

On the Characterization of Trace Class Representations and Schwartz Operators

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Abstract. In this note we collect several characterizations of unitary representations (π, \mathcal{H}) of a finite dimensional Lie group G which are trace class, i.e., for each compactly supported smooth function f on G , the operator $\pi(f)$ is trace class. In particular we derive the new result that, for some $m \in \mathbb{N}$, all operators $\pi(f)$, $f \in C_c^m(G)$, are trace class. As a consequence the corresponding distribution character θ_π is of finite order. We further show π is trace class if and only if every operator A , which is smoothing in the sense that $A\mathcal{H} \subseteq \mathcal{H}^\infty$, is trace class and that this in turn is equivalent to the Fréchet space \mathcal{H}^∞ being nuclear, which in turn is equivalent to the realizability of the Gaussian measure of \mathcal{H} on the space $\mathcal{H}^{-\infty}$ of distribution vectors. Finally we show that, even for infinite dimensional Fréchet–Lie groups, A and A^* are smoothing if and only if A is a Schwartz operator, i.e., all products of A with operators from the derived representation are bounded.

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Introduction

Let (π, \mathcal{H}) be a (strongly continuous) unitary representation of the (possibly infinite dimensional) Lie group G (with an exponential function). Let \mathcal{H}^∞ be its subspace of smooth vectors. On this space we obtain by

$$\mathrm{d}\pi(x)v = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp tx)v$$

the *derived representation* of \mathfrak{g} which we extend naturally to a representation of the enveloping algebra $U(\mathfrak{g})$, also denoted $\mathrm{d}\pi$. We call an operator $A \in B(\mathcal{H})$ *smoothing* if $A\mathcal{H} \subseteq \mathcal{H}^\infty$ ([NSZ15]). A closely related concept is that of a *Schwartz operator*, which means that, for all $D_1, D_2 \in U(\mathfrak{g})$ (the enveloping algebra of the Lie algebra \mathfrak{g} of G), the sesquilinear form

$$(v, w) \mapsto \langle \mathrm{Ad}\pi(D_2)v, \mathrm{d}\pi(D_1)w \rangle$$

on \mathcal{H}^∞ extends continuously to $\mathcal{H} \times \mathcal{H}$ ([Ho77, Thm. 3.4, p. 349], [KKW15]). This note grew out of the question to understand the relation between smoothing and Schwartz operators. This is completely answered by Theorem 2.4 which asserts, for any smooth representation of a Fréchet–Lie group G and $S \in B(\mathcal{H})$, the following are equivalent:

- S is Schwartz.
- S and S^* are smoothing.
- The map $G \times G \rightarrow B(\mathcal{H})$, $(g, h) \mapsto \pi(g)S\pi(h)$ is smooth.

Smoothing operators are of particular importance for unitary representations of finite dimensional Lie groups which are *trace class* in the sense that, for each $f \in C_c^\infty(G)$, the operator $\pi(f) = \int_G f(g)\pi(g) dg$ is trace class. Actually we show in Proposition 1.6 that every smoothing operator is trace class if and only if π is trace class. This connection was our motivation to compile various characterizations of trace class representations scattered in the literature, mostly without proofs ([Ca76]). Surprisingly, this also led us to some new insights, such as the fact that, if π is trace class, then there exists an $m \in \mathbb{N}$ such that all operators $\pi(f)$, $f \in C_c^m(G)$, are trace class. As a consequence, the corresponding distribution character θ_π is of finite order. This is contained in Theorem 1.3 which collects various characterizations of trace class representations. One of them is that, for every basis X_1, \dots, X_n of \mathfrak{g} and $\Delta := \sum_{j=1}^n X_j^2$, the positive selfadjoint operator $\mathbf{1} - \overline{\mathfrak{d}\pi(\Delta)}$ has some negative power which is trace class. This is analogous to the Nelson–Stinespring characterization of CCR representations (all operators $\pi(f)$, $f \in L^1(G)$, are compact) by the compactness of the inverse of $\mathbf{1} - \overline{\mathfrak{d}\pi(\Delta)}$. It is well known that all these representations are direct sums of irreducible representations with finite multiplicities ([Bo72, Prop. 5.9]). Locally compact groups for which all irreducible unitary representations are trace class, so-called *trace class groups*, have recently been studied in [DD16], and for a characterization of groups for which all irreducible unitary representations are CCR (*type I groups*), we refer to [Pu78, Thm. 2]. For a connected, simply connected Lie group G , the type I property can be characterized in terms of a regularity property of the coadjoint action. The trace class property is much more restrictive, f.i., the groups $\mathbb{R}^n \rtimes \mathrm{SL}_n(\mathbb{R})$, $n \geq 2$, are type I but not trace class. All connected nilpotent and reductive Lie groups are trace class ([DD16, Prop. 1.9, Thm. 2.1]).

In the measure theoretic approach to second quantization, the Fock space of a real Hilbert space is realized as the L^2 -space for the Gaussian measure γ on a suitable enlargement of \mathcal{H} . Combining our characterization of trace class representations with results in [JNO15], we see that the trace class condition is equivalent to \mathcal{H}^∞ being nuclear, which in turn is equivalent to the realizability of the Gaussian measure on the dual space $\mathcal{H}^{-\infty}$ of distribution vectors.

Notation: Throughout this article, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a unitary representation (π, \mathcal{H}) of G , let $\overline{\mathfrak{d}\pi}(x)$ for $x \in \mathfrak{g}$ denote the infinitesimal generator of the one-parameter group $t \mapsto \pi(\exp(tx))$ by Stone’s Theorem. Set $\mathcal{D}^n = \mathcal{D}^n(\pi) := \bigcap_{x_1, \dots, x_n \in \mathfrak{g}} \mathcal{D}(\overline{\mathfrak{d}\pi}(x_1) \cdots \overline{\mathfrak{d}\pi}(x_n))$ and $\mathcal{D}^\infty = \mathcal{D}^\infty(\pi) := \bigcap_{n=1}^\infty \mathcal{D}^n$.

1. Characterizing trace class representations

In this section G will be a finite dimensional Lie group and \mathfrak{g} will be the Lie algebra of G . We fix a basis X_1, \dots, X_n of \mathfrak{g} and consider the corresponding Nelson–Laplacian $\Delta := X_1^2 + \dots + X_n^2$, considered as an element of the enveloping algebra $U(\mathfrak{g})$. We write $B_p(\mathcal{H})$ for the p th Schatten ideal in the algebra $B(\mathcal{H})$ of bounded operators on a Hilbert space \mathcal{H} and $K(\mathcal{H})$ for the ideal of compact operators.

Recall that a unitary representation (π, \mathcal{H}) is called trace class if $\pi(f) \in B_1(\mathcal{H})$ for every $f \in C_c^\infty(G)$. For every unitary representation (π, \mathcal{H}) , the subspace \mathcal{H}^∞ of smooth vectors can naturally be endowed with a Fréchet space structure obtained from the embedding $\mathcal{H}^\infty \rightarrow C^\infty(G, \mathcal{H}), v \mapsto \pi^v$, where $\pi^v(g) = \pi(g)v$. Its range is the closed subspace of smooth equivariant maps in the Fréchet space $C^\infty(G, \mathcal{H})$. This Fréchet topology on \mathcal{H}^∞ is identical to the topology obtained by the family of seminorms $\{\|\cdot\|_D : D \in U(\mathfrak{g})\}$, where $\|v\|_D := \|\mathfrak{d}\pi(D)v\|$ for $v \in \mathcal{H}^\infty$.

Lemma 1.1. *If (π, \mathcal{H}) is a unitary representation of the Lie group G , then $\pi(f)\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$ for every $f \in C_c(G)$.*

Proof. In view of [Ne10, Thm. 4.4], the representation π^∞ of G on the Fréchet space \mathcal{H}^∞ is smooth. Hence, for every $v \in \mathcal{H}^\infty$ and $f \in C_c(G)$, the continuous compactly supported map

$$G \rightarrow \mathcal{H}^\infty, \quad g \mapsto f(g)\pi(g)v$$

has a weak integral I . Then, for every $w \in \mathcal{H}$,

$$\langle I, w \rangle = \int_G f(g) \langle \pi(g)v, w \rangle dg = \langle \pi(f)v, w \rangle,$$

and therefore $I = \pi(f)v \in \mathcal{H}^\infty$. ■

Lemma 1.2. *Let V be a Fréchet space, W be a metrizable vector space and $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of continuous linear maps $V \rightarrow W$ for which $\lambda(v) = \lim_{n \rightarrow \infty} \lambda_n(v)$ exists for every $v \in V$. Then λ is continuous.*

Proof. Since V is a Baire space and W is metrizable, it follows from [Bou74, Ch. IX, §5, Ex. 22(a)] that the set of discontinuity points of λ is of the first category, hence its complement is non-empty. This implies that λ is continuous. ■

The following theorem generalizes [Ca76, Thm. 2.6] in a Bourbaki exposé of P. Cartier which states the equivalence of (iii) and (v), but unfortunately without giving a proof or a reference to one.

Theorem 1.3. *Suppose that G has at most countably many connected components.¹ For a unitary representation (π, \mathcal{H}) of G , the following are equivalent:*

¹This assumption ensures that $C_c^\infty(G)$ is an LF space, otherwise the topology on $C_c^\infty(G)$

- (i) *There exists an $m \in \mathbb{N}$ such that $\pi(C_c^m(G)) \subseteq B_1(\mathcal{H})$ and the corresponding map $\pi: C_c^m(G) \rightarrow B_1(\mathcal{H})$ is continuous.*
- (ii) *$\pi(C_c^\infty(G)) \subseteq B_1(\mathcal{H})$ and the map $\pi: C_c^\infty(G) \rightarrow B_1(\mathcal{H})$ is continuous.*
- (iii) *π is a trace class representation, i.e., $\pi(C_c^\infty(G)) \subseteq B_1(\mathcal{H})$.*
- (iv) *$\pi(C_c^\infty(G)) \subseteq B_2(\mathcal{H})$.*
- (v) *There exists a $k \in \mathbb{N}$ such that $(\mathbf{1} - \overline{\mathfrak{d}\pi(\Delta)})^{-k}$ is trace class.*

Proof. Let $D := \overline{\mathfrak{d}\pi(\Delta)}$, where $\mathfrak{d}\pi: U(\mathfrak{g}) \rightarrow \text{End}(\mathcal{H}^\infty)$ denotes the derived representation, extended to the enveloping algebra. Recall that D is a non-positive selfadjoint operator on \mathcal{H} ([NS59]).

(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are trivial implications.

(iv) \Rightarrow (iii): According to the Dixmier–Malliavin Theorem ([DM78, Thm. 3.1]), we can write every $f \in C_c^\infty(G)$ as a finite sum of products $a * b$ with $a, b \in C_c^\infty(G)$. Hence the assertion follows from $B_2(\mathcal{H})B_2(\mathcal{H}) \subseteq B_1(\mathcal{H})$.

(iii) \Rightarrow (ii): ² For every sequence $(C_m)_{m \in \mathbb{N}}$ of compact subsets of G with $C_m \subseteq C_{m+1}^0$ and $\bigcup_m C_m = G$ (which exists because G has at most countably many connected components), the space $C_c^\infty(G)$ is the direct limit of the Fréchet subspaces

$$C_{C_m}^\infty(G) := \{f \in C^\infty(G) : \text{supp}(f) \subseteq C_m\}.$$

Therefore it suffices to show that $\pi: C_{C_m}^\infty(G) \rightarrow B_1(\mathcal{H})$ is a continuous linear map for every $m \in \mathbb{N}$.

Let $(\delta_n)_{n \in \mathbb{N}}$ be a δ -sequence in $C_c^\infty(G)$, i.e., $\int_G \delta_n(g) dg = 1$ and $\text{supp}(\delta_n)$ converges to $\{\mathbf{1}\}$ in the sense that, for every $\mathbf{1}$ -neighborhood U in G , we eventually have $\text{supp}(\delta_n) \subseteq U$. Then $\delta_n * f \rightarrow f$ for every $f \in C_c^\infty(G)$ holds in $L^1(G)$ (and even in $C_c^\infty(G)$). For every $n \in \mathbb{N}$, the linear map

$$\pi_n: C_{C_m}^\infty(G) \rightarrow B_1(\mathcal{H}), \quad f \mapsto \pi(\delta_n)\pi(f) = \pi(\delta_n * f)$$

is continuous because the linear maps

$$C_{C_m}^\infty(G) \rightarrow B(\mathcal{H}), \quad f \mapsto \pi(f) \quad \text{and} \quad B(\mathcal{H}) \rightarrow B_1(\mathcal{H}), \quad A \mapsto \pi(\delta_n)A$$

are continuous. Here we use that $\|\pi(\delta_n)A\|_1 \leq \|\pi(\delta_n)\|_1 \|A\|$.

In view of Lemma 1.2, it suffices to show that, for every $f \in C_{C_m}^\infty(G)$, we have

$$\pi(f) = \lim_{n \rightarrow \infty} \pi_n(f) = \lim_{n \rightarrow \infty} \pi(\delta_n * f)$$

holds in $B_1(\mathcal{H})$. Using the Dixmier–Malliavin Theorem ([DM78, Thm. 3.1]), we write $f = \sum_{j=1}^k a_j * b_j$ with $a_j, b_j \in C_c^\infty(G)$. Then

$$\pi_n(f) = \pi(\delta_n * f) = \sum_{j=1}^k \pi(\delta_n * a_j * b_j) = \sum_{j=1}^k \pi(\delta_n * a_j)\pi(b_j).$$

becomes somewhat pathological. However, π is trace class if and only if the restriction to the identity component is, so that the main point is the characterization for the class of connected Lie groups.

²The assertion in [DD16, Prop. 1.4] comes close to this statement but does not assert the continuity of the $B_1(\mathcal{H})$ -valued map.

Since the right multiplication maps $B(\mathcal{H}) \rightarrow B_1(\mathcal{H}), A \mapsto A\pi(b_j)$ are continuous and $\lim_{n \rightarrow \infty} \pi(\delta_n * a_j) = \pi(a_j)$ in $B(\mathcal{H})$, it follows that $\pi_n(f) \rightarrow \pi(f)$ for every $f \in C_c^\infty(G)$. Now the assertion follows from Lemma 1.2.

(ii) \Rightarrow (v): Let $\Omega \subseteq G$ be a compact $\mathbf{1}$ -neighborhood in G and

$$C_\Omega^m(G) := \{f \in C^m(G) : \text{supp}(f) \subseteq \Omega\} \quad \text{for } m \in \mathbb{N}_0 \cup \{\infty\}.$$

Then $C_\Omega^m(G)$ is a Banach space for each $m \in \mathbb{N}_0$, and the Fréchet space $C_\Omega^\infty(G)$ is the projective limit of the Banach spaces $C_\Omega^m(G)$. Therefore the continuity of the seminorm $f \mapsto \|\pi(f)\|_1$ on $C_\Omega^\infty(G)$ implies the existence of some $m \in \mathbb{N}$ such that the map $\pi : C_\Omega^\infty(G) \rightarrow B_1(\mathcal{H})$ extends continuously to $C_\Omega^m(G)$. This implies that $\pi(C_\Omega^m(G)) \subseteq B_1(\mathcal{H})$.

Next we observe that by an argument similar to the proof of a Lemma by M. Duflo ([B72, Lemma 3.2.3, p. 250]), there exists for every $m \in \mathbb{N}$ a positive integer k , an open $\mathbf{1}$ -neighborhood $U \subseteq \Omega$ in G , and functions $\beta, \gamma \in C_c^m(U)$ such that

$$(\mathbf{1} - \Delta)^k \beta = \delta_{\mathbf{1}} + \gamma, \tag{1}$$

where $\delta_{\mathbf{1}}$ is the Dirac distribution in $\mathbf{1}$. Then

$$\pi(\beta) = (\mathbf{1} - D)^{-k} (\mathbf{1} - D)^k \pi(\beta) = (\mathbf{1} - D)^{-k} \pi((\mathbf{1} - \Delta)^k \beta) = (\mathbf{1} - D)^{-k} (\mathbf{1} + \pi(\gamma))$$

holds as an identity of linear operators on \mathcal{H}^∞ (Lemma 1.1), and since both sides are bounded on \mathcal{H} , we obtain

$$(\mathbf{1} - D)^{-k} = \pi(\beta) - (\mathbf{1} - D)^{-k} \pi(\gamma). \tag{2}$$

By the preceding argument, both summands on the right are trace class, so that $(\mathbf{1} - D)^{-k}$ is trace class as well.

(v) \Rightarrow (i): For $f \in C_c^\infty(G)$, we have

$$\pi(f) = (\mathbf{1} - D)^{-k} (\mathbf{1} - D)^k \pi(f) = (\mathbf{1} - D)^{-k} \pi((\mathbf{1} - \Delta)^k f). \tag{3}$$

Since the first factor on the right is trace class and $\pi((\mathbf{1} - \Delta)^k f) \in B(\mathcal{H})$, it follows that $\pi(C_c^\infty(G)) \subseteq B_1(\mathcal{H})$. Moreover, the continuity of the linear operator $(\mathbf{1} - \Delta)^k : C_c^{2k}(G) \rightarrow L^1(G)$ and the density of $C_c^\infty(G)$ in $C_c^{2k}(G)$ imply that the identity (3) holds for all $f \in C_c^{2k}(G)$. We conclude that $\pi(C_c^{2k}(G)) \subseteq B_1(\mathcal{H})$, and continuity of the integrated representation $\pi : L^1(G) \rightarrow B(\mathcal{H})$ implies that the corresponding map $C_c^{2k}(G) \rightarrow B_1(\mathcal{H})$ is continuous. ■

Along the same lines one obtains the following characterization of completely continuous representations (CCR) from [NS59, Thm. 4.1].

Theorem 1.4. (Nelson–Stinespring) *For a unitary representation (π, \mathcal{H}) of G , the following are equivalent:*

- (i) $\pi(L^1(G)) \subseteq K(\mathcal{H})$.
- (ii) $\pi(C_c^\infty(G)) \subseteq K(\mathcal{H})$.
- (iii) $(\mathbf{1} - \overline{\mathfrak{d}\pi(\Delta)})^{-1}$ is a compact operator.

Proof. The equivalence of (i) and (ii) follows from the density of $C_c^\infty(G)$ in $L^1(G)$. We now use the same notation as in the preceding proof.

(i) \Rightarrow (iii): From the relation

$$(\mathbf{1} - D)^{-k} = \pi(\beta) - (\mathbf{1} - D)^{-k}\pi(\gamma)$$

we derive the existence of some $k \in \mathbb{N}$ for which $(\mathbf{1} - D)^{-k}$ is compact, but this implies that $(\mathbf{1} - D)^{-1}$ is compact as well.

(iii) \Rightarrow (ii): For $f \in C_c^\infty(G)$, we have

$$\pi(f) = (\mathbf{1} - D)^{-1}(\mathbf{1} - D)\pi(f) = (\mathbf{1} - D)^{-1}\pi((\mathbf{1} - \Delta)f). \quad (4)$$

Therefore the compactness of $(\mathbf{1} - D)^{-1}$ implies (ii). \blacksquare

1.1. Application to smoothing operators.

Definition 1.5. For a unitary representation (π, \mathcal{H}) of a Lie group G , an operator $A \in B(\mathcal{H})$ is called *smoothing* if $A\mathcal{H} \subseteq \mathcal{H}^\infty$. We write $B(\mathcal{H})^\infty$ for the subspace of smoothing operators in $B(\mathcal{H})$.

It is shown in [NSZ15, Thm. 2.11] that for the class of Fréchet–Lie groups, which contains in particular all finite dimensional ones, an operator A is smoothing if and only if it is a smooth vector for the representation $\lambda(g)A := \pi(g)A$ of G on $B(\mathcal{H})$. If π is not norm continuous, then this representation is not continuous because the orbit map of the identity operator is not continuous, but it defines a continuous representation by isometries on the norm-closed subspace

$$B(\mathcal{H})_c := \{A \in B(\mathcal{H}) : \lim_{g \rightarrow \mathbf{1}} \pi(g)A = A\}.$$

By Gårding’s Theorem, $\pi(f) \in B(\mathcal{H})^\infty$ for every $f \in C_c^\infty(G)$. Applying the Dixmier–Malliavin Theorem [DM78, Thm. 3.3] to the continuous representation $(\lambda, B(\mathcal{H})_c)$, we see that

$$B(\mathcal{H})^\infty = \text{span}\{\pi(f)A : f \in C_c^\infty(G), A \in B(\mathcal{H})\}. \quad (5)$$

It follows in particular that all smoothing operators are trace class if π is a trace class representation. Alternatively one can use the factorization

$$A = (\mathbf{1} - D)^{-k}(\mathbf{1} - D)^k A$$

for every smoothing operator A to see that A is trace class because $(\mathbf{1} - D)^{-k}$ is trace class for some k .

From Gårding’s Theorem we obtain another characterization of trace class representations:

Proposition 1.6. *A unitary representation (π, \mathcal{H}) of G is trace class if and only if $B(\mathcal{H})^\infty \subseteq B_1(\mathcal{H})$, i.e., all smoothing operators are trace class.*

Proposition 1.7. *If (π, \mathcal{H}) is a trace class representation of G , then the space of smoothing operators coincides with the subspace of smooth vectors of the unitary representation $(\lambda, B_2(\mathcal{H}))$ defined by $\lambda(g)A := \pi(g)A$.*

Proof. Since the inclusion $B_2(\mathcal{H}) \rightarrow B(\mathcal{H})$ is smooth, every $A \in B_2(\mathcal{H})^\infty$ has a smooth orbit map $G \rightarrow B(\mathcal{H}), g \mapsto \pi(g)A$, hence is smoothing.

If, conversely, A is smoothing, then (5) shows that A is a finite sum of operators of the form $\pi(f)B, f \in C_c^\infty(G), B \in B(\mathcal{H})$. Since $\pi: C_c^\infty(G) \rightarrow B_2(\mathcal{H}), f \mapsto \pi(f)$, is a continuous linear map by Theorem 1.3, the right multiplication map $B_2(\mathcal{H}) \rightarrow B_2(\mathcal{H}), C \mapsto CB$ is continuous, and the map $G \rightarrow C_c^\infty(G), g \mapsto \delta_g * f$ is smooth, the relation

$$\pi(g)\pi(f)B = \pi(\delta_g * f)B$$

implies that $\pi(f)B$ has a smooth orbit map in $B_2(\mathcal{H})$. We conclude that the same holds for every smoothing operator. ■

The equivalence of the statements in the first two parts of the following corollary can also be derived from the vastly more general Theorem 2.4, but it may be instructive to see the direct argument for trace class representations as well.

Corollary 1.8. *For a trace class representation of G and $A \in B(\mathcal{H})$, the following are equivalent:*

- (i) A is a Schwartz operator.
- (ii) A and A^* are smoothing.
- (iii) $A \in B_2(\mathcal{H})$ and the map $\alpha^A: G \times G \rightarrow B_2(\mathcal{H}), (g, h) \mapsto \pi(g)A\pi(h^{-1})$ is smooth.

Proof. (i) \Rightarrow (ii): If A is Schwartz, then in particular the operators $Ad\pi(D), D \in U(\mathfrak{g})$, are bounded on \mathcal{H}^∞ , and thus from [NSZ15, Thm 2.11] it follows that A^* is smoothing. Furthermore, boundedness of $d\pi(D)A$ for every $D \in U(\mathfrak{g})$ entails in particular that $A\mathcal{H} \subseteq \mathcal{D}^\infty$, so that by [NSZ15, Thm 2.11] we obtain that A is also smoothing.

(ii) \Rightarrow (iii): Next assume that A and A^* are smoothing. Then Proposition 1.6 implies that $A, A^* \in B_2(\mathcal{H})$, and Proposition 1.7 implies that the maps

$$G \rightarrow B_2(\mathcal{H}), \quad g \mapsto \pi(g)A \quad \text{and} \quad G \rightarrow B_2(\mathcal{H}), \quad g \mapsto A\pi(g)$$

are smooth. For the unitary representation of $G \times G$ on $B_2(\mathcal{H})$ defined by $\alpha(g, h)M := \pi(g)M\pi(h)^{-1}$ this implies that the matrix coefficient

$$(g, h) \mapsto \langle \alpha(g, h)A, A \rangle = \langle \pi(g)A\pi(h)^*, A \rangle = \langle \pi(g)A, A\pi(h) \rangle$$

is smooth, so that A is a smooth vector for α by [Ne10, Thm. 7.2].

(iii) \Rightarrow (i): Finally, assume that $A \in B_2(\mathcal{H})$ and the map α^A is smooth. Since the linear embedding $B_2(\mathcal{H}) \rightarrow B(\mathcal{H})$ is continuous, the orbit map

$$G \times G \rightarrow B(\mathcal{H}), \quad (g, h) \mapsto \pi(g)A\pi(h) \tag{6}$$

is also smooth. From [NSZ15, Lem 2.9] and [NSZ15, Lem 2.10], and by considering suitable partial derivatives at $(\mathbf{1}, \mathbf{1})$ of the map (6), we obtain boundedness of the operators

$$d\pi(X_1) \cdots d\pi(X_n) \text{Ad}\pi(Y_1) \cdots d\pi(Y_m),$$

where $X_1, \dots, X_n, Y_1, \dots, Y_m \in \mathfrak{g}$. ■

For the last result of this section we need the following lemma, which appears in [Ca76, Thm 1.3(b)] without proof.

Lemma 1.9. *Let (π, \mathcal{H}) be a unitary representation of G and let $\mathcal{H}^{-\infty}$ denote the space of distribution vectors, i.e., the anti-dual of \mathcal{H}^∞ . Then every $\lambda \in \mathcal{H}^{-\infty}$ is a sum of finitely many anti-linear functionals $\lambda_{D,v} \in \mathcal{H}^{-\infty}$ of the form $\lambda_{D,v}(w) := \langle v, d\pi(D)w \rangle$, where $v \in \mathcal{H}$ and $D \in U(\mathfrak{g})$. In fact we can write λ as a linear combination of finitely many $\lambda_{D,v}$ with $D = (\mathbf{1} - \Delta)^k$ where $k \in \mathbb{N}$.*

Proof. Continuity of $\lambda_{D,v}$ is straightforward. Next fix $\lambda \in \mathcal{H}^{-\infty}$. The map

$$\mathcal{H}^\infty \rightarrow \mathcal{H}^{U(\mathfrak{g})}, \quad v \mapsto (d\pi(D)v)_{D \in U(\mathfrak{g})}$$

is a topological embedding, where $\mathcal{H}^{U(\mathfrak{g})}$ is equipped with the product topology. Thus by the Hahn–Banach Theorem, we can extend λ to a continuous anti-linear functional on $\mathcal{H}^{U(\mathfrak{g})}$. Since the continuous anti-dual of \mathcal{H} is identical to the continuous dual of the complex conjugate Hilbert space $\overline{\mathcal{H}}$, and the continuous dual of a direct product is isomorphic to the direct sum of the continuous duals, we obtain that $\lambda = \sum_{i=1}^m \lambda_{D_i, v_i}$ for some $D_i \in U(\mathfrak{g})$ and $v_i \in \mathcal{H}$.

For the refinement claimed at the end of the statement of the lemma, we can modify the above argument using the fact that the Fréchet topology of \mathcal{H}^∞ is given by the seminorms $v \mapsto \|(\mathbf{1} - d\pi(\Delta))^k v\|$, where $k \in \mathbb{N}$. ■

Let $\mathcal{S}(\pi, \mathcal{H}) \subset B(\mathcal{H})$ denote the space of Schwartz operators of a unitary representation (π, \mathcal{H}) . If (π, \mathcal{H}) is trace class, then from Corollary 1.8 it follows that $\mathcal{S}(\pi, \mathcal{H})$ is the space of smooth vectors of the unitary representation of $G \times G$ on the Hilbert space $B_2(\mathcal{H})$, defined by $\alpha(g, h)M := \pi(g)M\pi(h)^{-1}$. In this case we equip $\mathcal{S}(\pi, \mathcal{H})$ with the usual Fréchet topology of the space of smooth vectors. The next proposition characterizes the topological dual of $\mathcal{S}(\pi, \mathcal{H})$.

Proposition 1.10. *Let (π, \mathcal{H}) be a trace class representation of G . Every continuous linear functional on the Fréchet space $\mathcal{S}(\pi, \mathcal{H})$ of Schwartz operators is of the form*

$$\lambda_{A,k}(T) := \text{tr}((\mathbf{1} - D)^k T (\mathbf{1} - D)^k A),$$

where $D = \overline{d\pi(\Delta)}$, $A \in B_1(\mathcal{H})$, and $k \in \mathbb{N}$.

Proof. From Corollary 1.8 we know that the space $\mathcal{S}(\pi, \mathcal{H})$ of Schwartz operators coincides with the space of smooth vectors of the unitary representation $(\alpha, B_2(\mathcal{H}))$ given by $\alpha(g, h)A := \pi(g)A\pi(h)^{-1}$. For $x_1, \dots, x_n, y_1, \dots, y_m \in \mathfrak{g}$ and

every smooth vector T for α , we have

$$\begin{aligned} & \mathrm{d}\alpha((x_1, 0) \cdots (x_n, 0)(0, y_1) \cdots (0, y_m))T \\ & = (-1)^m \mathrm{d}\pi(x_1) \cdots \mathrm{d}\pi(x_n)T \mathrm{d}\pi(y_m) \cdots \mathrm{d}\pi(y_1). \end{aligned}$$

Set $N := 1 - \overline{\mathrm{d}\pi(\Delta)}$. Since $\dim(G) < \infty$, by Lemma 1.9 it now follows that, for every $T \in \mathcal{S}(\pi, \mathcal{H})$, we can write λ as a linear combination of finitely many linear functionals $\lambda_{p,q,A}(T) := \mathrm{tr}(N^p T N^q A)$, where $p, q \in \mathbb{N}_0$ and $A \in B_2(\mathcal{H})$. Furthermore, for every $r > p, q$ such that N^{p-r} and N^{q-r} are both trace class (see Theorem 1.3), we have

$$\lambda_{p,q,A}(T) = \mathrm{tr}(N^p T N^q A) = \mathrm{tr}(N^{p-r} N^r T N^r N^{q-r} A) = \lambda_{r,r,B}(T),$$

where $B = N^{q-r} A N^{p-r}$ is trace class. Since r can be chosen to be arbitrarily large, every linear combination of the functionals $\lambda_{p,q,A}$ can be expressed as a single linear functional $\lambda_{r,r,B}$ for a suitable $B \in B_1(\mathcal{H})$. ■

1.2. Nuclearity of the space of smooth vectors. Combining Theorem 1.3 with [JNO15, Cor. 4.18] we obtain:

Proposition 1.11. *For a unitary representation (π, \mathcal{H}) , the following are equivalent:*

- (a) π is trace class.
- (b) The Fréchet space \mathcal{H}^∞ is nuclear.
- (c) There exists a measure γ on the real dual space $\mathcal{H}^{-\infty}$ of \mathcal{H}^∞ , endowed with the σ -algebra generated by the evaluations in smooth vectors, whose Fourier transform is $\widehat{\gamma}(v) = \int_{\mathcal{H}^{-\infty}} e^{i\alpha(v)} d\gamma(\alpha) = e^{-\|v\|^2/2}$ for $v \in \mathcal{H}^\infty$.

The main idea in the proof of [JNO15, Cor. 4.18] is that \mathcal{H}^∞ coincides with the space of smooth vectors of the selfadjoint operator $\mathrm{d}\pi(\Delta)$ and that properties (b) and (c) can now be investigated in terms of the spectral resolution of this operator. The equivalence of (a) and (b) is also stated in [Ca76, Thm. 2.6] without proof.

1.3. Tensor product structure of $\mathcal{S}(\pi, \mathcal{H})$. In this subsection we show that, for any trace class representation (π, \mathcal{H}) of G , the space $\mathcal{S}(\pi, \mathcal{H})$ of Schwartz operators can be identified with the tensor product of the nuclear space \mathcal{H}^∞ with its complex conjugate space.

Remark 1.12. (a) For a unitary representation $\pi : G \rightarrow U(\mathcal{H})$ we have $\mathcal{H}^\infty = \mathcal{D}^\infty(\overline{\mathrm{d}\pi(\Delta)})$ and the natural Fréchet topology on \mathcal{H}^∞ is generated by the norms $v \mapsto \|N^k v\|, k \in \mathbb{N}_0$, where $N := \mathbf{1} - \mathrm{d}\pi(\Delta) \geq \mathbf{1}$.

(b) For $k \in \mathbb{Z}$ we define \mathcal{H}_k as the completion of the dense subspace $\mathcal{D}(N^k)$ of \mathcal{H} with respect to the Hilbert norm

$$\|v\|_k := \|N^k v\| = \langle N^k v, N^k v \rangle^{1/2}.$$

For $k \geq 0$ we have $\mathcal{H}_k = \mathcal{D}(N^k)$. Furthermore the scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{D}(N^k) \times \mathcal{H}$ extends to a sesquilinear map $\mathcal{H}_k \times \mathcal{H}_{-k} \rightarrow \mathbb{C}$ which exhibits \mathcal{H}_{-k} as the dual space of \mathcal{H}_k . The unitary transformations $N^k : \mathcal{H}_k \rightarrow \mathcal{H}, k \in \mathbb{Z}$, yield isomorphisms of Banach spaces

$$B(\mathcal{H}_k, \mathcal{H}_\ell) \rightarrow B(\mathcal{H}), \quad A \mapsto N^\ell AN^{-k}, \quad k, \ell \in \mathbb{Z}.$$

It is straightforward to show that these restrict to isomorphisms of the Schatten classes $B_p(\mathcal{H}_k, \mathcal{H}_\ell) \rightarrow B_p(\mathcal{H}), p \geq 1$.

The following description of the space of Schwartz operators is well-known for the Schrödinger representation of the Heisenberg group ([KKW15]), but it actually holds for all trace class representations.

Proposition 1.13. *Let G be a finite-dimensional Lie group and $\pi : G \rightarrow U(\mathcal{H})$ be a trace-class representation. Then we obtain an isomorphism of complex Fréchet spaces*

$$\eta : \mathcal{H}^\infty \widehat{\otimes}_\pi \overline{\mathcal{H}^\infty} \xrightarrow{\cong} \mathcal{S}(\pi, \mathcal{H}), \quad (x, y) \mapsto P_{x,y} \quad \text{with } P_{x,y}(v) := \langle v, y \rangle x,$$

where $\overline{\mathcal{H}^\infty}$ denotes the complex conjugate space of \mathcal{H}^∞ . In particular, $\mathcal{S}(\pi, \mathcal{H})$ is a nuclear space.

Proof. By Corollary 1.8 $\mathcal{S}(\pi, \mathcal{H})$ is the space of smooth vectors of the unitary representation $\alpha : G \times G \rightarrow U(B_2(\mathcal{H})), \alpha(g, h)M = \pi(g)M\pi(h)^{-1}$. Moreover $\mathcal{S}(\pi, \mathcal{H})$ is equipped with the usual Fréchet topology of the space of smooth vectors which is specified by the norms $M \mapsto \|N^k M N^\ell\|, k, \ell \in \mathbb{N}_0$, with $N := \mathbf{1} - \overline{\text{d}\pi(\Delta)}$ as in Remark 1.12(a). Since $N^k P_{x,y} N^\ell = P_{N^k x, N^\ell y}$ and $N : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty$ is continuous, the bilinear map

$$b : \mathcal{H}^\infty \times \overline{\mathcal{H}^\infty} \rightarrow \mathcal{S}(\pi, \mathcal{H}), \quad (x, y) \mapsto P_{x,y}$$

is continuous and therefore induces a continuous linear map

$$\eta : \mathcal{H}^\infty \widehat{\otimes}_\pi \overline{\mathcal{H}^\infty} \rightarrow \mathcal{S}(\pi, \mathcal{H}).$$

Recall from Proposition 1.11 that \mathcal{H}^∞ is nuclear. From [GV64, Thm. 4 in Sect. I.3.5] and Remark 1.12(b) we obtain that the continuous dual $(\mathcal{H}^\infty \widehat{\otimes}_\pi \overline{\mathcal{H}^\infty})'$ consists of the elements

$$\lambda_{A,k,\ell} : \mathcal{H}^\infty \widehat{\otimes}_\pi \overline{\mathcal{H}^\infty} \rightarrow \mathbb{C}, \quad x \otimes y \mapsto \text{tr}(N^k P_{x,y} N^\ell A),$$

where $A \in B_2(\mathcal{H}), k, \ell \in \mathbb{N}_0$. With Proposition 1.10 we conclude that

$$\eta^* : \mathcal{S}(\pi, \mathcal{H})' \rightarrow (\mathcal{H}^\infty \widehat{\otimes}_\pi \overline{\mathcal{H}^\infty})'$$

is surjective which implies that η is injective.

Using the identification $B_2(\mathcal{H}) = \mathcal{H} \widehat{\otimes} \overline{\mathcal{H}}$, the representation α corresponds to $\pi \otimes \overline{\pi}$. Let γ_N denote the unitary one-parameter group on \mathcal{H} generated

by N . Then $T := N \otimes \mathbf{1} + \mathbf{1} \otimes N$ is the self-adjoint generator of $\gamma_N \otimes \gamma_{-N}$. By Remark 1.12(a) we have $\mathcal{H}^\infty = \mathcal{D}^\infty(N)$ and $\mathcal{S}(\pi, \mathcal{H}) = \mathcal{D}^\infty(T)$. Therefore $\mathcal{H}^\infty \otimes \overline{\mathcal{H}^\infty}$ is invariant under $\gamma_N \otimes \gamma_{-N}$ and hence $\mathcal{H}^\infty \otimes \overline{\mathcal{H}^\infty}$ is dense in $\mathcal{D}^\infty(T) = \mathcal{S}(\pi, \mathcal{H})$ by [NZ13, Lem. 4.1(b)]. This implies that $\text{im}(\eta) \subset \mathcal{S}(\pi, \mathcal{H})$ is dense.

We now show that η is an open map. The topology of \mathcal{H}^∞ is generated by the seminorms $p_k(v) := \|N^k v\|, k \in \mathbb{N}_0$. Since $p_k \leq p_{k+1}$, the topology of $\mathcal{H}^\infty \widehat{\otimes}_\pi \overline{\mathcal{H}^\infty}$ is generated by the seminorms

$$(p_k \otimes p_k)(x) := \inf \left\{ \sum_{i=1}^n p_k(v_i) p_k(w_i) : x = \sum_{i=1}^n v_i \otimes w_i \right\}, \quad k \in \mathbb{N}_0.$$

Recall from Remark 1.12(b) the Hilbert spaces \mathcal{H}_k . Then each p_k extends to \mathcal{H}_k and we have an isomorphism of Banach spaces

$$\eta_k : \mathcal{H}_k \widehat{\otimes}_\pi \overline{\mathcal{H}_k} \rightarrow B_1(\mathcal{H}_k) \quad \text{with} \quad \eta_k(x \otimes y)(v) = \langle v, y \rangle_{\mathcal{H}_k} x = \langle N^k v, N^k y \rangle \cdot x$$

([Tr67, Thm. 48.4]). Then $\eta_k(x) = \eta(x) N^{2k}$ for $x \in \mathcal{H}^\infty \otimes_\pi \overline{\mathcal{H}^\infty}$, and we thus obtain with Remark 1.12(b):

$$(p_k \otimes p_k)(x) = \|\eta_k(x)\|_{B_1(\mathcal{H}_k)} = \|N^k \eta(x) N^{2k} N^{-k}\|_1 \leq \|N^k \eta(x) N^{\ell+k}\| \cdot \|N^{-\ell}\|_1$$

for $\ell \in \mathbb{N}$ such that $N^{-\ell}$ is a trace-class operator. (Here on the right hand side, the operator $N^k \eta(x) N^{\ell+k}$ can be considered as an element of $B(\mathcal{H})$, because $\eta(x) \in \mathcal{S}(\pi, \mathcal{H})$.) By continuity of both sides, the same estimate holds for elements x of the completion $\mathcal{H}^\infty \widehat{\otimes}_\pi \overline{\mathcal{H}^\infty}$. Hence η is a continuous linear bijection onto its image with continuous inverse. Therefore the image of η in $\mathcal{S}(\pi, \mathcal{H})$ is closed and since it is dense we conclude that η is a topological isomorphism from $\mathcal{H}^\infty \widehat{\otimes}_\pi \overline{\mathcal{H}^\infty}$ onto $\mathcal{S}(\pi, \mathcal{H})$. ■

2. Characterizing Schwartz operators

In this section we prove a characterization of Schwartz operators in terms of smoothing operators, namely that S is Schwartz if and only if S and S^* are smoothing for any smooth unitary representation of a Fréchet–Lie group.

We shall need the following result from interpolation theory ([RS75, Prop. 9, p. 44]):

Proposition 2.1. *Let \mathcal{H} be a Hilbert space and A, B be positive selfadjoint operators on \mathcal{H} with possibly unbounded inverses. Suppose that the bounded operator $T \in B(\mathcal{H})$ satisfies*

$$T(\mathcal{D}(A^2)) \subseteq \mathcal{D}(B^2) \quad \text{with} \quad \|B^2 T v\| \leq C \|A^2 v\| \quad \text{for} \quad v \in \mathcal{D}(A^2).$$

Then $T(\mathcal{D}(A)) \subseteq \mathcal{D}(B)$ with

$$\|B T v\| \leq \sqrt{\|T\| C} \cdot \|A v\| \quad \text{for} \quad v \in \mathcal{D}(A).$$

We consider a smooth unitary representation (π, \mathcal{H}) of the (locally convex) Lie group G and we assume that G has a smooth exponential function.

For $x \in \mathfrak{g}$ and $n \in \mathbb{N}$, we consider the selfadjoint operator

$$N_{x,n} := \mathbf{1} + (-1)^n \overline{\mathfrak{d}\pi}(x)^{2n} \geq \mathbf{1}.$$

Note that [NZ13, Lemma 4.1(b)] implies that $N_{x,n}$ coincides with the closure of the operator $\mathbf{1} + (-1)^n \mathfrak{d}\pi(x)^{2n}$ on \mathcal{H}^∞ .

Proposition 2.2. *If $S \in B(\mathcal{H})$ is a smoothing operator whose adjoint S^* is smoothing as well, then S is a Schwartz operator, i.e., for $D_1, D_2 \in U(\mathfrak{g})$, the operators $\mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2)$ defined on \mathcal{H}^∞ are bounded, i.e., extend to bounded operators on \mathcal{H} .*

Proof. Since $U(\mathfrak{g})$ is spanned by the elements of the form x^n , $x \in \mathfrak{g}$, we have to show that, for $x, y \in \mathfrak{g}$ and $n, m \in \mathbb{N}_0$, the operator $\mathfrak{d}\pi(x)^n S \mathfrak{d}\pi(y)^m$ is bounded. From [NSZ15, Thm. 2.11] we know that $N_{x,n}S$ is bounded, and from [NSZ15, Lem. 2.8(a)] it follows that $SN_{y,m}$ is bounded on $\mathcal{D}(N_{y,m})$. Next we observe that the operators

$$T := N_{x,n}S, \quad A := N_{y,m}^{-1/2} \quad \text{and} \quad B := N_{x,n}^{-1/2}$$

are all bounded. Writing $v \in \mathcal{H}$ as $v = N_{y,m}w$, we obtain the estimate

$$\|B^2Tv\| = \|N_{x,n}^{-1}TN_{y,m}w\| = \|SN_{y,m}w\| \leq \|SN_{y,m}\| \|N_{y,m}^{-1}v\| = \|SN_{y,m}\| \|A^2v\|.$$

Therefore Proposition 2.1 implies that for $c := \|N_{x,n}S\|^{1/2}\|SN_{y,m}\|^{1/2}$ we have

$$\|N_{x,n}^{1/2}Sv\| = \|BTv\| \leq c\|Av\| = c\|N_{y,m}^{-1/2}v\| \quad \text{for} \quad v \in \mathcal{D}(A) = \mathcal{R}(N_{y,m}^{1/2}) = \mathcal{H}.$$

For $v = N_{y,m}^{1/2}w$, this leads to

$$\|N_{x,n}^{1/2}SN_{y,m}^{1/2}w\| \leq c\|w\| \quad \text{for} \quad w \in \mathcal{D}(N_{y,m}^{1/2}),$$

so that $N_{x,n}^{1/2}SN_{y,m}^{1/2}$ is bounded on $\mathcal{D}(N_{y,m}^{1/2})$. As $N_{x,n}^{-1/2}\mathfrak{d}\pi(x)^n$ is bounded, it follows that the following operator is bounded:

$$\begin{aligned} & (N_{x,n}^{-1/2}\mathfrak{d}\pi(x)^n)^*(N_{x,n}^{1/2}SN_{y,m}^{1/2})(N_{y,m}^{-1/2}\mathfrak{d}\pi(y)^m) \\ & \supseteq (\mathfrak{d}\pi(x)^n)^*(N_{x,n}^{-1/2}N_{x,n}^{1/2}SN_{y,m}^{1/2})(N_{y,m}^{-1/2}\mathfrak{d}\pi(y)^m) \\ & = (\mathfrak{d}\pi(x)^n)^*S\mathfrak{d}\pi(y)^m \supseteq (-1)^n\mathfrak{d}\pi(x)^nS\mathfrak{d}\pi(y)^m, \end{aligned}$$

and this implies the boundedness of $\mathfrak{d}\pi(x)^nS\mathfrak{d}\pi(y)^m$, more precisely

$$\|\mathfrak{d}\pi(x)^nS\mathfrak{d}\pi(y)^m\| \leq \|N_{x,n}^{1/2}SN_{y,m}^{1/2}\| \leq \|N_{x,n}S\|^{1/2}\|SN_{y,m}\|^{1/2}. \tag{7}$$

■

We now consider the representation of $G \times G$ on $B(\mathcal{H})$ by

$$\begin{aligned} \alpha(g, h)A &:= \pi(g)A\pi(h)^{-1} = \lambda(g)\rho(h)A, \\ \lambda(g)A &= \pi(g)A, \quad \rho(g)A = A\pi(g)^{-1}. \end{aligned}$$

Remark 2.3. (a) Suppose that A is a continuous vector for the left multiplication representation λ and also for the right multiplication action ρ . Then

$$\begin{aligned} \|\pi(g)A\pi(h) - A\| &\leq \|\pi(g)A\pi(h) - A\pi(h)\| + \|A\pi(h) - A\| \\ &\leq \|\pi(g)A - A\| + \|A\pi(h) - A\| \end{aligned}$$

implies that A is a continuous vector for α .

We write $B(\mathcal{H})_c(\alpha)$ for the closed subspace of α -continuous vectors in $B(\mathcal{H})$ and note that since α acts by isometries, it defines a continuous action of G on the Banach space $B(\mathcal{H})_c(\alpha)$.

(b) Suppose that A is a C^1 -vector for λ and ρ and $x, y \in \mathfrak{g}$. Since all operators $\pi(\exp tx)A, A\pi(\exp ty)$ are contained in $B(\mathcal{H})_c(\alpha)$, the closedness of $B(\mathcal{H})_c(\alpha)$ implies that $\overline{d\pi}(x)A$ and $A\overline{d\pi}(y)$ are also α -continuous.

We claim that A is a C^1 -vector for α . In fact, the map

$$F: G \times G \rightarrow B(\mathcal{H}), \quad F(g, h) := \alpha(g, h)A = \pi(g)A\pi(h)^{-1}$$

is partially C^1 , so that its differential $dF: TG \times TG \rightarrow B(\mathcal{H})$ exists. This map is given by

$$\begin{aligned} dF(g.x, h.y) &= \pi(g)\overline{d\pi}(x)A\pi(h)^{-1} - \pi(g)A\overline{d\pi}(y)\pi(h)^{-1} \\ &= \pi(g)(\overline{d\pi}(x)A - A\overline{d\pi}(y))\pi(h)^{-1}. \end{aligned}$$

Since α defines a continuous action on $B(\mathcal{H})_c(\alpha)$, the continuity of dF follows from the continuity of the corresponding linear map

$$\mathfrak{g} \times \mathfrak{g} \rightarrow B(\mathcal{H}), \quad (x, y) \mapsto \overline{d\pi}(x)A - A\overline{d\pi}(y),$$

which follows from the assumption that A is a C^1 -vector for λ and ρ . This shows that dF is continuous and hence that A is a C^1 -vector for α .

Theorem 2.4. *For a smooth unitary representation of a Fréchet–Lie group G and $S \in B(\mathcal{H})$, the following are equivalent:*

- (i) S and S^* are smoothing.
- (ii) S is Schwartz.
- (iii) S is a smooth vector for α , i.e., the map $G \times G \rightarrow B(\mathcal{H})$, $(g, h) \mapsto \pi(g)S\pi(h)^{-1}$ is smooth.

Proof. That (i) implies (ii) is Proposition 2.2.

(ii) \Rightarrow (iii): For $D \in U(\mathfrak{g}_{\mathbb{C}})$, the operators $Sd\pi(D)$ and $S^*d\pi(D)$ on \mathcal{H}^∞ are bounded, so that [NSZ15, Thm. 2.11] implies that S and S^* are smoothing, hence in particular C^1 -vectors for α by Remark 2.3 and

$$\overline{d\alpha}(x, y)S = \overline{d\pi}(x)S - S\overline{d\pi}(y).$$

It follows in particular that $\overline{d\alpha}(x, y)S$ is Schwartz as well (because S is Schwartz if and only if $d\pi(D_1)Sd\pi(D_2)$ is bounded on \mathcal{H}^∞ for every $D_1, D_2 \in \mathcal{H}^\infty$). Thus we

obtain inductively that $S \in \mathcal{D}^n(\alpha)$ for every $n \in \mathbb{N}$. Since G is Fréchet, [NSZ15, Thm. 1.6(ii), Cor. 1.7] now imply that S is a smooth vector for α .

(iii) \Rightarrow (i) follows from the characterization of smoothing operators ([NSZ15, Thm. 2.11]). ■

If the Lie group G is only assumed to be metrizable, the additional quantitative information from Proposition 2.2 can still be used to obtain the equivalence of (i) and (iii) in the preceding theorem. This is done in Theorem 2.6 below. First we need a lemma.

Remark 2.5. Let (π, \mathcal{H}) be a smooth unitary representation of a (locally convex) Lie group G with a smooth exponential map. Let $S \in B(\mathcal{H})$ be a Schwartz operator, and set $A := \mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2)$ with domain \mathcal{H}^∞ , where $D_1, D_2 \in U(\mathfrak{g})$. Then A is bounded, and therefore $\overline{A} \in B(\mathcal{H})$. We now show that $\overline{A}(\mathcal{H}) \subseteq \mathcal{D}(\overline{\mathfrak{d}\pi}(x)^2)$. Indeed for $v \in \mathcal{H}$, if $(v_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}^\infty$ is a sequence such that $\lim_{n \rightarrow \infty} v_n = v$ in \mathcal{H} , then $\lim_{n \rightarrow \infty} Av_n = \overline{A}v$ and from boundedness of $\mathfrak{d}\pi(x)^2A$ with domain \mathcal{H}^∞ (recall that S is Schwartz) it follows that the sequence $(\overline{\mathfrak{d}\pi}(x)^2Av_n)_{n \in \mathbb{N}}$ is convergent. But $\overline{\mathfrak{d}\pi}(x)^2$ is closed, hence $\overline{A}v \in \mathcal{D}(\overline{\mathfrak{d}\pi}(x)^2)$.

Theorem 2.6. Let (π, \mathcal{H}) be a smooth unitary representation of the Lie group G and assume that \mathfrak{g} is metrizable. For $S \in B(\mathcal{H})$, the following are equivalent:

- (i) S and S^* are smoothing.
- (ii) S is a smooth vector for α , i.e., the map $G \times G \rightarrow B(\mathcal{H})$, $(g, h) \mapsto \pi(g)S\pi(h)^{-1}$ is smooth.

Proof. (ii) \Rightarrow (i) follows from [NSZ15, Thm. 2.11]. Now assume that S and S^* are smoothing. Then S is a Schwartz operator by Proposition 2.2. According to (7) we have

$$\begin{aligned} \|\mathfrak{d}\pi(x)^n S \mathfrak{d}\pi(y)^m\| &\leq \|N_{x,n}S\|^{1/2} \|SN_{y,m}\|^{1/2} \leq \frac{1}{2}(\|N_{x,n}S\| + \|SN_{y,m}\|) \\ &\leq \|S\| + \frac{1}{2}(\|\mathfrak{d}\pi(x)^{2n}S\| + \|\mathfrak{d}\pi(y)^{2m}\|). \end{aligned} \tag{8}$$

By [NSZ15, Thm. 2.11] the map $g \mapsto \lambda(g)S = \pi(g)S$ is smooth with

$$\overline{\mathfrak{d}\lambda}(x_1) \cdots \overline{\mathfrak{d}\lambda}(x_n)S = \mathfrak{d}\pi(x_1) \cdots \mathfrak{d}\pi(x_n)S.$$

In particular, $\mathfrak{g}^k \rightarrow B(\mathcal{H})$, $(x_1, \dots, x_k) \mapsto \mathfrak{d}\pi(x_1) \cdots \mathfrak{d}\pi(x_k)S$ is k -linear and continuous. Similarly the smoothness of $g \mapsto \rho(g)S = (\pi(g)S^*)^*$ shows that $S\mathfrak{d}\pi(x_1) \cdots \mathfrak{d}\pi(x_k)$ is continuous in (x_1, \dots, x_k) . Therefore (8) entails that $\|\mathfrak{d}\pi(x)^n S \mathfrak{d}\pi(y)^m\|$ is bounded for (x, y) in a neighborhood of $(0, 0) \in \mathfrak{g}^2$. Since $U(\mathfrak{g})$ is spanned by the elements of the form x^k , $x \in \mathfrak{g}$, $k \in \mathbb{N}_0$, and

$$U(\mathfrak{g}) \times U(\mathfrak{g}) \rightarrow B(\mathcal{H}), \quad (D_1, D_2) \mapsto \mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2)$$

is bilinear, polarization implies that the $(n + m)$ -linear map $f_{n,m} : \mathfrak{g}^{n+m} \rightarrow B(\mathcal{H})$

$$f_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m) := \mathfrak{d}\pi(x_1) \cdots \mathfrak{d}\pi(x_n)S\mathfrak{d}\pi(y_1) \cdots \mathfrak{d}\pi(y_m)$$

is bounded near 0 and therefore continuous for every $n, m \in \mathbb{N}$.

Next we show that (the unique extension to \mathcal{H} of) $\mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2)$ lies in $B(\mathcal{H})_c(\alpha)$ for every $D_1, D_2 \in U(\mathfrak{g})$. The proof is inductive, namely, we assume that $\overline{A} \in B(\mathcal{H})_c(\alpha)$ where $A := \mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2)$, and we show that for all $x, y \in \mathfrak{g}$, the unique extensions to \mathcal{H} of $\mathfrak{d}\pi(x)A$ and $A\mathfrak{d}\pi(y)$ are in $B(\mathcal{H})_c(\alpha)$. Remark 2.5 and [NSZ15, Lem. 2.9] imply that $\overline{A} \in \mathcal{D}(\overline{\mathfrak{d}\lambda}(x))$ for any $x \in \mathfrak{g}$, and $\overline{\mathfrak{d}\lambda}(x)\overline{A} = \overline{\mathfrak{d}\pi}(x)\overline{A}$. Now an argument similar to Remark 2.3(b) yields $\overline{\mathfrak{d}\pi}(x)\overline{A} \in B(\mathcal{H})_c(\alpha)$. Furthermore, $(\overline{A})^* = A^*$ and it is straightforward to verify that

$$A^*|_{\mathcal{H}^\infty} = \mathfrak{d}\pi(D_2^\dagger)S^*\mathfrak{d}\pi(D_1^\dagger),$$

where \dagger is the principal anti-involution of $U(\mathfrak{g})$ defined by $x^\dagger := -x$ for $x \in \mathfrak{g}$. Since obviously S^* is Schwartz, the operator A^* is the unique extension to \mathcal{H} of the bounded operator $\mathfrak{d}\pi(D_2^\dagger)S^*\mathfrak{d}\pi(D_1^\dagger)$, hence for any $y \in \mathfrak{g}$ we have by Remark 2.5 that $A^*(\mathcal{H}) \subseteq \mathcal{D}(\overline{\mathfrak{d}\pi}(y)^2)$. Now [NSZ15, Lem 2.8(a)] yields boundedness of $\overline{A\mathfrak{d}\pi}(y)^2$, and [NSZ15, Lem 2.10] implies that $\overline{A\mathfrak{d}\pi}(y) \in B(\mathcal{H})_c(\alpha)$.

Next we observe that for $x, y \in \mathfrak{g}$, the partial derivatives of

$$\mathbb{R}^2 \rightarrow B(\mathcal{H}), \quad (t, s) \mapsto \pi(\exp(tx))\mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2)\pi(\exp(-sy))$$

exist and are continuous (see [NSZ15, Lemmas 2.9/10] and Remark 2.5, and recall from above that for $A = \mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2)$, the operator $\overline{A\mathfrak{d}\pi}(y)^2$ is bounded). This yields $\mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2) \in \mathcal{D}^1(\alpha)$ and

$$\overline{\mathfrak{d}\alpha}(x, y)(\mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2)) = \mathfrak{d}\pi(x)\mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2) - \mathfrak{d}\pi(D_1)S\mathfrak{d}\pi(D_2)\mathfrak{d}\pi(y).$$

Hence we can prove $S \in \mathcal{D}^\infty(\alpha)$ by induction. The continuity of the maps $f_{n,m}$ and [NSZ15, Cor. 1.7(ii)] now implies that S is a smooth vector for α . ■

Recall that $\mathcal{S}(\pi, \mathcal{H})$ denotes the space of Schwartz operators of a unitary representation (π, \mathcal{H}) . The next proposition is an application of Theorem 2.4.

Proposition 2.7. *Let (π, \mathcal{H}) be a smooth unitary representation of a Fréchet-Lie group G . Let $T \in \mathcal{S}(\pi, \mathcal{H})$. Assume that T is a non-negative self-adjoint operator. Then $\sqrt{T} \in \mathcal{S}(\pi, \mathcal{H})$.*

Proof. Since \sqrt{T} is self-adjoint, by Theorem 2.4 it is enough to show that it is smoothing. Next choose $v \in \mathcal{H}^\infty$ such that $\|v\| = 1$. Let \dagger denote the principal anti-involution of $U(\mathfrak{g})$, defined by $x^\dagger := -x$ for $x \in \mathfrak{g}$. Then

$$\begin{aligned} \|\sqrt{T}\mathfrak{d}\pi(D)v\|^2 &= \langle \mathfrak{d}\pi(D)v, T\mathfrak{d}\pi(D)v \rangle \\ &= \langle v, \mathfrak{d}\pi(D^\dagger)T\mathfrak{d}\pi(D)v \rangle \leq \|\mathfrak{d}\pi(D^\dagger)T\mathfrak{d}\pi(D)\|. \end{aligned}$$

Thus the operator $\sqrt{T}\mathfrak{d}\pi(D)$ is bounded on \mathcal{H}^∞ . From [NSZ15, Thm 2.11] it follows that \sqrt{T} is smoothing. ■

3. Relation to literature on Schwartz operators

Schwartz operators have also been studied in [Pe94] for nilpotent Lie groups, and more generally in [Be10]. Note that from [NSZ15, Thm. 2.11] it follows that there is redundancy in the definitions given in [Pe94, Sec. 1.2] and [Be10, Def. 3.1]. From [Be10, Thm. 3.1] it follows that smooth vectors of the $G \times G$ -action on $B_2(\mathcal{H})$ are Schwartz operators. This is weaker than Theorem 2.4 above. Furthermore, [Be10, Thm. 3.1] gets close to Proposition 1.6 and Corollary 1.8(iii), but in [Be10] it is not proved that being trace class is equivalent to nuclearity of the space of smooth vectors (see Proposition 1.11). Finally, Proposition 1.6 implies that what is proved in [Be10, Cor. 3.1] for irreducible unitary representations of nilpotent Lie groups indeed holds for all trace class representations of general finite dimensional Lie groups.

3.1. The Schrödinger representation. In this section we investigate the connection between our results and those of [KKW15] more closely. In particular, we will show that several of the results of [KKW15] are special cases of the results of our paper, when applied to the Schrödinger representation.

Let (V, ω) be a $2n$ -dimensional real symplectic space and let $H_{V, \omega}$ denote the Heisenberg group associated to (V, ω) , that is, $H_{V, \omega} := V \times \mathbb{R}$ with the multiplication

$$(v, s)(w, t) := (v + w, s + t + \frac{1}{2}\omega(v, w)).$$

Let $\mathfrak{h}_{V, \omega}$ denote the Lie algebra of $H_{V, \omega}$, and let $U(\mathfrak{h}_{V, \omega})$ denote the universal enveloping algebra of $\mathfrak{h}_{V, \omega}$. By the Stone–von Neumann Theorem, to every non-trivial unitary character $\chi : \mathbb{R} \rightarrow \mathbb{C}^\times$ we can associate a unique irreducible unitary representation π_χ of $H_{V, \omega}$ for which the center acts by χ . In the Schrödinger realization, π_χ acts on the Hilbert space $\mathcal{H} := L^2(Y, \mu)$, where $V = X \oplus Y$ is a polarization of V , and μ is the Lebesgue measure on $Y \cong \mathbb{R}^n$. The action of π_χ is given by

$$(\pi_\chi(x, 0)\varphi)(y) := \chi(\omega(x, y))\varphi(y), \quad (\pi_\chi(y_0, 0)\varphi)(y) := \varphi(y - y_0),$$

and $\pi_\chi(0, t)\varphi := \chi(t)\varphi$, where $x \in X$, $\varphi \in L^2(Y, \mu)$, $y, y_0 \in Y$, and $t \in \mathbb{R}$. The following result is a special case of the general theory of unitary representations of nilpotent Lie groups (e.g., see [Ho80] and [Ki62, Thm. 7.1, Cor.]).

Proposition 3.1. *The representation π_χ is trace class, $d\pi(U(\mathfrak{h}_{V, \omega}))$ is equal to the algebra of polynomial coefficient differential operators on Y , and the space of smooth vectors of π_χ is the Schwartz space $\mathcal{S}(Y)$.*

From Proposition 3.1 it follows that the operators defined in [KKW15, Def. 3.1] are the Schwartz operators for π_χ in the sense of our paper. From Corollary 1.8 it follows that $\mathcal{S}(\pi_\chi, \mathcal{H})$ is the space of smooth vectors for the action of $H_{V, \omega} \times H_{V, \omega}$ on $B_2(\mathcal{H})$, and therefore it can be equipped with a canonical Fréchet topology. It is straightforward to verify that this Fréchet topology is identical to the one described in [KKW15, Prop. 3.3].

Next we show that if $S, T \in \mathcal{S}(\pi_\chi, \mathcal{H})$, then $SAT \in \mathcal{S}(\pi_\chi, \mathcal{H})$ for every $A \in B(\mathcal{H})$. This is proved in [KKW15, Lemma 3.5(b)], but the argument that

will be given below applies to any trace class representation. From Proposition 1.7 it follows that the map $G \rightarrow B_2(\mathcal{H})$ given by $g \mapsto \lambda(g)S$ is smooth. Since the bilinear map

$$B_2(\mathcal{H}) \times B(\mathcal{H}) \rightarrow B_2(\mathcal{H}), (P, Q) \mapsto PQ$$

is continuous, the map $g \mapsto \lambda(g)SAT$ is also smooth. Thus by Proposition 1.7, the operator SAT is smoothing. A similar argument shows that $T^*A^*S^*$ is also smoothing, and Corollary 1.8 implies that SAT is Schwartz.

Proposition 1.6 implies that every Schwartz operator for π_χ is trace class. This is also obtained in [KKW15, Lemma 3.6]. Theorem 2.4 applied to π_χ gives [KKW15, Thm. 3.12]. Proposition 2.7 implies [KKW15, Prop. 3.15], and Proposition 1.10 implies [KKW15, Prop. 5.12].

It is possible that the relation between the Weyl transform and Schwartz operators that is investigated in [KKW15, Sec. 3.6] is a special case of more general results in the spirit of our paper, at least for nilpotent Lie groups. Finally, it is worth mentioning that the paper [Be11] studies (among several other things) the class of representations of infinite dimensional Lie groups with the property that their space of smooth vectors is nuclear. By Proposition 1.11, when G is finite dimensional this condition is equivalent to the representation being trace class. In the infinite dimensional case this is an interesting class of representations which deserves further investigation. We hope to come back to these problems in the near future.

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