}}(X) = \{y \in \mathcal{L} \mid [y,x] = 0 \text{ for all } x \in X\}$, the centralizer of a subset X of $\mathcal{L}$. The center $Z(\mathcal{L})$ of $\mathcal{L}$ is simply $C_{\mathcal{L}}(\mathcal{L})$. A bilinear map $\varphi: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is called *symmetric* if $\varphi(x,y) = \varphi(y,x)$ for all $x, y \in \mathcal{L}$; *skew-symmetric* if $\varphi(x,y) = -\varphi(y,x)$ for all $x, y \in \mathcal{L}$. For any bilinear map $\varphi: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, we set

$$F(x,y,u,v) = [\varphi(x,y),[u,v]] - [[x,y],\varphi(u,v)] \quad \text{for} \quad x,y,u,v \in \mathcal{L}.$$

We have the following general results on biderivations of Lie algebras. The original idea is due to Brešar [4] (see also [5, 8]). Here we still include the proof for the sake of completeness.

Lemma 2.1. *Let φ be a biderivation of $\mathcal{L}$. Then*

$$F(x,y,u,v) = F(x,v,u,y) \quad \textit{for} \quad x,y,u,v \in \mathcal{L}.$$

Proof. Let us consider $\varphi([x,u],[y,v]])$ for arbitrary $x,y,u,v \in \mathcal{L}$. On one hand, since φ is a derivation in the first argument, we have

$$\varphi([x,u],[y,v]]) = [x, \varphi(u,[y,v])] + [\varphi(x,[y,v]),u],$$

and since φ is also a derivation in the second argument, it becomes

$$\varphi([x,u],[y,v]) = [x,[\varphi(u,y),v]] + [x,[y,\varphi(u,v)]] + [[\varphi(x,y),v],u] + [[y,\varphi(x,v)],u].$$

On the other hand, first using the derivation law in the second and then in the first argument, we get

$$\varphi([x,u],[y,v]) = [[x,\varphi(u,y)],v] + [[\varphi(x,y),u],v] + [y,[x,\varphi(u,v)]] + [y,[\varphi(x,v),u]].$$

Comparing the above two relations, we obtain

$$[\varphi(x,y),[u,v]] - [[x,y],\varphi(u,v)] = [\varphi(x,v),[u,y]] - [[x,v],\varphi(u,y)].$$

Namely, $F(x,y,u,v) = F(x,v,u,y)$. ■

Corollary 2.2. *Let φ be a skew-symmetric biderivation of $\mathcal{L}$.*

(1) $F(x,y,u,v) = 0$ *for* $x,y,u,v \in \mathcal{L}$.

(2) $[\varphi(x,y),[x,y]] = 0$ *for* $x,y \in \mathcal{L}$.

(3) *For* $x,y \in \mathcal{L}$, *if* $[x,y]=0$, *then* $\varphi(x,y) \in C_{\mathcal{L}}([\mathcal{L},\mathcal{L}])$.

Proof. (1) By Lemma 2.1, we have relation $F(x,y,u,v) = F(x,v,u,y)$. On the other hand, by the skewsymmetry of φ, we have another relation $F(x,y,u,v) = -F(x,y,v,u)$. Using the two relations alternately, we have

$$F(x,y,u,v) = F(x,v,u,y) = -F(x,v,y,u) = -F(x,u,y,v),$$

and

$$F(x,y,u,v) = -F(x,y,v,u) = -F(x,u,v,y) = F(x,u,y,v).$$

Hence, $F(x,y,u,v) = 0$, as desired.

(2) By (1), we immediately have

$$[\varphi(x,y),[x,y]] = [[x,y],\varphi(x,y)] = -[\varphi(x,y),[x,y]],$$

which implies that $[\varphi(x,y),[x,y]] = 0$.

(3) Suppose $[x,y]=0$. By (1), we have $[\varphi(x,y),[u,v]] = [[x,y],\varphi(u,v)] = 0$ for any $u,v \in \mathcal{L}$. Thus $\varphi(x,y) \in C_{\mathcal{L}}([\mathcal{L},\mathcal{L}])$. ■

Remark 2.3. Similar statements to the above corollary with $\mathcal{L}$ being some special Lie algebras were also given in [17, 18] but with a gap (without the assumption of skewsymmetry of φ), which was first found and filled in [10].

3. Biderivatons of $\mathcal{W}(a,b)$

From now on, we shall denote $\mathcal{W}(a,b)$ by $\mathcal{W}$ for simplicity. In this section, we shall give a description on the skew-symmetric biderivatons of $\mathcal{W}$.

For all $\mathcal{W}$, the map $(x,y) \mapsto [x,y]$ is an inner biderivaton as we mentioned in the Introduction. Obviously, it is skew-symmetric. For $\mathcal{W}$ with $a \in \mathbb{Z}$, $b=-1$, consider the following skew-symmetric bilinear map:

$$\varphi_0 : \mathcal{W}\times\mathcal{W} \to \mathcal{W}, \quad (L_\alpha, L_\beta) \mapsto (\beta-\alpha)W_{\alpha+\beta-a}, \; (L_\alpha, W_\beta) \mapsto 0, \; (W_\alpha, W_\beta) \mapsto 0. \tag{3.1}$$

One can easily check that φ_0 is a non-inner biderivation of $\mathcal{W}$. The main result in this section is as follows.

Theorem 3.1. *Let φ be a skew-symmetric biderivation of $\mathcal{W}$. We have*

$$\varphi(x,y) = \begin{cases} \lambda[x,y] + \mu\varphi_0(x,y) & \text{if } a \in \mathbb{Z},\ b=-1, \\ \lambda[x,y] & \text{otherwise}, \end{cases}$$

for $x, y \in \mathcal{W}$, where $\lambda, \mu \in \mathbb{C}$, and φ_0 is given by (3.1).

Before proving this result, we need more information on the structural features of $\mathcal{W}$.

Lemma 3.2. (1) *The center $Z(\mathcal{W})$ of $\mathcal{W}$ is given by*

$$Z(\mathcal{W}) = \begin{cases} \mathbb{C}W_{-a} & \text{if } a \in \mathbb{Z},\ b=0, \\ 0 & \text{otherwise}. \end{cases}$$

(2) *The Lie algebra $\mathcal{W}$ is nonperfect if and only if $a \in \mathbb{Z}$, $b=1$. More precisely, we have*

$$\mathcal{W}\backslash[\mathcal{W},\mathcal{W}] = \begin{cases} \mathbb{C}W_{-a} & \text{if } a \in \mathbb{Z},\ b=1, \\ \emptyset & \text{otherwise}. \end{cases}$$

(3) *The centralizer of $[\mathcal{W},\mathcal{W}]$ coincides with $Z(\mathcal{W})$, i.e., $C_{\mathcal{W}}([\mathcal{W},\mathcal{W}]) = Z(\mathcal{W})$.*

Proof. (1) and (2) can be easily checked by (1.3)–(1.5). For (3), by (2) we only need to consider the case $a \in \mathbb{Z}$, $b=1$. In fact, in this case, from (1.3)–(1.5) we immediately have $C_{\mathcal{W}}([\mathcal{W},\mathcal{W}]) = 0$, which also coincides with $Z(\mathcal{W})$ by (1). ■

Now, we can give the proof of Theorem 3.1.

Proof of Theorem 3.1. We first prove the following three claims.

Claim 1. $\varphi(W_\alpha, W_\beta) \equiv 0 \pmod{Z(\mathcal{W})}$ for $\alpha, \beta \in \mathbb{Z}$.

For any fixed $\alpha, \beta \in \mathbb{Z}$, since $[W_\alpha, W_\beta] = 0$, by Corollary 2.2(3) we have $\varphi(W_\alpha, W_\beta) \in C_{\mathcal{W}}([\mathcal{W},\mathcal{W}])$. Then this claim holds by Lemma 3.2(3).

Claim 2. There exist $\lambda, \mu \in \mathbb{C}$ such that

$$\varphi(L_\alpha, L_\beta) \equiv \begin{cases} \lambda[L_\alpha, L_\beta] + \mu\varphi_0(L_\alpha, L_\beta) \pmod{Z(\mathcal{W})} & \text{if } a \in \mathbb{Z},\ b=-1, \\ \lambda[L_\alpha, L_\beta] \pmod{Z(\mathcal{W})} & \text{otherwise}, \end{cases}$$

for $\alpha, \beta \in \mathbb{Z}$, where φ_0 is given by (3.1).

For any fixed $\alpha, \beta \in \mathbb{Z}$, we suppose that

$$\varphi(L_\alpha, L_\beta) = \sum_{i\in\mathbb{Z}} m_i L_i + \sum_{j\in\mathbb{Z}} n_j W_j.$$

If $\beta=\alpha$, then $[L_\alpha,L_\beta]=0$. By Corollary 2.2(3) and Lemma 3.2(3), $\varphi(L_\alpha,L_\beta)\in Z(\mathcal{W})$. This claim holds. Next, we assume that $\alpha\neq\beta$. By Corollary 2.2(2), we have

$$\begin{aligned}0&=\frac{1}{\beta-\alpha}[[L_\alpha,L_\beta],\varphi(L_\alpha,L_\beta)]\\&=\Big[L_{\alpha+\beta},\sum_{i\in\mathbb{Z}}m_iL_i+\sum_{j\in\mathbb{Z}}n_jW_j\Big]\\&=\sum_{i\in\mathbb{Z}}m_i(i-\alpha-\beta)L_{\alpha+\beta+i}+\sum_{j\in\mathbb{Z}}n_j(a+j+b(\alpha+\beta))W_{\alpha+\beta+j},\end{aligned}$$

which implies that

$$m_i=0\ \text{ if }\ i\neq\alpha+\beta,\qquad n_j=0\ \text{ if }\ j\neq -a-b(\alpha+\beta). \tag{3.2}$$

Furthermore, by Corollary 2.2(1) and (3.2), we have

$$\begin{aligned}0&=[\varphi(L_\alpha,L_\beta),[L_1,L_0]]-[[L_\alpha,L_\beta],\varphi(L_1,L_0)]\\&=[m_{\alpha+\beta}L_{\alpha+\beta}+n_{-a-b(\alpha+\beta)}W_{-a-b(\alpha+\beta)},-L_1]\\&\quad-[(\beta-\alpha)L_{\alpha+\beta},m_1L_1+n_{-a-b}W_{-a-b}]\\&=(m_{\alpha+\beta}-(\alpha-\beta)m_1)(\alpha+\beta-1)L_{\alpha+\beta+1}\\&\quad-b(\alpha+\beta-1)\big(n_{-a-b(\alpha+\beta)}W_{1-a-b(\alpha+\beta)}-(\alpha-\beta)n_{-a-b}W_{-a-b+\alpha+\beta}\big),\end{aligned} \tag{3.3}$$

$$\begin{aligned}0&=[\varphi(L_\alpha,L_\beta),[L_2,L_0]]-[[L_\alpha,L_\beta],\varphi(L_2,L_0)]\\&=[m_{\alpha+\beta}L_{\alpha+\beta}+n_{-a-b(\alpha+\beta)}W_{-a-b(\alpha+\beta)},-2L_2]\\&\quad-[(\beta-\alpha)L_{\alpha+\beta},m_2L_2+n_{-a-2b}W_{-a-2b}]\\&=(2m_{\alpha+\beta}-(\alpha-\beta)m_2)(\alpha+\beta-2)L_{\alpha+\beta+2}\\&\quad-b(\alpha+\beta-2)\big(2n_{-a-b(\alpha+\beta)}W_{2-a-b(\alpha+\beta)}-(\alpha-\beta)n_{-a-2b}W_{-a-2b+\alpha+\beta}\big).\end{aligned} \tag{3.4}$$

First, considering the coefficient of term "L" in (3.3), by the arbitrariness of α and β, we must have $m_{\alpha+\beta}=(\alpha-\beta)m_1$. Taking $\lambda=-m_1$, by (3.2) we have

$$\varphi(L_\alpha,L_\beta)=\lambda[L_\alpha,L_\beta]+n_{-a-b(\alpha+\beta)}W_{-a-b(\alpha+\beta)}.$$

Here we adopt the convention that $n_{-a-b(\alpha+\beta)}=0$ if $-a-b(\alpha+\beta)\notin\mathbb{Z}$. If $b=0$, by this convention, we only need to consider the case $a\in\mathbb{Z}$. In fact, this claim holds in this case because $W_{-a-b(\alpha+\beta)}=W_{-a}\in Z(\mathcal{W})$ by Lemma 3.2(1). Next, we assume that $b\neq0$. Considering the coefficients of terms "W" in (3.3), if $b\neq-1$ and $\alpha+\beta\neq1$, then the subscripts of two "W" are different, and so $n_{-a-b(\alpha+\beta)}=0$. Similarly, by (3.4) we have $n_{-a-b(\alpha+\beta)}=0$ if $b\neq-1$ and $\alpha+\beta\neq2$. Hence this claim holds in case $b\neq0,-1$. At last, consider the case $b=-1$. In this case the subscripts of two "W" in (3.3) are the same, by the arbitrariness of α and β, one can deduce that $n_{-a+\alpha+\beta}=(\alpha-\beta)n_{-a+1}$ (as above, only need to consider the case $a\in\mathbb{Z}$). Taking $\mu=-n_{-a+1}$, we have $\varphi(L_\alpha,L_\beta)=\lambda[L_\alpha,L_\beta]+\mu\varphi_0(L_\alpha,L_\beta)$. This claim still holds.

Claim 3. $\varphi(L_\alpha,W_\beta)\equiv\lambda[L_\alpha,W_\beta]\ (\mathrm{mod}\ Z(\mathcal{W}))$ for $\alpha,\beta\in\mathbb{Z}$.

For any fixed $\alpha, \beta \in \mathbb{Z}$, we suppose that

$$\varphi(L_\alpha, W_\beta) = \sum_{i\in\mathbb{Z}} p_i L_i + \sum_{j\in\mathbb{Z}} q_j W_j.$$

For any fixed $0 \neq k \in \mathbb{Z}$, by Corollary 2.2(1) and Claim 2, we have (in the following second equality one should note that $\varphi_0(L_0, L_k)$ (if appears) commutes with $W_{\alpha+\beta}$)

$$\begin{aligned} 0 &= [\varphi(L_\alpha, W_\beta), [L_0, L_k]] - [[L_\alpha, W_\beta], \varphi(L_0, L_k)] \\ &= \Big[\sum_{i\in\mathbb{Z}} p_i L_i + \sum_{j\in\mathbb{Z}} q_j W_j, kL_k\Big] - [(a+\beta+b\alpha)W_{\alpha+\beta}, \lambda k L_k] \\ &= \Big(\sum_{i\in\mathbb{Z}} p_i k(k-i)L_{i+k} - \sum_{j\in\mathbb{Z}} q_j k(a+j+bk)W_{j+k}\Big) \\ &\quad + \lambda k(a+\beta+b\alpha)(a+\alpha+\beta+bk)W_{\alpha+\beta+k}, \end{aligned}$$

which implies that

$$p_i = 0 \ \text{ if } \ i \neq k, \tag{3.5}$$
$$q_j(a+j+bk) = 0 \ \text{ if } \ j \neq \alpha+\beta, \tag{3.6}$$
$$(\lambda(a+\beta+b\alpha) - q_j)(a+\alpha+\beta+bk) = 0 \ \text{ if } \ j = \alpha+\beta. \tag{3.7}$$

First, by the arbitrariness of k in (3.5), we must have $p_i = 0$. If $b \neq 0$, by the arbitrariness of k in (3.6) and (3.7), we have $q_j = 0$ if $j \neq \alpha+\beta$ and $q_{\alpha+\beta} = \lambda(a+\beta+b\alpha)$. Hence, $\varphi(L_\alpha, W_\beta) = \lambda[L_\alpha, W_\beta]$. This claim holds. Next assume that $b = 0$. If $\alpha+\beta = -a$ (only need to consider the case $a \in \mathbb{Z}$), it follows from (3.6) and Lemma 3.2(1) that $\varphi(L_\alpha, W_\beta) \in Z(\mathcal{W})$. This claim holds. If $\alpha+\beta \neq -a$, it follows from (3.6), (3.7) and Lemma 3.2(1) that $\varphi(L_\alpha, W_\beta) - \lambda[L_\alpha, W_\beta] = \varphi(L_\alpha, W_\beta) - \lambda(a+\beta)W_{\alpha+\beta} \in Z(\mathcal{W})$. This claim still holds.

Now, by Claims 1–3, for any $x, y \in \mathcal{W}$, we have

$$\varphi(x,y) \equiv \begin{cases} \lambda[x,y] + \mu\varphi_0(x,y) \pmod{Z(\mathcal{W})} & \text{if } a \in \mathbb{Z},\ b = -1, \\ \lambda[x,y] \pmod{Z(\mathcal{W})} & \text{otherwise.} \end{cases} \tag{3.8}$$

If $a \notin \mathbb{Z}$ or $b \neq 0$, then by Lemma 3.2(1) we have $Z(\mathcal{W}) = 0$, and so Theorem 3.1 holds. Next, consider the case $a \in \mathbb{Z}$, $b = 0$. By Lemma 3.2(1) and (3.8), we may assume that $\varphi(x,y) = \lambda[x,y] + f(x,y)W_{-a}$, where f is a bilinear function from $\mathcal{W} \times \mathcal{W}$ to $\mathbb{C}$. We only need to show that f is the zero function. In fact, by

$$\varphi([x,y],z) = [x, \varphi(y,z)] + [\varphi(x,z), y],$$

we have that $f([x,y],z) = 0$ for all $x, y, z \in \mathcal{W}$. Note that $\mathcal{W}$ is perfect in this case (by Lemma 3.2(2)). It follows that f is exactly the zero function, as desired. ∎

4. Linear commuting maps on $\mathcal{W}(a, b)$

In this section, we shall describe the linear commuting maps on $\mathcal{W}$ based on Theorem 3.1.

Let $\mathcal{L}$ be an arbitrary Lie algebra with center $Z(\mathcal{L})$. For an arbitrary complex number $\lambda \in \mathbb{C}$ and an arbitrary linear function $f : \mathcal{L} \to Z(\mathcal{L})$, clearly, the following map

$$\psi(x) = \lambda x + f(x) \quad \text{for all} \quad x \in \mathcal{L}$$

is a linear commuting map on $\mathcal{L}$. We call such a map a *standard linear commuting map on* $\mathcal{L}$ (note that it is analog to a standard linear commuting map on an associative algebra, see Introduction). A linear commuting map on $\mathcal{L}$ in other forms is said to be *non-standard*.

Obviously, for all $\mathcal{W}$, the identity map $x \mapsto x$ is a standard linear commuting map. For $\mathcal{W}$ with $a \in \mathbb{Z}$, $b = 0$ and some linear function f from $\mathcal{W}$ to $\mathbb{C}$, by Lemma 3.2(1), the map $x \mapsto f(x)W_{-a}$ is also a standard linear commuting map. For $\mathcal{W}$ with $a \in \mathbb{Z}$, $b = -1$, consider the following 2-step nilpotent map:

$$\psi_0 : \mathcal{W} \to \mathcal{W}, \qquad L_\alpha \mapsto W_{\alpha-a}, \quad W_\alpha \mapsto 0. \tag{4.1}$$

It is not difficult to check that it is a non-standard linear commuting map (we leave the verification details to readers). The main result in this section is as follows.

Theorem 4.1. *Each linear commuting map ψ on $\mathcal{W}$ is one of the following forms:*

$$\psi(x) = \begin{cases} \lambda x + \mu\psi_0(x) & \text{if } a \in \mathbb{Z},\ b = -1, \\ \lambda x + f(x)W_{-a} & \text{if } a \in \mathbb{Z},\ b = 0, \\ \lambda x & \text{otherwise}, \end{cases}$$

for $x \in \mathcal{W}$, where $\lambda, \mu \in \mathbb{C}$, ψ_0 is given by (4.1)*, and f is a linear function from $\mathcal{W}$ to $\mathbb{C}$.*

Proof. Suppose that ψ is a linear commuting map on $\mathcal{W}$. Define

$$\varphi : \mathcal{W} \times \mathcal{W} \to \mathcal{W} \quad (x, y) \mapsto [\psi(x), y] \quad \text{for all} \quad x, y \in \mathcal{W}. \tag{4.2}$$

By (4.2), we have

$$\varphi(x, [y, z]) = [\varphi(x, y), z] + [y, \varphi(x, z)] \quad \text{for all} \quad x, y, z \in \mathcal{W}.$$

Namely, φ is a derivation with respect to the second component. Since $[\psi(x), y] = [x, \psi(y)]$ (by (1.2) and the linearity of ψ), one can easily see that φ is also a derivation with respect to the first component. Thus φ is a biderivation of $\mathcal{W}$. In addition, φ defined by (4.2) is automatically skew-symmetric. By Theorem 3.1, there exist $\lambda, \mu \in \mathbb{C}$ such that

$$\varphi(x, y) = \begin{cases} \lambda[x, y] + \mu\varphi_0(x, y) & \text{if } a \in \mathbb{Z},\ b = -1, \\ \lambda[x, y] & \text{otherwise}, \end{cases}$$

for all $x, y \in \mathcal{W}$, where φ_0 is given by (3.1). Then, by (4.2), we have

$$[\psi(x) - \lambda x, y] = \begin{cases} \mu\varphi_0(x, y) & \text{if } a \in \mathbb{Z},\ b = -1, \\ 0 & \text{otherwise}. \end{cases}$$

If $a \in \mathbb{Z}$, $b = -1$, by the definition of φ_0 given by (3.1), we must have $\psi(x) - \lambda x = \mu\psi_0(x)$. The conclusion holds. If $a \notin \mathbb{Z}$ or $b \neq -1$, then $\psi(x) - \lambda x \in Z(\mathcal{W})$. By Lemma 3.2(1), we only need to consider the case $a \in \mathbb{Z}$, $b = 0$. Define f by setting $\psi(x) - \lambda x = f(x)W_{-a}$, then f is a linear function from $\mathcal{W}$ to $\mathbb{C}$, and $\psi(x) = \lambda x + f(x)W_{-a}$, as desired. ■

Remark 4.2. From Theorem 4.1, we see that the Lie algebras $\mathcal{W} = \mathcal{W}(a, b)$ can be divided into three classes: (i) $a \in \mathbb{Z}$, $b = -1$; (ii) $a \in \mathbb{Z}$, $b = 0$; (iii) other classes. The two meaningful Lie algebras $\mathcal{W}(0, -1)$ (i.e., the W-algebra $W(2, 2)$) and $\mathcal{W}(0, 0)$ (i.e., the twisted Heisenberg-Virasoro Lie algebra) we mentioned in the Introduction are typical examples of classes (i) and (ii), respectively.

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