Upper Bound for the Gromov Width
of Coadjoint Orbits of Compact Lie Groups

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Abstract.   We find an upper bound for the Gromov width of coadjoint orbits of compact Lie groups with respect to the Kostant–Kirillov–Souriau symplectic form by computing certain Gromov–Witten invariants. The approach presented here is closely related to the one used by Gromov in his celebrated non-squeezing theorem.
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1.   Introduction

The Gromov width of a symplectic manifold (M^{2n}, ω) is defined as

$$\mathrm{Gwidth}(M^{2n}, \omega) := \sup \left\{ \pi r^2 : \begin{array}{l} \exists \text{ a symplectic embedding} \\ (B^{2n}(r), \omega_{\mathrm{st}}) \hookrightarrow (M^{2n}, \omega) \end{array} \right\},$$

where $B^{2n}(r) := \{(x_1, \cdots, x_n, y_1, \cdots, y_n) : \sum_{i=1}^n (x_i^2 + y_i^2) < r^2\}$ denotes the open ball in $\mathbb{R}^{2n}$ of radius r and center the origin and $\omega_{\mathrm{st}} := \sum_{i=1}^n dx_i \wedge dy_i$ denotes the standard symplectic form defined on $B^{2n}(r)$.

The Darboux theorem in symplectic geometry implies that the Gromov width of any symplectic manifold is always positive.

In this paper, we are interested in finding upper bounds for the Gromov width of coadjoint orbits of compact Lie groups with respect to the Kostant–Kirillov–Souriau form. The main result in this paper is summarized in the following theorem:

Main Theorem.   *Let G be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a maximal torus and let $\check{R}$ be the corresponding system of coroots. We identify the dual Lie algebra $\mathfrak{t}^*$ of T with the fixed points of the coadjoint action of T on $\mathfrak{g}^*$. For $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$, let $\mathcal{O}_\lambda$ be the coadjoint orbit passing*

through λ and ω_λ be the Kostant–Kirillov–Souriau form defined on $\mathcal{O}_\lambda$. Then

$$\mathrm{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq \min_{\substack{\check{\alpha} \in \check{R} \\ \langle \lambda, \check{\alpha} \rangle \neq 0}} |\langle \lambda, \check{\alpha} \rangle|$$

The proof of the Main theorem follows the same line of thought as the one given by Gromov in his non-squeezing theorem [19]. It follows from Gromov's work that J-holomorphic curves can be used to bound from above the Gromov width of a symplectic manifold. Roughly speaking, if for a compact symplectic manifold (M, ω) there is a non-vanishing Gromov-Witten invariant of the form $\mathrm{GW}_{A,k}(\mathrm{PD}[\mathrm{pt}], \alpha_2, \cdots, \alpha_k)$ for some degree $A \in H_2(M; \mathbb{Z})$ and some cohomology classes $\alpha_2, \cdots, \alpha_k \in H^*(M; \mathbb{Z})$, then

$$\mathrm{Gwidth}(M, \omega) \leq \omega(A)$$

In this paper we use this approach to bound from above the Gromov width of arbitrary coadjoint orbits of compact Lie groups. We adopt Cielibak and Mohnke's definition of Gromov-Witten invariants for integral symplectic manifolds [8] and we use curve neighborhoods (Buch, Mihalcea and Perrin [4]) to compute them. We explain why in our considerations these notions are consistent and imply the bound appearing in the main theorem.

The Main theorem extends a result of Zoghi for *regular* coadjoint orbits of compact Lie groups that in addition satisfy a condition that Zoghi named *indecomposable* [53]. Recall that a coadjoint orbit of a compact Lie group G is regular if it is isomorphic to G/T where T is a maximal torus of G.

In contrast to the problem of bounding the Gromov width of a coadjoint orbit from above, Pabiniak has considered the problem of bounding it from below [44]. For instance, Pabiniak has proved that the upper bound appearing in the Main theorem is indeed an equality for arbitrary coadjoint orbits of $U(n)$ and for coadjoint orbits of $SO(n)$ that satisfy certain technical assumptions made on λ. Together with our result, this yields the following theorem for coadjoint orbits of $U(n)$:

Theorem.   *Let us identify the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ with its dual $\mathfrak{u}(n)^*$ via the ad–invariant inner product*

$$\mathfrak{u}(n) \times \mathfrak{u}(n) \to \mathbb{R}$$
$$(A, B) \mapsto -i \operatorname{tr}(A \cdot B)$$

For $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, let $\lambda = i \operatorname{diag}(\lambda_1, \cdots, \lambda_n) \in \mathfrak{u}(n) \cong \mathfrak{u}(n)^*$. Let

$$\mathcal{O}_\lambda = \{A \in \mathfrak{u}(n) : \operatorname{spectrum} A = \{i\lambda_1, \cdots, i\lambda_n\}\}$$

be the coadjoint orbit of $U(n)$ passing through $\lambda \in \mathfrak{u}(n) \cong \mathfrak{u}(n)^*$ and ω_λ be the Kostant-Kirillov-Souriau form defined on the coadjoint orbit, then

$$\mathrm{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) = \min_{\lambda_i \neq \lambda_j} |\lambda_i - \lambda_j|$$

A coadjoint orbit of $U(n)$ can be identified with a partial flag manifold. Two special cases are the full flag manifold and the Grassmannian manifold. The above theorem extends the results of G. Lu [36]; Karshon and Tolman [27] for Grassmannian manifolds and Zoghi [53] for *indecomposable* flag manifolds to partial flag manifolds.

In addition to the examples mentioned above, several authors have used Gromov's method for bounding the Gromov width of other families of symplectic manifolds such as G. Lu [37], McDuff and Polterovich [39] and Biran [2].

This paper is organized as follows: in the second, third and fourth sections, we review the J-holomorphic tools that we use throughout the text, and then we explain how upper bounds for the Gromov width of symplectic manifolds can be obtained by a non-vanishing Gromov–Witten invariant. In the fifth section, we recall background on the geometry of coadjoint orbits of compact Lie groups. In the sixth section, we define the concept of curve neighborhood and indicate how it is used to compute Gromov-Witten invariants on homogeneous spaces. In the seventh section, we show the upper bound appearing in the Main theorem for Grassmannian manifolds. In the eighth section, the Main theorem is proven for non-regular coadjoint orbits of compact Lie groups. In the appendix, we discuss about fibrations of homogeneous spaces and show two technical lemmas that are needed in the proof of the Main theorem for Grassmannian manifolds.

2. Gromov's width and pseudoholomorphic curves

In this section we give a short review of pseudoholomorphic theory, and we show how pseudoholomorphic curves are related with the Gromov width of a symplectic manifold. Most of the material presented here is adapted from McDuff and Salamon [40].

Let (M^{2n}, ω) be a compact symplectic manifold. An *almost complex structure* J on (M, ω) is a smooth operator $J : TM \to TM$ such that $J^2 = -\operatorname{Id}$. We say that an almost complex structure J is *compatible* with ω if

$$g(\cdot, \cdot) := \omega(\cdot, J\cdot)$$

defines a Riemannian metric on M. We denote the space of almost complex structures compatible with ω by $\mathcal{J}(M, \omega)$.

Let $(\mathbb{CP}^1, j)$ be the Riemann sphere with its standard complex structure j. Let $J \in \mathcal{J}(M, \omega)$. A map $u : \mathbb{CP}^1 \to M$ is a *J-holomorphic curve of genus zero* or simply a *J-holomorphic sphere* if for every $z \in \mathbb{CP}^1$

$$du_z + J_{u(z)} \circ du_z \circ j = 0$$

The *degree* of a J-holomorphic sphere $u : \mathbb{CP}^1 \to M$ is

$$\deg u := u_*[\mathbb{CP}^1] \in H_2(M; \mathbb{Z})$$

A homology class $A \in H_2(M; \mathbb{Z})$ is *spherical* if it is in the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M; \mathbb{Z})$.

A curve $u : \mathbb{CP}^1 \to M$ is said to be *multiply covered* if it is the composite of a holomorphic branched covering map $(\mathbb{CP}^1, j) \to (\mathbb{CP}^1, j)$ of degree greater than one with a J-holomorphic map $\mathbb{CP}^1 \to M$. It is *simple* if it is not multiply covered.

Let k be a nonnegative integer. A *nodal curve of genus zero with k-marked points* is a tuple $(\Sigma; z_1, \cdots, z_k)$ consisting of a compact nodal Riemman surface Σ of arithmetic genus zero together with a set of k-pairwise different points $z_1, \cdots, z_k$ that are not nodes of Σ. We denote a nodal curve with k-marked points $(\Sigma; z_1, \cdots, z_k)$ by $\mathbf{z}$ and call $\Sigma_{\mathbf{z}} := \Sigma$ the nodal surface of $\mathbf{z}$. A nodal curve with k-marked points $\mathbf{z}$ is *stable* if the number of nodes and marked points in any irreducible component of $\Sigma_{\mathbf{z}}$ is greater or equal to three. Two nodal curves with k-marked points $\mathbf{z} = (\Sigma; z_1, \cdots, z_k), \mathbf{z}' = (\Sigma'; z_1', \cdots, z_k')$ are *equivalent* if there exists an isomorphism of curves $\phi : \Sigma \to \Sigma'$ such that ϕ maps $\{z_1, \cdots, z_k\}$ bijectively onto $\{z_1', \cdots, z_k'\}$ (respecting the order of the marked points). We denote by $\overline{\mathcal{M}}_k$ the moduli space of equivalent classes of stable nodal curves of genus zero with k-marked points.

A *smooth curve with k-marked points* is a pair $(u; \mathbf{z})$ consisting of a nodal curve with k-marked points $\mathbf{z}$ and a smooth map $u : \Sigma_{\mathbf{z}} \to M$. The degree of a smooth curve with k-marked points $(u; \mathbf{z})$ is defined as

$$\deg(u; \mathbf{z}) := u_*[\Sigma_{\mathbf{z}}] \in H_2(M; \mathbb{Z})$$

Two smooth curves with k-marked points $(u; \mathbf{z}), (u'; \mathbf{z}')$ are *equivalent* if there exists an isomorphism $\phi : \Sigma_{\mathbf{z}} \to \Sigma_{\mathbf{z}'}$ of curves such that $u = u' \circ \phi$ and ϕ maps the marked points of $\mathbf{z}$ bijectively onto the marked points of $\mathbf{z}'$ (respecting the order of the marked points). A smooth curve with k-marked points $(u; \mathbf{z})$ is *stable* if the number of marked and nodal points in any component of $\Sigma_{\mathbf{z}}$ is greater or equal to three whenever the restriction of u on the component is constant. A *smooth sphere with k-marked points* is a smooth curve with k-marked points $(u; \mathbf{z})$ such that $\Sigma_{\mathbf{z}}$ is isomorphic with $\mathbb{CP}^1$.

A smooth curve with k-marked points $(u; \mathbf{z})$ is *J-holomophirc* if the restriction of u on each of the irreducible components of $\Sigma_{\mathbf{z}}$ is J-holomorphic. A J-holomorphic curve $(u; \mathbf{z})$ is *simple* if $\Sigma_{\mathbf{z}}$ is isomorphic with $\mathbb{CP}^1$ and u is simple.

Let $A \in H_2(M; \mathbb{Z})$. We denote by $\overline{\mathcal{M}}_{A,k}(M, J)$ the moduli space of equivalent classes of stable J-holomorphic curves with k-marked points of degree A. We denote by $\mathcal{M}_{A,k}(M, J)$ the moduli space of equivalent classes of J-holomorphic spheres in $\overline{\mathcal{M}}_{A,k}(M, J)$ and by $\mathcal{M}^*_{A,k}(M, J)$ the moduli space of equivalent classes of simple J-holomorphic spheres in $\overline{\mathcal{M}}_{A,k}(M, J)$. The moduli space $\overline{\mathcal{M}}_{A,k}(M, J)$ is compact with respect to the Gromov topology (see McDuff and Salamon [40]). The *evaluation map*

$$\mathrm{ev}^k_J = (\mathrm{ev}_1, \ldots, \mathrm{ev}_k) : \overline{\mathcal{M}}_{A,k}(M, J) \to M^k$$

is defined by

$$\mathrm{ev}^k_J[u; (\Sigma; z_1, \cdots, z_k)] = (u(z_1), \cdots, u(z_k)).$$

We denote by $\mathcal{J}_{\mathrm{reg}}(M, \omega) \subset \mathcal{J}(M, \omega)$ the set of almost complex structures that are regular in the sense of McDuff and Salamon [40, Definition 3.1.4, Section

6.2]. When J is a regular almost complex structure, the moduli space $\mathcal{M}^*_{A,k}(M, J)$ is a smooth oriented manifold of dimension equal to

$$\dim M + 2c_1(A) + 2k - 6,$$

where c_1 denotes the first Chern class of TM (see e.g. McDuff and Salamon [40]).

For $r > 0$, let

$$B^{2n}(r) := \left\{ (x_1, \cdots, x_n, y_1, \cdots, y_n) : \sum_{i=1}^{n}(x_i^2 + y_i^2) < r^2 \right\}$$

denote the open ball in $\mathbb{R}^{2n}$ of radius r and center the origin. Let

$$\omega_{\mathrm{st}} := \sum_{i=1}^{n} dx_i \wedge dy_i$$

be the standard symplectic form defined on $B^{2n}(r)$. Given a symplectic manifold (M^{2n}, ω), its *Gromov's width* is defined as

$$\mathrm{Gwidth}(M^{2n}, \omega) := \sup \left\{ \pi r^2 : \begin{array}{c} \exists \text{ a symplectic embedding} \\ (B^{2n}(r), \omega_{\mathrm{st}}) \hookrightarrow (M^{2n}, \omega) \end{array} \right\}$$

The Darboux theorem implies that the Gromov width of a symplectic manifold is always positive. Moreover, if the symplectic manifold is compact, its Gromov width is finite. In this paper, we always assume that any symplectic manifold is compact. For a symplectic manifold (M, ω) and a spherical class $A \in H_2(M; \mathbb{Z})$ we denote by $\omega(A)$ the symplectic area of A.

The following statement goes back to Gromov and shows the relation between the Gromov width of a symplectic manifold and its pseudoholomorphic curves.

Theorem 2.1 (Gromov [19]). *Let (M^{2n}, ω) be a compact symplectic manifold and $A \in H_2(M; \mathbb{Z})$ be a nontrivial spherical class. Suppose that for any almost complex structure $\tilde{J} \in \mathcal{J}(M, \omega)$ and for any point $p \in M$ there exists a $\tilde{J}$-holomorphic sphere of degree $B \in H_2(M; \mathbb{Z})$ passing through p with $0 < \omega(B) \leq \omega(A)$. Then for any symplectic embedding $B^{2n}(r) \hookrightarrow M$, we have*

$$\pi r^2 \leq \omega(A)$$

In particular,

$$\mathrm{Gwidth}(M, \omega) \leq \omega(A).$$

Proof. Let us suppose that there exists a symplectic embedding

$$\rho : B^{2n}(r) \hookrightarrow M$$

Let J_{st} be the standard almost complex structure defined on the open ball $B^{2n}(r)$. Given $\epsilon \in (0, r)$, let $\tilde{J}$ be an almost complex structure that is equal to $\rho_*(J_{\mathrm{st}})$

on the open set $D := \rho(B^{2n}(r - \epsilon)) \subset M$. By assumption, there exists a $\tilde{J}$-holomorphic sphere $u : \mathbb{CP}^1 \to M$ of degree $B \in H_2(M; \mathbb{Z})$ passing through $\rho(0)$ with $0 < \omega(B) \le \omega(A)$. The restriction of u to $S := u^{-1}(D) \subset \mathbb{CP}^1$ gives a proper holomorphic map

$$u' : S \to B^{2n}(r - \epsilon)$$

that passes through the origin. The monotonicity theorem implies that the area of this curve is bounded from below by $\pi(r - \epsilon)^2$. As a consequence,

$$\pi(r - \epsilon)^2 \le \text{area}(u') \le \text{area}(u) = \omega(B) \le \omega(A)$$

Since this inequality is true for every $\epsilon \in (0, r)$,

$$\pi r^2 \le \omega(A) \qquad \blacksquare$$

3. Gromov's width and Gromov-Witten invariants of semipositive symplectic manifolds

In this section we define the Gromov-Witten invariants of a *semipositive* symplectic manifold and explain how they can be used to bound from above the Gromov width of the symplectic manifold. We first review the notion of pseudocycle.

A subset $\Omega \subset X$ of a manifold X has *dimension at most d* if Ω is contained in the image of a smooth map $g : N \to X$ from a manifold of dimension $\dim N \le d$. A smooth map $f : M \to X$ from an oriented d-manifold M to a manifold X is called *d-dimensional pseudocycle* if $f(M)$ has compact closure and its omega limit set

$$\Omega_f := \bigcap_{K \subset M \text{ compact}} \overline{f(M - K)}$$

has dimension at most $d - 2$. Roughly speaking this means that $f(M)$ can be compactified by adding strata of codimension at least two.

Two d-dimensional pseudocycles $f_i : M_i \to X, i = 0, 1$, are called *cobordant* if there exists a smooth map $F : W \to X$ from an oriented $(d + 1)$-manifold W with boundary $\partial W = M_1 - M_0$ such that $F|_{M_i} = f_i$ and $\dim \Omega_F \le d - 1$.

Two pseudocycles $f_i : M_i \to X, i = 0, 1$, are called *strongly transverse* if

$$\Omega_{f_0} \cap \overline{f_1(M_1)} = \emptyset, \quad \overline{f_0(M_0)} \cap \Omega_{f_1} = \emptyset$$

and $f_1(m_1) = f_2(m_2) = x$ implies $T_x X = \text{im } df_1|_{m_1} + \text{im } df_2|_{m_2}$.

There exists a set $\text{Diff}_{\text{reg}}(X, f_1, f_2) \subset \text{Diff}(X)$ of the second category such that f_1 is strongly transverse to $\phi \circ f_2$ for every $\phi \in \text{Diff}_{\text{reg}}(X, f_1, f_2)$. If f_1 and f_2 are strongly transverse, the set $\{(m_1, m_2) \in M_1 \times M_2 : f_1(m_1) = f_2(m_2)\}$ is a compact manifold of dimension $\dim M_1 + \dim M_2 - \dim X$. In particular, this set is finite if M_1 and M_2 are of complementary dimension. In this case, the *intersection number* of f_1 and f_2 is defined as

$$\sharp(f_1 \pitchfork f_2) := \sum_{\substack{m_1 \in M_1, m_2 \in M_2 \\ f_1(m_1) = f_2(m_2)}} f_1(m_1) \cdot f_2(m_2),$$

where $f_1(m_1) \cdot f_2(m_2)$ denotes the intersection number of $f_1(M_1)$ and $f_2(M_2)$ at $f_1(m_1) = f_2(m_2)$. The intersection number $f_1 \pitchfork f_2$ depends only on the bordism classes of f_1 and f_2 (McDuff and Salamon [40][Section 6.5]).

A symplectic manifold (M^{2n}, ω) is *semipositive* if, for a spherical class A with positive symplectic area, $c_1(A) \geq 3 - n$ implies $c_1(A) \geq 0$.

Let (M, ω) be a semipositive symplectic manifold. Let A be a spherical class and k be a nonnegative integer. For $J \in \mathcal{J}_{\text{reg}}(M, \omega)$, the evaluation map

$$\text{ev}_J^k : \mathcal{M}_{A,k}^*(M, J) \to M^k$$

defines a pseudocycle of dimension equal to

$$\dim M + 2c_1(A) + 2k - 6$$

The pseudocycle $\text{ev}_J^k : \mathcal{M}_{A,k}^*(M, J) \to M^k$, up to cobordism, is independent of the regular almost complex structure $J \in \mathcal{J}_{\text{reg}}(M, \omega)$ (see e.g. McDuff and Salamon [40, Chapter 6]).

Let J be a regular almost complex structure and $a_1, \cdots, a_k$ be cohomology classes of total degree

$$\sum_{i=1}^k \deg a_i = \dim M + 2c_1(A) + 2k - 6$$

If the cohomology classes a_i are Poincaré dual to the fundamental classes of cycles $X_i \subset M$ so the evaluation map $\text{ev}_J^k : \mathcal{M}_{A,k}^*(M, J) \to M^k$ is transversal to $X_1 \times \cdots \times X_k \subset M^k$, the *Gromov-Witten invariant* $\text{GW}_{A,k}^J(a_1, \ldots, a_k)$ is the intersection number

$$\text{GW}_{A,k}^J(a_1, \ldots, a_k) := \sharp \, \text{ev}_J^k \pitchfork (X_1 \times \cdots \times X_k)$$

The Gromov-Witten invariant $\text{GW}_{A,k}^J(a_1, \ldots, a_k)$ is well-defined, finite and independent of the regular almost complex structure J and of generic perturbation of $X_1 \times \cdots \times X_k$ (McDuff and Salamon [40, Theorem 6.6.1, Theorem 7.1.1]).

Remark 3.1. Let (M, ω) be a compact symplectic manifold that is not necessarily semipositive. For $J \in \mathcal{J}(M, \omega)$, we say that a homology class $B \in H_2(M; \mathbb{Z})$ is *J-indecomposable* if it can not be decomposed as a sum $B = B_1 + \cdots + B_l$, where $l \geq 2$ and each B_i has a nonconstant spherical J-holomorphic representative.

Let $A \in H_2(M; \mathbb{Z})$ and $k \in \mathbb{Z}_{\geq 0}$. There exists a subset $\mathcal{J}_{\text{reg}}(A) \subset \mathcal{J}(M, \omega)$ of the second category such that for any $J \in \mathcal{J}_{\text{reg}}(A)$ the moduli space $\mathcal{M}_{A,k}^*(M, J)$ is a smooth manifold. If $J \in \mathcal{J}_{\text{reg}}(A)$ and A is J-indecomposable, the evaluation map

$$\text{ev}_J^k : \mathcal{M}_{A,k}^*(M, J) \to M^k$$

defines a pseudocycle and we define the Gromov-Witten invariant $\text{GW}_{A,k}^J$ using ev_J^k as it is done in the semipositive case. The Gromov-Witten invariant $\text{GW}_{A,k}^J$ is defined *exclusively* for the regular almost complex structure J, the J-indecomposable class A and the integer number k (see e.g. McDuff and Salamon [40][Lemma 7.1.8]).

The following theorem allows us to bound the Gromov width of a semipositive symplectic compact manifold in terms of its Gromov-Witten invariants.

Theorem 3.2.   *Let (M, ω) be a compact semipositive symplectic manifold. If there exist a regular almost complex structure J, a spherical class A and cohomology classes $a_2, \ldots, a_k$ Poincaré dual to the fundamental classes of cycles $X_2, \ldots, X_k \subset M$ such that*

$$\mathrm{GW}_{A,k}^{J}(\mathrm{PD}[\mathrm{pt}], a_2, \ldots, a_k) \neq 0,$$

then

$$\mathrm{Gwidth}(M, \omega) \leq \omega(A)$$

Proof.   For any $\tilde{J} \in \mathcal{J}_{\mathrm{reg}}(M, \omega)$, the evaluation map

$$\mathrm{ev}_{\tilde{J}}^{k} : \mathcal{M}_{A,k}^{*}(M, \tilde{J}) \to M^{k}$$

defines a pseudocycle. For a generic perturbation $\tilde{\mathrm{pt}} \times \tilde{X}_2 \times \cdots \times \tilde{X}_k$ of $\mathrm{pt} \times X_2 \times \cdots \times X_k$ such that the evaluation map $\mathrm{ev}_{\tilde{J}}^{k}$ is transversal to $\tilde{\mathrm{pt}} \times \tilde{X}_2 \times \cdots \times \tilde{X}_k$, the Gromov-Witten invariant $\mathrm{GW}_{A,k}^{\tilde{J}}(\mathrm{PD}[\mathrm{pt}], a_2, \cdots, a_k)$ counts with appropriate orientations the number of simple $\tilde{J}$-holomorphic spheres of degree A passing through $\tilde{\mathrm{pt}}, \tilde{X}_2, \cdots, \tilde{X}_k$. By assumption, this number is different from zero and in particular for a generic point $\tilde{\mathrm{pt}}$ in M there exists a $\tilde{J}$-holomorphic sphere of degree A passing through $\tilde{\mathrm{pt}}$.

Let $\tilde{J} \in \mathcal{J}(M, \omega)$. The set of regular almost complex structure $\mathcal{J}_{\mathrm{reg}}(M, \omega)$ is dense in $\mathcal{J}(M, \omega)$ with its C^∞-topology. Thus, there is a sequence of regular almost complex structures $\tilde{J}_n \in \mathcal{J}(M, \omega)$ that C^∞-converges to $\tilde{J}$. Also, for any point $p \in M$, we can find a sequence of points $p_n \in M$ that converges to p and a sequence of $\tilde{J}_n$-holomorphic spheres

$$u_n : \mathbb{CP}^1 \to M$$

of degree A such that u_n passes through p_n. The Gromov compactness theorem implies that there exists a $\tilde{J}$-holomorphic sphere $u : \mathbb{CP}^1 \to M$ of degree $B \in H_2(M; \mathbb{Z})$ passing through p with $0 < \omega(B) \leq \omega(A)$. Therefore, by Theorem 2.1

$$\mathrm{Gwidth}(M, \omega) \leq \omega(A) \qquad \blacksquare$$

4.   Gromov's width and Gromov-Witten invariants of integral symplectic manifolds

In this section we review the definition of Gromov-Witten invariants provided by Cieliebak and Mohnke for integral symplectic manifold in terms of Donaldson hypersurfaces (Cieliebak and Mohnke [8]). We give a slight extension of Theorem 3.2 to integral symplectic manifold that is used in this paper to bound the Gromov width of coadjoint orbits of compact Lie groups.

Let l be a nonnegative integer and $\overline{\mathcal{M}}_l$ be the moduli space of equivalent classes of stable nodal curves of genus zero with l-marked points. Let st be the

stabilization map that makes any nodal curve with l-marked points $\mathbf{z}$ into a stable nodal curve st($\mathbf{z}$) by removing an irreducible component of $\Sigma_{\mathbf{z}}$ if the number of nodes and marked points contained in the irreducible component is fewer than three and placing the extra marked points on neighboring irreducible components as necessary. The stabilization map determines a holomorphic surjection on the corresponding nodal curves

$$\mathrm{st} : \Sigma_{\mathbf{z}} \to \Sigma_{\mathrm{st}\,(\mathbf{z})}$$

We denote by

$$\pi : \overline{\mathcal{M}}_{l+1} \to \overline{\mathcal{M}}_l$$

the map that forgets the last marked point and stabilizes the result.

Let (M, ω) be a symplectic manifold. A *domain-dependent almost complex structure* is a smooth function

$$K : \overline{\mathcal{M}}_{l+1} \to \mathcal{J}(M, \omega)$$
$$[\mathbf{z}] \mapsto K_{[\mathbf{z}]}$$

A *domain-independent almost complex structure* is an almost complex structure $J \in \mathcal{J}(M, \omega)$ considered as a constant function

$$J : \overline{\mathcal{M}}_{l+1} \to \mathcal{J}(M, \omega)$$
$$[\mathbf{z}] \mapsto J$$

We interpret $K \in C^\infty(\overline{\mathcal{M}}_{l+1}, \mathcal{J}(M, \omega))$ in terms of the forgetful and stabilization maps as follows: for any nodal curve $\mathbf{z}$ with l-marked points, the fiber $\pi^{-1}([\mathrm{st}(\mathbf{z})])$ can be identified with the nodal curve $\Sigma_{\mathrm{st}(\mathbf{z})}$ by parameterizing $\pi^{-1}([\mathrm{st}(\mathbf{z})])$ via the position of the extra marked point. The restriction of K to $\pi^{-1}([\mathrm{st}(\mathbf{z})])$ yields a map

$$K^{\mathrm{st}}_{[\mathbf{z}]} : \Sigma_{\mathrm{st}(\mathbf{z})} \to \mathcal{J}(M, \omega)$$

We denote by $\mathcal{J}_{l+1}(M, \omega)$ the set of domain-dependent almost complex structures that are *coherent* in the sense of Cielibak and Mohnke [8]. Let $K \in \mathcal{J}_{l+1}(M, \omega)$. A smooth curve $(u, \mathbf{z})$ with l-marked points is K-*holomorphic* if for every $z \in \Sigma_{\mathbf{z}}$

$$du_z + K^{\mathrm{st}}_{[\mathbf{z}]}(\mathrm{st}(z))_{u(z)} \circ du_z \circ j = 0,$$

where j denotes the standard complex structure defined on $\Sigma_{\mathbf{z}}$. For $A \in H_2(M; \mathbb{Z})$, we denote by $\mathcal{M}_{A,l}(M, K)$ the moduli space of equivalence classes of K-holomorphic spheres of degree A with l-marked points and by $\mathrm{ev}^l_K : \mathcal{M}_{A,l}(M, K) \to M^l$ the corresponding evaluation map.

Let us assume now that (M, ω) is a symplectic manifold such that the symplectic form ω represents a cohomology class $[\omega]$ with integer coefficients. We call such symplectic manifold *integral*. A codimension two submanifold $Y \subset M$ (*a hypersurface*) has *degree D* if it represents the homology class Poincaré dual to $D[\omega]$.

A result of Donaldson states that given an almost complex structure $J \in \mathcal{J}(M, \omega)$ there exists a hypersurface $Y \subset M$ *nearly* J-holomorphic in the sense that the *Kähler angle* of Y with respect to J is sufficiently small if the degree of Y

is sufficiently large. This in particular implies that if the degree of Y is sufficiently large, for any $\epsilon > 0$ one can find $\tilde{J} \in \mathcal{J}(M, \omega)$ such that $\|J - \tilde{J}\|_{C^0} < \epsilon$ and Y is $\tilde{J}$-holomorphic (see e.g. Donaldson [10]). In this case, we call Y a *Donaldson hypersurface* and (J, Y) a *Donaldson pair*.

Let (J, Y) be a Donaldson pair and D be the degree of Y. For $\epsilon > 0$ and $l \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{J}_{l+1}(M, \omega, J, Y, \epsilon)$ the set of domain-dependent almost complex structure $K \in \mathcal{J}_{l+1}(M, \omega)$ that are ϵ-close to J in the C^0-norm and leave TY invariant.

Let $k \in \mathbb{Z}_{\geq 0}, A \in H_2(M; \mathbb{Z}), l := D\omega(A)$ and $K \in \mathcal{J}_{l+1}(M, \omega, J, Y, \epsilon)$. The domain-dependent almost complex structure K induces an element $\pi_k^* K \in \mathcal{J}_{k+l+1}(M, \omega)$ by composition with the map

$$\pi_k : \overline{\mathcal{M}}_{k+l+1} \to \overline{\mathcal{M}}_{l+1}$$

that forgets the last k marked points and stabilizes the result. We denote by $\mathcal{M}_{A,k+l}(M, K, Y)$ the moduli space of equivalence classes of $\pi_k^* K$-holomorphic spheres of degree A with $(k + l)$-marked points that map the last l marked points to Y. We denote by $\mathrm{ev}_K^k : \mathcal{M}_{A,k+l}(M, K, Y) \to M^k$ the evaluation map $[u; (\Sigma; z_1, \cdots, z_k, z_{k+1}, \cdots, z_{k+l})] \mapsto (u(z_1), \cdots, u(z_k))$.

Theorem 4.1 (Cieliebak and Mohnke [8]). *Let (M, ω) be an integral symplectic manifold and (J, Y) be a Donaldson pair consisting of an almost complex structure $J \in \mathcal{J}(M, \omega)$ and a Donaldson hypersurface $Y \subset M$ of degree D sufficiently large. Let $A \in H_2(M; \mathbb{Z})$ and $l = D\omega(A)$. For $\epsilon > 0$ small enough there exists a subset $\mathcal{J}_{l+1}^{\mathrm{reg}}(M, \omega, J, Y, \epsilon) \subset \mathcal{J}_{l+1}(M, \omega, J, Y, \epsilon)$ of the second category such that for every $k \in \mathbb{Z}_{\geq 0}$ and $K \in \mathcal{J}_{l+1}^{\mathrm{reg}}(M, \omega, J, Y, \epsilon)$ the evaluation map*

$$\mathrm{ev}_K^k : \mathcal{M}_{A,k+l}(M, K, Y) \to M^k$$

defines a pseudocycle. The bordism class of this pseudocycle is independent of (J, K, Y).

The previous theorem allows us to define Gromov-Witten invariants for arbitrary integral symplectic manifolds. Let (J, K, Y) and

$$\mathrm{ev}_K^k : \mathcal{M}_{A,k+l}(M, K, Y) \to M^k$$

be as in the previous theorem. Let $a_1, \cdots, a_k$ be cohomology classes Poincaré dual to the fundamental classes of cycles $X_1, \cdots, X_k \subset M$ such that the pseudocycle $\mathrm{ev}_K^k : \mathcal{M}_{A,k+l}(M, K, Y) \to M^k$ is transversal to $X_1 \times \cdots \times X_k$. The *Gromov-Witten invariant defined by Cieliebak and Mohnke* passing through the cycles $X_1, \cdots, X_k$ is the intersection number

$$\mathrm{GW}_{A,k}^{\mathrm{cm}}(a_1, \ldots, a_k) := \frac{1}{l!} \sharp \, \mathrm{ev}_K^k \pitchfork (X_1 \times \cdots \times X_k)$$

The Gromov-Witten invariant $\mathrm{GW}_{A,k}^{\mathrm{cm}}$ is well defined. It does not depend on the choice of (J, Y, K) and of generic perturbation of $X_1, \cdots, X_k \subset M$ (Cieliebak and Mohnke [8]).

Remark 4.2. The Gromov-Witten defined by Cieliebak and Mohnke coincides with the Gromov-Witten invariant defined in the previous section for (integral) semipositive symplectic manifolds (Cieliebak and Mohnke [8]).

Now we show that for an integral symplectic manifold (M, ω), the Gromov-Witten invariant $\mathrm{GW}^{\mathrm{cm}}_{A,k}$ coincides with the Gromov-Witten invariant $\mathrm{GW}^{J}_{A,k}$ when J is a regular domain-independent almost complex structure and A is a J-indecomposable spherical class (recall Remark 3.1) .

Lemma 4.3. *Let (M, ω) be a compact integral symplectic manifold. Let $A \in H_2(M; \mathbb{Z})$ and $J \in \mathcal{J}_{\mathrm{reg}}(A)$. Assume that A is a J-indecomposable class. Then for any $k \in \mathbb{Z}_{\geq 1}$, the Gromov-Witten invariant $\mathrm{GW}^{J}_{A,k}$ coincides with the Gromov-Witten invariant $\mathrm{GW}^{\mathrm{cm}}_{A,k}$*

Proof. Let $\tilde{k} \in \mathbb{Z}_{\geq 1}$. There exists a subset $\mathcal{J}^{\mathrm{reg}}_{\tilde{k}+1}(A) \subset \mathcal{J}_{\tilde{k}+1}(M, \omega)$ of the second category such that for every $K \in \mathcal{J}^{\mathrm{reg}}_{\tilde{k}+1}(A)$, the moduli space of equivalence classes of K-holomorphic spheres $\mathcal{M}_{A,\tilde{k}}(M, K)$ is a smooth manifold of dimension

$$\dim M + 2c_1(A) + 2\tilde{k} - 6$$

(see e.g. Cieliebak and Mohnke [8][Corollary 5.8]). A standard transversality argument implies that if $K_0, K_1 \in \mathcal{J}^{\mathrm{reg}}_{\tilde{k}+1}(A)$ can be joined by a path of domain-dependent almost complex structures, we can find a path $\{K_t\}_{t \in [0,1]} \subset \mathcal{J}_{\tilde{k}+1}(M, \omega)$ that joins K_0 with K_1 such that the moduli space

$$\mathcal{W}_{A,\tilde{k}}(M, \{K_t\}) := \bigcup_{t \in [0,1]} \mathcal{M}_{A,\tilde{k}}(M, K_t)$$

is a smooth oriented manifold with boundary

$$\partial(\mathcal{W}_{A,\tilde{k}}(M, \{K_t\})) = \mathcal{M}_{A,\tilde{k}}(M, K_1) - \mathcal{M}_{A,\tilde{k}}(M, K_0)$$

We claim that if $K \in \mathcal{J}^{\mathrm{reg}}_{\tilde{k}+1}(A)$ is close enough to J and can be joined to J by a path of domain-dependent almost complex structures close enough to J, the evaluation maps

$$\mathrm{ev}^{\tilde{k}}_K : \mathcal{M}_{A,\tilde{k}}(M, K) \to M^{\tilde{k}}, \ \mathrm{ev}^{\tilde{k}}_J : \mathcal{M}_{A,\tilde{k}}(M, J) \to M^{\tilde{k}}$$

define cobordant pseudocycles: first, for any almost complex structure $K \in \mathcal{J}_{\tilde{k}+1}(M, \omega)$, we say that a homology class $B \in H_2(M; \mathbb{Z})$ is K-indecomposable if it can not be decomposed as a sum $B = B_1 + \cdots + B_l$, where $l \geq 2$ and each B_i has a nonconstant spherical K-holomorphic representative. The set

$$\mathcal{J}^{\mathrm{ind}}_{A,\tilde{k}+1} := \{K \in \mathcal{J}_{\tilde{k}+1}(M, \omega) : A \text{ is } K\text{-indecomposable}\}$$

is an open subset of $\mathcal{J}_{\tilde{k}+1}(M, \omega)$. If $K \in \mathcal{J}^{\mathrm{reg}}_{\tilde{k}+1}(A)$ is close enough to J and satisfies the assumption, we can assume that A is K-indecomposable and that we can find

a path $\{K_t\}_{t\in[0,1]} \subset \mathcal{J}^{\mathrm{ind}}_{A,\tilde{k}+1}$ that joins K with J such that $\mathcal{W}_{A,\tilde{k}}(M, \{K_t\})$ is a smooth oriented manifold with boundary

$$\partial \mathcal{W}_{A,\tilde{k}}(M, \{K_t\}) = \mathcal{M}_{A,\tilde{k}}(M, K) - \mathcal{M}_{A,\tilde{k}}(M, J)$$

The indecomposability assumption made on A implies that the evaluation maps

$$\mathrm{ev}^{\tilde{k}}_K : \mathcal{M}_{A,\tilde{k}}(M, K) \to M^{\tilde{k}}, \ \mathrm{ev}^{\tilde{k}}_J : \mathcal{M}_{A,\tilde{k}}(M, J) \to M^{\tilde{k}}$$

are pseudocycles and that the evaluation map

$$\mathrm{ev}^{\tilde{k}}_{K_t} : \mathcal{W}_{A,\tilde{k}}(M, \{K_t\}) \to M^{\tilde{k}}$$

defines a cobordism between the pseudocycles $\mathrm{ev}^{\tilde{k}}_K$ and $\mathrm{ev}^{\tilde{k}}_J$. This proves the claim.

Now, let Y be a Donaldson hypersurface of degree D sufficiently large such that (J, Y) is a Donaldson pair. Let $k \in \mathbb{Z}_{\geq 0}$ and $l = D\omega(A)$. For $\epsilon > 0$ small enough, let $K \in \mathcal{J}^{\mathrm{reg}}_{l+1}(M, \omega, J, Y, \epsilon) \subset \mathcal{J}^{\mathrm{reg}}_{l+1}(A)$ be such that the evaluation map

$$\mathrm{ev}^k_K : \mathcal{M}_{A,k+l}(M, K, Y) \to M^k$$

defines a pseudocycle.

If K is close enough to J, we can always join K with J by a path of domain-dependent almost complex structures close enough to J (see e.g. Cieliebak and Mohnke [8][Definition 9.4, Proposition 10.1]). Thus the evaluation maps

$$\mathrm{ev}^{k+l}_{\pi^*_l K} : \mathcal{M}_{A,k+l}(M, \pi^*_l K) \to M^{k+l}, \ \mathrm{ev}^{k+l}_J : \mathcal{M}_{A,k+l}(M, J) \to M^{k+l}$$

are cobordant pseudocycles. As a consequence, for cycles in general position $X_1, \cdots, X_k \subset M$ whose fundamental classes are Poincaré dual to cohomology classes $a_1, \cdots, a_k \in H^*(M; \mathbb{Z})$

$$\mathrm{GW}^{\mathrm{cm}}_{A,k}(a_1, \cdots, a_k) := \frac{1}{l!} \sharp \, \mathrm{ev}^{k+l}_{\pi^*_l K} \pitchfork (X_1 \times \cdots \times X_k \times Y \times \cdots \times Y)$$

$$= \frac{1}{l!} \sharp \, \mathrm{ev}^{k+l}_J \pitchfork (X_1 \times \cdots \times X_k \times Y \times \cdots \times Y)$$

Finally, the divisor axiom implies that

$$\mathrm{GW}^J_{A,k}(a_1, \cdots, a_k) := \sharp \, \mathrm{ev}^k_J \pitchfork (X_1 \times \cdots \times X_k)$$

$$= \frac{1}{l!} \sharp \, \mathrm{ev}^{k+l}_J \pitchfork (X_1 \times \cdots \times X_k \times Y \times \cdots \times Y),$$

and we are done.   ∎

For our purposes, the following two results are enough to bound the Gromov width of a symplectic manifold in the forthcoming sections.

Lemma 4.4. *Let (M, ω) be a compact integral symplectic manifold. If there exist a spherical class A and cohomology classes $a_2, \cdots, a_k$ such that*

$$\mathrm{GW}^{\mathrm{cm}}_{A,k}(\mathrm{PD}[\mathrm{pt}], a_2, \ldots, a_k) \neq 0,$$

then for any almost complex structure $\tilde{J} \in \mathcal{J}(M, \omega)$ and for any point $p \in M$ there exists a $\tilde{J}$-holomorphic sphere of degree B passing through p with $0 < \omega(B) \leq \omega(A)$.

Proof.   Assume that there exist a spherical class $A \in H_2(M; \mathbb{Z})$ and cohomology classes $a_2, \cdots, a_k$ Poincaré dual to the fundamental classes of cycles $X_2, \cdots, X_k$ in general position such that

$$\mathrm{GW}_{A,k}^{\mathrm{cm}}(\mathrm{PD[pt]}, a_2, \ldots, a_k) \neq 0$$

The definition of the Gromov-Witten invariant $\mathrm{GW}_{A,k}^{\mathrm{cm}}$ implies that for any almost complex structure $\tilde{J} \in \mathcal{J}(M, \omega)$ we can construct a domain-dependent almost complex structure K sufficiently close to $\tilde{J}$ such that for a generic point in M there exists a K-holomorphic sphere

$$u : \mathbb{CP}^1 \to M$$

of degree A passing through the generic point. The Gromov compactness theorem implies that for any point $p \in M$ there exists a $\tilde{J}$-holomorphic curve of degree B passing through p with $0 < \omega(B) \leq \omega(A)$.   ∎

Theorem 4.5.   *Let (M, ω) be a compact symplectic manifold such that the symplectic form ω represents a rational cohomology class $[\omega] \in H^2(M; \mathbb{Q})$. Let $A \in H_2(M; \mathbb{Z})$ and $J \in \mathcal{J}_{\mathrm{reg}}(A)$. Assume that A is a J-indecomposable spherical class. If there exist cohomology classes $a_2, \cdots, a_k \in H^*(M; \mathbb{Z})$ such that*

$$\mathrm{GW}_{A,k}^{J}(\mathrm{PD[pt]}, a_2, \ldots, a_k) \neq 0,$$

then for any almost complex structure $\tilde{J} \in \mathcal{J}(M, \omega)$ and for any point $p \in M$ there exists a $\tilde{J}$-holomorphic sphere of degree B passing through p with $0 < \omega(B) \leq \omega(A)$. Moreover,

$$\mathrm{Gwidth}(M, \omega) \leq \omega(A)$$

Proof.   Let c be a positive integer such that $[c\omega] \in H^2(M; \mathbb{Z})$. The first part of the theorem follows from the previous two lemmas when we apply them to the integral symplectic manifold $(M, c\omega)$ and the fact that $\mathcal{J}(M, \omega) = \mathcal{J}(M, c\omega)$.

Finally, Theorem 2.1 implies the following upper bound for the Gromov width of the integral symplectic manifold $(M, c\omega)$

$$\mathrm{Gwidth}(M, c\omega) \leq c\omega(A),$$

and we are done.   ∎

Remark 4.6.   For a general compact symplectic manifold (M, ω) more involved constructions are needed to define its Gromov-Witten invariants. In such constructions, usually one associates to the moduli space of J-holomorphic curves $\overline{\mathcal{M}}_{A,k}(M, J)$ a *virtual fundamental class* $[\overline{\mathcal{M}}_{A,k}(M, J)]_{\mathrm{virt}}$ with *rational* coefficients. The virtual fundamental class $[\overline{\mathcal{M}}_{A,k}(M, J)]_{\mathrm{virt}}$ is usually well defined and independent of the almost complex structure, and Gromov-Witten invariants on (M, ω) are defined by integrating over this fundamental class (see for example B. Chen, A.M. Li and B. L. Wang [7], Fukaya and Ono [11], Fukaya, Ohta, Oh and Ono [12], [13], [14], [15], Hofer, Wysocki and Zehnder [20], [21], [22], [23], [24],

[25], J. Li and G. Tian [32], G. Liu and G. Tian [34], [35], G. Lu and G. Tian [38], McDuff and Wehrheim [41], [42], Pardon [45], Ruan [46], Siebert [49]).

It is expected that many of the applications of Gromov-Witten invariants in symplectic topology that work for the semipositive case can be extended to the general case when virtual fundamental classes are used to define them. For instance, G. Lu has shown with G. Liu and G. Tian's definition of Gromov-Witten invariants that the Gromov width of a general compact symplectic manifold is bounded from above by the symplectic area of a spherical class that has a non-vanishing Gromov-Witten invariant with one of its constrains being Poincaré dual to the class of a point [36][Section 1.5]. In this paper we avoid using this result to bound the Gromov width of a symplectic manifold and instead we use Theorem 4.5 that is a consequence of the Cielibak-Mohnke definition of Gromov-Witten invariants.

5. Coadjoint Orbits of Compact Lie groups

In this section we recall some general facts about homogeneous spaces $G_{\mathbb{C}}/P$, coadjoint orbits and its geometry. Most of the material shown here can be found in the classical literature such as Bernstein, Gelfand and Gelfand [1] and Kirillov [28]. The material presented about Chern classes and stable curves is adapted from Fulton and Woodward [18, Chapters 2, 3].

Let G be a connected compact Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{g}^*$ be the dual of the Lie algebra $\mathfrak{g}$. The compact Lie group G acts on its Lie algebra $\mathfrak{g}^*$ by the coadjoint action. Let $\lambda \in \mathfrak{g}^*$ and $\mathcal{O}_\lambda$ be the coadjoint orbit through λ. As a homogeneous space $\mathcal{O}_\lambda \cong G/G_\lambda$, where $G_\lambda \subset G$ denotes the stabilizer of $\lambda \in \mathfrak{g}^*$ under the coadjoint action.

The coadjoint orbit $\mathcal{O}_\lambda$ carries a symplectic form defined as follows: for $\lambda \in \mathfrak{g}^*$ we define a skew bilinear form on $\mathfrak{g}$ by

$$\omega_\lambda : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$$
$$(X, Y) \mapsto \langle \lambda, [X, Y] \rangle$$

The kernel of ω_λ is the Lie algebra $\mathfrak{g}_\lambda$ of the stabilizer G_λ of $\lambda \in \mathfrak{g}^*$. In particular, ω_λ defines a nondegenerate skew-symmetric bilinear form on $\mathfrak{g}/\mathfrak{g}_\lambda$, a vector space that can be identified with $T_\lambda(\mathcal{O}_\lambda) \subset \mathfrak{g}^*$. Using the coadjoint action, the bilinear form ω_λ induces a closed, invariant, nondegenerate 2-form on the orbit $\mathcal{O}_\lambda$, therefore defining a symplectic structure on $\mathcal{O}_\lambda$. This symplectic form is known as the *Kostant-Kirillov-Souriau form* of the coadjoint orbit $\mathcal{O}_\lambda$.

The compact Lie group G admits a *complexification* $G_{\mathbb{C}}$. There exists a parabolic subgroup $P \subset G_{\mathbb{C}}$ (see definition below) such that the homogeneous spaces G/G_λ and $G_{\mathbb{C}}/P$ are diffeomorphic. We can use the complex structure defined on the quotient of complex Lie groups $G_{\mathbb{C}}/P$ to define a complex structure J on the coadjoint orbit $G/G_\lambda \cong \mathcal{O}_\lambda$ which is invariant under the $G_{\mathbb{C}}$-action. Together with the Kostant-Kirillov-Souriau form, this makes the coadjoint orbit $\mathcal{O}_\lambda$ a Kähler manifold.

Let $T \subset G$ be a maximal torus and $\mathfrak{t}$ denote its Lie algebra. Let $R \subset \mathfrak{t}^*$

be the root system of T in G so

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha},$$

where

$$\mathfrak{g}_{\alpha} := \{x \in \mathfrak{g}_{\mathbb{C}} : [\,h\,,x\,] = \alpha(h)\,x \text{ for all } h \in \mathfrak{t}_{\mathbb{C}}\}$$

is the root space associated with the root $\alpha \in R$.

Let $R^+ \subset R$ be a choice of positive roots with simple roots $S \subset R^+$. Let $W := N_G(T)/T$ be the *Weyl group* of G. For every root $\alpha \in R$, let $s_\alpha \in W$ be the reflection associated to it.

Let $B \subset G_{\mathbb{C}}$ be the Borel subgroup with Lie algebra

$$\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\alpha}$$

We call a subgroup of $G_{\mathbb{C}}$ *parabolic* if it is conjugate to a subgroup of $G_{\mathbb{C}}$ that contains B. Let us fix a parabolic subgroup $P \supset B$. Let $W_P := N_P(T)/T$ be the *Weyl group of P* and $S_P := \{\alpha \in S : s_\alpha \in W_P\} \subset S$ be the set of simple roots whose corresponding reflections are in W_P. The group W_P is the subgroup of W generated by the set of simple reflections $\{s_\alpha : \alpha \in S_P\}$. The parabolic subgroup P is generated by the Borel subgroup B and $N_P(T)$. Set $R_P = R \cap \mathbb{Z}S_P$ and $R_P^+ = R^+ \cap \mathbb{Z}S_P$, where $\mathbb{Z}S_P = \operatorname{span}_{\mathbb{Z}}(S_P)$ is the Abelian group spanned by S_P in $\mathfrak{t}^*$. The Lie algebra of P is

$$\mathfrak{p} = \mathfrak{b} \oplus \bigoplus_{\alpha \in R_P^+} \mathfrak{g}_{-\alpha}$$

Remark 5.1.   The map $\tilde{P} \mapsto S_{\tilde{P}}$ establishes a bijection between the set of all parabolic subgroups of $G_{\mathbb{C}}$ containing B and the set of all subsets of simple roots contained in S (see for instance Kumar [30, Chapter 5]).

For each $w \in W$, the *length* $l(w)$ of w is defined as the minimum number of simple reflections $s_\alpha \in W, \alpha \in S$, whose product is w.

For $w', w \in W$, write $w' \to w$ if there exists simple reflections $s \in S$ such that $w = w' \cdot s$ and $l(w) = l(w') + 1$. Then define $w' \leq_B w$ if there is a sequence

$$w' \to w_1 \to \ldots \to w_m = w.$$

The *Bruhat order* on W is the partial ordering defined by the relation $\leq_B$.

Let $W^P \subset W$ be the set of all *minimum length representatives* for cosets in W/W_P. Each element $w \in W$ can be written uniquely as $w = w^P w_P$ where $w^P \in W^P$ and $w_P \in W_P$. Their lengths satisfy $l(w) = l(w^P) + l(w_P)$ (see e.g. Humphreys [26][Chapter 1]). The *Bruhat order* $\leq_B$ on W^P is the restriction to W^P of the Bruhat order on W. The *Bruhat order* $\leq_B$ on W/W_P is defined by $w'W_P \leq_B wW_P$ if and only if $w'^P \leq_B w^P$ on W^P.

The Weyl group W has a unique longest element that we denote by w_0. Let $B^{\mathrm{op}} := w_0 B w_0 \subset G_{\mathbb{C}}$ be the *Borel subgroup opposite* to B. For $w \in W^P$ we define the *Schubert cell*

$$C_P(w) := BwP/P \subset G_{\mathbb{C}}/P$$

and the *opposite Schubert cell*

$$C_P^{\mathrm{op}}(w) := B^{\mathrm{op}} wP/P \subset G_{\mathbb{C}}/P$$

associated with w. The *Schubert variety* $X_P(w)$ associated with w and its opposite $X_P^{\mathrm{op}}(w)$ are by definition the closures of the Schubert cells $C_P(w)$ and $C_P^{\mathrm{op}}(w)$, respectively.

The Bruhat order can be written in terms of the inclusion relation of Schubert varieties, i.e., for $w', w \in W^P \cong W/W_P$,

$$X_P(w') \subset X_P(w)$$

if and only if $w' \leq_B w$. Indeed,

$$X_P(w) = \bigsqcup_{w' \leq_B w} C_P(w')$$

For $w \in W^P$, the Schubert cell $C_P(w)$ is isomorphic to an affine space of complex dimension equal to $l(w)$. The set of Schubert cells $\{X_P(w)\}_{w \in W^P}$ defines a CW-complex for $G_{\mathbb{C}}/P$ with cells occurring only in even dimension. Thus, the set of fundamental classes $\{\sigma_P(w) := [X_P(w)]\}_{w \in W^P}$ is a free basis of $H_*(G_{\mathbb{C}}/P; \mathbb{Z})$ as a $\mathbb{Z}$-module. Likewise, the set of cohomology classes $\{\mathrm{PD}(\sigma_P(w))\}_{w \in W^P}$ is a free basis of $H^*(G_{\mathbb{C}}/P; \mathbb{Z})$ as a $\mathbb{Z}$-module. Similar statements hold for the fundamental classes of the opposite Schubert varieties $\check{\sigma}_P(w) := [X_P^{\mathrm{op}}(w)] \in H_*(G_{\mathbb{C}}/P; \mathbb{Z})$ (note that $\check{\sigma}_P(w) = \sigma_P(\check{w})$ where $\check{w} := w_0 w w_p \in W^P$ and w_p denotes the longest element in W_P).

The Poincaré intersection pairing is the map

$$\langle\,,\,\rangle : H_*(G_{\mathbb{C}}/P; \mathbb{Z}) \otimes H_*(G_{\mathbb{C}}/P; \mathbb{Z}) \to \mathbb{Z}$$

that associates to a pair of homology classes $a, b \in H_*(G_{\mathbb{C}}/P; \mathbb{Z})$ the coefficient at the class of a point $[\mathrm{pt}]$ of the homological intersection product $a \cdot b \in H_*(G_{\mathbb{C}}/P; \mathbb{Z})$. The *duality theorem* states that for any $w', w \in W^P$,

$$\langle \check{\sigma}_P(w), \sigma_P(w') \rangle = \delta_{ww'},$$

and in particular

$$\dim_{\mathbb{C}}(G_{\mathbb{C}}/P) = \dim_{\mathbb{C}}(X_P(w)) + \dim_{\mathbb{C}}(X_P^{\mathrm{op}}(w))$$

The collection of cosets $\{wP\}_{w \in W^P}$ is the set of all T-fixed points in $G_{\mathbb{C}}/P$. For each positive root $\alpha \in R^+ - R_P^+$ there is a unique irreducible T-invariant curve C_α that contains $1 \cdot P$ and $s_\alpha \cdot P$. Indeed, $C_\alpha := Sl(2, \mathbb{C})_\alpha \cdot P/P$ where $Sl(2, \mathbb{C})_\alpha \subset G_{\mathbb{C}}$ is the subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

The curve C_α is unique because there is a neighborhood of $1 \cdot P/P$ that is T-equivariantly isomorphic to $\mathfrak{g}_\mathbb{C}/\mathfrak{p}$. The T-invariant curves in $\mathfrak{g}_\mathbb{C}/\mathfrak{p}$ correspond to weight spaces $\mathfrak{g}_{-\alpha}$ for $\alpha \in R^+ - R_P^+$.

Let $(\cdot, \cdot)$ denote an ad-invariant inner product defined on $\mathrm{Lie}(G) = \mathfrak{g}$. We identify the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ via this inner product. The inner product $(\cdot, \cdot)$ defines an inner product in $\mathfrak{t}^* = \mathbb{R}R$. Each root $\alpha \in R$ has a *coroot* $\check{\alpha} \in \mathfrak{t}$ that is identified with $\dfrac{2\alpha}{(\alpha, \alpha)}$ via the inner product $(\cdot, \cdot)$. The coroots form the dual root system $\check{R} = \{\check{\alpha} : \alpha \in R\}$, with basis of simple coroots $\check{S} = \{\check{\alpha} : \alpha \in S\}$. For the parabolic subgroup $P \subset G_\mathbb{C}$, let $\check{S}_P := \{\check{\alpha} : \alpha \in S_P\} \subset \check{S}$.

The set of *fundamental weights* $\{\omega_\alpha\}_{\alpha \in S} \subset \mathfrak{t}^*$ is defined by the relation $(\omega_\alpha, \check{\beta}) = \delta_{\alpha, \beta}$ for every $\alpha, \beta \in S$. A *weight* is an element in the Abelian group spanned by the set of fundamental weights.

The cohomology group $H^2(G_\mathbb{C}/P; \mathbb{Z})$ can be identified with the span

$$\mathbb{Z}\{\omega_\alpha : \alpha \in S - S_P\}$$

and the homology group $H_2(G_\mathbb{C}/P; \mathbb{Z})$ with the quotient

$$\mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$$

For each $\alpha \in S - S_P$ we identify the Schubert class $\sigma_P(s_\alpha) \in H_2(G_\mathbb{C}/P; \mathbb{Z})$ with $\check{\alpha} + \mathbb{Z}\check{S}_P \in \mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$ and the Poincaré dual class $\mathrm{PD}(\check{\sigma}_P(s_\beta)) \in H^2(G_\mathbb{C}/P; \mathbb{Z})$ with ω_β. The pairing

$$H^2(G_\mathbb{C}/P; \mathbb{Z}) \otimes H_2(G_\mathbb{C}/P; \mathbb{Z}) \to \mathbb{Z}$$
$$(\sigma, \alpha) \mapsto \int_\alpha \sigma$$

is then given by the ad-invariant inner product $(\cdot, \cdot)$ on $\mathfrak{t}$.

The following localization lemma, due to Bott [3], allows us to compute the first Chern classes of line bundles on T-invariant curves of $G_\mathbb{C}/P$:

Lemma 5.2. *Suppose that a torus T acts on a curve $C \cong \mathbb{CP}^1$, with fixed points $p \neq q$, and suppose L is a T-equivariant line bundle on C. Let η_p and η_q be the weights of T acting on the fibers L_p and L_q, and let ψ_p be the weight of T acting on the tangent space to C at p. Then*

$$\eta_p - \eta_q = n\psi_p$$

where $n = \int_C c_1(L)$ is the degree of L.

If η is a weight that vanishes on all β in S_P, it determines a character on P, and so a line bundle $L(\eta) := G_\mathbb{C} \times_P \mathbb{C}(-\eta)$ over $G_\mathbb{C}/P$. The Chern class $c_1(L(\eta)) \in H^2(G_\mathbb{C}/P; \mathbb{Z})$ is identified with the weight $\eta \in \mathbb{Z}\{w_\alpha : \alpha \in S - S_P\}$. Indeed, if L is any holomorphic line bundle over $G_\mathbb{C}/P$, there exists a unique weight $\eta \in \mathbb{Z}\{w_\alpha : \alpha \in S - S_P\}$ such that $L \cong L(\eta)$, and thus $\mathrm{Pic}(G_\mathbb{C}/P) \cong \mathbb{Z}\{w_\alpha : \alpha \in S - S_P\} \cong H^2(G_\mathbb{C}/P; \mathbb{Z})$ (see e.g. Borel-Weil [47]).

Proposition 5.3. *For any root $\alpha \in R^+ - R_P^+$,*

$$[C_\alpha] = \check{\alpha} + \mathbb{Z}\check{S}_P \in H_2(G_\mathbb{C}/P; \mathbb{Z}) \cong \mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$$

Proof. The localization lemma implies that for any weight $\eta \in \mathbb{Z}\{w_\alpha : \alpha \in S - S_P\}$

$$\int_{C_\alpha} c_1(L(\eta)) \cdot (-\alpha) = -\eta - s_\alpha(-\eta) = s_\alpha(\eta) - \eta$$

and thus

$$(\eta, \check{\alpha}) = \int_{C_\alpha} c_1(L(\eta))$$

The nondegeneracy of the pair $(\,,)$ implies the proposition. $\blacksquare$

Proposition 5.4. *The Chern class $c_1(T(G_\mathbb{C}/P))$ is identified with $\sum_{\gamma \in R^+ - R_P^+} \gamma$ via the isomorphism*

$$\mathbb{Z}\{w_\alpha : \alpha \in S - S_P\} \to H^2(G_\mathbb{C}/P; \mathbb{Z})$$
$$\eta \mapsto c_1(L(\eta))$$

Proof. The tangent space of $G_\mathbb{C}/P$ at the point $1 \cdot P \in G_\mathbb{C}/P$ is

$$\mathfrak{g}_\mathbb{C}/\mathfrak{p} = \bigoplus_{\alpha \in R^+ - R_P^+} \mathfrak{g}_{-\alpha}$$

The weight of the line bundle $\bigwedge^n T(G_\mathbb{C}/P)$, $n = \dim_\mathbb{C}(G_\mathbb{C}/P)$, at the point $1 \cdot P \in G_\mathbb{C}/P$ is $-\sum_{\gamma \in R^+ - R_P^+} \gamma$. The proposition follows from the fact that $c_1(T(G_\mathbb{C}/P)) = c_1\left(\bigwedge^n T(G_\mathbb{C}/P)\right)$. $\blacksquare$

Let

$$\eta = \sum_{\beta \in S - S_P} l_\beta w_\beta \in \mathbb{Z}_{\geq 0}\{w_\alpha : \beta \in S - S_P\}$$

be an integral weight and V_η be the irreducible representation of $G_\mathbb{C}$ with highest weight η. The Borel-Weil-Bott theorem states that the set of holomorphic sections $H^0(G_\mathbb{C}/P, L(\eta))$ of the line bundle $L(\eta)$ is isomorphic as a $G_\mathbb{C}$-representation to the irreducible representation V_η.

Let v_η be a highest weight vector of V_η. We can embed $G_\mathbb{C}/P$ in the projective space $\mathbb{P}V_\eta$ by the transformation

$$G_\mathbb{C}/P \hookrightarrow \mathbb{P}V_\eta$$
$$[g] \mapsto [g \cdot v_\eta]$$

Remark 5.5. For $\lambda \in \mathbb{R}_{\geq 0}\{w_\alpha : \beta \in S - S_P\} \subset \mathfrak{t}^*$, let $\mathcal{O}_\lambda$ be the coadjoint orbit passing through λ and ω_λ its Kostant-Kirillov-Souriau form. The cohomology class of ω_λ is identified with $\lambda \in H^2(G_\mathbb{C}/P; \mathbb{R}) \cong \mathbb{R}\{w_\alpha : \beta \in S - S_P\} \subset \mathfrak{t}^*$, and for any positive root $\alpha \in R^+ - R_P^+$

$$\omega_\lambda(C_\alpha) = \int_{C_\alpha} \omega_\lambda = \langle \lambda, \check{\alpha} \rangle$$

If λ is integral, the projective embedding $G_{\mathbb{C}}/P \hookrightarrow \mathbb{P}V_\lambda$ is symplectic, i.e., the pullback of the Fubini-Study form defined on $\mathbb{P}V_\lambda$ is the Kostant-Kirillov-Souriau form ω_λ defined on $\mathcal{O}_\lambda \cong G_{\mathbb{C}}/P$.

Let J be the invariant complex structure defined on the quotient of complex Lie groups $G_{\mathbb{C}}/P$. Let $A \in H_2(G_{\mathbb{C}}/P; \mathbb{Z})$ and $k \in \mathbb{Z}_{\geq 0}$. The moduli space $\overline{\mathcal{M}}_{A,k}(G_{\mathbb{C}}/P, J)$ of stable J-holomorphic curves of degree A with k-marked points is a normal projective variety of dimension equal to

$$\dim \overline{\mathcal{M}}_{A,k}(G_{\mathbb{C}}/P, J) = \dim(G_{\mathbb{C}}/P) + 2c_1(A) + 2k - 6$$

(see e.g. Fulton and Pandharipande [17]). Let

$$\mathrm{ev}_J^k = (\mathrm{ev}_1, \cdots, \mathrm{ev}_k) : \overline{\mathcal{M}}_{A,k}(G_{\mathbb{C}}/P, J) \to (G_{\mathbb{C}}/P)^k$$

be the evaluation map. The *algebraic Gromov-Witten invariant* $\mathrm{GW}_{A,k}^{\mathrm{alg}}$ on $G_{\mathbb{C}}/P$ is defined by

$$\mathrm{GW}_{A,k}^{\mathrm{alg}}(a_1, \cdots, a_k) := \int_{\overline{\mathcal{M}}_{A,k}(G_{\mathbb{C}}/P, J)} \mathrm{ev}_1^* a_1 \cup \cdots \cup \mathrm{ev}_k^* a_k$$

for cohomology classes $a_1, \cdots, a_k \in H^*(G_{\mathbb{C}}/P; \mathbb{Z})$. If $X_1, \ldots, X_k \subset G_{\mathbb{C}}/P$ are irreducible varieties whose fundamental classes are Poincaré dual to $a_1, \ldots, a_k \in H^*(G_{\mathbb{C}}/P; \mathbb{Z})$ and

$$\dim \overline{\mathcal{M}}_{A,k}(G_{\mathbb{C}}/P, J) = \sum_{i=1}^{k} \deg a_i,$$

the number of J-holomorphic spheres of degree A passing through $g_1 X_1, \cdots, g_k X_k$ coincides with the Gromov–Witten invariant $\mathrm{GW}_{A,k}^{\mathrm{alg}}(a_1, \ldots, a_k)$ for generic $g_1, \ldots, g_k \in G_{\mathbb{C}}$ (see e.g. Fulton and Pandharipande [17, Lemma 14]).

Remark 5.6. Let $\mathcal{O}_\lambda$ be a coadjoint orbit passing through $\lambda \in \mathfrak{g}^*$ with Kostant-Kirillov-Souriau form ω_λ. Assume that $P \subset G_{\mathbb{C}}$ is a parabolic subgroup such that $\mathcal{O}_\lambda \cong G_{\mathbb{C}}/P$. Let $A \in H_2(G_{\mathbb{C}}/P; \mathbb{Z})$ and $k \in \mathbb{Z}_{\geq 0}$.

If $J \in \mathcal{J}(\mathcal{O}_\lambda, \omega_\lambda)$ is a regular almost complex structure and A is a J-indecomposable class, we denote by $\mathrm{GW}_{A,k}^J$ the Gromov-Witten invariant defined exclusively for J and A (see Remark 3.1). If $(\mathcal{O}_\lambda, \omega_\lambda)$ is an integral coadjoint orbit, we denote by $\mathrm{GW}_{A,k}^{\mathrm{cm}}$ the symplectic Gromov-Witten invariant defined by Cielibak and Mohnke (see Section 4).

When the coadjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ is an integral symplectic manifold, J is the almost complex structure coming from the quotient of complex Lie groups $G_{\mathbb{C}}/P$ and A is a J-indecomposable class, all the three Gromov-Witten invariants

$$\mathrm{GW}_{A,k}^{\mathrm{alg}}, \quad \mathrm{GW}_{A,k}^J, \quad \mathrm{GW}_{A,k}^{\mathrm{cm}}$$

coincide.

6. Curve Neighborhoods and Gromov Witten invariants

In this section we define the concept of curve neighborhood and explain its relation with the Gromov-Witten invariants of a homogeneous space $G_{\mathbb{C}}/P$. The material presented here is mostly adapted from Buch and Mihalcea [5].

Let G be a compact connected simple Lie group. Let B be a Borel subgroup and P be a parabolic subgroup of $G_{\mathbb{C}}$ with $B \subset P$. Let $T \subset G$ be a maximal torus with $T_{\mathbb{C}} \subset B \subset P$. Let $W = N(T)/T$ be the associated Weyl group. Let R be the set of roots and S be the choice of simple roots compatible with B. For the parabolic subgroup $P \subset G_{\mathbb{C}}$, let W_P be the Weyl group of P and S_P be the subset of simple roots whose corresponding reflections are in W_P.

Let J be the complex structure defined on $G_{\mathbb{C}}/P$ coming from its presentation as a quotient of complex Lie groups. Let $A \in H_2(G_{\mathbb{C}}/P; \mathbb{Z})$ be a spherical class. Let $\overline{\mathcal{M}}_{A,2}(G_{\mathbb{C}}/P, J)$ be the moduli space of equivalence classes of stable J-holomorphic curves of degree A with two marked points and

$$\mathrm{ev}_J^2 = (\mathrm{ev}_1, \mathrm{ev}_2) : \overline{\mathcal{M}}_{A,2}(G_{\mathbb{C}}/P, J) \to (G_{\mathbb{C}}/P)^2$$

be its corresponding evaluation map.

Given any subvariety $Z \subset G_{\mathbb{C}}/P$, define the *degree A neighborhood* of Z to be

$$\Gamma_A(Z) := \mathrm{ev}_2(\mathrm{ev}_1^{-1}(Z)) \subset G_{\mathbb{C}}/P,$$

i.e., the union of all stable J-holomorphic curves of degree A that meet Z. The *Gromov-Witten variety* of Z is

$$\mathrm{GW}_A(Z) := \mathrm{ev}_1^{-1}(Z) \subset \overline{\mathcal{M}}_{A,2}(G_{\mathbb{C}}/P, J)$$

By definition,

$$\Gamma_A(Z) = \mathrm{ev}_2(\mathrm{GW}_A(Z))$$

When $Z \subset G_{\mathbb{C}}/P$ is a irreducible variety, then $\Gamma_A(Z)$ is also a irreducible variety. In particular, if Z is a B-stable irreducible variety, i.e. a B-stable Schubert variety, then so is $\Gamma_A(Z)$ (Buch, Chaput, Mihalcea and Perrin [4]).

The following lemma is enough for our computations of Gromov-Witten invariants:

Lemma 6.1. *Let $A \in H_2(G_{\mathbb{C}}/P; \mathbb{Z})$ be a J-indecomposable class and $\Gamma_A(\mathrm{pt})$ be the degree A neighborhood of a point in $G_{\mathbb{C}}/P$. If*

$$c_1(A) = 1 + \dim_{\mathbb{C}} \Gamma_A(\mathrm{pt}),$$

then

$$\mathrm{GW}_{A,2}^{\mathrm{alg}}(\mathrm{PD}[\mathrm{pt}], \mathrm{PD}[\Gamma_A(\mathrm{pt})]^{\mathrm{op}}) > 0$$

Proof. If the opposite Schubert variety $\Gamma_A(\mathrm{pt})^{\mathrm{op}}$ satisfies the dimensional constraint

$$\dim_{\mathbb{C}} \Gamma_A(\mathrm{pt})^{\mathrm{op}} + \dim_{\mathbb{C}} \overline{\mathcal{M}}_{A,2}(G_{\mathbb{C}}/P) = 2 \dim_{\mathbb{C}} G_{\mathbb{C}}/P$$

or equivalently

$$c_1(A) = 1 + \dim_{\mathbb{C}} \Gamma_A(\mathrm{pt}),$$

and if the evaluation map

$$\mathrm{ev}_J : \overline{\mathcal{M}}_{A,2}(G_{\mathbb{C}}/P, J) \to (G_{\mathbb{C}}/P)^2$$

is transverse to $\{g \cdot P\} \times h \cdot \Gamma_A(1 \cdot P)^{\mathrm{op}} \subset (G_{\mathbb{C}}/P)^2$ and there exists a J-holomorphic sphere $u : \mathbb{CP}^1 \to G_{\mathbb{C}}/P$ passing through $g \cdot P$ and $h \cdot \Gamma_A(1 \cdot P)^{\mathrm{op}}$ for generic $g, h \in G_{\mathbb{C}}$, then the Gromov-Witten invariant $\mathrm{GW}_{A,2}^{\mathrm{alg}}(\mathrm{PD}[\mathrm{pt}], \mathrm{PD}[\Gamma_A(\mathrm{pt})]^{\mathrm{op}})$ is finite and positive (see e.g. McDuff and Salamon [40][Proposition 7.4.5]).

On one side, the Bertini-Kleiman transversality theorem implies that for generic $g, h \in G_{\mathbb{C}}$ the evaluation map

$$\mathrm{ev}_J : \overline{\mathcal{M}}_{A,2}(G_{\mathbb{C}}/P, J) \to (G_{\mathbb{C}}/P)^2$$

is transverse to $\{g \cdot P\} \times h \cdot \Gamma_A(1 \cdot P)^{\mathrm{op}} \subset (G_{\mathbb{C}}/P)^2$.

On the other side, the duality theorem implies that for generic $g, h \in G_{\mathbb{C}}$, the Schubert varieties $g \cdot \Gamma_A(1 \cdot P)$ and $h \cdot \Gamma_A(1 \cdot P)^{\mathrm{op}}$ intersect transversally at one point say $q \cdot P$. Hence, there exists a J-holomorphic stable curve $u : \Sigma \to G_{\mathbb{C}}/P$ with two marked points (z_1, z_2) such that $u(z_1) = g \cdot P$ and $u(z_2) = q \cdot P \in h \cdot \Gamma_A(1 \cdot P)^{\mathrm{op}}$. The indecomposability condition of A implies that $\Sigma \cong \mathbb{CP}^1$. ∎

Remark 6.2.   For any two B-stable Schubert varieties Z_1, Z_2 and any degree $A \in H_2(G_{\mathbb{C}}/P; \mathbb{Z})$ the following formula due to Buch and Mihalcea

$$\mathrm{GW}_{A,2}^{\mathrm{alg}}(\mathrm{PD}[Z_1], \mathrm{PD}[Z_2]) = \begin{cases} 1 & \text{if } c_1(A) - 1 + \dim_{\mathbb{C}} Z_1 = \dim_{\mathbb{C}} \Gamma_A(Z_1) \\ & \text{and } [\Gamma_A(Z_1)] \text{ is the Schubert class} \\ & \text{opposite to } [Z_2] \\ 0 & \text{otherwise} \end{cases}$$

extends the previous lemma [5]. The above formula is a consequence of the *projection formula* and the fact that the pushforward of $\mathrm{PD}[\mathrm{GW}_A(Z_1)]$ under the evaluation map ev_2 is equal to $\mathrm{PD}[\Gamma_A(Z_1)]$ if $\dim_{\mathbb{C}} \mathrm{GW}_A(Z_1) = \dim_{\mathbb{C}} \Gamma_A(Z_1)$ and zero otherwise (Buch and Mihalcea [5]). We briefly explain how the above formula is deduced from these two facts:

$$\mathrm{GW}_{A,2}^{\mathrm{alg}}(\mathrm{PD}[Z_1], \mathrm{PD}[Z_2]) := \int_{\overline{\mathcal{M}}_{A,2}(G_{\mathbb{C}}/P, J)} \mathrm{ev}_1^* \, \mathrm{PD}[Z_1] \cup \mathrm{ev}_2^* \, \mathrm{PD}[Z_2]$$

$$= \int_{G_{\mathbb{C}}/P} \mathrm{ev}_{2*}(\mathrm{ev}_1^* \, \mathrm{PD}[Z_1]) \cup \mathrm{PD}[Z_2] = \int_{G_{\mathbb{C}}/P} \mathrm{ev}_{2*}(\mathrm{PD}[\mathrm{GW}_A(Z_1)]) \cup \mathrm{PD}[Z_2]$$

$$= \begin{cases} \int_{G_{\mathbb{C}}/P} \mathrm{PD}[\Gamma_A(Z_1)] \cup \mathrm{PD}[Z_2] & \text{if } \dim_{\mathbb{C}} \mathrm{GW}_A(Z_1) = \dim_{\mathbb{C}} \Gamma_A(Z_1) \\ 0 & \text{otherwise} \end{cases}$$

Example 6.3.   Let $G(k, n)$ denote the Grassmannian manifold of k-dimensional vector spaces in $\mathbb{C}^n$. Let A be the Schubert class that cyclic generates the homology group $H_2(G(k, n); \mathbb{Z})$. Let J be the invariant complex structure defined on $G(k, n)$.

For a J-holomorphic curve $u : \mathbb{CP}^1 \to G(k,n)$ of degree A, there exist linearly independent vectors $v_0, v_1, \ldots, v_k \in \mathbb{C}^n$ such that

$$u : \mathbb{CP}^1 \to G(k,n)$$
$$[z_0, z_1] \mapsto \mathrm{span}\{z_0 v_0 + z_1 v_1, v_2, \ldots, v_k\}$$

If we let $W^{k-1} = \mathrm{span}\{v_2, \ldots, v_k\}$ and $W^{k+1} = \mathrm{span}\{v_0, v_1, v_2, \ldots, v_k\}$, then

$$u(\mathbb{CP}^1) = \{V^k \in G(k,n) : W^{k-1} \subset V^k \subset W^{k+1}\}$$

Note that two vector subspaces can be joined by a J-holomorphic curve of degree A if and only if their intersection has dimension equal to $k-1$. Thus, for a fixed vector subspace $V^k \in G(k,n)$, the degree A neighborhood of V^k is the set

$$\Gamma_A(V^k) = \{V'^k \in G(k,n) : k-1 \leq \dim(V^k \cap V'^k) \leq \dim(V^k + V'^k) \leq k+1\}$$

The complex dimension of $\Gamma_A(V^k)$ equals to $n-1 = c_1(A) - 1$. By the previous lemma

$$\mathrm{GW}_{A,2}^{\mathrm{alg}}(\mathrm{PD}[\mathrm{pt}], \mathrm{PD}[\Gamma_A(\mathrm{pt})]^{\mathrm{op}}) = 1$$

The following lemma allows us to give an explicit description of the degree A neighborhood $\Gamma_A(1 \cdot P)$ of the point $1 \cdot P \in G_{\mathbb{C}}/P$ when P is a maximal parabolic subgroup of $G_{\mathbb{C}}$ and A is the Schubert class that cyclic generates $H_2(G_{\mathbb{C}}/P; \mathbb{Z})$.

Lemma 6.4. *Let $\alpha \in S$ and $P \subset G_{\mathbb{C}}$ be the maximal parabolic subgroup such that $S_P = S - \{\alpha\}$. Let $A = \sigma_P(s_\alpha)$ be the Schubert class that cyclic generates $H_2(G_{\mathbb{C}}/P; \mathbb{Z})$. Let*

$$Z_A^P := \{s_\beta \cdot P \in W/W_P : \beta \in R^+ - R_P^+, \ \check{\beta} = \check{\alpha} + \mathbb{Z}\check{S}_P\}$$

The set Z_A^P has a unique maximal element z_A^P with respect to the Bruhat order defined on W/W_P and

$$\Gamma_A(1 \cdot P) = X_P(z_A^P)$$

Proof. The curve neighborhood $\Gamma_A(1 \cdot P)$ is a B-stable Schubert variety and thus determined by the T-fixed points that it contains. Note that if there exists a J-holomorphic curve of degree A passing through two T-fixed points, the curve is T-invariant. Hence, the set of T-fixed points in $\Gamma_A(1 \cdot P)$ is the set of all T-fixed points that can be joined with $1 \cdot P$ by a T-invariant curve of degree A. This corresponds to the set

$$Z_A^P = \{s_\beta \cdot P \in W/W_P : \beta \in R^+ - R_P^+, \ \check{\beta} = \check{\alpha} + \mathbb{Z}\check{S}_P\}$$

The set Z_A^P has a unique maximal element z_A^P with respect to the Bruhat order because $\Gamma_A(1 \cdot P)$ is a Schubert variety and thus

$$\Gamma_A(1 \cdot P) = X_P(z_A^P) \qquad \blacksquare$$

7. Upper Bound for the Gromov width of Grassmannian Manifolds

Let G be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a maximal torus and let $B \subset G_\mathbb{C}$ be a Borel subgroup with $T_\mathbb{C} \subset B \subset P$. Let $W = N(T)/T$ be the associated Weyl group. Let R be the set of roots and S be the choice of simple roots compatible with B. For a parabolic subgroup $P \subset G_\mathbb{C}$ that contains B, let W_P be the Weyl group of P and S_P be the subset of simple roots whose corresponding reflections are in W_P.

The *maximal parabolic subgroup of $G_\mathbb{C}$ associated with a simple root $\alpha \in S$* is the parabolic subgroup P_α such that $S_{P_\alpha} = S - \{\alpha\}$. We call the corresponding homogeneous space $G_\mathbb{C}/P_\alpha$ a *Grassmannian manifold*. The second homology group $H_2(G_\mathbb{C}/P_\alpha; \mathbb{Z})$ is generated as a $\mathbb{Z}$-module by the fundamental class $[X_{P_\alpha}(s_\alpha)]$. We denote the class $[X_{P_\alpha}(s_\alpha)]$ by A.

Let $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$. Let us assume that the coadjoint orbit $\mathcal{O}_\lambda$ passing through λ is isomorphic with the Grassmannian manifold $G_\mathbb{C}/P_\alpha$ for some $\alpha \in S$. This means that $\langle \lambda, \check{\beta} \rangle = 0$ for every $\beta \in S - \{\alpha\}$ and $\langle \lambda, \check{\alpha} \rangle \neq 0$. In this section we will show that

$$\mathrm{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq |\langle \lambda, \check{\alpha} \rangle|,$$

where ω_λ denotes the Kostant-Kirillov-Souriau form defined on $\mathcal{O}_\lambda$. We obtain this upper bound by computing a non-vanishing Gromov-Witten invariant with one of its constraints being Poincaré dual to a point. More precisely, we show that if $\Gamma_A(\mathrm{pt})$ is the degree A neighborhood of a point in $G_\mathbb{C}/P_\alpha$, then

$$\mathrm{GW}^{\mathrm{alg}}_{A,2}(\mathrm{PD}[\mathrm{pt}], \mathrm{PD}[\Gamma_A(\mathrm{pt})]^{\mathrm{op}}) > 0$$

First, we give an explicit description of the degree A neighborhood $\Gamma_A(1 \cdot P_\alpha)$ of the point $1 \cdot P_\alpha$ when $P_\alpha \subset G_\mathbb{C}$ is a maximal parabolic subgroup associated with a *long* simple root $\alpha \in S$.

Theorem 7.1.   *Let $\alpha \in S$ be a simple root, $P \subset G_\mathbb{C}$ be the maximal parabolic subgroup associated with α and A be the Schubert class that cyclic generates $H_2(G_\mathbb{C}/P; \mathbb{Z})$. Let $N(\alpha) \subset S$ denotes the neighbors of α in the Dynkin diagram of G. Let $R \subset G_\mathbb{C}$ be the parabolic subgroup with $S_R = S - (N(\alpha) \cup \{\alpha\})$. If α is a long simple root, then the degree A neighborhood $\Gamma_A(1 \cdot P)$ of the point $1 \cdot P \in G_\mathbb{C}/P$ is a B-stable Schubert variety and*

$$\Gamma_A(1 \cdot P) = X_P(w^r_p s_\alpha),$$

where w^r_p is the longest element in the set of minimum length representatives of cosets in W_P/W_R.

Proof.   Let J be the invariant complex structure defined on $G_\mathbb{C}/P$. For any non-negative integer k, we denote by $\overline{\mathcal{M}}_{A,k}(G_\mathbb{C}/P, J)$ the moduli space of equivalent classes of stable J-holomorphic curves of degree A with k-marked points.

Let $f : \overline{\mathcal{M}}_{A,1}(G_\mathbb{C}/P, J) \to \overline{\mathcal{M}}_{A,0}(G_\mathbb{C}/P, J)$ be the forgetful map that maps a class $[u; (\mathbb{CP}^1; z)]$ to $[u]$ and $\mathrm{ev}^1_J : \overline{\mathcal{M}}_{A,1}(G_\mathbb{C}/P, J) \to G_\mathbb{C}/P$ be the evaluation

map that maps a class $[u; (\mathbb{CP}^1; z)]$ to $u(z)$. We have a diagram of arrows

$$\overline{\mathcal{M}}_{A,1}(G_\mathbb{C}/P, J) \xrightarrow{f} \overline{\mathcal{M}}_{A,0}(G_\mathbb{C}/P, J)$$

$$\downarrow{\text{ev}_J^1}$$

$$G_\mathbb{C}/P$$

Note that the curve neighborhood $\Gamma_A(1 \cdot P)$ of the point $1 \cdot P \in G_\mathbb{C}/P$ is the same as

$$\Gamma_A(1 \cdot P) = \text{ev}_{J*}^1(f^*(f_*(\text{ev}_J^{1*}(1 \cdot P))))$$

Let $Q \subset G_\mathbb{C}$ be the parabolic subgroup with $S_Q = S - N(\alpha)$. The complex group $G_\mathbb{C}$ acts holomorphically on $G_\mathbb{C}/P$ and trivially on $H_*(G_\mathbb{C}/P; \mathbb{Z})$, as a consequence there is a group action of $G_\mathbb{C}$ on $\overline{\mathcal{M}}_{A,k}(G_\mathbb{C}/P, J)$. This action is transitive when $k = 0, 1$ and α is a long simple root. Under this action, the moduli spaces $\overline{\mathcal{M}}_{A,0}(G_\mathbb{C}/P, J)$, $\overline{\mathcal{M}}_{A,1}(G_\mathbb{C}/P, J)$ are isomorphic to the homogeneous spaces $G_\mathbb{C}/Q$, $G_\mathbb{C}/R$, respectively. Via these isomorphisms, the diagram of arrows shown above is compatible with the diagram

$$G_\mathbb{C}/R \xrightarrow{\pi_q} G_\mathbb{C}/Q$$

$$\downarrow{\pi_p}$$

$$G_\mathbb{C}/P$$

where π_p and π_q denote the projection quotient maps (see e.g. Manivel and Landsberg [31], Strickland [50]). Thus,

$$\Gamma_A(1 \cdot P) = \pi_{p*}(\pi_q^*(\pi_{q*}(\pi_p^*(1 \cdot P))))$$

From Lemma 9.5 in the appendix, we have that

$$\Gamma_A(1 \cdot P) = \pi_{p*}(\pi_q^*(X_Q(w_p^r))) = X_P(w_p^r s_\alpha),$$

and we are done. ∎

Now we show that when $P \subset G_\mathbb{C}$ is a maximal parabolic subgroup and A is the class that cyclic generates $H_2(G_\mathbb{C}/P; \mathbb{Z})$, the Gromov-Witten invariant $\text{GW}_{A,2}^{\text{alg}}(\text{PD}[1 \cdot P], \text{PD}[\Gamma_A(1 \cdot P)^{\text{op}}])$ is different from zero.

Theorem 7.2. *Let $P \subset G_\mathbb{C}$ be a maximal parabolic subgroup, A be the Schubert class that cyclic generates $H_2(G_\mathbb{C}/P; \mathbb{Z})$ and $\Gamma_A(1 \cdot P)$ be the degree A neighborhood of $1 \cdot P$. Then*

$$\text{GW}_{A,2}^{\text{alg}}(\text{PD}[1 \cdot P], \text{PD}[\Gamma_A(1 \cdot P)^{\text{op}}]) = 1$$

Proof.   According to Lemma 6.1, it is enough to check that the curve neighborhood $\Gamma_A(1 \cdot P)$ satisfies the dimensional constrain

$$c_1(A) = 1 + \dim_{\mathbb{C}}(\Gamma_A(1 \cdot P))$$

We split the proof in several cases:

- Long root case: Assume that $P \subset G_{\mathbb{C}}$ is a maximal parabolic subgroup associated with a long simple root $\alpha \in S$. Let $R \subset G_{\mathbb{C}}$ be the parabolic subgroup with $S_R = S - (N(\alpha) \cup \{\alpha\})$, where $N(\alpha) \subset S$ denotes the neighbors of α in the Dynkin diagram of G. By the previous theorem and Lemma 9.5 in the appendix,

$$\dim_{\mathbb{C}} \Gamma_A(1 \cdot P) = l(w_p^r s_\alpha),$$

  where w_p^r is the longest element in the set of minimum length representatives of cosets in W_P/W_R. We have that

$$l(w_p^r s_\alpha) = l(w_p^r) + 1 = \dim_{\mathbb{C}}(G_{\mathbb{C}}/R) - \dim_{\mathbb{C}}(G_{\mathbb{C}}/P) + 1$$
$$= \dim_{\mathbb{C}} \overline{\mathcal{M}}_{A,1}(G_{\mathbb{C}}/P) - \dim_{\mathbb{C}}(G_{\mathbb{C}}/P) + 1$$

  So in conclusion,

$$\dim_{\mathbb{C}} \Gamma_A(1 \cdot P)^{\mathrm{op}} = 2\dim_{\mathbb{C}} G_{\mathbb{C}}/P - \dim_{\mathbb{C}} \overline{\mathcal{M}}_{A,1}(G_{\mathbb{C}}/P) - 1$$
$$= 2\dim_{\mathbb{C}} G_{\mathbb{C}}/P - \dim_{\mathbb{C}} \overline{\mathcal{M}}_{A,2}(G_{\mathbb{C}}/P),$$

  and we are done

- Type B and D Grassmannians: For a positive integer m, we will write m as $2n$ if m is even, and as $2n+1$ if m is odd (here n is a non-negative integer number). Let $SO(m, \mathbb{C})$ be the group of special complex orthogonal matrices which preserves the standard nondegenerate symmetric bilinear form defined on $\mathbb{C}^m$. Let $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{R}^n$.

  The standard root system for the group $SO(2n+1, \mathbb{C})$ is identified with the set of vectors $R = \{\pm e_i, \pm(e_j \pm e_k) : j \neq k\}_{1 \leq i,j \leq n} \subset \mathbb{R}^n$ with a choice of simple roots given by $S = \{\alpha_1 = e_1 - e_2, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}$, and with Dynkin diagram shown in Figure 1
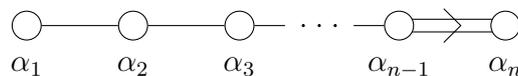

Figure 1: Dynkin Diagram B_n

  The standard root system for the group $SO(2n, \mathbb{C})$ is identified with the set of vectors $R = \{\pm(e_j \pm e_k) : j \neq k\}_{1 \leq i,j \leq n}$ with simple roots given by $S = \{\alpha_1 = e_1 - e_2, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$ and with Dynkin diagram shown in Figure 2

  Let $k \leq m/2$ be a positive integer. We denote by $OG(k, m)$ the *Orthogonal Grassmannian manifold* of k-dimensional isotropic subspaces in $\mathbb{C}^m$ with respect to the standard nondegenerate symmetric bilinear form defined on $\mathbb{C}^m$.
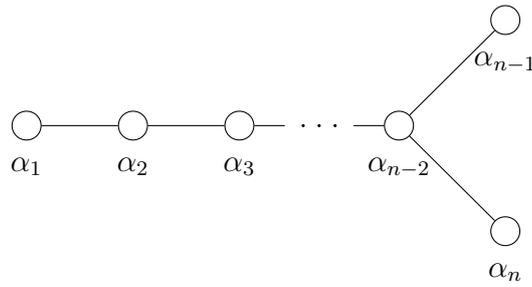
Figure 2: Dynkin Diagram D_n

When $k \neq m/2$, the group $SO(m, \mathbb{C})$ acts transitively on $OG(k, m)$ and the orthogonal Grassmannian $OG(k, m)$ is isomorphic to the quotient $SO(m, \mathbb{C})/P_{\alpha_k}$. The two orthogonal Grassmannians $OG(k, 2n)$ and $OG(k, 2n+1)$ are isomorphic.

When $k = m/2 = n$, $OG(n, 2n)$ is the union of two $SO(2n, \mathbb{C})$-orbits. These two $SO(2n, \mathbb{C})$-orbits are isomorphic to $SO(2n, \mathbb{C})/P_{\alpha_{n-1}}$ and $SO(2n, \mathbb{C})/P_{\alpha_n}$. The two $SO(2n, \mathbb{C})$-orbits of the orthogonal Grassmannian $OG(n, 2n)$ are isomorphic to the orthogonal Grassmannian $OG(n, 2n+1)$.

In summary, Grassmannian manifolds of type B can be identified with Grassmannian manifolds of type D. The statement in this case follows from the long root case.

- Short root case (type G): Let $G = G_2$ and $T \subset G$ be the maximal torus whose Lie algebra $\mathfrak{t}$ is identified with $\mathbb{R}^2$ and such that the set

$$S = \left\{ \alpha_1 = \left(-\frac{3}{2}, \frac{\sqrt{3}}{2} \right), \alpha_2 = (1, 0) \right\} \subset \mathfrak{t}^* \cong \mathbb{R}^2$$

defines a set of simple root systems for G with Dynkin diagram shown in Figure 3
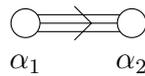

Figure 3: Dynkin Diagram G_2

Let us assume that $P \subset G_{\mathbb{C}}$ is the maximal parabolic subgroup associated with the short simple root $\alpha_2 \in S$. The homogeneous space $G_{\mathbb{C}}/P$ can be considered as a homogenous space of $SO(7, \mathbb{C})$: Note first that the complex dimension of $G_{\mathbb{C}}/P$ is equal to 5. Let $w_1 \in \mathfrak{t}^*$ be the fundamental weight associated with α_1. Let $L(w_1) = G_{\mathbb{C}} \times_P \mathbb{C}(-w_1)$ be the line bundle defined over $G_{\mathbb{C}}/P$ associated with the fundamental weight w_1. The irreducible representation $H^0(G_{\mathbb{C}}/P, L(w_1))$ has dimension 7 (this computation can be made by using for instance the Weyl dimensional formula). Thus, $G_{\mathbb{C}}/P$ is embedded in the 6 dimensional projective space $\mathbb{P}(H^0(G_{\mathbb{C}}/P, L(w_1))) \cong \mathbb{CP}^6$. Under this embedding, $G_{\mathbb{C}}/P$ is a G_2-homogenous hypersurface and thus a nondegenerate quadric. A quadric in $\mathbb{CP}^6$ is a complete homogeneous space for the special

orthogonal group $SO(7, \mathbb{C})$. The result in this case follows from the long root case for type B and D Grassmannians.

- Type C Grassmannians: Let $(\mathbb{C}^{2n}, \Omega)$ be the standard complex symplectic vector space with complex coordinates $(z_1, \cdots, z_n, w_1, \cdots, w_n)$ and with complex bilinear skew-symmetric form

$$\Omega = \sum_{i=1}^{n} dz_i \wedge dw_i$$

Let $Sp(n, \mathbb{C})$ be the complex Lie group of linear transformation on $\mathbb{C}^{2n}$ that preserves Ω. Let $\{e_1, \cdots, e_n\}$ denotes the standard basis of $\mathbb{R}^n$.

The standard root system of $Sp(n, \mathbb{C})$ is identified with the set $R = \{\pm e_i \pm e_j \ (i \neq j), \ \pm 2e_i\}_{1 \leq i,j \leq n} \subset \mathbb{R}^n$, with a choice of simple roots given by $S = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n\}$ and with Dynkin diagram shown in Figure 4
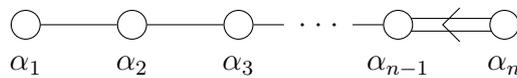

Figure 4: Dynkin Diagram C_n

For an integer $0 < k \leq n$, let $IG(k, 2n)$ denote the space of k-dimensional isotropic subspaces of $\mathbb{C}^{2n}$, i.e,

$$IG(k, 2n) := \{V^k \in G(k, 2n) : \Omega|_{V^k} = 0\}.$$

When $k = n$, the isotropic Grassmannian $IG(n, 2n)$ is the space of Lagrangian subspaces of $\mathbb{C}^{2n}$. The isotropic Grassmannian manifold $IG(k, 2n)$ has dimension equal to

$$2k(n - k) + \frac{k(k + 1)}{2}$$

and is isomorphic to the quotient of complex Lie groups $Sp(2n, \mathbb{C})/P_{\alpha_k}$, where $P_{\alpha_k} \subset Sp(n, \mathbb{C})$ denotes the maximal parabolic subgroup associated with the simple root $\alpha_k \in S$.

Given a J-holomorphic curve $u : \mathbb{CP}^1 \to IG(k, 2n)$ of degree $A := \sigma_{P_{\alpha_k}}(s_{\alpha_k})$, there exist linearly independent vectors $v_0, v_1, v_2, \cdots, v_k \in \mathbb{C}^{2n}$ such that

$$u : \mathbb{CP}^1 \to IG(k, 2n)$$
$$[z_0, z_1] \mapsto \mathrm{span}\{z_0 v_0 + z_1 v_1, v_2, \cdots, v_k\}$$

In particular, for every $[z_0, z_1] \in \mathbb{CP}^1$, the subspace

$$\mathrm{span}\{z_0 v_0 + z_1 v_1, v_2, \cdots, v_k\} \subset \mathbb{C}^{2n}$$

is isotropic. We can associate a pair (V^{k-1}, V^{k+1}) of vector subspaces to the curve u such that
$$V^{k-1} \subset V^{k+1} \subset (V^{k-1})^{\Omega} \subset \mathbb{C}^{2n}$$

Here, $V^{k-1} = \mathrm{span}\{v_2, \cdots, v_k\}$ and $V^{k+1} = \mathrm{span}\{v_0, v_1, v_2, \cdots, v_k\}$. The pair of vector subspaces (V^{k-1}, V^{k+1}) determines uniquely, up to reparametrization, the curve u. This means that if $v : \mathbb{CP}^1 \to IG(k, 2n)$ is another J-holomorphic curve of degree A such that for any $W^k \in v(\mathbb{CP}^1) \subset IG(k, 2n)$

$$V^{k-1} \subset W^k \subset V^{k+1},$$

then there exists $\varphi \in \mathrm{PSL}(2, \mathbb{C})$ such that $v \circ \varphi = u$. This implies that the moduli space $\overline{\mathcal{M}}_{A,0}(IG(k, 2n), J)$ of unparameterized J-holomorphic curves of degree A in $IG(k, 2n)$ can be identified with the set of pairs of subspaces

$$\{(V^{k-1}, V^{k+1}) : V^{k-1} \subset V^{k+1} \subset (V^{k-1})^{\Omega} \subset \mathbb{C}^{2n}\}$$

Note that two different isotropic subspaces $V_1^k, V_2^k \in IG(k, 2n)$ are joined by a J-holomorphic curve of degree A if

$$\dim_{\mathbb{C}}(V_1^k \cap V_2^k) = k - 1 \quad \text{and} \quad \dim_{\mathbb{C}}(V_1^k + V_2^k) = k + 1$$

In particular, the degree A neighborhood of the isotropic subspace $\mathbb{C}^k \subset \mathbb{C}^{2n}$ is given by

$$\Gamma_A(\mathbb{C}^k) = \{V^k \in IG(k, 2n) : k - 1 \leq \dim(\mathbb{C}^k \cap V^k) \leq \dim(\mathbb{C}^k + V^k) \leq k + 1\}$$

We can compute the dimension of $\Gamma_A(\mathbb{C}^k)$ if we consider the fibration

$$\Gamma_A(\mathbb{C}^k) - \{\mathbb{C}^k\} \to \{V^{k-1} : V^{k-1} \subset \mathbb{C}^k\} \cong G(k-1, k)$$
$$V^k \mapsto \mathbb{C}^k \cap V^k$$

The fiber of this fibration over any $(k-1)$-dimensional vector subspace $V^{k-1} \subset \mathbb{C}^k$ is the set

$$\{V^k \subset \mathbb{C}^{2n} : V^{k-1} \subset V^k \subset (V^{k-1})^{\Omega}\} - \{\mathbb{C}^k\}$$

of complex dimension $2n - 2k + 1$. Thus,

$$\dim_{\mathbb{C}} \Gamma_A(\mathbb{C}^k) = 2n - k$$

Likewise, the dimension of $\overline{\mathcal{M}}_{A,0}(IG(k, 2n))$ can be computed by considering the fibration

$$\overline{\mathcal{M}}_{A,0}(IG(k, 2n)) \to IG(k-1, 2n)$$
$$(V^{k-1}, V^{k+1}) \mapsto V^{k-1}$$

This fibration has fiber isomorphic to $G(2, 2n - 2k + 2)$, so that

$$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{A,0}(IG(k, 2n)) = \dim_{\mathbb{C}} IG(k-1, 2n) + \dim_{\mathbb{C}} G(2, 2n - 2k + 2)$$
$$= \dim_{\mathbb{C}} IG(k, 2n) - 2 + 2n - k,$$
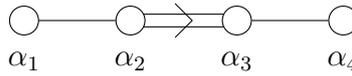
and we are done.

Figure 5: Dynkin Diagram F_4

- Short root case (Type F): Let G be a compact Lie group of type F_4 and $T \subset G$ be the maximal torus whose Lie algebra is identified with $\mathbb{R}^4$ and such that the set $S \subset \mathfrak{t}^* \cong \mathbb{R}^4$ defined by

$$\left\{ \alpha_1 = (0,1,-1,0), \alpha_2 = (0,0,1,-1), \alpha_3 = (0,0,0,1), \alpha_4 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right) \right\},$$

corresponds to the standard set of simple roots of G with Dynkin diagram shown in Figure 5

Let $P \subset G_{\mathbb{C}}$ be the maximal parabolic associated to a simple root $\alpha \in S$ and

$$Z_A^P := \{ s_\beta \cdot P \in W/W_P : \beta \in R^+ - R_P^+, \ \check{\beta} = \check{\alpha} + \mathbb{Z} \check{S}_P \}$$

be the set of T-fixed points in $\Gamma_A(1 \cdot P)$. According to Lemma 6.4, the set Z_A^P contains a unique maximal element z_A^P with respect to the Bruhat order and

$$\Gamma_A(1 \cdot P) = X_P(z_A^P)$$

Now we check that for the parabolic subgroup P associated with either the short simple root α_3 or α_4 we have that

$$c_1(A) = \dim_{\mathbb{C}}(\Gamma_A(1 \cdot P)) + 1 = l(z_A^P) + 1$$

1. Let P be the maximal parabolic subgroup associated with $\alpha = \alpha_4$. Figure 6 shows the minimum length representatives of cosets in Z_A^P ordered with respect to the Bruhat order:
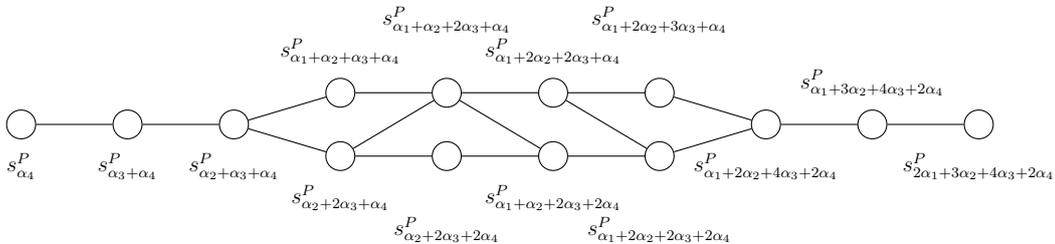

Figure 6: Poset $Z_A^P, P \longleftrightarrow \alpha_4, \dim_{\mathbb{C}} \Gamma_A(1 \cdot P) = 10$

The maximal element of Z_A^P is the coset $z_A^P := s_{2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4} \cdot P$. The minimum length representative of z_A^P in W^P is

$$s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}$$

with length equal to 10. Thus $\dim_{\mathbb{C}} \Gamma_A(1 \cdot P) = 10$. Using Proposition 5.4 we get

$$\begin{aligned} c_1(A) &= \langle c_1(T(G_{\mathbb{C}}/P)), \check{\alpha}_4 \rangle = \langle 11(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4), \check{\alpha}_4 \rangle \\ &= 11(1 \cdot 0 + 2 \cdot 0 + 3 \cdot (-1) + 2 \cdot 2) = 11 \\ &= \dim_{\mathbb{C}} \Gamma_A(1 \cdot P) + 1 \end{aligned}$$

2. Let P be the maximal parabolic subgroup associated with $\alpha = \alpha_3$. The Hasse diagram of minimum length representatives of cosets in Z_A^P ordered with respect to the Bruhat order is shown in Figure 7
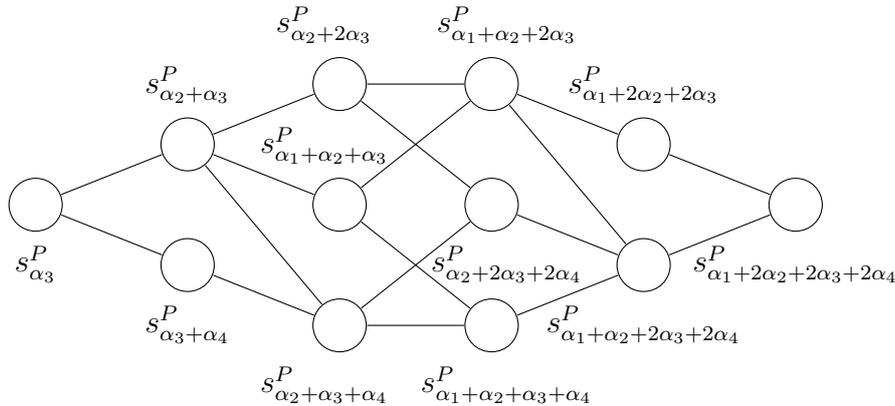

Figure 7: Poset $Z_A^P, P \leftrightsquigarrow \alpha_3, \dim_{\mathbb{C}} \Gamma_A(1 \cdot P) = 6$

The maximal element of Z_A^P is $z_A^P := s_{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4} \cdot P$. The minimum length representative of z_A^P in W^P is

$$s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}$$

with length equal to 6. Thus $\dim_{\mathbb{C}} \Gamma_A(1 \cdot P) = 6$. Using Proposition 5.4 we get

$$
\begin{aligned}
c_1(A) &= \langle c_1(T(G_{\mathbb{C}}/P)), \check{\alpha}_3 \rangle = \langle 7(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4), \check{\alpha}_3 \rangle \\
&= 7(2 \cdot 0 + 4 \cdot (-2) + 6 \cdot 2 + 3 \cdot (-1)) = 7 \\
&= \dim_{\mathbb{C}}(\Gamma_A(1 \cdot P)) + 1,
\end{aligned}
$$

and we are done.   ∎

Now we are ready to state our upper bound for the Gromov width of Grassmannian manifolds.

Theorem 7.3.   *Let G be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a maximal torus with a choice of simple roots $S \subset \mathfrak{t}^*$. For $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$, let $\mathcal{O}_\lambda$ be the coadjoint orbit passing through λ and ω_λ be the Kostant-Kirillov-Souriau form defined on $\mathcal{O}_\lambda$. Assume that there is a maximal parabolic subgroup $P \subset G_{\mathbb{C}}$ associated with a simple root $\alpha \in S$ such that $\mathcal{O}_\lambda \cong G_{\mathbb{C}}/P$, then*

$$\mathrm{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq |\langle \lambda, \check{\alpha} \rangle|$$

Proof.   Let J be the complex structure defined on $(\mathcal{O}_\lambda, \omega_\lambda)$ coming from the quotient of complex Lie groups $G_{\mathbb{C}}/P$. The complex structure J is regular (see e.g. McDuff and Salamon [40][Proposition 7.4.3]). Let A be the Schubert class that cyclic generates $H_2(G_{\mathbb{C}}/P; \mathbb{Z})$ and $\Gamma_A(1 \cdot P)$ be the degree A neighborhood of $1 \cdot P$.

The symplectic manifold $(\mathcal{O}_\lambda, \omega_\lambda)$ is semipositive because when $P \subset G_\mathbb{C}$ is a maximal parabolic subgroup, the homology group $H_2(G/P; \mathbb{Z})$ is cyclic and the numbers $\omega_\lambda(A), c_1(A)$ are both positive.

The previous theorem states that

$$\mathrm{GW}_{A,2}^{\mathrm{alg}}\big(\mathrm{PD}[1 \cdot P], \mathrm{PD}[\Gamma_A(1 \cdot P)^{\mathrm{op}}]\big) = \mathrm{GW}_{A,2}^{J}\big(\mathrm{PD}[1 \cdot P], \mathrm{PD}[\Gamma_A(1 \cdot P)^{\mathrm{op}}]\big) = 1$$

Thus, by Lemma 3.2

$$\mathrm{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq \omega_\lambda(A) = |\langle \lambda, \check{\alpha} \rangle| \qquad \blacksquare$$

8.   Upper bound for the Gromov width of coadjoint orbits of compact Lie groups

The problem of finding upper bounds for the Gromov width of coadjoint orbits of compact Lie groups has already been addressed by Masrour Zoghi in his Ph.D thesis [53] where he has considered the problem of determining the Gromov width of *regular* coadjoint orbits of compact Lie groups. In this section, we extend Zoghi's theorem to coadjoint orbits of compact Lie groups that are not necessarily regular. We estimate from above the Gromov width of arbitrary coadjoint orbits of compact Lie group by computing Gromov-Witten invariants on holomorphic fibrations whose fibers are isomorphic to Grassmannian manifolds.

Let G be a compact Lie group, $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}^*$ be the dual of this Lie algebra. Let $\lambda \in \mathfrak{g}^*$ and $\mathcal{O}_\lambda \subset \mathfrak{g}^*$ be the coadjoint orbit passing through λ. Let us assume that $\mathcal{O}_\lambda \cong G_\mathbb{C}/P$, where $P \subset G_\mathbb{C}$ is an arbitrary parabolic subgroup of $G_\mathbb{C}$. We endow $G_\mathbb{C}/P$ with a Kähler structure coming from its identification with $\mathcal{O}_\lambda$. This Kähler structure and the one defined on $\mathcal{O}_\lambda$ are denoted indistinguishably by (ω_λ, J).

Let $T \subset G$ be a maximal torus and let $B \subset G_\mathbb{C}$ be a Borel subgroup with $T_\mathbb{C} \subset B \subset P$. Let $W = N(T)/T$ be the associated Weyl group. Let R be the corresponding set of roots and S be the corresponding system of simple roots. Let W_P be the Weyl group of P and S_P be the subset of simple roots whose corresponding reflections are in W_P.

The second homology group $H_2(G_\mathbb{C}/P; \mathbb{Z})$ is freely generated as a $\mathbb{Z}$-module by the set of Schubert classes $\{[X_P(s_\alpha)]\}_{\alpha \in S - S_P}$. We denote the Schubert class $[X_P(s_\alpha)]$ by A_α. The symplectic area of A_α is equal to $\omega_\lambda(A_\alpha) = |\langle \lambda, \check{\alpha} \rangle|$.

Now we show that for every class A_α there is a non-vanishing Gromov-Witten invariant with one its constrains being Poincaré dual to the class of a point.

Theorem 8.1.   *Let $P \subset G_\mathbb{C}$ be a parabolic subgroup, $A_\alpha \in H_2(G_\mathbb{C}/P, \mathbb{Z})$ be the spherical class associated with a simple root $\alpha \in S - S_P$ and $\Gamma_{A_\alpha}(1 \cdot P)$ be the degree A_α neighborhood of $1 \cdot P$. Then*

$$\mathrm{GW}_{A_\alpha,2}^{\mathrm{alg}}\big(\mathrm{PD}[1 \cdot P], \mathrm{PD}[\Gamma_{A_\alpha}(1 \cdot P)^{\mathrm{op}}]\big) = 1$$

Proof.   We show that

$$c_1(T(G_\mathbb{C}/P))(A_\alpha) = 1 + \dim_\mathbb{C}(\Gamma_{A_\alpha}(1 \cdot P)),$$

and the result will follow from Lemma 6.1.

For the simple root $\alpha \in S - S_P$, we have a parabolic subgroup $Q \supset P$ with $S_Q = S_P \sqcup \{\alpha\}$ and a fibration

$$\pi_\alpha : G_\mathbb{C}/P \to G_\mathbb{C}/Q$$

The fibration π_α is holomorphic with respect to the complex structures defined on the quotients of complex Lie groups $G_\mathbb{C}/P$ and $G_\mathbb{C}/Q$.

The fiber Q/P can be identified with the quotient of a simple complex Lie group and a maximal parabolic subgroup. We also have an inclusion map

$$\iota_\alpha : Q/P \hookrightarrow G_\mathbb{C}/P.$$

Let A be the Schubert class that cyclic generates $H_2(Q/P; \mathbb{Z})$. Note that $\iota_{\alpha*}(A) = A_\alpha \in H_2(G_\mathbb{C}/P; \mathbb{Z})$ and $\pi_{\alpha*}(A_\alpha) = 0 \in H_2(G_\mathbb{C}/Q; \mathbb{Z})$.

We have an exact sequence of vector bundles over Q/P

$$0 \to T(Q/P) \xrightarrow{d\iota_\alpha} T(G_\mathbb{C}/P)|_{Q/P} \xrightarrow{d\pi_\alpha} Q/P \times \mathfrak{g}_\mathbb{C}/\mathfrak{q} \to 0$$

Thus,

$$c_1(T(G_\mathbb{C}/P)|_{Q/P}) = c_1(T(Q/P)) + c_1(Q/P \times \mathfrak{g}_\mathbb{C}/\mathfrak{q}) = c_1(T(Q/P))$$

The projection formula of Chern classes implies that

$$\begin{aligned}
c_1(T(G_\mathbb{C}/P)|_{Q/P})(A) &= c_1(\iota_\alpha^*(T(G_\mathbb{C}/P)))(A) \\
&= c_1(T(G_\mathbb{C}/P))(\iota_{\alpha*}(A)) \\
&= c_1(T(G_\mathbb{C}/P))(A_\alpha)
\end{aligned}$$

and we get

$$c_1(T(Q/P))(A) = c_1(T(G_\mathbb{C}/P))(A_\alpha)$$

Now we describe the degree A_α neighborhood $\Gamma_{A_\alpha}(1 \cdot P)$ of $1 \cdot P \in G_\mathbb{C}/P$. Let $u : \mathbb{CP}^1 \to G_\mathbb{C}/P$ be a J-holomorphic map of degree A_α. The map

$$\pi_\alpha \circ u : \mathbb{CP}^1 \to G_\mathbb{C}/Q$$

is holomorphic and its degree is equal to $(\pi_\alpha \circ u)_*[\mathbb{CP}^1] = (\pi_\alpha)_*[A_\alpha] = 0 \in H_2(G_\mathbb{C}/Q; \mathbb{Z})$. Therefore, the map $\pi \circ u : \mathbb{CP}^1 \to G_\mathbb{C}/Q$ is constant and the image of $u : \mathbb{CP}^1 \to G_\mathbb{C}/P$ is totally contained in a fiber of $\pi_\alpha : G_\mathbb{C}/P \to G_\mathbb{C}/Q$. If the curve u passes through $1 \cdot P \in G_\mathbb{C}/P$, we can identify u with a curve of degree A in the fiber Q/P. As a consequence, the degree A_α neighborhood $\Gamma_{A_\alpha}(1 \cdot P)$ of $1 \cdot P$ in $G_\mathbb{C}/P$ can be identified with the degree A neighborhood $\Gamma_A(1 \cdot P)$ of $1 \cdot P$ in Q/P, and in particular they share the same dimension. By the proof of Theorem 7.2,

$$c_1(T(Q/P))(A) = 1 + \dim_\mathbb{C}(\Gamma_A(1 \cdot P)),$$

and thus

$$c_1(T(G_\mathbb{C}/P))(A_\alpha) = 1 + \dim_\mathbb{C}(\Gamma_{A_\alpha}(1 \cdot P)),$$

and we are done.   ∎

The result that follows is the main theorem of this paper and gives an upper bound for the Gromov width of coadjoint orbits of compact Lie groups that are not necessarily regular.

Theorem 8.2. *Let G be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a maximal torus and let $\check{R} \subset \mathfrak{t}$ be the corresponding system of coroots. We identify the dual Lie algebra $\mathfrak{t}^*$ with the fixed points of the coadjoint action of T on $\mathfrak{g}^*$. Let $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$, $\mathcal{O}_\lambda$ be the coadjoint orbit passing through λ and ω_λ be the Kostant–Kirillov–Souriau form defined on $\mathcal{O}_\lambda$, then*

$$\mathrm{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq \min_{\substack{\check{\alpha} \in \check{R} \\ \langle \lambda, \check{\alpha} \rangle \neq 0}} |\langle \lambda, \check{\alpha} \rangle|$$

Proof. Let $P \subset G_\mathbb{C}$ be a parabolic subgroup such that $\mathcal{O}_\lambda \cong G_\mathbb{C}/P$. We establish the following convention: for every $\tilde{\lambda} \in \mathfrak{t}^*$ such that the coadjoint orbit $\mathcal{O}_{\tilde{\lambda}}$ is isomorphic with $G_\mathbb{C}/P$, we are going to see the Kostant-Kirillov-Souriau form $\omega_{\tilde{\lambda}}$ defined on $\mathcal{O}_{\tilde{\lambda}}$ as a symplectic form on $G_\mathbb{C}/P$.

Let $\{\lambda_n\}_{n \in \mathbb{Z}_{>0}} \subset \mathfrak{t}^*$ be a sequence that converges to λ. Assume that for every $n \in \mathbb{Z}_{>0}$, the coadjoint orbit $\mathcal{O}_{\lambda_n}$ is isomorphic with $G_\mathbb{C}/P$ and the Kostant-Kirillov-Souriau form ω_{λ_n} represents a cohomology class $[\omega_{\lambda_n}]$ with rational coefficients.

Let $\tilde{J} \in \mathcal{J}(G_\mathbb{C}/P, \omega_\lambda)$. Let $\tilde{J}_n \in \mathcal{J}(G_\mathbb{C}/P, \omega_{\lambda_n})$ be a sequence that converges to $\tilde{J}$ in the C^∞-topology.

For every $n \in \mathbb{Z}_{>0}$, the symplectic manifold $(G_\mathbb{C}/P, \omega_{\lambda_n})$ meets all the requirements of Theorem 4.5: the almost complex structure J coming from the quotient of complex Lie groups $G_\mathbb{C}/P$ is regular and compatible with the Kostant-Kirillov-Souriau form ω_{λ_n}. For any simple root $\alpha \in S - S_P$, the homology class $A_\alpha \in H_2(G_\mathbb{C}/P; \mathbb{Z})$ is J-indecomposable and according to the previous theorem there exists a cohomology class $a \in H^*(G_\mathbb{C}/P; \mathbb{Z})$ such that

$$\mathrm{GW}_{A_\alpha, 2}^J \big(\mathrm{PD}[1 \cdot P], a \big) \neq 0$$

Thus by Theorem 4.5, for every point $p \in G_\mathbb{C}/P$, we can find a $\tilde{J}_n$-holomorphic sphere $u_n : \mathbb{CP}^1 \to G_\mathbb{C}/P$ of degree $B_n \in H_2(G_\mathbb{C}/P; \mathbb{Z})$ passing through p with $0 < \omega_{\lambda_n}(B_n) \leq \omega_{\lambda_n}(A_\alpha)$. The Gromov compactness theorem implies that there exists a $\tilde{J}$-holomorphic curve of degree B passing through p with $0 < \omega_\lambda(B) \leq \omega_\lambda(A_\alpha)$. By Theorem 2.1,

$$\mathrm{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq \omega_\lambda(A_\alpha) = |\langle \lambda, \check{\alpha} \rangle|$$

The above inequality holds for any $\alpha \in S - S_P$, and as consequence for any $\alpha \in R^+ - R_P^+$, and we are done. ∎

9. Appendix: Fibrations

In this section we show the technical lemma involving fibrations of homogeneous spaces that is used in the proof of Theorem 7.1. Let G be a compact simple Lie group. Let $T \subset G$ be a maximal torus, $B \subset G_\mathbb{C}$ be a Borel subgroup with

$T_{\mathbb{C}} \subset B$ and S be the corresponding system of simple roots. Let $W = N(T)/T$ be the associated Weyl group. For a parabolic subgroup $P \subset G_{\mathbb{C}}, B \subset P$, let $W_P = N_P(T)/T$ be the Weyl group of P and S_P be the subset of simple roots of S whose corresponding reflections are in W_P. Let $W^P \subset W$ be the set of all minimum length representatives for cosets in W/W_P. For $w \in W^P$, let $X_P(w) \subset G_{\mathbb{C}}/P$ be the Schubert variety associated with $w \in W^P$.

For a pair of parabolic subgroups $P, Q \subset G_{\mathbb{C}}$, such that $B \subset P \subset Q$, we have a quotient map $G_{\mathbb{C}}/P \to G_{\mathbb{C}}/Q$. We want to study the images and preimages of Schubert varieties under these quotient maps.

Lemma 9.1 (*Stumbo [51]*). *For parabolic subgroups $P, Q \subset G_{\mathbb{C}}$ such that $B \subset P \subset Q$ define*

$$W_Q^P := \{w \in W_Q : l(ws) > l(w) \text{ for } s \in S_P\}$$
$$= \text{ minimum length representatives of cosets in } W_Q/W_P$$

Given $w \in W^P$, there is a unique $w^Q \in W^Q$ and a unique $w_Q^P \in W_Q^P$ such that $w = w^Q w_Q^P$. Their lengths satisfy $l(w) = l(w^Q) + l(w_Q^P)$. In other words, the map

$$W^Q \times W_Q^P \to W^P$$
$$(w^Q, w_Q^P) \mapsto w^Q w_Q^P$$

is a length preserving bijection

Lemma 9.2. *For parabolic subgroups $P, Q \subset G_{\mathbb{C}}$ such that $B \subset P \subset Q$, let w_q^p, w_p and w_q be the longest elements in W_Q^P, W_P and W_Q, respectively. Then, $w_q^p = w_q w_p$.*

Proof. Let w_0 be the longest element in W. The quotient map $\pi : W \to W^P \cong W/W_P$ is order preserving and thus the longest element in W^P is $\pi(w_0)$. By the previous lemma $w_0 = \pi(w_0)w_p$, so that $\pi(w_0) = w_0 w_p^{-1}$. Similarly, for the quotient map $\pi' : W \to W^Q \cong W/W_Q$, we have that $\pi'(w_0) = w_0 w_q^{-1}$ is the longest element in W^Q. Using again the previous lemma, we have that the permutation $\pi'(w_0)w_q^p$ is the longest permutation in W^P. So that $\pi'(w_0)w_q^p = \pi(w_0)$, and thus $w_0 w_q^{-1} w_q^p = w_0 w_p^{-1}$ or $w_q^p = w_q w_p^{-1}$. The longest element in any finite Coxeter group is idempotent. Thus $w_p^2 = e$, and $w_q^p = w_q w_p$. $\blacksquare$

Proposition 9.3. *For parabolic subgroups $P, Q \subset G_{\mathbb{C}}$ such that $P \subset Q \subset G_{\mathbb{C}}$, let*

$$\pi_q : G_{\mathbb{C}}/P \to G_{\mathbb{C}}/Q$$

be the quotient fibration. If we decompose $w \in W^P$ as $w^Q w_Q^P$, where $w^Q \in W^Q$ and $w_Q^P \in W_Q^P$, then $\pi_{q*}(X_P(w)) = X_Q(w^Q)$. On the other hand, if $\tilde{w} \in W^Q$, then $\pi_q^*(X_Q(\tilde{w})) = X_P(\tilde{w}w_q^p)$, where w_q^p is the longest element in W_Q^P.

Proof. The map $\pi_q : G_{\mathbb{C}}/P \to G_{\mathbb{C}}/Q$ is B-equivariant and closed (this is a consequence of for example the closed map lemma). This implies that Schubert

cells, which are B-orbits, and Schubert varieties, which are their closures, in $G_\mathbb{C}/P$ are mapped to Schubert cells and Schubert varieties in $G_\mathbb{C}/Q$, respectively.

For $w \in W^P \subset W$, there exist unique $w^Q \in W^Q$ and $w_Q^P \in W_Q^P \subset W_Q$ such that $w = w^Q w_Q^P$ and $l(w) = l(w^Q) + l(w_Q^P)$. The Schubert cell $C_P(w) = BwP/P \subset G_\mathbb{C}/P$ is mapped to the Schubert cell $C_Q(w^Q) = Bw^Q Q/Q \subset G_\mathbb{C}/Q$ via π, and

$$\pi_{q*}(X_P(w)) = \pi_{q*}(\overline{C_P(w)}) = \overline{\pi_{q*}(C_P(w))} = \overline{C_Q(w^Q)} = X_Q(w^Q).$$

On the other hand, if $\tilde{w} \in W^Q$, then

$$\pi_q^*(C_Q(\tilde{w})) = \bigsqcup_{\substack{v \in W^P \\ v^Q = \tilde{w}}} C_P(v).$$

The maximum element, with respect to the Bruhat order defined on W^P, in the set $\{v \in W^P : v^Q = \tilde{w}\}$ is $\tilde{w}w_q^p$, where w_q^p denotes the longest element in W_Q^P. Since π is a continuous map, we have that

$$\pi_q^*(X_Q(\tilde{w})) = \pi_q^*(\overline{C_Q(\tilde{w})}) = \overline{\bigsqcup_{\substack{v \in W^P \\ v^Q = \tilde{w}}} C_P(v)} = \bigsqcup_{\substack{v \in W^P \\ v \leq_B \tilde{w}w_q^p}} C_P(v) = X_P(\tilde{w}w_q^p). \qquad \blacksquare$$

The following two technical lemmas are needed it in the proof of Theorem 7.3:

Lemma 9.4. *Let $\alpha \in S$ be a simple root and $N(\alpha) \subset S$ be the neighbors of α in the Dynkin diagram of G, i.e., the simple roots connected to α by an edge in the Dynkin diagram of G. Let $P, R \subset G_\mathbb{C}$ be the parabolic subgroups such that $S_P = S - \{\alpha\}, S_R = S - (N(\alpha) \cup \{\alpha\})$, respectively. Then*

$$W_P^R \cdot s_\alpha \subset W^P$$

Proof. Let $w \in W_P^R$. We write $w = s_1 \cdot \ldots \cdot s_r$ where $s_1, \ldots, s_r$ are simple reflections in S_P. Suppose that there exists a simple reflection t in S_P such that $l(ws_\alpha t) < l(ws_\alpha)$. By the Exchange Principle (see e.g. Humphreys [26]),

$$ws_\alpha t = s_1 \cdot \ldots \cdot \hat{s}_i \cdot \ldots \cdot s_r s_\alpha$$

for some i, in particular $s_\alpha t s_\alpha \in W_P$. We now consider two cases and see that this is not possible:

1. Suppose that $s_\alpha t = t s_\alpha$. Thus $t \notin N(\alpha)$, but $t \neq s_\alpha$ so $t \in S - (N(\alpha) \cup \{s_\alpha\}) = S_R$. As $w \in W_P^R$
$$l(wt) > l(w),$$

   hence
$$l(ws_\alpha t) = l(wts_\alpha) = l(wt) + 1 > l(w) + 1 = l(ws_\alpha),$$

   which contradicts our asumption of having $l(ws_\alpha t) < l(ws_\alpha)$.

2. Suppose that $s_\alpha t \neq t s_\alpha$. If $l(s_\alpha t s_\alpha) \neq 3$, by the Deletion Principle (see e.g. Humphreys [26]) either $s_\alpha t s_\alpha = s_\alpha$, or $s_\alpha t s_\alpha = t$, which are not possible. So $l(s_\alpha t s_\alpha) = 3$. Now, clearly $l(s_\alpha t) = l(s_\alpha t s_\alpha s_\alpha) = 2 < l(s_\alpha t s_\alpha)$, so if $s_\alpha t s_\alpha = s_1 s_2 s_3$, for some simple reflections $s_1, s_2, s_3 \in S_P$, by the Exchange Principle $s_\alpha t \in W_P$ which would imply that $s_\alpha \in W_P$, a contradiction. ∎

Lemma 9.5. *Let $P, Q, R \subset G_\mathbb{C}$ be the parabolic subgroups with $S_P = S - \{\alpha\}, S_Q = S - N(\alpha)$ and $S_R = S - (N(\alpha) \cup \{\alpha\})$, where $N(\alpha) \subset S$ denotes the neighbors of α in the Dynkin diagram of G. Let $\pi_p : G_\mathbb{C}/R \to G_\mathbb{C}/P$ and $\pi_q : G_\mathbb{C}/R \to G_\mathbb{C}/Q$ be the corresponding quotient maps. Then*

$$\pi_{q*}(\pi_p^*(1 \cdot P)) = X_Q(w_p w_r)$$

where w_p and w_r are the longest elements in W_P and W_R, respectively. In addition,

$$\pi_{p*}(\pi_q^*(X_Q(w_p w_r))) = X_P(w_p w_r s_\alpha),$$

where s_α is the simple reflection associated to $\alpha \in S$.

Proof. By the Proposition 9.3, we have that $\pi_p^*(X_P(e)) = X_R(w_p^r) = X_R(w_p w_r)$, so

$$\pi_{q*}(\pi_p^*(1 \cdot P) = \pi_{q*}(X_R(w_p^r))$$

Note that $W_P^R \subset W^Q$. In particular $w_p^r \in W^Q$, and thus

$$\pi_{q*}(\pi_p^*(1 \cdot P)) = X_Q(w_p^r)$$

Finally, Lemma 9.4 implies that $w_p^r s_\alpha \in W^P$, thus

$$\pi_{p*}(\pi_q^*(X_Q(w_p^r))) = \pi_{p*}(X_R(w_p^r s_\alpha)) = X_P(w_p^r s_\alpha),$$

and we are done. ∎

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