

A 1-Differentiable Cohomology Induced by a Vector Field

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Communicated by P. W. Michor

Abstract. Using the Lie derivative of a vector field, we define a cohomology on spaces of pairs of differential forms (or 1-differentiable forms) in a manifold. We provide a link to the classical de Rham cohomology and to a 1-differentiable cohomology of Lichnerowicz type associated to an one-form. We discuss also the case of a complex manifold and a holomorphic vector field. Finally, an application to the harmonicity of 1-differentiable forms is studied in a particular case. *Mathematics Subject Classification 2010:* 14F40, 57R99, 58A10, 58A12. *Key Words and Phrases:* 1-differentiable form, Lie derivative, vector field, cohomology, harmonic form.

1. Introduction

Let us consider the field $\Omega^0(M) = \mathcal{F}(M)$ of smooth real valued functions defined on a smooth manifold M . For each $p = 1, \dots, n = \dim M$ denote by $\Omega^p(M)$ the module of p -forms on M and by $\Omega(M) = \bigoplus_{p \geq 0} \Omega^p(M)$ the exterior algebra of M . The cohomology of some pair differential forms (or tensor fields) called *1-differentiable p -cochain* $\tilde{\varphi} = (\varphi, \psi)$, where φ is a p -form and ψ is a $(p - 1)$ -form was initiated and intensively studied by Lichnerowicz and all in [3, 11] in the context of symplectic and contact geometry and in [12] in the context of Poisson geometry. Further significant developments of a such cohomology in the context of Lichnerowicz-Jacobi cohomology can be found, for instance in [8, 9, 10]. Also, we notice that an harmonic and C -harmonic theory of 1-differentiable forms on Sasakian manifolds was recently studied in [6]. Another approach concerning calculus on 1-differentiable forms (called also generalized forms) is proposed in [15, 16, 17] where some interesting applications in mechanical and physical field theories are illustrated.

On the other hand, another cohomology of some similar pair forms is so called *relative cohomology* with respect to a smooth map $f : M' \rightarrow M$ between

* The second author is supported by the Sectorial Operational Program Human Resources Development (SOP HRD), financed from the European Social Fund and by the Romanian Government under the Project number POSDRU/159/1.5/S/134378.

two smooth manifolds, which is defined in [1, p. 78] as follows: if we consider the set $\Omega^p(f) = \Omega^p(M) \oplus \Omega^{p-1}(M')$ and the operator

$$d_f : \Omega^p(f) \rightarrow \Omega^{p+1}(f), \quad d_f(\varphi, \psi) = (d\varphi, f^*\varphi - d\psi),$$

then $d_f^2 = (0, 0)$. The cohomology of the differential complex $(\Omega^\bullet(f), d_f)$ is called the *relative cohomology with respect to f* . We notice that the identity $d_f^2 = (0, 0)$ is based on the identity $[f^*, d] = 0$, where we have denoted by d the exterior derivative both on M and M' and $[\cdot, \cdot]$ denotes the graded commutator of two operators which is defined by $[D_1, D_2] = D_1D_2 - (-1)^{\deg D_1 \cdot \deg D_2} D_2D_1$, see for instance [13]. This fact suggest that we can consider cohomology for a smooth manifold M with respect to the differential complex $(\tilde{\Omega}^\bullet(M), \tilde{d})$, where $\tilde{\Omega}^p(M) = \Omega^p(M) \oplus \Omega^{p-1}(M)$ and

$$\tilde{d} : \tilde{\Omega}^p(M) \rightarrow \tilde{\Omega}^{p+1}(M), \quad \tilde{d}(\varphi, \psi) = (D_1\varphi, D_2\varphi - D_1\psi),$$

where D_1 is an operator of degree 1 which satisfies $D_1^2 = 0$ and D_2 is an operator of degree 0 such that $[D_1, D_2] = 0$.

Taking into account the remarkable identity $[\mathcal{L}_X, d] = 0$, where \mathcal{L}_X denotes the Lie derivative with respect to a vector field $X \in \mathcal{X}(M)$, then we can consider $D_1 = d$ and $D_2 = \mathcal{L}_X$ in the above definition of \tilde{d} and the resulting cohomology is named by us the *1-differentiable cohomology induced by the vector field X* .

The aim of the present paper is the study of this new cohomology as well as some of its possible applications to harmonic theory for 1-differentiable forms. The paper is organized as follows.

In the second section we define some differential operators associated to a vector field X on M which act on the exterior graded algebra $(\tilde{\Omega}^\bullet(M), \tilde{\wedge})$, where $\tilde{\wedge}$ is the exterior product of 1-differentiable forms $\tilde{\varphi} = (\varphi, \psi)$ defined in [11]. These operators, which are denoted by \tilde{d}_X, \tilde{i}_X and $\tilde{\mathcal{L}}_X$, represent a natural extension of the operators d, ι_X and \mathcal{L}_X , respectively on $\tilde{\Omega}^\bullet(M)$. We prove some natural properties of these operators in relation with the classical properties of d, ι_X and \mathcal{L}_X , respectively. Next, a cohomology induced by the vector field X , denoted by $\tilde{H}_X^\bullet(M)$ is defined as the cohomology of the differential complex $(\tilde{\Omega}^\bullet, \tilde{d}_X)$ and we prove an isomorphism of $\tilde{H}_X^\bullet(M)$ with $H_{dR}^\bullet(M) \oplus H_{dR}^{\bullet-1}(M)$, where $H_{dR}^\bullet(M)$ denotes de Rham cohomology of M . In this way we recover an isomorphism of our cohomology with a classical 1-differentiable cohomology of Lichnerowicz type associated to an one form. Also, some examples on symplectic manifolds are considered with respect to a symplectic, hamiltonian or Liouville vector field and a Hamilton type equation for 1-differentiable forms is described. Next, when the manifold is complex and the vector field is holomorphic, a similar approach is considered, and when the manifold is compact Kählerian we establish an isomorphism with $H^{p,\bullet}(M) \oplus H^{p,\bullet-1}(M)$, where $H^{p,\bullet}(M)$ denotes the p -th Dolbeault cohomology of M . In the third section we consider a generalization of this cohomology with respect to a smooth map $f : M' \rightarrow M$, obtaining a *relative cohomology induced by the vector field X* . Then, some similar properties as for \tilde{d}_X -cohomology are investigated in this more general case.

In the last section we discuss some aspects concerning harmonicity of 1-differentiable forms with respect to our operator associated to a Killing vector

field in a particular case when M is a n -dimensional compact smooth manifold endowed with a closed 1-form ω and with a Riemannian metric g such that ω is parallel with respect to g .

2. 1-differentiable cohomology induced by a vector field

2.1. Calculus on 1-differentiable forms associated to a vector field. Let us consider the set $\tilde{\Omega}^p(M) = \Omega^p(M) \oplus \Omega^{p-1}(M)$. As in formula (5.5) from [12], we can define a *pair wedge product* $\tilde{\wedge} : \tilde{\Omega}^p(M) \times \tilde{\Omega}^{p'}(M) \rightarrow \tilde{\Omega}^{p+p'}(M)$ by

$$(\varphi, \psi)\tilde{\wedge}(\varphi', \psi') = (\varphi \wedge \varphi', (-1)^p\varphi \wedge \psi' + \psi \wedge \varphi') \tag{1}$$

to be the exterior product on the space $\tilde{\Omega}^\bullet(M)$, where $(\varphi, \psi) \in \tilde{\Omega}^p(M)$, $(\varphi', \psi') \in \tilde{\Omega}^{p'}(M)$. By this definition, we notice that for an 1-differentiable 0-form $(f, 0)$, where f is a smooth function on M , we have $(f, 0) \cdot (\varphi, \psi) = (f\varphi, f\psi)$. Also, one easily verifies that

$$\begin{aligned} (\varphi, \psi)\tilde{\wedge}((\varphi', \psi') + (\varphi'', \psi'')) &= (\varphi, \psi)\tilde{\wedge}(\varphi', \psi') + (\varphi, \psi)\tilde{\wedge}(\varphi'', \psi''), \\ (\varphi, \psi)\tilde{\wedge}((\varphi', \psi')\tilde{\wedge}(\varphi'', \psi'')) &= ((\varphi, \psi)\tilde{\wedge}(\varphi', \psi'))\tilde{\wedge}(\varphi'', \psi'') \end{aligned}$$

and

$$(\varphi, \psi)\tilde{\wedge}(\varphi', \psi') = (-1)^{pp'}(\varphi', \psi')\tilde{\wedge}(\varphi, \psi),$$

which say that $(\tilde{\Omega}^\bullet(M), \tilde{\wedge})$ is a graded algebra.

Now, we consider $X \in \mathcal{X}(M)$ a fixed vector field on M , and we define the following differential operators acting on $\tilde{\Omega}^\bullet(M)$:

$$\tilde{d}_X : \tilde{\Omega}^p(M) \rightarrow \tilde{\Omega}^{p+1}(M), \quad \tilde{d}_X(\varphi, \psi) = (d\varphi, \mathcal{L}_X\varphi - d\psi). \tag{2}$$

$$\tilde{i}_X : \tilde{\Omega}^p(M) \rightarrow \tilde{\Omega}^{p-1}(M), \quad \tilde{i}_X(\varphi, \psi) = (\iota_X\varphi, -\iota_X\psi) \tag{3}$$

and

$$\tilde{\mathcal{L}}_X : \tilde{\Omega}^p(M) \rightarrow \tilde{\Omega}^p(M), \quad \tilde{\mathcal{L}}_X(\varphi, \psi) = (\mathcal{L}_X\varphi, \mathcal{L}_X\psi). \tag{4}$$

By direct calculus, using (1)-(4) and the classical relations concerning the operators d , ι_X and \mathcal{L}_X , the following relations hold:

$$\tilde{d}_X((\varphi, \psi)\tilde{\wedge}(\varphi', \psi')) = \tilde{d}_X(\varphi, \psi)\tilde{\wedge}(\varphi', \psi') + (-1)^p(\varphi, \psi)\tilde{\wedge}\tilde{d}_X(\varphi', \psi'), \tag{5}$$

$$\tilde{i}_X((\varphi, \psi)\tilde{\wedge}(\varphi', \psi')) = \tilde{i}_X(\varphi, \psi)\tilde{\wedge}(\varphi', \psi') + (-1)^p(\varphi, \psi)\tilde{\wedge}\tilde{i}_X(\varphi', \psi'), \tag{6}$$

and

$$\tilde{\mathcal{L}}_X((\varphi, \psi)\tilde{\wedge}(\varphi', \psi')) = \tilde{\mathcal{L}}_X(\varphi, \psi)\tilde{\wedge}(\varphi', \psi') + (\varphi, \psi)\tilde{\wedge}\tilde{\mathcal{L}}_X(\varphi', \psi'), \tag{7}$$

which say that \tilde{d}_X is an antiderivation of degree 1, \tilde{i}_X is an antiderivation of degree -1 and $\tilde{\mathcal{L}}_X$ is a derivation of degree 0, respectively, in the exterior graded algebra $(\tilde{\Omega}^p(M), \tilde{\wedge})$.

Using the above definitions and the classical properties concerning the operators d , ι_X and \mathcal{L}_X , we easily obtain:

$$\tilde{d}_X^2 = (0, 0), \quad \tilde{i}_X^2 = (0, 0), \quad [\tilde{\mathcal{L}}_X, \tilde{\mathcal{L}}_Y] = \tilde{\mathcal{L}}_{[X, Y]}, \quad [\tilde{\mathcal{L}}_X, \tilde{i}_Y] = \tilde{i}_{[X, Y]}. \tag{8}$$

For instance, the first relation from (8) it follows by direct calculus in (2) using $[\mathcal{L}_X, d] = 0$.

Also, we have

Proposition 2.1. *If $Y \in \mathcal{X}(M)$ is a Lie symmetry of X , that is $[X, Y] = 0$, then the following remarkable identities hold:*

$$\tilde{\mathcal{L}}_Y = \tilde{d}_X \tilde{\iota}_Y + \tilde{\iota}_Y \tilde{d}_X, [\tilde{\mathcal{L}}_Y, \tilde{d}_X] = (0, 0). \tag{9}$$

As a consequence, one gets

$$\tilde{\mathcal{L}}_X = \tilde{d}_X \tilde{\iota}_X + \tilde{\iota}_X \tilde{d}_X, [\tilde{\mathcal{L}}_X, \tilde{d}_X] = (0, 0). \tag{10}$$

Let $f : M' \rightarrow M$ be a smooth map between two smooth manifolds. We define $\tilde{f}^* : \tilde{\Omega}^p(M) \rightarrow \tilde{\Omega}^p(M')$ by

$$\tilde{f}^*(\varphi, \psi) = (f^*\varphi, f^*\psi), \forall (\varphi, \psi) \in \tilde{\Omega}^p(M). \tag{11}$$

It is well known that if $f : M' \rightarrow M$ is a diffeomorphism, then we have

$$f^*d\varphi = df^*\varphi, f^*(\iota_{f_*X}\varphi) = \iota_X(f^*\varphi), f^*(\mathcal{L}_{f_*X}\varphi) = \mathcal{L}_X(f^*\varphi), \tag{12}$$

for any $X \in \mathcal{X}(M')$, $\varphi \in \Omega^\bullet(M)$, and by direct calculus we obtain

Proposition 2.2. *If $f : M' \rightarrow M$ is a diffeomorphism the following relations hold:*

$$\tilde{d}_X \tilde{f}^* = \tilde{f}^* \tilde{d}_{f_*X}, \tilde{f}^* \tilde{\iota}_{f_*X} = \tilde{\iota}_X \tilde{f}^*, \tilde{f}^* \tilde{\mathcal{L}}_{f_*X} = \tilde{\mathcal{L}}_X \tilde{f}^*, X \in \mathcal{X}(M'). \tag{13}$$

2.2. \tilde{d}_X -cohomology.

As well as we seen, by the first relation of (8) we have a differential complex $(\tilde{\Omega}^\bullet(M), \tilde{d}_X)$, and denote by $\tilde{H}_X^\bullet(M)$ the cohomology of this complex. We will call it the *1-differentiable cohomology induced by the vector field X* of the smooth manifold M or \tilde{d}_X -cohomology of M .

We have

Proposition 2.3. *The exterior product from (1) induces, on the space of \tilde{d}_X -cohomology classes, a cohomology algebra structure.*

If $f : M' \rightarrow M$ is a diffeomorphism between two smooth manifolds then the first relation (13) says that the map \tilde{f}^* from (11) induces the following map in the cohomology level:

$$\tilde{f}^\bullet : \tilde{H}_{f_*X}^\bullet(M) \rightarrow \tilde{H}_X^\bullet(M'), \tilde{f}^\bullet[(\varphi, \psi)] = [\tilde{f}^*(\varphi, \psi)], \forall X \in \mathcal{X}(M'). \tag{14}$$

We notice that a cohomology class in $\tilde{H}_X^\bullet(M)$ is represented by a closed form φ whose Lie derivative with respect to X is an exact form. Because for any

closed form φ we have $\mathcal{L}_X\varphi = d\iota_X\varphi$ (that is $\mathcal{L}_X\varphi$ is an exact form), then it is easy to see that we have the natural maps in cohomology:

$$H_{dR}^\bullet(M) \mapsto \tilde{H}_X^\bullet(M), [\varphi] \mapsto [(\varphi, \iota_X\varphi)],$$

$$\tilde{H}_X^\bullet(M) \mapsto H_{dR}^{\bullet-1}(M), [(\varphi, \psi)] \mapsto [\iota_X\varphi - \psi].$$

Moreover, the map

$$\tilde{H}_X^\bullet(M) \mapsto H_{dR}^\bullet(M) \oplus H_{dR}^{\bullet-1}(M), [(\varphi, \psi)] \mapsto ([\varphi], [\iota_X\varphi - \psi])$$

with the inverse

$$H_{dR}^\bullet(M) \oplus H_{dR}^{\bullet-1}(M) \mapsto \tilde{H}_X^\bullet(M), ([\varphi], [\psi]) \mapsto [(\varphi, \iota_X\varphi - \psi)]$$

establishes the following isomorphism:

$$\tilde{H}_X^\bullet(M) \cong H_{dR}^\bullet(M) \oplus H_{dR}^{\bullet-1}(M). \tag{15}$$

Remark 2.4. This cohomology is isomorphic with an 1-differentiable cohomology associated to an 1-form of Lichnerowicz type. More exactly, for any $\eta \in \Omega^1(M)$ we consider the operator defined by

$$\tilde{d}_\eta : \tilde{\Omega}^p(M) \rightarrow \tilde{\Omega}^{p+1}(M) \quad \tilde{d}_\eta(\varphi, \psi) = (d\varphi - d\eta \wedge \psi, -d\psi). \tag{16}$$

By direct calculus we get $\tilde{d}_\eta^2 = (0, 0)$, hence we have a differential complex $(\tilde{\Omega}^\bullet(M), \tilde{d}_\eta)$. Denote by $\tilde{H}_\eta^\bullet(M)$ the cohomology of this complex and we will call it the 1-differentiable cohomology associated to the 1-form η of the smooth manifold M or \tilde{d}_η -cohomology of M . This cohomology is a particular case of a more general 1-differentiable cohomology firstly defined by Lichnerowicz in [11] in the context of symplectic and contact manifolds, and [12] in the context of Poisson manifolds, respectively. Also, according to [9, 11, 12] we have $\tilde{H}_\eta^\bullet(M) \cong H_{dR}^\bullet(M) \oplus H_{dR}^{\bullet-1}(M)$ and the isomorphism (15) says that $\tilde{H}_X^\bullet(M) \cong \tilde{H}_\eta^\bullet(M)$.

Consequently, $\dim \tilde{H}_X^p(M) = b_p(M) + b_{p-1}(M)$, where $b_p(M)$ is the p -th Betti number of M . In particular, the dimension of $\tilde{H}_X^p(M)$ is a topological invariant of M , for all p . Also, by applying the Poincaré duality for the de Rham cohomology $H_{dR}^\bullet(M)$ in (15) we obtain the following Poincaré duality for \tilde{d}_X -cohomology:

$$\tilde{H}_X^p(M) \cong \left(\tilde{H}_{c,X}^{n+1-p}(M) \right)^*, \quad p = 0, \dots, n + 1, \tag{17}$$

where the index c denotes *compactly supported*.

Indeed, taking into account the Poincaré duality $H_{dR}^p(M) \cong (H_{c,dR}^{n-p}(M))^*$ and the classical isomorphism $(V \oplus W)^* \cong V^* \oplus W^*$ then we easily obtain (17).

Also, we have

Proposition 2.5. $\tilde{H}_X^p(M) = \tilde{0}$ for any $p > n + 1$.

Proposition 2.6. For any vector fields $X, Y \in \mathcal{X}(M)$ there is an isomorphism:

$$\alpha_{X,Y} : \widetilde{H}_X^\bullet(M) \rightarrow \widetilde{H}_Y^\bullet(M), \alpha_{X,Y}([\varphi, \psi]) = [(\varphi, \iota_{Y-X}\varphi + \psi)] \tag{18}$$

with the inverse $\alpha_{X,Y}^{-1} = \alpha_{Y,X}$.

2.3. Examples.

Example 2.7. Let (M, ω) be a symplectic manifold, that is $\omega \in \Omega^2(M)$ is closed and nondegenerated. We recall that a vector field $X \in \mathcal{X}(M, \omega)$ is called *symplectic* if $\iota_X\omega$ is closed and it is called *hamiltonian* if $\iota_X\omega$ is exact. Taking into account that for every symplectic vector field $X \in \mathcal{X}(M, \omega)$ we have $\mathcal{L}_X\omega = 0$ it follows that $\widetilde{d}_X(\omega, \theta) = (0, 0)$ for every symplectic vector field $X \in \mathcal{X}(M, \omega)$ and every closed 1-form θ on M .

Moreover we have the following result:

Proposition 2.8. If $[(\omega, \theta)] = (0, 0)$ and θ is exact then ω is exact and X is hamiltonian.

Proof. Let $\theta = dg$, where $g \in C^\infty(M, \omega)$. Then $[(\omega, dg)] = (0, 0)$ if there exists $(\varphi, f) \in \widetilde{\Omega}^1(M)$ such that $(\omega, dg) = \widetilde{d}_X(\varphi, f)$ which implies

$$\omega = d\varphi \text{ and } \mathcal{L}_X\varphi - df = dg.$$

The first relation from above says that ω is exact and replacing $\omega = d\varphi$ in the second relation we obtain $\iota_X\omega = d(f + g - \iota_X\varphi)$, were we have used the Cartan identity $\mathcal{L}_X = d\iota_X + \iota_Xd$. Thus, $\iota_X\omega$ is exact i.e., X is hamiltonian. ■

Example 2.9. Let (M, ω) be a symplectic manifold and $X \in \mathcal{X}(M, \omega)$ be a symplectic vector field. Then $(\iota_X\omega, k)$ is \widetilde{d}_X -closed for every $k \in \mathbb{R}$. Moreover if $k \neq 0$ then $[(\iota_X\omega, k)] \neq (0, 0)$.

Indeed, if we suppose that $[(\iota_X\omega, k)] = (0, 0)$ then there exists $(f, 0) \in \widetilde{\Omega}^0(M)$ such that $(\iota_X\omega, k) = \widetilde{d}_X(f, 0)$ which implies

$$df = \iota_X\omega \text{ and } \mathcal{L}_Xf = k.$$

The first relation from above says that X is hamiltonian and if H is the hamiltonian function of X , that is $\iota_X\omega = -dH$, then we have $d(f + H) = 0$ which implies $f = -H + c$, where $c \in \mathbb{R}$. Now, replacing $f = -H + c$ in the second relation and taking into account that $\mathcal{L}_XH = 0$ we obtain $\mathcal{L}_Xf = 0 \neq k$.

Example 2.10. Suppose now that (M, ω) is a symplectic manifold with $\omega \in \Omega^2(M)$ an exact symplectic form i.e. there exists an 1-form θ (sometimes called *Liouville 1-form*) such that: $\omega = d\theta$. Then there exists a *potential symplectic vector field* ξ (sometimes called *Liouville vector field*) i.e. a vector field for which: $\theta = \iota_\xi\omega$, see [2, p. 22]. Then, we have $\widetilde{d}_\xi(\omega, \theta) = (0, \mathcal{L}_\xi\omega - \omega)$. But

$\mathcal{L}_\xi\omega = d\iota_\xi\omega + \iota_\xi d\omega = d\theta = \omega$, and then $\tilde{d}_\xi(\omega, \theta) = (0, 0)$ which means that $[(\omega, \theta)] \in \tilde{H}_\xi^2(M)$. The \tilde{d}_ξ -cohomology class $[(\omega, \theta)]$ from this example will be called the *1-differentiable cohomology class induced by the Liouville vector field ξ of the exact symplectic manifold (M, ω)* .

We have

Proposition 2.11. *The \tilde{d}_ξ -cohomology class $[(\omega, \theta)]$ induced by the Liouville vector field ξ of the exact symplectic manifold (M, ω) vanishes.*

Proof. Let $f \in C^\infty(M)$. Taking into account $d\theta = \omega$, $\mathcal{L}_\xi\theta = \theta$ and $\mathcal{L}_\xi df = d(\xi f)$, then it follows that $(\omega, \theta) = \tilde{d}_\xi(\theta + df, \xi f)$. ■

Remark 2.12. The vanishing of the \tilde{d}_ξ -cohomology class $[(\omega, \theta)]$ it follows also from the isomorphism (15), since $[(\omega, \theta)] \mapsto ([\omega], [\iota_\xi\omega - \theta]) = (0, 0)$.

Example 2.13. (Hamilton type equations.) Let $\tilde{\omega} = (\omega, \theta) \in \tilde{\Omega}^2(M)$ such that $\tilde{d}_X\tilde{\omega} = (0, 0)$ and $\omega \in \Omega^2(M)$ has a maximal rank. Since $d\omega = 0$ and $d\iota_X\omega = d\theta$, it follows that ω is a symplectic form and we can consider (at least locally) θ having the form $\theta = \iota_X\omega + dh$, where $h \in \Omega^0(M) = C^\infty(M)$. Also, for $\tilde{\Theta} = (f, 0) \in \tilde{\Omega}^0(M)$ we have $\tilde{d}_X\tilde{\Theta} = (df, Xf)$. The equation $-\tilde{d}_X\tilde{\Theta} = \iota_X\tilde{\omega}$, i.e. $-(df, Xf) = (\iota_X\omega, -\iota_X dh)$ reads $\iota_X\omega = -df$ and $Xh = Xf$.

It follows that given $\tilde{\omega} = (\omega, \theta)$ which is \tilde{d}_X -closed, ω of maximal rank, and $\tilde{\Theta} = (f, 0) \in \tilde{\Omega}^0(M)$, then the Hamilton type equation $-\tilde{d}_X\tilde{\Theta} = \iota_X\tilde{\omega}$ has as solution the ω -hamiltonian vector field X corresponding to f , provided that θ is an exact form, i.e. $\theta = dh'$, $h' \in C^\infty(M)$, and $Xh' = 0$. In the particular case when $\theta = 0$, then X is simply the hamiltonian vector field corresponding to f .

2.4. The complex case.

The main purpose of this subsection is to formulate a cohomology induced by a holomorphic vector field on complex manifolds and possible relation with classical Dolbeault cohomology.

Let M be a complex n -dimensional manifold and X a holomorphic vector field on M . Denote by $\Omega^{p,q}(M)$ the space of all complex differential forms on M of type (p, q) , briefly (p, q) -forms on M . Then we have a decomposition of the exterior derivative $d = \partial + \bar{\partial}$, where $\partial : \Omega^{p,q}(M) \rightarrow \Omega^{p+1,q}(M)$ and $\bar{\partial} : \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M)$. Similarly to previous notations we consider the set $\tilde{\Omega}^{p,q}(M) = \Omega^{p,q}(M) \oplus \Omega^{p,q-1}(M)$. Since for every holomorphic vector field X on M the Lie derivative \mathcal{L}_X preserves the (p, q) type of forms on which this act, we define the following operator:

$$\bar{\partial}_X : \tilde{\Omega}^{p,q}(M) \rightarrow \tilde{\Omega}^{p,q+1}(M), \bar{\partial}_X(\varphi, \psi) = (\bar{\partial}\varphi, \mathcal{L}_X\varphi - \bar{\partial}\psi), \tag{19}$$

for every $(\varphi, \psi) \in \tilde{\Omega}^{p,q}(M)$.

Also, taking the same components in relation $\mathcal{L}_X(\partial + \bar{\partial}) = (\partial + \bar{\partial})\mathcal{L}_X$ we get $[\mathcal{L}_X, \bar{\partial}] = 0$.

It is easy to see that $\bar{\partial}_X^2 = (0, 0)$. Then, for any $p \geq 0$ we have the differential complex $(\tilde{\Omega}^{p,\bullet}, \bar{\partial}_X)$, and denote by $\tilde{H}_X^{p,\bullet}(M)$ the cohomology groups of this complex, called *p-th Dolbeault 1-differentiable cohomology induced by the holomorphic vector field X* of M .

We notice that the exterior product from (1) can be extended to $\tilde{\Omega}^{p,q}(M)$ and this induces, on the space of $\bar{\partial}_X$ -cohomology classes, a cohomology algebra structure. Also, if we consider a biholomorphic map $f : M' \rightarrow M$ between two complex manifolds, taking into account that for any holomorphic vector field X on M' , f_*X is a holomorphic vector field on M and f^* preserves the (p, q) -forms, then the map \tilde{f}^* from (11) induces the following map in the cohomology level:

$$\tilde{f}^\bullet : \tilde{H}_{f_*X}^{p,\bullet}(M) \rightarrow \tilde{H}_X^{p,\bullet}(M'), \quad \tilde{f}^\bullet[(\varphi, \psi)] = [\tilde{f}^*(\varphi, \psi)] \tag{20}$$

for any holomorphic vector field X on M' .

In the following we are interested to find an isomorphism of type (15) for Dolbeault 1-differentiable cohomology induced by the holomorphic vector field X of M . We have

Theorem 2.14. *If (M, g) is a compact Kähler manifold then we have the following isomorphism:*

$$\tilde{H}_X^{p,\bullet}(M) \cong H^{p,\bullet}(M) \oplus H^{p,\bullet-1}(M), \tag{21}$$

where $H^{p,\bullet}(M)$ denotes the classical *p-th Dolbeault cohomology* of M .

Proof. Using the Hodge theory on compact Kähler manifolds we obtain that any Dolbeault class in $H^{p,q}(M)$ is represented by a form $\varphi \in \Omega^{p,q}(M)$ that is closed both with respect to d and $\bar{\partial}$, see for instance [4, 7, 14, 18]. Thus, for any Dolbeault class $[\varphi] \in H^{p,q}(M)$ we have $\mathcal{L}_X\varphi = d(\iota_X\varphi)$ is again a (p, q) -form on M . Since X is holomorphic we have $\iota_X\varphi \in \Omega^{p-1,q}(M)$ and so

$$\mathcal{L}_X\varphi = \partial(\iota_X\varphi) \text{ and } \bar{\partial}(\iota_X\varphi) = 0$$

for degree reasons. As $\bar{\partial}(\iota_X\varphi) = 0$ we find a $(p-1, q-1)$ -form φ' on M such that $\iota_X\varphi + \bar{\partial}\varphi'$ is $\bar{\partial}$ -closed which implies according to our hypothesis that $\iota_X\varphi + \bar{\partial}\varphi'$ is d -closed. Hence

$$\partial(\iota_X\varphi) = -\partial\bar{\partial}\varphi' = \bar{\partial}\partial\varphi'$$

and so

$$\mathcal{L}_X\varphi = \bar{\partial}\partial\varphi', \tag{22}$$

which shows that the induced action on Dolbeault cohomology $H^{p,q}(M)$ is trivial. Then, the map

$$\tilde{H}_X^{p,\bullet}(M) \mapsto H^{p,\bullet}(M) \oplus H^{p,\bullet-1}(M), \quad [(\varphi, \psi)] \mapsto ([\varphi], [\partial\varphi' - \psi])$$

with the inverse

$$H^{p,\bullet}(M) \oplus H^{p,\bullet-1}(M) \mapsto \tilde{H}_X^{p,\bullet}(M), \quad ([\varphi], [\psi]) \mapsto [(\varphi, \partial\varphi' - \psi)]$$

establishes the isomorphism from (21). ■

Remark 2.15. Using the Serre duality for Dolbeault cohomology on compact Kähler manifolds we obtain the following Serre duality for $\bar{\partial}_X$ -cohomology:

$$\tilde{H}_X^{p,q}(M) \cong \tilde{H}_X^{n-p,n+1-q}(M). \tag{23}$$

3. A relative cohomology induced by a vector field

In this section we present a generalization of the \tilde{d}_X -cohomology relative to a smooth map $f : M' \rightarrow M$ between two manifolds.

As in the case of relative cohomology, we consider the set $\Omega^p(f) = \Omega^p(M) \oplus \Omega^{p-1}(M')$ and the operator $d_{X,f}$ defined by

$$d_{X,f} : \Omega^p(f) \rightarrow \Omega^{p+1}(f), \quad d_{X,f}(\varphi, \psi) = (d\varphi, \mathcal{L}_X f^* \varphi - d\psi), \quad X \in \mathcal{X}(M'). \tag{24}$$

An easy calculation shows that $d_{X,f}^2 = (0, 0)$. Denote the cohomology groups of the complex $(\Omega^\bullet(f), d_{X,f})$ by $H_X^\bullet(f)$ and we call it the *relative cohomology induced by the vector field X*.

Remark 3.1. If $M' = M$ and $f = Id|_M$ then $H_X^\bullet(Id|_M) = \tilde{H}_X^\bullet(M)$.

As in formula (1), we can define a *pair wedge product* adapted to our study by

$$\begin{aligned} \wedge_f : \Omega^p(f) \times \Omega^{p'}(f) &\rightarrow \Omega^{p+p'}(f), \\ (\varphi, \psi) \wedge_f (\varphi', \psi') &= (\varphi \wedge \varphi', (-1)^p f^* \varphi \wedge \psi' + \psi \wedge f^* \varphi'), \end{aligned} \tag{25}$$

where $(\varphi, \psi) \in \Omega^p(f)$ and $(\varphi', \psi') \in \Omega^{p'}(f)$, respectively.

One easily verifies that this product is anticomutative, namely

$$(\varphi, \psi) \wedge_f (\varphi', \psi') = (-1)^{pp'} (\varphi', \psi') \wedge_f (\varphi, \psi)$$

and also $(\Omega^p(f), \wedge_f)$ is an exterior graded algebra. By direct calculus we have that $d_{X,f}$ is an antiderivation with respect to the exterior product from (25). We have

Proposition 3.2. *The exterior product from (25) induces, on the space of $d_{X,f}$ -cohomology classes, a cohomology algebra structure.*

Proposition 3.3. *Let $\varphi \in \Omega^p(M)$ be a closed p -form on M , i.e. $d\varphi = 0$. Then for every vector field $X \in \mathcal{X}(M')$ we have $(\varphi, \iota_X f^* \varphi) \in \Omega^p(f)$ is $d_{X,f}$ -closed and, moreover if φ is exact then the $d_{X,f}$ -cohomology class $[(\varphi, \iota_X f^* \varphi)]$ vanishes.*

Proof. Taking into account $d\varphi = 0$ and Cartan identity $\mathcal{L}_X = d\iota_X + \iota_X d$, we have

$$d_{X,f}(\varphi, \iota_X f^* \varphi) = (d\varphi, \mathcal{L}_X f^* \varphi - d\iota_X f^* \varphi) = (0, d\iota_X f^* \varphi - \iota_X d f^* \varphi) = (0, 0).$$

Now, if $\varphi = d\psi$, by direct calculus we have $d_{X,f}(\psi, \iota_X f^* \psi + d\zeta) = (\varphi, \iota_X f^* \varphi)$, for every $\zeta \in \Omega^{p-2}(M')$, which says that $[(\varphi, \iota_X f^* \varphi)] \in H_X^p(f)$ vanishes. ■

Example 3.4. Let (M, ω) and (M', ω') be two exact symplectic manifolds with θ, ξ and θ', ξ' the corresponding Liouville 1-forms and vector fields, respectively. We also consider $f : (M', \omega') \rightarrow (M, \omega)$ a symplectomorphism between two exact symplectic manifolds, that is $\omega' = f^*\omega$. Then $d_{\xi', f}(\omega, \theta') = (0, \mathcal{L}_{\xi'} f^*\omega - d\theta')$. But $\mathcal{L}_{\xi'} f^*\omega = \mathcal{L}_{\xi'} \omega' = d\theta'$, which implies $d_{\xi', f}(\omega, \theta') = (0, 0)$, that is $[(\omega, \theta')] \in H_{\xi'}^2(f)$, which is called the *cohomology class induced by the Liouville vector field ξ' relative to the symplectomorphism $f : (M', \omega') \rightarrow (M, \omega)$ between two exact symplectic manifolds*. Moreover if $f^*\theta = \theta'$ then $[(\omega, \theta')]$ vanishes.

As in the classical case of relative cohomology, [1], we can consider the mappings $\alpha : \Omega^{p-1}(M') \rightarrow \Omega^p(f)$, $\alpha(\psi) = (0, \psi)$ and $\beta : \Omega^p(f) \rightarrow \Omega^p(M)$, $\beta(\varphi, \psi) = \varphi$ for all $\varphi \in \Omega^p(M)$ and $\psi \in \Omega^{p-1}(M')$, respectively. Then, we have the following result which relates the cohomology $H_X^\bullet(f)$ with the de Rham cohomology of M and M' , respectively.

Proposition 3.5. *Let $f : M' \rightarrow M$ be a smooth map between two manifolds. Then:*

(i) *The mappings α and β induce an exact sequence of complexes*

$$0 \longrightarrow (\Omega^{\bullet-1}(M'), d) \xrightarrow{\alpha} (\Omega^\bullet(f), d_{X,f}) \xrightarrow{\beta} (\Omega^\bullet(M), d) \longrightarrow 0.$$

(ii) *This exact sequence induces a long exact cohomology sequence*

$$\dots \longrightarrow H_{dR}^{p-1}(M') \xrightarrow{\alpha^*} H_X^p(f) \xrightarrow{\beta^*} H_{dR}^p(M) \xrightarrow{\delta_p^*} H_{dR}^p(M') \longrightarrow \dots, \quad (26)$$

where the connecting homomorphism δ_p^* is defined by

$$\delta_p^*([\varphi]) = [\mathcal{L}_X f^*\varphi], \text{ for any } [\varphi] \in H_{dR}^p(M). \quad (27)$$

Since for any closed form $\varphi \in \Omega^p(M)$ we have $\mathcal{L}_X f^*\varphi = d(i_X f^*\varphi)$, the connecting homomorphism $\delta_p^* = 0$ and so for every $p \geq 1$ we have an exact sequence:

$$0 \longrightarrow H_{dR}^{p-1}(M') \xrightarrow{\alpha^*} H_X^p(f) \xrightarrow{\beta^*} H_{dR}^p(M) \xrightarrow{\delta_p^*} 0$$

which leads to the following isomorphism

$$H_X^\bullet(f) \cong H_{dR}^\bullet(M) \oplus H_{dR}^{\bullet-1}(M'). \quad (28)$$

In the following, we present a link between our cohomology and a generalization of a classical 1-differentiable cohomology associated to an one form of Lichnerowicz type.

For a smooth map $f : M' \rightarrow M$ we can consider the set $\Omega^p(f) = \Omega^p(M') \oplus \Omega^{p-1}(M)$, and define the following operator

$$d_{\eta, f} : \Omega^p(f) \rightarrow \Omega^{p+1}(f), \quad d_{\eta, f}(\varphi, \psi) = (d\varphi - f^*(d\eta \wedge \psi), -d\psi). \quad (29)$$

An easy calculation leads to $d_{\eta,f}^2 = (0, 0)$. We denote the cohomology groups of the complex $(\Omega^\bullet(f), d_{\eta,f})$ by $H_\eta^\bullet(f)$ and we call it the *relative 1-differentiable cohomology associated to the 1-form η* . We also notice that if f is diffeomorphism then $\Omega^\bullet(f) = \Omega^\bullet(f^{-1})$.

As in the case of $d_{X,f}$ -cohomology, we can consider the mappings $\mu : \Omega^p(M') \rightarrow \Omega^p(f)$, $\mu(\varphi) = (\varphi, 0)$ and $\nu : \Omega^{p-1}(f) \rightarrow \Omega^{p-1}(M)$, $\nu(\varphi, \psi) = \psi$ for all $\varphi \in \Omega^p(M')$ and $\psi \in \Omega^{p-1}(M)$, respectively. Then, we have the following result which relates $H_\eta^\bullet(f)$ with the de Rham cohomologies of M and M' , respectively.

Proposition 3.6. *Let $f : M' \rightarrow M$ be a smooth map between two manifolds. Then:*

(i) *The mappings μ and ν induce an exact sequence of complexes*

$$0 \longrightarrow (\Omega^\bullet(M'), d) \xrightarrow{\mu} (\Omega^\bullet(f), d_{\eta,f}) \xrightarrow{\nu} (\Omega^{\bullet-1}(M), -d) \longrightarrow 0.$$

(ii) *This exact sequence induces a long exact cohomology sequence*

$$\dots \longrightarrow H_{dR}^p(M') \xrightarrow{\mu^*} H_\eta^p(f) \xrightarrow{\nu^*} H_{dR}^{p-1}(M) \xrightarrow{\Delta_{p-1}^*} H_{dR}^{p+1}(M') \longrightarrow \dots, \quad (30)$$

where the connecting homomorphism Δ_{p-1}^* is defined by

$$\Delta_{p-1}^*[\psi] = [-f^*(d\eta \wedge \psi)], \text{ for any } [\psi] \in H_{dR}^{p-1}(M). \quad (31)$$

Since for any closed form $\psi \in \Omega^{p-1}(M)$ we have $f^*(d\eta \wedge \psi) = df^*(\eta \wedge \psi)$ the connecting homomorphism Δ_p^* vanishes and so we obtain the isomorphism:

$$H_\eta^\bullet(f) \cong H_{dR}^\bullet(M') \oplus H_{dR}^{\bullet-1}(M). \quad (32)$$

Now, taking into account the isomorphisms (28) and (32), we obtain

Theorem 3.7. *If $f : M' \rightarrow M$ is a diffeomorphism then the following isomorphism hold:*

$$H_X^\bullet(f) \cong H_\eta^\bullet(f^{-1}). \quad (33)$$

Finally, we notice that the 1-differentiable cohomology induced by a holomorphic vector field can be generalized in the complex case with respect to a holomorphic map $f : M' \rightarrow M$. More exactly, we can consider a more general definition of $\bar{\partial}_X$ on the set $\Omega^{p,q}(f) = \Omega^{p,q}(M) \oplus \Omega^{p,q-1}(M')$ by setting

$$\bar{\partial}_{X,f} : \Omega^{p,q}(f) \rightarrow \Omega^{p,q+1}(f), \bar{\partial}_{X,f}(\varphi, \psi) = (\bar{\partial}\varphi, \mathcal{L}_X f^* \varphi - \bar{\partial}\psi), \quad X \in \mathcal{X}(M'). \quad (34)$$

An easy calculation shows that $\bar{\partial}_{X,f}^2 = (0, 0)$. Denote the cohomology groups of the complex $(\Omega^{p,\bullet}(f), \bar{\partial}_{X,f})$ by $H_X^{p,\bullet}(f)$ and we call it the *relative p -th Dolbeault cohomology induced by the holomorphic vector field X of M'* . As well as we seen if $M' = M$ and $f = Id|_M$ then $\bar{\partial}_X = \bar{\partial}_{X,Id|_M}$ and $\tilde{H}_X^{p,\bullet}(M) \equiv H_X^{p,\bullet}(Id|_M)$. We also notice that $\bar{\partial}_{X,f}$ is an antiderivation with respect to the exterior product from (25) extended to $\tilde{\Omega}^{p,q}(f)$.

Using the same technique as above, we can consider the mappings $\bar{\alpha} : \Omega^{p,q-1}(M') \rightarrow \Omega^{p,q}(f)$, $\bar{\alpha}(\psi) = (0, \psi)$ and $\bar{\beta} : \Omega^{p,q}(f) \rightarrow \Omega^{p,q}(M)$, $\bar{\beta}(\varphi, \psi) = \varphi$ for all $\varphi \in \Omega^{p,q}(M)$ and $\psi \in \Omega^{p,q-1}(M')$, respectively. Then, we have the following exact sequence of complexes:

$$0 \longrightarrow (\Omega^{p,\bullet-1}(M'), \bar{\partial}) \xrightarrow{\bar{\alpha}} (\Omega^{p,\bullet}(f), \bar{\partial}_{X,f}) \xrightarrow{\bar{\beta}} (\Omega^{p,\bullet}(M), \bar{\partial}) \longrightarrow 0.$$

which induces a long exact cohomology sequence

$$\dots \longrightarrow H^{p,q-1}(M') \xrightarrow{\bar{\alpha}^*} H_X^{p,q}(f) \xrightarrow{\bar{\beta}^*} H^{p,q}(M) \xrightarrow{\bar{\delta}_{p,q}^*} H^{p,q}(M') \longrightarrow \dots, \tag{35}$$

where the connecting homomorphism $\bar{\delta}_{p,q}^*$ is defined by

$$\bar{\delta}_{p,q}^*([\varphi]) = [\mathcal{L}_X f^* \varphi], \text{ for any } [\varphi] \in H^{p,q}(M). \tag{36}$$

Taking into account that $X \in \mathcal{X}(M')$ is holomorphic and f^* preserves the type of complex (p, q) forms, a similar calculation as in the proof of Theorem 2.14 leads to

Corollary 3.8. *If (M, g) is a compact Kähler manifold then we have the following isomorphism:*

$$H_X^{p,\bullet}(f) \cong H^{p,\bullet}(M) \oplus H^{p,\bullet-1}(M'). \tag{37}$$

4. An application to harmonicity of 1-differentiable forms

In this section we discuss some aspects concerning harmonicity of 1-differentiable forms with respect to our operator in a particular case.

Let us consider a n -dimensional compact smooth manifold M , a closed 1-form ω on M and a Riemannian metric g on M such that ω is parallel with respect to g . We consider the vector field $U \in \mathcal{X}(M)$ characterized by the condition

$$\omega(X) = g(X, U), \quad \forall X \in \mathcal{X}(M). \tag{38}$$

Also, we consider the codifferential operator δ given by

$$\delta\varphi = (-1)^{np+n+1}(\star \circ d \circ \star)\varphi, \quad \forall \varphi \in \Omega^p(M), \tag{39}$$

where \star is the Hodge star operator, see for instance [4, 18].

Using that ω is parallel and that U is Killing, we have the well known relations (see [9])

$$\mathcal{L}_U = -\delta \circ e(\omega) - e(\omega) \circ \delta, \quad \delta \circ \mathcal{L}_U = \mathcal{L}_U \circ \delta, \tag{40}$$

where $e(\omega)\varphi = \omega \wedge \varphi$ for every $\varphi \in \Omega^\bullet(M)$.

Now, we define the following codifferential induced by the vector field U associated to the operator \tilde{d}_U on the space $\tilde{\Omega}^p(M)$ of 1-differentiable p -forms:

$$\tilde{\delta}_U : \tilde{\Omega}^p(M) \rightarrow \tilde{\Omega}^{p-1}(M), \quad \tilde{\delta}_U(\varphi, \psi) = (\delta\varphi + \mathcal{L}_U\psi, -\delta\psi). \tag{41}$$

By straightforward calculus, using the second relation of (40), we obtain $\tilde{\delta}_U^2 = (0, 0)$. Moreover, using the natural scalar product \langle, \rangle on the space $\Omega^\bullet(M)$ given by

$$\langle, \rangle : \Omega^p(M) \times \Omega^p(M) \rightarrow \mathbb{R}, (\varphi, \varphi') \mapsto \langle \varphi, \varphi' \rangle = \int_M \varphi \wedge \star \varphi', \tag{42}$$

we also define a scalar product $\langle\langle, \rangle\rangle$ on $\tilde{\Omega}^\bullet(M)$ by

$$\langle\langle(\varphi, \psi), (\varphi', \psi')\rangle\rangle = \langle \varphi, \varphi' \rangle + \langle \psi, \psi' \rangle, \forall (\varphi, \psi), (\varphi', \psi') \in \tilde{\Omega}^p(M). \tag{43}$$

We have

Proposition 4.1. *The operator \tilde{d}_U is the $\langle\langle, \rangle\rangle$ -adjoint of $\tilde{\delta}_U$ and conversely.*

Proof. It is sufficient to prove that $\langle\langle\tilde{d}_U(\varphi, \psi), (\varphi', \psi')\rangle\rangle = \langle\langle(\varphi, \psi), \tilde{\delta}_U(\varphi', \psi')\rangle\rangle$, for every $(\varphi, \psi) \in \tilde{\Omega}^{p-1}(M)$ and $(\varphi', \psi') \in \tilde{\Omega}^p(M)$. By (43) we have

$$\langle\langle\tilde{d}_U(\varphi, \psi), (\varphi', \psi')\rangle\rangle = \langle d\varphi, \varphi' \rangle + \langle \mathcal{L}_U \varphi, \psi' \rangle - \langle d\psi, \psi' \rangle \tag{44}$$

and

$$\langle\langle(\varphi, \psi), \tilde{\delta}_U(\varphi', \psi')\rangle\rangle = \langle \varphi, \delta \varphi' \rangle + \langle \varphi, \mathcal{L}_U \psi' \rangle - \langle \psi, \delta \psi' \rangle. \tag{45}$$

Now, the result follows taking into account that d is the \langle, \rangle -adjoint of δ and \mathcal{L}_U is \langle, \rangle -self-adjoint. ■

Let us define now a $\tilde{\Delta}_U$ -Laplacian associated to the vector field U on the space $\tilde{\Omega}^p(M)$ by $\tilde{\Delta} = \tilde{d}_U \tilde{\delta}_U + \tilde{\delta}_U \tilde{d}_U$. Then by direct calculations, we obtain

$$\tilde{\Delta}_U(\varphi, \psi) = (\Delta \varphi + \mathcal{L}_U^2 \varphi, \Delta \psi + \mathcal{L}_U^2 \psi), \forall (\varphi, \psi) \in \tilde{\Omega}^p(M). \tag{46}$$

Hence, the 1-differentiable form $(\varphi, \psi) \in \tilde{\Omega}^p(M)$ is $\tilde{\Delta}_U$ -harmonic, that is, we have $\tilde{\Delta}_U(\varphi, \psi) = (0, 0)$ if and only if

$$\Delta \varphi + \mathcal{L}_U^2 \varphi = 0 \text{ and } \Delta \psi + \mathcal{L}_U^2 \psi = 0. \tag{47}$$

Proposition 4.2. *The 1-differentiable form (φ, ψ) is $\tilde{\Delta}_U$ -harmonic if and only if $\tilde{d}_U(\varphi, \psi) = (0, 0)$ and $\tilde{\delta}_U(\varphi, \psi) = (0, 0)$.*

Proof. It follows directly from

$$\begin{aligned} \langle\langle\tilde{\Delta}_U(\varphi, \psi), (\varphi, \psi)\rangle\rangle &= \langle\langle\tilde{d}_U \tilde{\delta}_U(\varphi, \psi) + \tilde{\delta}_U \tilde{d}_U(\varphi, \psi), (\varphi, \psi)\rangle\rangle \\ &= \langle\langle\tilde{\delta}_U(\varphi, \psi), \tilde{\delta}_U(\varphi, \psi)\rangle\rangle + \langle\langle\tilde{d}_U(\varphi, \psi), \tilde{d}_U(\varphi, \psi)\rangle\rangle. \end{aligned}$$

■

Also, if we continue the calculus from the proof of above proposition, we obtain

$$\langle\langle\tilde{\Delta}_U(\varphi, \psi), (\varphi, \psi)\rangle\rangle = \langle\langle(\varphi, \psi), \tilde{\Delta}_U(\varphi, \psi)\rangle\rangle \tag{48}$$

which says that the $\tilde{\Delta}_U$ -Laplacian is $\langle\langle, \rangle\rangle$ -self-adjoint.

Example 4.3. Let us consider a smooth function $f \in C^\infty(M)$ with the property that $\mathcal{L}_U f = kf$, $k \in \mathbb{R}$. Then, the 1-differentiable 0-form $(f, 0) \in \widetilde{\Omega}^0(M)$ is $\widetilde{\Delta}_U$ -harmonic iff the function f satisfies the Klein-Gordon type equation

$$\Delta f + k^2 f = 0, \tag{49}$$

with mass term $m^2 = k^2$.

A first interpretation of formula (46) in terms of usual harmonicity can be obtained as follows. As U is a Killing vector field, it is well known that in this case $\mathcal{L}_U \varphi = 0$ for every harmonic form $\varphi \in \Omega^\bullet(M)$. Thus, we have

Proposition 4.4. *If $\varphi \in \Omega^p(M)$ and $\psi \in \Omega^{p-1}(M)$ are both harmonic then the 1-differentiable p -form (φ, ψ) is $\widetilde{\Delta}_U$ -harmonic.*

Remark 4.5. We notice that a similar Laplacian can be defined for compact Sasakian manifolds (or more generally for K -contact manifolds) with respect to the Reeb vector field ξ which is a Killing vector field.

Another interpretation of formula (46) will be given bellow involving the Lichnerowicz-Laplacian. More exactly, as ω is a closed 1-form on M there is an associated Lichnerowicz differential, see [5, 8, 12], $d_\omega : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ defined by

$$d_\omega \varphi = d\varphi + e(\omega)\varphi, \forall \varphi \in \Omega^p(M). \tag{50}$$

By direct calculus, we easily obtain that $d_\omega^2 = 0$ and the cohomology $H_\omega^\bullet(M)$ of the resulting complex $(\Omega^\bullet(M), d_\omega)$ is called the Lichnerowicz cohomology of (M, ω) . Also, a codifferential operator $\delta_\omega : \Omega^p(M) \rightarrow \Omega^{p-1}(M)$ associated to ω is given by (see [5])

$$\delta_\omega = \delta + \iota_U. \tag{51}$$

Then, the Lichnerowicz-Laplacian is defined as $\Delta_\omega = d_\omega \delta_\omega + \delta_\omega d_\omega$ and by direct calculus, taking into account the first relation of (40), we get

$$\Delta_\omega \varphi = \Delta \varphi + \varphi, \forall \varphi \in \Omega^p(M). \tag{52}$$

Also, using an associated Hodge decomposition (with respect to the operators Δ_ω , d_ω and δ_ω , see [5]), it is proved, see [9], that for every Δ_ω -harmonic form φ we have $\mathcal{L}_U \varphi = -\varphi$.

Hence, we have the following result

Proposition 4.6. *If $\varphi \in \Omega^p(M)$ and $\psi \in \Omega^{p-1}(M)$ are both Δ_ω -harmonic then the 1-differentiable p -form (φ, ψ) is $\widetilde{\Delta}_U$ -harmonic.*

References

- [1] Bott, R., and L. W. Tu, “Differential Forms in Algebraic Topology,” Graduate Text in Math. **82**, Springer-Verlag, Berlin, 1982.
- [2] Cannas da Silva, A., “Lectures on symplectic geometry,” Lecture Notes in Mathematics **1764**, Springer-Verlag, Berlin, 2001.
- [3] Flato, M., A. Lichnerowicz, and D. Sternheimer, *Déformations 1-différentiables des algèbres de Lie attachées à une variété symplectique ou de contact*, Compositio Math. **31** (1975), 47–82.
- [4] Goldberg, S. I., “Curvature and Homology,” Revised Edition, ISBN 0-486-40207-X, Dover Publication, Inc. Mineola, New-York 1998.
- [5] Guédira, F., and A. Lichnerowicz, *Géométrie des algèbres de Lie locales de Kirillov*, J. Math. Pures et Appl. **63** (1984), 407–484.
- [6] Ida, C., and S. Merçeşan, *On harmonic and C-harmonic 1-differentiable forms on Sasakian manifolds*, Mediterr. J. Math. **11** (2014), 155–171.
- [7] Kobayashi, S., “Differential geometry of complex vector bundles,” Publ. of the Math. Soc. of Japan **15**, Kanô Memorial Lectures 5. Princeton Univ. Press, Princeton, NJ; Iwanami Shoten, Tokyo, 1987.
- [8] de León, M., J. C. Marrero, and E. Padrón, *Lichnerowicz-Jacobi cohomology*, J. Phys. A: Math. Gen. **30** (1997), 6029–6055.
- [9] de León, M., B. López, J. C. Marrero, and E. Padrón, *Lichnerowicz-Jacobi cohomology and homology of Jacobi manifolds: modular class and duality*, arXiv: math/9910079v1 [math.DG] 1999.
- [10] —, *On the computation of the Lichnerowicz-Jacobi cohomology*, J. Geom. Phys. **44** (2003), 507–522.
- [11] Lichnerowicz, A., *Cohomologie 1-différentiable des algèbres de Lie attachées à une variété symplectique ou de contact*, C. R. Acad. Sc. Paris **277 A** (1973), 215–219; J. Math. Pures et Appl. **53** (1974), 459–484.
- [12] —, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Diff. Geom. **12** (1977), 253–300.
- [13] Michor, P. W., “Topics in differential geometry,” Graduate Studies in Mathematics **93**, Amer. Math. Soc., Providence, RI, 2008.
- [14] Morrow, J., and K. Kodaira, “Complex Manifolds,” Amer. Math. Soc. Chelsea Publ., New York, 1971.
- [15] Nurowski, P., and D. C. Robinson, *Generalized exterior forms, geometry and space-time*, Class. Quantum Grav. **18** (2001), L81–L86.
- [16] Nurowski, P., and D. C. Robinson, *Generalized forms and their applications*, Class. Quantum Grav. **19** (2002), 2425–2436.

- [17] Robinson, D. C., *Generalized forms, Chern-Simons and Einstein-Yang-Mills theory*, *Class. Quantum Grav.* **26** (2009), 075019.
- [18] Vaisman, I., “Cohomology and differential forms,” *Pure and Applied Mathematics* **21**, M. Dekker Inc., New York, 1973.

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Received June 22, 2014
and in final form February 18, 2016