

Algebraic Dirac Induction for Nonholomorphic Discrete Series of $SU(2, 1)$

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Abstract. In a joint paper P. Pandžić and D. Renard proved that holomorphic and antiholomorphic discrete series representations can be constructed via algebraic Dirac induction. The group $SU(2, 1)$, except for those two types, also has a third type of discrete series representations that are neither holomorphic nor antiholomorphic. In this paper we show that nonholomorphic discrete series representations of the group $SU(2, 1)$ can also be constructed using algebraic Dirac induction.

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1. Introduction

Dirac operators were introduced into representation theory by Parthasarathy in [P] as a tool for constructing discrete series representations. An algebraic version of the Dirac operator was studied by D. Vogan [V].

Let G be a connected real reductive Lie group with Cartan involution Θ such that $K = G^\Theta$ is a maximal compact subgroup of G and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the complexified Lie algebra of G corresponding to Θ . The Dirac operator D is an element of the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, where $U(\mathfrak{g})$ denotes the universal enveloping algebra of \mathfrak{g} and $C(\mathfrak{p})$ denotes the Clifford algebra of \mathfrak{p} with respect to the Killing form. For a (\mathfrak{g}, K) -module X Vogan defines Dirac cohomology as

$$H_V^D(X) = \text{Ker}D / \text{Im}D \cap \text{Ker}D.$$

It is a \tilde{K} -module, where \tilde{K} is the spin double cover of K . If X is unitary, then $H_V^D(X) = \text{Ker}D = \text{Ker}D^2$.

The main result about Dirac cohomology was conjectured by Vogan [V], and proved by Huang and Pandžić in [HP]. Roughly speaking, it asserts that the Dirac cohomology, if nonzero, determines the infinitesimal character of the

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representation. By now, Dirac cohomology has been calculated for many (unitary) modules (see [DH], [HKP], [HPP], [BP1], [BP2]).

In [PR] the authors describe certain constructions in the opposite direction which give representations with prescribed Dirac cohomology. As the name fails to indicate, Dirac cohomology, as defined by Vogan, is not a cohomological theory. In particular, it is a functor which admits no adjoint. In [PR], two alternative definitions were proposed, which both coincide with the Vogan's definition for unitary and finite dimensional representations. They are called Dirac cohomology and homology. The functor of the Dirac cohomology is left exact and admits a right adjoint, while the functor of the Dirac homology is right exact and admits a left adjoint. These adjoints are called Dirac induction functors. One gets a representation with the Dirac cohomology W by tensoring the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ with W over a certain subalgebra. There are several options for the choice of the algebra over which we tensor. A minimal option is the subalgebra of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ generated by the diagonal version of the Lie algebra of the group K and by the Dirac operator. Modules obtained in this way are typically not irreducible. A maximal option is to include in the subalgebra all K -invariants in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. A problem with this approach is that, in general, the algebra $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ is complicated to describe. Therefore, an "intermediate" option is also introduced, where we tensor over the subalgebra that does not include all K -invariants but only K -invariants in the algebra $C(\mathfrak{p})$ which is easy to describe. In [PR], the representations of holomorphic and antiholomorphic discrete series are constructed in this way.

The group $SU(2, 1)$, except for holomorphic and antiholomorphic discrete series representations, also has a third type of discrete series which are neither highest weight modules nor lowest weight modules, the nonholomorphic discrete series representations. The irreducible unitary representations of the group $SU(n, 1)$, $n \geq 2$ which are not principal series representations are considered in [K2]. Each of those representations is uniquely determined by its minimal K -type.

The algebra $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ for the group $SU(2, 1)$ is generated by five elements, see [Pr]. Two of them are in the center $Z(\mathfrak{k})$ of $U(\mathfrak{k})$ – the Casimir element and the element spanning the center of \mathfrak{k} . One of the generators is in another abelian algebra, $C(\mathfrak{p})^K$ (which is three-dimensional in this case). The fourth generator is the Dirac operator, and the fifth generator is another distinguished element that can be thought of as a \mathfrak{k} -version of the Dirac operator. We will prove that the nonholomorphic discrete series representations can be obtained using algebraic Dirac induction where we tensor over the algebra that is generated by the Lie algebra of the group K and by the whole algebra $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$.

The paper is organized as follows. In Section 2, we give a brief review of the main definitions and results of [PR]. In Section 3, using results of [K2], we describe nonholomorphic discrete series representation of $SU(2, 1)$ in terms of the highest weight vectors of their minimal K -types. In Section 4, we describe the K -module structure of the algebra $U(\mathfrak{g})$. Using this and the action of the algebra $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ on the Dirac cohomology, we will reduce the obvious set of generators for the induced module to the set of generators which will later be shown to be a basis for the induced module. Finally, in Section 5 we prove that

nonholomorphic discrete series can be obtained via algebraic Dirac induction.

In future, we hope to generalize our results to nonholomorphic discrete series representations of the group $SU(n, 1)$ and to some other more complicated examples.

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2. Dirac induction

Let \mathfrak{g}_0 be the Lie algebra of G and let B be an invariant nondegenerate symmetric bilinear form on \mathfrak{g}_0 . The Dirac operator $D = U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is given by

$$D = \sum_i b_i \otimes d_i,$$

where b_i is a basis of \mathfrak{p} and d_i is the dual basis with respect to B . The operator D is independent of the choice of basis b_i and K -invariant for the adjoint action on both factors. The adjoint action of \mathfrak{k} on \mathfrak{p} defines a map $\text{ad} : \mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{p})$. Composing it with the usual embedding of the Lie algebra $\mathfrak{so}(\mathfrak{p})$ into the Clifford algebra $C(\mathfrak{p})$, we get a Lie algebra map $\alpha : \mathfrak{k} \rightarrow C(\mathfrak{p})$. Using α we can embed the Lie algebra \mathfrak{k} diagonally into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, by

$$X \mapsto X_\Delta = X \otimes 1 + 1 \otimes \alpha(X).$$

We will denote $\Delta(\mathfrak{k})$ by \mathfrak{k}_Δ . Let us denote $\mathcal{A} = U(\mathfrak{g}) \otimes C(\mathfrak{p})$. Let \mathcal{I} be the two-sided ideal in the algebra of K -invariants \mathcal{A}^K generated by D and let \mathcal{B} be the K -invariant subalgebra of \mathcal{A} with unit, generated by \mathfrak{k}_Δ and \mathcal{I} . Let us recall the definitions of the Dirac cohomology and homology from [PR].

Definition 2.1. Let X be a (\mathfrak{g}, K) -module. Dirac cohomology of X is the space of \mathcal{I} -invariants in $X \otimes S$. Therefore,

$$H^D(X) = \{v \in X \otimes S \mid av = 0, \forall a \in \mathcal{I}\}.$$

Dirac homology of X is the space of \mathcal{I} -coinvariants in $X \otimes S$

$$H_D(X) = X \otimes S / \mathcal{I}(X \otimes S).$$

In the most interesting cases when the module X is finite-dimensional or unitary, $H_D(X)$, $H^D(X)$ and $H_V^D(X)$ all coincide and they are all equal to $\text{Ker}(D) = \text{Ker}(D^2)$.

The functor of Dirac cohomology $H^D : \mathcal{M}(\mathcal{A}, \tilde{K}) \rightarrow \mathcal{M}(\mathfrak{k}_\Delta, \tilde{K})$ has a left adjoint, the functor

$$\text{Ind}_D : W \mapsto \mathcal{A} \otimes_{\mathcal{B}} W.$$

The functor of Dirac homology $H_D : \mathcal{M}(\mathcal{A}, \tilde{K}) \rightarrow \mathcal{M}(\mathfrak{k}_\Delta, \tilde{K})$ has a right adjoint, the functor

$$\text{Ind}^D : W \mapsto \text{Hom}_{\mathcal{B}}(\mathcal{A}, W)_{\tilde{K}\text{-finite}}.$$

Furthermore, $H^D(X)$ and $H_D(X)$ consist of full \tilde{K} -isotypic components of $X \otimes S$, which does not seem to be true for $H_V^D(X)$.

Modules we get by the first version of induction are typically not irreducible. Therefore, we would like to tensor over a “bigger” algebra \mathcal{B} . The algebra of K -invariants, \mathcal{A}^K , acts on each \tilde{K} -isotypic component of $X \otimes S$. Furthermore, from Theorem [PR, Theorem 4.10.] follows that the action of the algebra \mathcal{A}^K on any nontrivial \tilde{K} -isotypic component of $X \otimes S$ determines the irreducible (\mathcal{A}, \tilde{K}) -module $X \otimes S$ up to isomorphism. This is a version of a theorem of Harish-Chandra (see [HC], [LMC]).

Now we can consider H^D and H_D as functors from the category of (\mathcal{A}, \tilde{K}) -modules to the category of $(U(\mathfrak{k}_\Delta)\mathcal{A}^K, \tilde{K})$ -modules on which \mathcal{I} acts by zero. Then the functor $\widetilde{\text{Ind}}_D = \mathcal{A} \otimes_{U(\mathfrak{k}_\Delta)\mathcal{A}^K} \cdot$ is left adjoint to the functor H^D , while the functor $\widetilde{\text{Ind}}^D = \text{Hom}_{U(\mathfrak{k}_\Delta)\mathcal{A}^K}(\mathcal{A}, \cdot)_{\tilde{K}\text{-finite}}$ is right adjoint to the functor H_D . The problem with this approach is that the algebra \mathcal{A}^K contains $U(\mathfrak{g})^K$ which is in general very hard to describe. The same can be expected for the algebra \mathcal{A}^K .

Fortunately, the algebra of K -invariants contains the algebra $C(\mathfrak{p})^K$ which is very easy to describe. It can be easily seen (see [PR]) that $C(\mathfrak{p})^K \simeq \text{End}_{\tilde{K}}(S)$ which is spanned by the projections onto \tilde{K} -types in S . Moreover, on each irreducible $C(\mathfrak{p})^K$ -module exactly one of the projections acts by 1 and all other act by 0.

Therefore, the third approach is to consider an “intermediate” version of induction where we tensor over the algebra $U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I})$.

Let W be a $U(\mathfrak{k}_\Delta)$ -module. Let $U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I})$ act on W so that \mathcal{I} acts by zero, one of the projections p_1, p_2, \dots, p_m acts by 1, and the rest of them by zero. Then we define

$$\begin{aligned} \widetilde{\text{Ind}}_D(W) &= \mathcal{A} \otimes_{U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I})} W \\ \widetilde{\text{Ind}}^D(W) &= \text{Hom}_{U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I})}(\mathcal{A}, W)_{\tilde{K}\text{-finite}}. \end{aligned}$$

The functor $\widetilde{\text{Ind}}_D = \mathcal{A} \otimes_{U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I})} \cdot$ is left adjoint to the functor H^D considered as a functor from the category of (\mathcal{A}, \tilde{K}) -modules to the category of $(U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I}), \tilde{K})$ -modules on which \mathcal{I} acts by zero, while the functor $\widetilde{\text{Ind}}^D = \text{Hom}_{U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I})}(\mathcal{A}, \cdot)_{\tilde{K}\text{-finite}}$ is right adjoint to the functor H_D .

All holomorphic discrete series representations can be constructed via the intermediate version of Dirac induction. Let us assume that the pair (\mathfrak{g}, K) is Hermitian so we have a K -invariant decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, where \mathfrak{p}^+ and \mathfrak{p}^- are abelian subalgebras of \mathfrak{p} . We fix a choice of positive roots for \mathfrak{k} and add the roots corresponding to \mathfrak{p}^+ to obtain a positive root system for \mathfrak{g} . We can choose the spin module to be $S = \bigwedge \mathfrak{p}^+$. Let us denote by $p_1 \in C(\mathfrak{p})^K$ the projection of the spin module S to its \tilde{K} -submodule $\mathbb{C}1$ and by $p_2, \dots, p_n \in C(\mathfrak{p})^K$ the projections to other \tilde{K} -types of the spin module. Let W_1 be an irreducible K -module. Let X be the (\mathfrak{g}, K) -module $X = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} W_1$, where $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^-$ is a maximal parabolic subalgebra of \mathfrak{g} which acts on W_1 so that \mathfrak{p}^- acts by 0. The action of \mathfrak{g} on X is given by the left multiplication in the first factor, while the action of K is the adjoint action in the first factor tensored by the given action

on W_1 . Let us denote by $W = W_1 \otimes 1$ and by $\tilde{\mathcal{B}} = U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I})$. Then W is a $\tilde{\mathcal{B}}$ -module, with \mathcal{I} acting by 0, p_1 by 1 and p_2, \dots, p_n by 0. We consider the case when X belongs to the holomorphic discrete series. Now we have the following result proved by Pandžić and Renard in [PR]:

Theorem 2.2. *Let $X = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} W_1$ be a holomorphic discrete series representation, and let $W = W_1 \otimes 1$. Then the (\mathcal{A}, \tilde{K}) -module $\widetilde{\text{Ind}}_D(W) = \mathcal{A} \otimes_{\tilde{\mathcal{B}}} W$ is isomorphic to $X \otimes S$.*

Remark 2.3. The Dirac cohomology of X is equal to W .

3. Nonholomorphic discrete series representations of the group $SU(2, 1)$

We will denote by G the Lie group

$$SU(2, 1) = \{g \in GL(3, \mathbb{C}) \mid \det g = 1, g^* \Gamma g = \Gamma\},$$

where $\Gamma = \text{diag}(1, 1, -1)$. Its Cartan involution is given by $\Theta(g) = (g^{-1})^*$ and the corresponding maximal compact subgroup is

$$K = \{g \in SU(2, 1) \mid \Theta(g) = g\} = \left\{ \begin{bmatrix} A & \\ & (\det(A))^{-1} \end{bmatrix} \mid A \in U(2) \right\} \cong U(2).$$

The Lie algebra \mathfrak{g}_0 of the group $SU(2, 1)$ is given by

$$\mathfrak{su}(2, 1) = \{x \in \mathfrak{gl}(3, \mathbb{C}) \mid \text{tr}(X) = 0, x^* \Gamma + \Gamma x = 0\}.$$

Its complexification is $\mathfrak{g} \cong \mathfrak{sl}(3, \mathbb{C})$. The complexified Lie algebra \mathfrak{k} of the Lie algebra of K is isomorphic to the Lie algebra $\mathfrak{gl}(2, \mathbb{C})$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra \mathfrak{g} corresponding to the Cartan involution $\theta(x) = -x^*$. Then

$$\mathfrak{k} = \text{span}_{\mathbb{C}}\{H_1, H_2, E, F\}, \quad \mathfrak{p} = \text{span}_{\mathbb{C}}\{E_1, E_2, F_1, F_2\},$$

where $H_1 = e_{11} - e_{33}$, $H_2 = e_{22} - e_{33}$, $E = e_{12}$, $F = e_{21}$, $E_i = e_{i3}$, $F_i = e_{3i}$ for $i = 1, 2$. Here e_{ij} denotes the matrix in \mathfrak{g} with ij entry equal to one and all other entries equal to 0.

The Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 (and \mathfrak{k}_0) consists of diagonal matrices in \mathfrak{g}_0 . We identify \mathfrak{h}^* with the set

$$\mathbb{C}^3 = \{s \in \mathbb{C}^3 \mid s_1 + s_2 + s_3 = 0\},$$

where $s \in \mathbb{C}^3$ corresponds to the functional on \mathfrak{h} given by $H_j \mapsto s_j - s_3$, $j = 1, 2$. The root system of $(\mathfrak{g}, \mathfrak{h})$ is

$$R = \{\epsilon_j - \epsilon_k \mid 1 \leq j, k \leq 3, j \neq k\},$$

where ϵ_i denotes the functional $h \mapsto h_i$ on \mathfrak{h} . Furthermore, the sets of compact and noncompact roots are given by

$$R_K = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_1\}, \quad R_P = \{\epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_1, \epsilon_3 - \epsilon_2\}.$$

The Weyl group is the group of permutations of coordinates, i.e. $W(R) = S_3$. We choose the R_K -Weyl chamber to be $C = \{s \in \mathbb{R}_0^3 \mid s_1 > s_2\}$. The corresponding set of positive compact roots is given by $R_K^C = \{\epsilon_1 - \epsilon_2\}$. There are three R -Weyl chambers contained in the R_K -Weyl chamber C , and they are:

$$\begin{aligned} D_0 &= \{s \in \mathbb{R}_0^3 \mid s_3 > s_1 > s_2\} \\ D_1 &= \{s \in \mathbb{R}_0^3 \mid s_1 > s_3 > s_2\} \\ D_2 &= \{s \in \mathbb{R}_0^3 \mid s_1 > s_2 > s_3\}. \end{aligned}$$

The corresponding noncompact positive roots are given by

$$\begin{aligned} R_P^{D_0} &= R^{D_0} \setminus R_K^C = \{\epsilon_3 - \epsilon_1, \epsilon_3 - \epsilon_2\} \\ R_P^{D_1} &= R^{D_1} \setminus R_K^C = \{\epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_2\} \\ R_P^{D_2} &= R^{D_2} \setminus R_K^C = \{\epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3\}. \end{aligned}$$

Let \hat{K} denote the set of equivalence classes of irreducible representations of K . In [K2] \hat{K} is identified with

$$\{(q_1, q_2, -q_1 - q_2) \mid q_1, q_2 \in \frac{1}{3}\mathbb{Z}, q_1 - q_2 \in \mathbb{Z}_+\}$$

For each $p \in \hat{K}$ there is a unique irreducible unitary representation of G which is not a principal series representation with minimal K -type p . We will denote that representation by π_p . It is also proved that the representation π_p belongs to a discrete series representation if and only if $p + \rho_K - \rho_P^{D_j} \in D_j$, for some $j \in \{0, 1, 2\}$. Here ρ_K is the half sum of compact positive roots and $\rho_P^{D_j}$ is the half sum of noncompact positive roots $R_P^{D_j}$, $j \in \{0, 1, 2\}$. If $p + \rho_K - \rho_P^{D_2} \in D_2$, then π_p is a lowest weight module i.e., belongs to the holomorphic discrete series. If $p + \rho_K - \rho_P^{D_0} \in D_0$, then π_p is a highest weight module i.e., belongs to the anti-holomorphic discrete series. In the case when $p + \rho_K - \rho_P^{D_1} \in D_1$, π_p is neither a highest nor a lowest weight module and its K -spectrum is given by

$$\Gamma(\pi_p) = \{q \in \hat{K} \mid q_1 \in \{p_1, p_1 + 1, p_1 + 2, \dots\}, q_2 \in \{p_2, p_2 - 1, p_2 - 2, \dots\}\}.$$

Let us denote $h_1 = \frac{2H_1 - H_2}{3}$ and $h_2 = \frac{2H_2 - H_1}{3}$. For $(q_1, q_2) \in (\frac{1}{3}\mathbb{Z})^2$ and $q_1 - q_2 \in \mathbb{Z}_+$, the irreducible representation of the group K with the highest weight (q_1, q_2) is the representation with the basis $\{v_{q_1, q_2}^{q_1 - q_2 - 2s} \mid s \in \{0, 1, \dots, q_1 - q_2\}\}$ and the following action of \mathfrak{k} :

$$\begin{aligned} h_1 v_{q_1, q_2}^{q_1 - q_2 - 2s} &= (q_1 - s) v_{q_1, q_2}^{q_1 - q_2 - 2s}, \quad \text{for } s \in \{0, 1, \dots, q_1 - q_2\} \\ h_2 v_{q_1, q_2}^{q_1 - q_2 - 2s} &= (q_2 + s) v_{q_1, q_2}^{q_1 - q_2 - 2s}, \quad \text{for } s \in \{0, 1, \dots, q_1 - q_2\} \\ E v_{q_1, q_2}^{q_1 - q_2 - 2s} &= s \cdot (q_1 - q_2 - s + 1) v_{q_1, q_2}^{q_1 - q_2 - 2(s-1)}, \quad \text{for } s \in \{1, \dots, q_1 - q_2\} \\ F v_{q_1, q_2}^{q_1 - q_2 - 2s} &= v_{q_1, q_2}^{q_1 - q_2 - 2(s+1)}, \quad \text{for } s \in \{0, 1, \dots, q_1 - q_2 - 1\} \\ E v_{q_1, q_2}^{q_1 - q_2} &= 0 \\ F v_{q_1, q_2}^{-q_1 + q_2} &= 0. \end{aligned} \tag{1}$$

Furthermore, if π_p is a nonholomorphic discrete series representation, then it follows from [K1] and [K2] that there exists a basis

$$\{v_{q_1, q_2}^{q_1 - q_2 - 2s} \mid (q_1, q_2) \in \Gamma(\pi_p), s \in \{0, 1, \dots, q_1 - q_2\}\}$$

such that

$$\begin{aligned} E_1 v_{q_1, q_2}^{q_1 - q_2} &= A_1^{\pi_p}(q) v_{q_1 + 1, q_2}^{(q_1 + 1) - q_2} \\ F_2 v_{q_1, q_2}^{q_1 - q_2} &= B_2^{\pi_p}(q) v_{q_1, q_2 - 1}^{q_1 - (q_2 - 1)}, \end{aligned} \tag{2}$$

where $A_1^{\pi_p}(q) \neq 0$ and $B_2^{\pi_p}(q) \neq 0$ for each $q = (q_1, q_2) \in \Gamma(\pi_p)$. From this, it easily follows

Theorem 3.1. *If π_p is nonholomorphic discrete series representation with representation space X , and if*

$$\{v_{q_1, q_2}^{q_1 - q_2 - 2s} \mid (q_1, q_2) \in \Gamma(\pi_p), s \in \{0, 1, \dots, q_1 - q_2\}\}$$

is a basis for X as in (2), then

$$\{F^t E_1^n F_2^m v_{p_1, p_2}^{p_1 - p_2} \mid n, m \in \mathbb{N}_0, t \in \{0, 1, \dots, (p_1 + n) - (p_2 - m)\}\}.$$

is a basis for X .

4. Generating system for the induced module

We will consider reduced version of the algebraic Dirac induction where we tensor over the algebra \mathcal{B} generated by the algebra \mathfrak{k}_Δ and the algebra \mathcal{A}^K . It follows from [Pr] that the algebra \mathcal{A}^K is generated by $Z(\mathfrak{k})$, $C(\mathfrak{p})^K$, by the Dirac operator and by the element

$$\begin{aligned} D^\mathfrak{k} &= E \otimes \alpha(F) + \frac{1}{2}(h_1 - h_2) \otimes \alpha(h_1 - h_2) + \frac{3}{2}(h_1 + h_2) \otimes \alpha(h_1 + h_2) + F \otimes \alpha(E) \\ &= -\frac{1}{4}(2E \otimes E_2 F_1 + (h_1 - h_2) \otimes (E_1 F_1 - E_2 F_2) \\ &\quad + 3(h_1 + h_2) \otimes (E_1 F_1 + E_2 F_2) + 2F \otimes E_1 F_2) - \frac{3}{2}(h_1 + h_2) \otimes 1. \end{aligned} \tag{3}$$

Remark 4.1. Since the basis $(E, F, h_1 - h_2, h_1 + h_2)$ of \mathfrak{k} is dual to the basis $(F, E, \frac{1}{2}(h_1 - h_2), \frac{3}{2}(h_1 + h_2))$, the element (3) can be thought of as a \mathfrak{k} -version of the Dirac operator. We will call it the \mathfrak{k} -Dirac.

We will give one generating system for (\mathcal{A}, \tilde{K}) -module $\text{Ind}_D(W) = \mathcal{A} \otimes_{\mathcal{B}} W$, where $W = H^D(X)$.

One can easily check that

$$W = \text{span}_{\mathbb{C}} \{v_{p_1, p_2}^{p_1 - p_2 - 2s} \otimes E_1 - (p_1 - p_2 - (s - 1)) v_{p_1, p_2}^{p_1 - p_2 - 2(s - 1)} \otimes E_2 \mid s \in \{1, 2, \dots, p_1 - p_2\}\}.$$

Let us denote

$$w_s = v_{p_1, p_2}^{p_1 - p_2 - 2s} \otimes E_1 - (p_1 - p_2 - (s - 1)) v_{p_1, p_2}^{p_1 - p_2 - 2(s - 1)} \otimes E_2, \quad s \in \{1, 2, \dots, p_1 - p_2\}.$$

Then we have

$$\begin{aligned}
 h_{1\Delta} w_s &= (p_1 - s + \frac{1}{2}) w_s \\
 h_{2\Delta} w_s &= (p_2 + s - \frac{1}{2}) w_s \\
 E_{\Delta} w_s &= (s - 1)(p_1 - p_2 - (s - 1)) w_{s-1} \\
 F_{\Delta} w_s &= w_{s+1}.
 \end{aligned} \tag{4}$$

Let

$$Z = \text{span}_{\mathbb{C}}\{ab \otimes w - a \otimes bw \mid a \in \mathcal{A}, b \in \mathcal{B}, w \in W\}.$$

We are going to reduce the obvious generating system for $\mathcal{A} \otimes_{\mathcal{B}} W$ given by

$$\{a \otimes w + Z \mid a \in \mathcal{A}, w \in W\}.$$

The first step is to “remove” $U(\mathfrak{k})$.

For $x \in \mathfrak{k}$, $y \in C(\mathfrak{p})$ and $w \in W$ we have

$$\begin{aligned}
 (x \otimes y) \otimes w &= ((1 \otimes y) \cdot (x_{\Delta} - 1 \otimes \alpha(x))) \otimes w = \\
 &= ((1 \otimes y) \cdot x_{\Delta}) \otimes w - (1 \otimes y \cdot \alpha(x)) \otimes w \\
 &= \underbrace{((1 \otimes y) \cdot x_{\Delta}) \otimes w - (1 \otimes y) \otimes x_{\Delta} w}_{\in Z} + (1 \otimes y) \otimes x_{\Delta} w \\
 &\quad - (1 \otimes y \cdot \alpha(x)) \otimes w \\
 &\in (1 \otimes C(\mathfrak{p})) \otimes W + Z.
 \end{aligned}$$

It follows:

$$(U(\mathfrak{k}) \otimes C(\mathfrak{p})) \otimes W \subset (1 \otimes C(\mathfrak{p})) \otimes W + Z. \tag{5}$$

Now, we are going to describe the structure of the algebra $U(\mathfrak{g})$.

Proposition 4.2. *We have*

$$\begin{aligned}
 U(\mathfrak{g}) &= \text{span}_{\mathbb{C}}\{x(E_1 F_1 + E_2 F_2)^t y \mid x \in \{E_1^n F_2^m, (\text{ad}F)(E_1^n F_2^m), \dots, \\
 &\quad (\text{ad}F)^{n+m}(E_1^n F_2^m) \mid n, m \in \mathbb{N}_0\}, t \in \mathbb{N}_0, y \in U(\mathfrak{k})\}.
 \end{aligned}$$

Proof. Let us introduce the following notation

$$\begin{aligned}
 T &= \text{span}_{\mathbb{C}}\{x(E_1 F_1 + E_2 F_2)^t y \mid x \in \{E_1^n F_2^m, (\text{ad}F)(E_1^n F_2^m), \dots, \\
 &\quad (\text{ad}F)^{n+m}(E_1^n F_2^m) \mid n, m \in \mathbb{N}_0\}, t \in \mathbb{N}_0, y \in U(\mathfrak{k})\}
 \end{aligned}$$

We will show by induction that $U_l(\mathfrak{g}) \subset T$ for all $l \in \mathbb{N}_0$. The claim is obvious for $l = 0$ and $l = 1$. Let us assume that the claim is true for $l \in \mathbb{N}_0$ less than or equal to some fixed $k \geq 1$. By Poincare-Birkhoff-Witt’s theorem, it is enough to prove that elements of the form $E_1^p E_2^q F_1^r F_2^s$, $p, q, r, s \in \mathbb{N}_0, p + q + r + s = k + 1$ are in T . From [Pr, Lemma 2.3.] it follows that

$$S^n(\mathfrak{p}) = \left(\sum_{i=0}^n \oplus V_{(n-i, -i)} \right) \oplus (E_1 F_1 + E_2 F_2) S^{n-2}(\mathfrak{p}),$$

for all $n \in \mathbb{N}, n \geq 2$, where $V_{(n-i, -i)}$ is the irreducible \mathfrak{k} -module with the highest weight vector $E_1^{n-i}F_2^i$ and the highest weight $(n-i, -i)$. Now we have

$$E_1^p E_2^q F_1^r F_2^s \in \text{span}_{\mathbb{C}}\{E_1^{k+1-i}F_2^i, (\text{ad}F)(E_1^{k+1-i}F_2^i), \dots, (\text{ad}F)^{k+1}(E_1^{k+1-i}F_2^i) \mid 0 \leq i \leq k+1\} \oplus \text{span}_{\mathbb{C}}\{E_1^{a+1}E_2^b F_1^{c+1}F_2^d + E_1^a E_2^{b+1} F_1^c F_2^{d+1} \mid a, b, c, d \in \mathbb{N}_0, a+b+c+d = k-1\}.$$

Therefore, it is enough to show that $E_1^{a+1}E_2^b F_1^{c+1}F_2^d + E_1^a E_2^{b+1} F_1^c F_2^{d+1} \in T$ for $a, b, c, d \in \mathbb{N}_0, a+b+c+d = k-1$. By induction on c and d it can be easily seen that

$$E_1^{a+1}E_2^b F_1^{c+1}F_2^d + E_1^a E_2^{b+1} F_1^c F_2^{d+1} \in E_1^a E_2^b F_1^c F_2^d (E_1 F_1 + E_2 F_2) + U_k(\mathfrak{g}) \subset U_{k-1}(\mathfrak{g})(E_1 F_1 + E_2 F_2) + U_k(\mathfrak{g}).$$

Since $E_1 F_1 + E_2 F_2$ commutes with \mathfrak{k} , the proposition follows. ■

Let us denote $C = E_1 \otimes F_1 + E_2 \otimes F_2$ and $C^- = F_1 \otimes E_1 + F_2 \otimes E_2$.

Lemma 4.3. *The elements C and C^- are contained in the ideal \mathcal{I} .*

Proof. We have $D = C + C^-$. Furthermore, it can easily be seen that

$$D(-\frac{1}{2} \otimes (E_1 F_1 + E_2 F_2)) - (-\frac{1}{2} \otimes (E_1 F_1 + E_2 F_2))D = C - C^-.$$

Since $-\frac{1}{2} \otimes (E_1 F_1 + E_2 F_2) \in \mathcal{A}^K$, we have

$$C = \frac{1}{2}(D + D(-\frac{1}{2} \otimes (E_1 F_1 + E_2 F_2)) - (-\frac{1}{2} \otimes (E_1 F_1 + E_2 F_2))D) \in \mathcal{I} \\ C^- = D - C \in \mathcal{I}. \quad \blacksquare$$

Since the ideal I acts on W by zero, we get $CC^- \otimes W \subset Z$ and $C^-C \otimes W \subset Z$. Furthermore, we have

$$(E_1 F_1 + E_2 F_2) \otimes 1 = -\frac{1}{4}(2CC^- + 2C^-C + 3(h_1 + h_2) \otimes (E_1 F_1 + E_2 F_2) + 2E \otimes E_2 F_1 + (h_1 - h_2) \otimes (E_1 F_1 - E_2 F_2) + 2F \otimes E_1 F_2) \quad (6)$$

from where it follows that

$$((E_1 F_1 + E_2 F_2) \otimes 1) \otimes w \in (U(\mathfrak{k}) \otimes C(\mathfrak{p})) \otimes W + Z.$$

By (5), Proposition 4.2 and (6) we see that the vector space $\mathcal{A} \otimes_{\mathcal{B}} W$ is spanned by the elements of the form

$$(x \otimes y) \otimes w, \quad x \in V_{(n, -m)}, y \in C(\mathfrak{p}), w \in W, n, m \in \mathbb{N}_0. \quad (7)$$

Remark 4.4. From (6), (3), and from the identity

$$\begin{aligned} (h_1 + h_2) \otimes 1 &= (h_1 + h_2)_\Delta - 1 \otimes \alpha(h_1 + h_2) \\ &= (h_1 + h_2)_\Delta - 1 \otimes \left(-\frac{1}{2}(E_1F_1 + E_2F_2) - 1\right) \in \mathcal{B}, \end{aligned}$$

it follows that $(E_1F_1 + E_2F_2) \otimes 1 \in \mathcal{B}$.

The next step is to reduce a part of the algebra $C(\mathfrak{p})$.

Lemma 4.5. *If $T \in \text{End}_{\mathbb{C}}(S) \cong C(\mathfrak{p})$ is any linear operator such that*

$$T(E_1) = T(E_2) = 0,$$

then $T = Tp'$, where $p' = p_1 + p_3$ is the sum of the projections on $\text{span}_{\mathbb{C}}\{1\}$, respectively $\text{span}_{\mathbb{C}}\{E_1 \wedge E_2\}$.

Proof. For $s \in \text{span}_{\mathbb{C}}\{1, E_1 \wedge E_2\}$ we have $p'(s) = s$ and then $Tp'(s) = T(s)$. Furthermore, $T(E_1) = T(E_2) = 0 = Tp'(E_1) = Tp'(E_2)$. Since $S = \text{span}_{\mathbb{C}}\{1, E_1, E_2, E_1 \wedge E_2\}$ it follows that $T = Tp'$. ■

In the spin module S , the following identities hold

$$\begin{aligned} E_1E_2 \cdot E_1 &= E_1 \cdot (-E_1 \wedge E_2) = 0, & F_1F_2 \cdot E_1 &= F_1 \cdot 0 = 0, \\ E_1E_2 \cdot E_2 &= E_1 \cdot 0 = 0, & F_1F_2 \cdot E_2 &= F_1 \cdot (-2) = 0. \end{aligned}$$

By Lemma 4.5, we have $E_1E_2p' = E_1E_2$ and $F_1F_2p' = F_1F_2$. Since $1 \otimes p_1$ and $1 \otimes p_3$ act on W by zero, we have $(1 \otimes p')w = 0$ for $w \in W$. For $a \in \mathcal{A}$ and $s \in \text{span}_{\mathbb{C}}\{E_1E_2, F_1F_2\}$ we get

$$\begin{aligned} (a(1 \otimes s)) \otimes w &= (a(1 \otimes sp')) \otimes w = (a(1 \otimes s)(1 \otimes p')) \otimes w \\ &= (a(1 \otimes s)(1 \otimes p')) \otimes w - \underbrace{(a(1 \otimes s)) \otimes (1 \otimes p')w}_0 \in Z. \end{aligned}$$

From here it follows that

$$(U(\mathfrak{g}) \otimes C(\mathfrak{p})\mathfrak{p}^-\mathfrak{p}^-) \otimes W \subset Z \text{ and } (U(\mathfrak{g}) \otimes C(\mathfrak{p})\mathfrak{p}^+\mathfrak{p}^+) \otimes W \subset Z. \tag{8}$$

Furthermore, since the projection on the two-dimensional component in the spin module S is equal to $p_2 = -\frac{1}{2}(E_1F_1 + E_2F_2)$ and since $1 \otimes p_2$ acts on W by one, then the element $1 \otimes (E_1F_1 + E_2F_2) \in \mathcal{B}$ acts on W by -2 .

Therefore, for $a \in U(\mathfrak{g})$ and $w \in W$ we have

$$\begin{aligned} (a \otimes 1) \otimes w &= -\frac{1}{2}(a \otimes 1) \otimes (-2w) = -\frac{1}{2}(a \otimes 1) \otimes ((1 \otimes (E_1F_1 + E_2F_2)) \cdot w) \\ &= \frac{1}{2} \underbrace{((a \otimes 1)(1 \otimes (E_1F_1 + E_2F_2)) \otimes w - (a \otimes 1) \otimes ((1 \otimes (E_1F_1 + E_2F_2)) \cdot w))}_{\in Z} \\ &\quad - \frac{1}{2}(a \otimes (E_1F_1 + E_2F_2)) \otimes w \\ &\in -\frac{1}{2}(a \otimes E_1F_1) \otimes w - \frac{1}{2}(a \otimes E_2F_2) \otimes w + Z. \end{aligned} \tag{9}$$

From (7), (8) and (9) it follows that the vector space $\mathcal{A} \otimes_{\mathcal{B}} W$ is spanned by the set

$$\{(x \otimes y) \otimes w \mid x \in \{E_1^n F_2^m, (\text{ad}F)(E_1^n F_2^m), \dots, (\text{ad}F)^{n+m}(E_1^n F_2^m) \mid n, m \in \mathbb{N}_0\}, \\ y \in \{E_1, E_2, F_1, F_2, E_1 F_1, E_1 F_2, E_2 F_1, E_2 F_2\}, w \in W\}. \quad (10)$$

Furthermore, one can easily check that

$$(F_1 \otimes E_1 + F_2 \otimes E_2)(1 \otimes F_1 F_2) - (1 \otimes F_1 F_2)(F_1 \otimes E_1 + F_2 \otimes E_2) = -2(F_1 \otimes F_2 - F_2 \otimes F_1)$$

and

$$(E_1 \otimes F_1 + E_2 \otimes F_2)(1 \otimes E_1 E_2) - (1 \otimes E_1 E_2)(E_1 \otimes F_1 + E_2 \otimes F_2) = -2(E_1 \otimes E_2 - E_2 \otimes E_1).$$

From here, (8) and $C \otimes W \subset Z$, $C^- \otimes W \subset Z$ we get

$$(E_1 \otimes E_2 - E_2 \otimes E_1) \otimes w \in Z \quad \text{and} \quad (F_1 \otimes F_2 - F_2 \otimes F_1) \otimes w \in Z. \quad (11)$$

A straightforward calculation shows that the element $(E_1 F_1 + E_2 F_2) \otimes 1$ acts on W by $(p_1 + 2p_2 - 1)$.

Proposition 4.6. *For $s \in \{1, 2, \dots, p_1 - p_2\}$ we have*

$$(1 \otimes F_1) \otimes (p_2 - p_1 + s)w_s \in (1 \otimes F_2) \otimes F_{\Delta} w_s + Z.$$

Proof. For $w \in W$ we have

$$F_{\Delta}(1 \otimes F_2) \otimes w \in (1 \otimes F_2) \otimes F_{\Delta} w - (1 \otimes F_1) \otimes w + Z. \quad (12)$$

Furthermore, using (8) we get

$$F_{\Delta}(1 \otimes F_2) \otimes w \in (F \otimes F_2) \otimes w + Z. \quad (13)$$

From (11), $C \otimes W \subset Z$ and $[E_1, F_1] = F$ we get

$$(F \otimes F_2) \otimes w \in ((E_2 F_2 + F_1 E_1) \otimes F_1) \otimes w + Z \\ = ((E_1 F_1 + E_2 F_2) \otimes F_1) \otimes w + ((F_1 E_1 - E_1 F_1) \otimes F_1) \otimes w + Z \\ = ((E_1 F_1 + E_2 F_2) \otimes F_1) \otimes w - (H_1 \otimes F_1) \otimes w + Z. \quad (14)$$

Since $H_1 = 2h_1 + h_2$, $\alpha(h_1) = -\frac{1}{2} - \frac{1}{2}E_1 F_1$ and $\alpha(h_2) = -\frac{1}{2} - \frac{1}{2}E_2 F_2$, we have $\alpha(H_1) = -E_1 F_1 - \frac{1}{2}E_2 F_2 - \frac{3}{2}$. Using (8) we get

$$(H_1 \otimes F_1) \otimes w = ((1 \otimes F_1)(H_1 \otimes 1 + 1 \otimes \alpha(H_1))) \otimes w - (1 \otimes F_1 \alpha(H_1)) \otimes w \\ \in (1 \otimes F_1) \otimes H_{1\Delta} w - \frac{1}{2}(1 \otimes F_1) \otimes w + Z. \quad (15)$$

Now from (12), (13), (14) and (15) it follows that

$$(1 \otimes F_1) \otimes w \in (1 \otimes F_2) \otimes F_{\Delta} w - F_{\Delta}(1 \otimes F_2) \otimes w + Z \\ = (1 \otimes F_2) \otimes F_{\Delta} w - ((E_1 F_1 + E_2 F_2) \otimes F_1) \otimes w \\ + (1 \otimes F_1) \otimes H_{1\Delta} w - \frac{1}{2}(1 \otimes F_1) \otimes w + Z.$$

From here we get

$$\begin{aligned}
 & (1 \otimes F_1) \otimes \left(\frac{3}{2}w - H_{1_\Delta}w\right) \in (1 \otimes F_2) \otimes F_\Delta w - ((E_1F_1 + E_2F_2) \otimes F_1) \otimes w + Z \\
 & = (1 \otimes F_2) \otimes F_\Delta w - (1 \otimes F_1) \otimes ((E_1F_1 + E_2F_2) \otimes 1)w \\
 & \quad - \underbrace{(((E_1F_1 + E_2F_2) \otimes F_1) \otimes w - (1 \otimes F_1) \otimes ((E_1F_1 + E_2F_2) \otimes 1)w)}_{\in Z} + Z. \quad (16)
 \end{aligned}$$

Since the element $(E_1F_1 + E_2F_2) \otimes 1 \in \mathcal{B}$ acts on W by the scalar $(p_1 + 2p_2 - 1)$ and $H_{1_\Delta}w_s = (2h_{1_\Delta} + h_{2_\Delta})w_s = (2p_1 + p_2 - s + \frac{1}{2})w_s$ (see (4)), the (16) implies

$$(1 \otimes F_1) \otimes (p_2 - p_1 + s)w_s \in (1 \otimes F_2) \otimes F_\Delta w_s + Z \text{ for } s \in \{1, 2, \dots, p_1 - p_2\}.$$

This finishes the proof. ■

Corollary 4.7. *For $s \in \{1, 2, \dots, p_1 - p_2\}$ we have*

$$\begin{aligned}
 & (1 \otimes E_1F_1) \otimes (p_2 - p_1 + s)w_s \in (1 \otimes E_1F_2) \otimes F_\Delta w_s + Z \\
 & (1 \otimes E_2F_1) \otimes (p_2 - p_1 + s)w_s \in (1 \otimes E_2F_2) \otimes F_\Delta w_s + Z \\
 & (1 \otimes E_2) \otimes (p_2 - p_1 + s)w_s \in -(1 \otimes E_1) \otimes F_\Delta w_s + Z
 \end{aligned}$$

Proof. The first two claims follow from the previous lemma. Furthermore, we have

$$(1 \otimes E_1E_2F_1) \otimes (p_2 - p_1 + s)w_s \in (1 \otimes E_1E_2F_2) \otimes F_\Delta w_s + Z.$$

The proof follows from $E_1E_2F_1 = -E_1F_1E_2 = -(-2 - F_1E_1)E_2 = 2E_2 + F_1E_1E_2$, $E_1E_2F_2 = E_1(-2 - F_2E_2) = -2E_1 - E_1F_2E_2 = -2E_1 + F_2E_1E_2$ and $(U(\mathfrak{g}) \otimes C(\mathfrak{p})\mathfrak{p}^+\mathfrak{p}^+) \otimes W \subset Z$. ■

From Lemma 4.6, Corollary 4.7 and from the previous conclusions it follows that the vector space $\mathcal{A} \otimes_{\mathcal{B}} W$ is spanned by the set

$$\begin{aligned}
 & \{(x \otimes F_2) \otimes w_s, (x \otimes E_1F_2) \otimes w_s, (x \otimes E_2F_2) \otimes w_s, (x \otimes E_1) \otimes w_s, \\
 & \quad (x \otimes F_1) \otimes w_{p_1-p_2}, (x \otimes E_1F_1) \otimes w_{p_1-p_2}, \\
 & \quad (x \otimes E_2F_1) \otimes w_{p_1-p_2}, (x \otimes E_2) \otimes w_{p_1-p_2} \mid s = 1, 2, \dots, p_1 - p_2, \\
 & \quad x \in \{E_1^n F_2^m, (\text{ad}F)E_1^n F_2^m, \dots, (\text{ad}F)^{n+m}E_1^n F_2^m \mid n, m \in \mathbb{N}_0\}\}.
 \end{aligned}$$

So far, we have reduced a big part of the Clifford algebra. Similarly, using the well known (and easy to prove) fact

$$D(1 \otimes X) + (1 \otimes X)D = -2X \otimes 1, \quad \text{for } X \in \mathfrak{p}$$

we can reduce a big part of the algebra $U(\mathfrak{g})$.

Corollary 4.8. *For $s \in \{1, 2, \dots, p_1 - p_2\}$ we have*

$$\begin{aligned}
 & (F_1 \otimes 1) \otimes (p_2 - p_1 + s)w_s \in (F_2 \otimes 1) \otimes F_\Delta w_s + Z \\
 & (E_2 \otimes 1) \otimes (p_2 - p_1 + s)w_s \in -(E_1 \otimes 1) \otimes F_\Delta w_s + Z.
 \end{aligned}$$

Proof. Since $D(1 \otimes F_i) + (1 \otimes F_i)D = -2F_i \otimes 1$, for $i = 1, 2$, we have

$$\begin{aligned} & (F_i \otimes 1) \otimes w = -\frac{1}{2}(D(1 \otimes F_i) + (1 \otimes F_i)D) \otimes w \\ = & -\frac{1}{2}(D(1 \otimes F_i)) \otimes w - \frac{1}{2} \underbrace{((1 \otimes F_i)D \otimes w - (1 \otimes F_i) \otimes Dw)}_{\in Z} - \frac{1}{2}(1 \otimes F_i) \otimes \underbrace{Dw}_{=0} \\ \in & -\frac{1}{2}D(1 \otimes F_i) \otimes w + Z. \end{aligned}$$

From Lemma 4.6 we get

$$(p_2 - p_1 + s)(F_1 \otimes 1) \otimes w_s \in (F_2 \otimes 1) \otimes F_\Delta w_s + Z.$$

Similarly, using $D(1 \otimes E_i) + (1 \otimes E_i)D = -2E_i \otimes 1$ for $i = 1, 2$ and corollary 4.7 we get

$$(p_2 - p_1 + s)(E_2 \otimes 1) \otimes w_s \in -(E_1 \otimes 1) \otimes F_\Delta w_s + Z.$$

■

One can easily show by induction that for each $n \in \mathbb{N}$ it holds

$$\begin{aligned} E_2 F_2^n &= n F_2^{n-1} H_2 - n(n-1) F_2^{n-1} + F_2^n E_2 \\ F_2^n E_1 &= E_1 F_2^n - n F_2^{n-1} E_1. \end{aligned} \tag{17}$$

Roughly speaking, the following proposition shows that we can replace the second highest weight vector of the K -type $V_{(n,-m)} \in U(\mathfrak{g})$ with the highest weight vectors of the K -types $V_{(n,-m)} \in U(\mathfrak{g})$ and $V_{(n-1,-(m-1))} \in U(\mathfrak{g})$.

Proposition 4.9. For $s \in \{1, 2, \dots, p_1 - p_2 - 1\}$ and for nonnegative integers m and n we have

$$\begin{aligned} ((\text{ad}F)(E_1^n F_2^m) \otimes F_2) \otimes w_s &\in \text{span}_{\mathbb{C}}\{(E_1^n F_2^m \otimes F_2) \otimes w_{s+1}, (E_1^{n-1} F_2^{m-1} \otimes F_2) \otimes w_s\} + Z, \\ \text{where } E_1^{-1} &= F_2^{-1} = 1. \end{aligned}$$

Proof. From $[F, E_1] = E_2$, $[F, F_2] = -F_1$, $[E_1, E_2] = 0$ and $[F_1, F_2] = 0$ we get

$$(\text{ad}F)(E_1^n F_2^m) = n \cdot E_1^{n-1} E_2 F_2^m - m E_1^n F_2^{m-1} F_1.$$

From here and from (17) it follows that

$$\begin{aligned} ((\text{ad}F)(E_1^n F_2^m) \otimes F_2) \otimes w_s &= nm(E_1^{n-1} F_2^{m-1} H_2 \otimes F_2) \otimes w_s \\ &\quad - nm(m-1)(E_1^{n-1} F_2^{m-1} \otimes F_2) \otimes w_s + n(E_1^{n-1} F_2^m E_2 \otimes F_2) \otimes w_s \\ &\quad - m(E_1^n F_2^{m-1} F_1 \otimes F_2) \otimes w_s. \end{aligned} \tag{18}$$

From $(p_2 - p_1 + s)(F_1 \otimes 1) \otimes w_s \in (F_2 \otimes 1) \otimes w_{s+1} + Z$ (see Corollary 4.8) we get

$$\begin{aligned} (p_2 - p_1 + s)(E_1^n F_2^{m-1} F_1 \otimes F_2) \otimes w_s \\ \in (p_2 - p_1 + s)(E_1^n F_2^m \otimes F_2) \otimes w_{s+1} + Z. \end{aligned} \tag{19}$$

Furthermore, using $H_{2\Delta} w_s = (h_{1\Delta} + 2h_{2\Delta})w_s = (p_1 + 2p_2 + s - \frac{1}{2})w_s$,

$$\alpha(H_2) = \alpha(h_1 + 2h_2) = -\frac{1}{2}E_1 F_1 - E_2 F_2 - \frac{3}{2}$$

and (8), we get

$$\begin{aligned} (H_2 \otimes F_2) \otimes w_s &= (1 \otimes F_2)(H_{2\Delta} - 1 \otimes \alpha(H_2)) \otimes w_s \\ &\in (p_1 + 2p_2 + s - 1) \cdot (1 \otimes F_2) \otimes w_s + Z. \end{aligned}$$

Therefore

$$\begin{aligned} (p_2 - p_1 + s)(E_1^{n-1}F_2^{m-1}H_2 \otimes F_2) \otimes w_s \\ \in (p_2 - p_1 + s)(p_1 + 2p_2 + s - 1)(E_1^{n-1}F_2^{m-1} \otimes F_2) \otimes w_s + Z. \end{aligned} \tag{20}$$

Similarly, using $(p_2 - p_1 + s)(E_2 \otimes 1) \otimes w_s \in -(E_1 \otimes 1) \otimes w_{s+1} + Z$ and (17), we get

$$\begin{aligned} (p_2 - p_1 + s)(E_1^{n-1}F_2^m E_2 \otimes F_2) \otimes w_s &\in -(E_1^{n-1}F_2^m E_1 \otimes F_2) \otimes w_{s+1} + Z \\ &= -(E_1^{n-1}(E_1 F_2^m - m F_2^{m-1} E) \otimes F_2) \otimes w_{s+1} + Z. \end{aligned}$$

Since

$$\begin{aligned} (E \otimes F_2) \otimes w_{s+1} &= (1 \otimes F_2)(E_\Delta - 1 \otimes \alpha(E)) \otimes w_{s+1} \\ &= (1 \otimes F_2) \underbrace{(E_\Delta \otimes w_{s+1} - (1 \otimes 1) \otimes E_\Delta w_{s+1})}_{\in Z} \\ &\quad + (1 \otimes F_2) \otimes E_\Delta w_{s+1} - 1 \otimes F_2 \cdot \left(-\frac{1}{2} E_1 F_2\right) \otimes w_{s+1} \\ &\in s(p_1 - p_2 - s)(1 \otimes F_2) \otimes w_s + Z, \end{aligned}$$

it follows that

$$\begin{aligned} (p_2 - p_1 + s)(E_1^{n-1}F_2^m E_2 \otimes F_2) \otimes w_s &\in -(E_1^n F_2^m \otimes F_2) \otimes w_{s+1} \\ &\quad + ms(p_1 - p_2 - s)(E_1^{n-1}F_2^{m-1} \otimes F_2) \otimes w_s + Z. \end{aligned} \tag{21}$$

The proof follows from (18), (19), (20) and (21). ■

The next proposition shows that a similar result holds for $s = p_1 - p_2$.

Proposition 4.10. *For nonnegative integers m and n we have*

$$\begin{aligned} ((\text{ad}F)(E_1^n F_2^m) \otimes F_2) \otimes w_{p_1-p_2} \\ \in \text{span}_{\mathbb{C}}\{(E_1^n F_2^m \otimes F_1) \otimes w_{p_1-p_2}, (E_1^{n-1}F_2^{m-1} \otimes F_2) \otimes w_{p_1-p_2}\} + Z, \end{aligned}$$

where $E_1^{-1} = F_2^{-1} = 1$.

Proof. From

$$(F_1 \otimes F_2) \otimes w_{p_1-p_2} = \underbrace{(F_1 \otimes F_2 - F_2 \otimes F_1) \otimes w_{p_1-p_2}}_{\in Z} + (F_2 \otimes F_1) \otimes w_{p_1-p_2}$$

we get

$$(E_1^n F_2^{m-1} F_1 \otimes F_2) \otimes w_{p_1-p_2} \in (E_1^n F_2^m \otimes F_1) \otimes w_{p_1-p_2} + Z \tag{22}$$

Furthermore, from the proof of the Proposition 4.9 we get

$$(E_1^{n-1}F_2^{m-1}H_2 \otimes F_2) \otimes w_{p_1-p_2} \in (2p_1+p_2-1) \cdot (E_1^{n-1}F_2^{m-1} \otimes F_2) \otimes w_{p_1-p_2} + Z. \quad (23)$$

Since

$$(E_2 \otimes F_2) \otimes w_{p_1-p_2} = \underbrace{(E_2 \otimes F_2 + E_1 \otimes F_1)}_{\in Z} \otimes w_{p_1-p_2} - (E_1 \otimes F_1) \otimes w_{p_1-p_2},$$

then

$$\begin{aligned} (E_1^{n-1}F_2^m E_2 \otimes F_2) \otimes w_{p_1-p_2} &\in - (E_1^{n-1}F_2^m E_1 \otimes F_1) \otimes w_{p_1-p_2} + Z \\ &= - (E_1^{n-1}(E_1 F_2^m - m F_2^{m-1} E) \otimes F_1) \otimes w_{p_1-p_2} + Z \\ &= - (E_1^n F_2^m \otimes F_1) \otimes w_{p_1-p_2} \\ &\quad + m(E_1^{n-1}F_2^{m-1} E \otimes F_1) \otimes w_{p_1-p_2} + Z. \end{aligned}$$

Furthermore, using (8) and $E_\Delta w_{p_1-p_2} = (p_1 - p_2 - 1)w_{p_1-p_2-1}$, we get

$$\begin{aligned} (E \otimes F_1) \otimes w_{p_1-p_2} &= (1 \otimes F_1)(E_\Delta - 1 \otimes \alpha(E)) \otimes w_{p_1-p_2} \\ &\in (p_1 - p_2 - 1)(1 \otimes F_1) \otimes w_{p_1-p_2-1} - (1 \otimes F_2) \otimes w_{p_1-p_2} + Z. \end{aligned}$$

Now, from

$$(p_2 - p_1 + (p_1 - p_2 - 1))(1 \otimes F_1) \otimes w_{p_1-p_2-1} \in (1 \otimes F_2) \otimes w_{p_1-p_2} + Z$$

we get

$$\begin{aligned} (E \otimes F_1) \otimes w_{p_1-p_2} &\in (p_2 - p_1 + 1)(1 \otimes F_2) \otimes w_{p_1-p_2} - (1 \otimes F_2) \otimes w_{p_1-p_2} + Z \\ &= (p_2 - p_1)(1 \otimes F_2) \otimes w_{p_1-p_2} + Z \end{aligned}$$

and therefore

$$\begin{aligned} (E_1^{n-1}F_2^m E_2 \otimes F_2) \otimes w_{p_1-p_2} \\ \in - (E_1^n F_2^m \otimes F_1) \otimes w_{p_1-p_2} + m(p_2 - p_1)(E_1^{n-1}F_2^{m-1} \otimes F_2) \otimes w_{p_1-p_2} + Z. \quad (24) \end{aligned}$$

The proof follows from (18), (22), (23) and (24). ■

Finally, from Propositions 4.9 and 4.10 and previous conclusions we get

Theorem 4.11. *One generating system of the vector space $\mathcal{A} \otimes_B W$ is given by*

$$\begin{aligned} &\{(E_1^n F_2^m \otimes F_2) \otimes w_s, (E_1^n F_2^m \otimes E_1 F_2) \otimes w_s, (E_1^n F_2^m \otimes E_2 F_2) \otimes w_s, \\ &\quad (E_1^n F_2^m \otimes E_1) \otimes w_s, \\ &(x \otimes F_1) \otimes w_{p_1-p_2}, (x \otimes E_1 F_1) \otimes w_{p_1-p_2}, (x \otimes E_2 F_1) \otimes w_{p_1-p_2}, (x \otimes E_2) \otimes w_{p_1-p_2} \mid \\ &\quad s = 1, 2, \dots, p_1 - p_2, \\ &x \in \{E_1^n F_2^m, (\text{ad}F)E_1^n F_2^m, \dots, (\text{ad}F)^{n+m} E_1^n F_2^m \mid n, m \in \mathbb{N}_0\}\}. \end{aligned}$$

There are no more obvious relations between the elements in 4.11. We will show that they form a basis of $\mathcal{A} \otimes_B W$ a little later.

5. The construction of the nonholomorphic discrete series representations of the group $SU(2, 1)$ via Dirac induction

From Theorem 3.1 it is easy to write down one basis for $X \otimes S$. However, we will give another basis for the space $X \otimes S$ which will be more useful to prove what we claim. First we will give bases for some finite-dimensional subspaces of the space $X \otimes 1$. Let us denote

$$V_{(p_1+n, p_2-m)} = \text{span}_{\mathbb{C}} \{v_{p_1+n, p_2-m}^{(p_1+n)-(p_2-m)-2s} \mid s \in \{0, 1, \dots, (p_1+n) - (p_2-m)\}\}.$$

From (2) it follows that one basis for the vector space

$$V_{k,l} = \sum_{0 \leq n \leq k, 0 \leq m \leq l} V_{(p_1+n, p_2-m)} \otimes 1$$

is given by

$$\{(F^s E_1^n F_2^m \otimes F_2) \cdot w_1 \mid 0 \leq n \leq k, \quad 0 \leq m \leq l, \quad 0 \leq s \leq (p_1+n) - (p_2-m)\}. \quad (25)$$

Our goal is to get a basis for $X \otimes S$ which is similar to the generating system from the Theorem 4.11. The next result is analogous to Proposition 4.6.

Lemma 5.1. *For $s \in \{1, 2, \dots, p_1 - p_2\}$ we have $(p_2 - p_1 + s)(1 \otimes F_1) \cdot w_s = (1 \otimes F_2) \cdot F_{\Delta} w_s$ and $(p_2 - p_1 + s)(1 \otimes E_2) \cdot w_s = -(1 \otimes E_1) \cdot F_{\Delta} w_s$.*

Proof. For $s \in \{1, 2, \dots, p_1 - p_2 - 1\}$ we have

$$\begin{aligned} (1 \otimes F_2) \cdot w_{s+1} &= (1 \otimes F_2)(v_{p_1, p_2}^{p_1-p_2-2(s+1)} \otimes E_1 - (p_1 - p_2 - s)v_{p_1, p_2}^{p_1-p_2-2s} \otimes E_2) \\ &= 2(p_1 - p_2 - s)v_{p_1, p_2}^{p_1-p_2-2s} \otimes 1, \\ (1 \otimes F_1) \cdot w_s &= (1 \otimes F_1)(v_{p_1, p_2}^{p_1-p_2-2s} \otimes E_1 - (p_1 - p_2 - (s-1))v_{p_1, p_2}^{p_1-p_2-2(s-1)} \otimes E_2) \\ &= -2v_{p_1, p_2}^{p_1-p_2-2s} \otimes 1, \\ (1 \otimes E_1) \cdot w_{s+1} &= (1 \otimes E_1)(v_{p_1, p_2}^{p_1-p_2-2(s+1)} \otimes E_1 - (p_1 - p_2 - s)v_{p_1, p_2}^{p_1-p_2-2s} \otimes E_2) \\ &= -(p_1 - p_2 - s)v_{p_1, p_2}^{p_1-p_2-2s} \otimes E_1 \wedge E_2, \\ (1 \otimes E_2) \cdot w_s &= (1 \otimes E_2)(v_{p_1, p_2}^{p_1-p_2-2s} \otimes E_1 - (p_1 - p_2 - (s-1))v_{p_1, p_2}^{p_1-p_2-2(s-1)} \otimes E_2) \\ &= -v_{p_1, p_2}^{p_1-p_2-2s} \otimes E_1 \wedge E_2. \end{aligned}$$

This implies the claim for $s \in \{1, 2, \dots, p_1 - p_2 - 1\}$. The case $s = p_1 - p_2$ is obvious. ■

Furthermore, we have the following analogue of Corollary 4.8:

Lemma 5.2. *For $s \in \{1, 2, \dots, p_1 - p_2\}$ we have*

$$(p_2 - p_1 + s)(F_1 \otimes 1) \cdot w_s = (F_2 \otimes 1) \cdot F_{\Delta} w_s$$

and

$$(p_2 - p_1 + s)(E_2 \otimes 1) \cdot w_s \in -(E_1 \otimes 1) \cdot F_{\Delta} w_s.$$

Proof. Since $D(1 \otimes F_i) + (1 \otimes F_i)D = -2F_i \otimes 1$, for $i = 1, 2$, then for all $w \in W$ we get

$$(F_i \otimes 1) \cdot w = -\frac{1}{2}(D(1 \otimes F_i) + (1 \otimes F_i)D) \cdot w = -\frac{1}{2}D(1 \otimes F_i) \cdot w.$$

From this and from Lemma 5.1 it follows that

$$(p_2 - p_1 + s)(F_1 \otimes 1) \cdot w_s = (F_2 \otimes 1) \cdot F_\Delta w_s.$$

Similarly, using $D(1 \otimes E_2) + (1 \otimes E_2)D = -2E_2 \otimes 1$ and Lemma 5.1 we get

$$(p_2 - p_1 + s)(E_2 \otimes 1) \cdot w_s = -(E_1 \otimes 1) \cdot F_\Delta w_s. \quad \blacksquare$$

Similarly as in the previous section, we would like to replace the second highest weight vector of the K -types in $U(\mathfrak{g})$ with the highest weight vectors of other K -types in $U(\mathfrak{g})$.

Proposition 5.3. For $n, m \in \mathbb{N}_0$ and $s \in \{1, 2, \dots, p_1 - p_2 - 1\}$ we have

$$(FE_1^n F_2^m \otimes F_2) \cdot w_s \in \text{span}_{\mathbb{C}}\{(E_1^{n-1} F_2^{m-1} \otimes F_2) \cdot w_s, (E_1^n F_2^m \otimes F_2) \cdot w_{s+1}\},$$

where $E_1^{-1} = F_2^{-1} = 1$.

Proof. From $\text{ad}(F)(E_1^n F_2^m) = FE_1^n F_2^m - E_1^n F_2^m F$, in a similar way to how it was done in (18), we get

$$\begin{aligned} (FE_1^n F_2^m \otimes F_2) \cdot w_s &= nm(E_1^{n-1} F_2^{m-1} H_2 \otimes F_2) \cdot w_s \\ &\quad - mn(m-1)(E_1^{n-1} F_2^{m-1} \otimes F_2) \cdot w_s + n(E_1^{n-1} F_2^m E_2 \otimes F_2) \cdot w_s \\ &\quad - m(E_1^n F_2^{m-1} F_1 \otimes F_2) \cdot w_s + (E_1^n F_2^m F \otimes F_2) \cdot w_s. \end{aligned} \quad (26)$$

Using Lemma 5.2 and (17), we get for $s \in \{1, 2, \dots, p_1 - p_2 - 1\}$

$$\begin{aligned} (p_2 - p_1 + s)(E_1^n F_2^{m-1} F_1 \otimes F_2) \cdot w_s &= (E_1^n F_2^m \otimes F_2) \cdot w_{s+1} & (27) \\ (p_2 - p_1 + s)(E_1^{n-1} F_2^m E_2 \otimes F_2) \cdot w_s &= -(E_1^{n-1} F_2^m E_1 \otimes F_2) \cdot w_{s+1} \\ &= -(E_1^{n-1}(E_1 F_2^m - mF_2^{m-1} E) \otimes F_2) \cdot w_{s+1} \\ &= -(E_1^n F_2^m \otimes F_2) \cdot w_{s+1} \\ &\quad + m(E_1^{n-1} F_2^{m-1} E \otimes F_2) \cdot w_{s+1}. \end{aligned}$$

Furthermore, for $s \in \{1, 2, \dots, p_1 - p_2 - 1\}$ the following identity holds

$$\begin{aligned} E \otimes F_2 \cdot w_{s+1} &= E \otimes F_2 \cdot (v_{p_1, p_2}^{p_1 - p_2 - 2(s+1)} \otimes E_1 - (p_1 - p_2 - s)v_{p_1, p_2}^{p_1 - p_2 - 2s} \otimes E_2) \\ &= 2(p_1 - p_2 - s)E v_{p_1, p_2}^{p_1 - p_2 - 2s} \otimes 1 \\ &= 2(p_1 - p_2 - s)s(p_1 - p_2 - s + 1)v_{p_1, p_2}^{p_1 - p_2 - 2(s-1)} \otimes 1 \\ &= (p_1 - p_2 - s)s \cdot (1 \otimes F_2) \cdot w_s. \end{aligned}$$

From this we get

$$(p_2 - p_1 + s)(E_1^{n-1}F_2^m E_2 \otimes F_2) \cdot w_s = -(E_1^n F_2^m \otimes F_2) \cdot w_{s+1} + ms(p_1 - p_2 - s)(E_1^{n-1}F_2^{m-1} \otimes F_2) \cdot w_s. \quad (28)$$

Now, from $H_2 = h_1 + 2h_2$ and (1) it follows that

$$\begin{aligned} (H_2 \otimes F_2) \cdot w_s &= 2(p_1 - p_2 - (s - 1))H_2 v_{p_1, p_2}^{p_1 - p_2 - 2(s-1)} \otimes 1 \\ &= 2(p_1 - p_2 - (s - 1))(p_1 + 2p_2 + s - 1)v_{p_1, p_2}^{p_1 - p_2 - 2(s-1)} \otimes 1 \\ &= (p_1 + 2p_2 + s - 1)(1 \otimes F_2) \cdot w_s. \end{aligned} \quad (29)$$

Furthermore, we have

$$\begin{aligned} (F \otimes F_2) \cdot w_s &= 2(p_1 - p_2 - (s - 1))F v_{p_1, p_2}^{p_1 - p_2 - 2(s-1)} \otimes 1 \\ &= 2(p_1 - p_2 - (s - 1))v_{p_1, p_2}^{p_1 - p_2 - 2s} \otimes 1 \\ &= \frac{p_1 - p_2 - (s - 1)}{p_1 - p_2 - s}(1 \otimes F_2) \cdot w_{s+1}, \end{aligned}$$

from where we get

$$(p_1 - p_2 - s)(F \otimes F_2) \cdot w_s = (p_1 - p_2 - (s - 1))(1 \otimes F_2) \cdot w_{s+1}. \quad (30)$$

Finally, from (26), (27), (28), (29) and (30) we get

$$\begin{aligned} (p_2 - p_1 + s)(F E_1^n F_2^m \otimes F_2) \cdot w_s &= (p_2 - p_1 + s)mn(p_1 + 2p_2 - m)(E_1^{n-1}F_2^{m-1} \otimes F_2) \cdot w_s \\ &\quad - (n + m + p_1 - p_2 - (s - 1))(E_1^n F_2^m \otimes F_2) \cdot w_{s+1}. \end{aligned}$$

Therefore, for $s \in \{1, 2, \dots, p_1 - p_2 - 1\}$ we have

$$(F E_1^n F_2^m \otimes F_2) \cdot w_s \in \text{span}_{\mathbb{C}}\{(E_1^{n-1}F_2^{m-1} \otimes F_2) \cdot w_s, (E_1^n F_2^m \otimes F_2) \cdot w_{s+1}\}. \quad \blacksquare$$

A similar result holds for $s = p_1 - p_2$.

Lemma 5.4. *Let n and m be nonnegative integers. Then*

$$(F E_1^n F_2^m \otimes F_2) \cdot w_{p_1 - p_2} \in \text{span}_{\mathbb{C}}\{(E_1^{n-1}F_2^{m-1} \otimes F_2) \cdot w_{p_1 - p_2}, (E_1^n F_2^m \otimes F_1) \cdot w_{p_1 - p_2}\}.$$

where $E_1^{-1} = F_2^{-1} = 1$.

Proof. From the facts $(E_1 \otimes F_1 + E_2 \otimes F_2) \cdot w_{p_1 - p_2} = 0$, $(F_1 \otimes F_2 - F_2 \otimes F_1) \cdot w_{p_1 - p_2} = 0$, $(F \otimes F_2) \cdot w_{p_1 - p_2} = 2v_{p_1, p_2}^{-p_1 + p_2} \otimes 1 = -(1 \otimes F_1) \cdot w_{p_1 - p_2}$, from (26), (29), (17) and $(E \otimes F_1) \cdot w_{p_1 - p_2} = (p_2 - p_1)(1 \otimes F_2) \cdot w_{p_1 - p_2}$ we get

$$\begin{aligned} (F E_1^n F_2^m \otimes F_2) \cdot w_{p_1 - p_2} &= mn(2p_2 + p_1 - m)(E_1^{n-1}F_2^{m-1} \otimes F_2) \cdot w_{p_1 - p_2} \\ &\quad - (m + n + 1)(E_1^n F_2^m \otimes F_1) \cdot w_{p_1 - p_2}. \end{aligned}$$

This finishes the proof. \blacksquare

Lemma 5.5. For $n, m, s \in \mathbb{N}_0$ we have

$$(\text{ad}F)^s(E_1^n F_2^m) = F^s E_1^n F_2^m - \sum_{i=1}^s \binom{s}{i} (\text{ad}F)^{s-i}(E_1^n F_2^m) F^i.$$

Proof. The proof follows easily by induction on s , for fixed $n, m \in \mathbb{N}_0$. ■

Lemma 5.6. For nonnegative integers n, m and s the following identity holds

$$(F^s E_1^n F_2^m \otimes F_1) \cdot w_{p_1-p_2} = ((\text{ad}F)^s(E_1^n F_2^m) \otimes F_1) \cdot w_{p_1-p_2}.$$

Proof. From Lemma 5.5 it follows that

$$\begin{aligned} (F^s E_1^n F_2^m \otimes F_1) \cdot w_{p_1-p_2} &= \left(\sum_{i=0}^s \binom{s}{i} ((\text{ad}F)^{s-i}(E_1^n F_2^m)) F^i \otimes F_1 \right) \cdot w_{p_1-p_2} \\ &= ((\text{ad}F)^s(E_1^n F_2^m) \otimes F_1) \cdot w_{p_1-p_2}, \end{aligned}$$

since $F^i \otimes F_1 \cdot w_{p_1-p_2} = 0$ for $i \geq 1$. ■

Now, from Lemmas 5.3, 5.4, 5.6 and from (25) we see that the set

$$\begin{aligned} B = \{ & (E_1^n F_2^m \otimes F_2) \cdot w_s, (x \otimes F_1) \cdot w_{p_1-p_2} \mid s \in \{1, 2, \dots, p_1 - p_2\}, \\ & x \in \{E_1^n F_2^m, (\text{ad}F)E_1^n F_2^m, \dots, (\text{ad}F)^{n+m} E_1^n F_2^m\}, \\ & 0 \leq n \leq k, 0 \leq m \leq l\} \end{aligned}$$

spans the vector space $V_{k,l}$. Since B has exactly

$$\sum_{0 \leq n \leq k, 0 \leq m \leq l} (p_1 - p_2) + n + m + 1$$

elements, which is the dimension of the vector space $V_{k,l}$, B is the basis of $V_{k,l}$. Since $X \otimes 1 = \sum_{n,m \in \mathbb{N}_0} V_{(p_1+n, p_2-m)} \otimes 1$, one basis for $X \otimes 1$ is given by

$$\begin{aligned} \{ & (E_1^n F_2^m \otimes F_2) \cdot w_s, (x \otimes F_1) \cdot w_{p_1-p_2} \mid s \in \{1, 2, \dots, p_1 - p_2\}, \\ & x \in \{E_1^n F_2^m, (\text{ad}F)E_1^n F_2^m, \dots, (\text{ad}F)^{n+m} E_1^n F_2^m \mid n, m \in \mathbb{N}_0\} \}, \end{aligned}$$

From here, using $(1 \otimes E_1 E_2) \cdot w_s = 0$, we get that one basis for the vector space $X \otimes S$ is given by

$$\begin{aligned} \{ & (E_1^n F_2^m \otimes F_2) \cdot w_s, (E_1^n F_2^m \otimes E_1 F_2) \cdot w_s, (E_1^n F_2^m \otimes E_2 F_2) \cdot w_s, (E_1^n F_2^m \otimes E_1) \cdot w_s, \\ & (x \otimes F_1) \cdot w_{p_1-p_2}, (x \otimes E_1 F_1) \cdot w_{p_1-p_2}, (x \otimes E_2 F_1) \cdot w_{p_1-p_2}, (x \otimes E_2) \cdot w_{p_1-p_2} \mid \\ & s = 1, 2, \dots, p_1 - p_2, \\ & x \in \{E_1^n F_2^m, (\text{ad}F)E_1^n F_2^m, \dots, (\text{ad}F)^{n+m} E_1^n F_2^m \mid n, m \in \mathbb{N}_0\} \}. \end{aligned}$$

The action map $\phi : \mathcal{A} \otimes W \longrightarrow X \otimes S$

$$\phi(a \otimes w) = a \cdot w, \quad a \in \mathcal{A}, w \in W$$

maps the generating system of $\mathcal{A} \otimes_B W$ from Theorem 4.11 into the basis of the space $X \otimes S$, which implies that generating system is a basis of the vector space $\mathcal{A} \otimes_B W$. Finally, we get:

Theorem 5.7. *Let X be the vector space of the nonholomorphic discrete series representation of the group $SU(2, 1)$ and let W be the Dirac cohomology of X . Then the (\mathcal{A}, \tilde{K}) -module $Ind_D(W) = \mathcal{A} \otimes_{\mathcal{B}} W$ is isomorphic to the (\mathcal{A}, \tilde{K}) -module $X \otimes S$.*

Remark 5.8. In this paper we obtain nonholomorphic discrete series representations by tensoring over the algebra $\mathcal{B} = U(\mathfrak{k}_\Delta)\mathcal{A}^K$, which contains the full algebra of invariants. Namely, in the proof of Proposition 4.6, which was essential in the proof of the main theorem, we used the fact that the element $(E_1F_1 + E_2F_2) \otimes 1$ is in the algebra $\mathcal{B} = U(\mathfrak{k}_\Delta)\mathcal{A}^K$ over which we tensor (see (16)). We show below that this element is not contained in the algebra $\tilde{\mathcal{B}} = U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I})$ which is used in the definition of the intermediate Dirac induction. In fact, we show the equivalent fact that the element $D^\mathfrak{k}$ of \mathcal{B} (see (3) and (6)) is not in $\tilde{\mathcal{B}}$.

So, a priori we are not using the intermediate Dirac induction here, but rather the “full” Dirac induction. It however remains unclear if tensoring \mathcal{A} with W over \mathcal{B} and over $\tilde{\mathcal{B}}$ gives the same result, even though the algebras \mathcal{B} and $\tilde{\mathcal{B}}$ are not the same.

To prove the claim that $D^\mathfrak{k} \notin \tilde{\mathcal{B}}$, we assume the contrary, i.e., $D^\mathfrak{k} \in U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I})$. Then it follows that

$$D^\mathfrak{k} = \sum_{i=1}^n \lambda_i h_{1\Delta}^{a_i} h_{2\Delta}^{b_i} F_\Delta^{c_i} E_\Delta^{d_i} x_i$$

for some $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $a_i, b_i, c_i, d_i \in \mathbb{N}_0$, $x_i \in C(\mathfrak{p})^K + \mathcal{I}$. If W is the Dirac cohomology of some nonholomorphic discrete series representation of the group $SU(2, 1)$, then each $x_i \in C(\mathfrak{p})^K + \mathcal{I}$ acts on W with some scalar μ_i which does not depend on the choice of the nonholomorphic discrete series. For $w \in W$ we thus have

$$D^\mathfrak{k}w = \sum_{i=1}^n \lambda_i \mu_i h_{1\Delta}^{a_i} h_{2\Delta}^{b_i} F_\Delta^{c_i} E_\Delta^{d_i} w. \tag{31}$$

In particular, for the highest weight vector w_1 of W , the terms with $d_i \neq 0$ do not contribute and we have

$$D^\mathfrak{k}w_1 = \sum_{1 \leq i \leq n, d_i=0} \lambda_i \mu_i h_{1\Delta}^{a_i} h_{2\Delta}^{b_i} F_\Delta^{c_i} w_1.$$

If W is the Dirac cohomology of the nonholomorphic discrete series $\pi_{(p_1, p_2)}$, then $D^\mathfrak{k}w_1 = (-\frac{1}{2}(p_1 - p_2) - 1)w_1$ and so

$$\left(-\frac{1}{2}(p_1 - p_2) - 1\right)w_1 = \sum_{1 \leq i \leq n, d_i=0} \lambda_i \mu_i h_{1\Delta}^{a_i} h_{2\Delta}^{b_i} F_\Delta^{c_i} w_1.$$

Varying different (p_1, p_2) , it easily follows from relations (4) that $d_i = 0$ implies that also $c_i = 0$. So we have

$$-\frac{1}{2}(p_1 - p_2) - 1 = \sum_{1 \leq i \leq n, c_i=d_i=0} \lambda_i \mu_i \left(p_1 - \frac{1}{2}\right)^{a_i} \left(p_2 + \frac{1}{2}\right)^{b_i}.$$

It now follows that $a_i + b_i \in \{0, 1\}$, and so $\lambda_i \mu_i = -\frac{1}{2}$ for $(a_i, b_i, c_i, d_i) = (1, 0, 0, 0)$, $\lambda_i \mu_i = \frac{1}{2}$ for $(a_i, b_i, c_i, d_i) = (0, 1, 0, 0)$, and $\lambda_i \mu_i = -\frac{3}{2}$ for $(a_i, b_i, c_i, d_i) = (0, 0, 0, 0)$. Applying (31) to w_s , we get

$$\begin{aligned} \left(-\frac{1}{2}(p_1 - p_2) - 1\right) w_s &= \left(-\frac{1}{2}\left(p_1 - s + \frac{1}{2}\right) + \frac{1}{2}\left(p_2 + s - \frac{1}{2}\right) - \frac{3}{2}\right) w_s \\ &+ \sum_{1 \leq i \leq n, d_i \neq 0} \lambda_i \mu_i h_{1\Delta}^{a_i} h_{2\Delta}^{b_i} F_{\Delta}^{c_i} E_{\Delta}^{d_i} w_s, \end{aligned}$$

from where it follows

$$(s - 1)w_s + \sum_{1 \leq i \leq n, d_i \neq 0} \lambda_i \mu_i h_{1\Delta}^{a_i} h_{2\Delta}^{b_i} F_{\Delta}^{c_i} E_{\Delta}^{d_i} w_s = 0.$$

In particular, for $s = 2$, we get

$$w_2 + (p_1 - p_2 - 1) \sum_{1 \leq i \leq n, d_i = 1} \lambda_i \mu_i h_{1\Delta}^{a_i} h_{2\Delta}^{b_i} F_{\Delta}^{c_i} w_1 = 0.$$

Varying different (p_1, p_2) , it easily follows from relations (4) that if $d_i = 1$ for some $i \in \{1, 2, \dots, n\}$, then also $c_i = 1$. Now we have

$$(p_1 - p_2 - 1) \sum_{1 \leq i \leq n, c_i = d_i = 1} \lambda_i \mu_i \left(p_1 - \frac{3}{2}\right)^{a_i} \left(p_2 + \frac{3}{2}\right)^{b_i} = -1$$

for all possible values of p_1, p_2 . This however can not be true, since the degree of the polynomial on the left-hand side is obviously greater than zero. This proves our claim that $D^{\mathfrak{k}} \notin U(\mathfrak{k}_{\Delta})(C(\mathfrak{p})^K + \mathcal{I})$.

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