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Filiform Lie Algebras Without Rational Structures

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Abstract. Hamrouni and Souissi gave a sufficient condition for a family of 7-dimensional filiform Lie algebras to have no rational structures and constructed an example of a 7-dimensional filiform Lie algebra without rational structures explicitly. In this paper, we give a sufficient condition for arbitrary-dimensional filiform Lie algebras to have no rational structures. Moreover, we consider a sufficient condition for products of filiform Lie algebras and arbitrary nilpotent Lie algebras to have no rational structures and give infinitely many new examples of nilpotent Lie algebras without rational structures.
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1. Introduction

An n-dimensional Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is said to be a Lie algebra with a rational structure if $\mathfrak{g}$ admits a basis such that the structure constants with respect to this basis are rational. In the case where $\mathfrak{g}$ is nilpotent, by a theorem of Mal'cev, a nilpotent Lie algebra $\mathfrak{g}$ has a rational structures if and only if the simply connected nilpotent Lie group G whose Lie algebra is $\mathfrak{g}$ has a uniform lattice. By the classification of nilpotent Lie algebra of dimension $n \leq 6$, it is known that every nilpotent Lie algebra of dimension $n \leq 6$ has a rational structure (cf. [8]).

The first example of a nilpotent Lie algebra without rational structures were given by Mal'cev [7], which is a 16-dimensional nilpotent Lie algebra. Chao [1] constructed 2-step nilpotent Lie algebras of dimension $n \geq 10$ without rational structures explicitly. Scheuneman [9] gave an example of a 2-step nilpotent Lie algebra of dimension 8 without rational structures. Hamrouni and Souissi [6] gave a sufficient condition for a family of 7-dimensional filiform Lie algebras to have no rational structures. By using this condition, they constructed 7-dimensional filiform Lie algebras without rational structures.

In this paper, we give a sufficient condition for almost all filiform Lie algebras to have no rational structures. Moreover, we give a sufficient condition for products of filiform Lie algebras and arbitrary nilpotent Lie algebras to have

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no rational structures and infinitely many new examples of nilpotent Lie algebras without rational structures.

2. Structures of filiform Lie algebras

Let $\mathfrak{g}$ be an n-dimensional nilpotent Lie algebra. The lower central series $\{\mathfrak{g}^{(k)}\}$ of $\mathfrak{g}$ is inductively defined by

$$\mathfrak{g}^{(1)} = \mathfrak{g} \text{ and}$$
$$\mathfrak{g}^{(k)} = [\mathfrak{g}, \mathfrak{g}^{(k-1)}].$$

The nilpotent Lie algebra $\mathfrak{g}$ is said to be r-step if $\mathfrak{g}^{(r)} \neq \{0\}$ and $\mathfrak{g}^{(r+1)} = \{0\}$. By the definition, $\mathfrak{g}$ is abelian if and only if $\mathfrak{g}$ is one-step.

Let d_k be the dimension of $\mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$, where $k = 1, \ldots, r$ and r is the step of $\mathfrak{g}$. The sequence $(d_1, \ldots, d_r)$ is called the type of $\mathfrak{g}$. An n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is said to be filiform if the step of $\mathfrak{g}$ is $n-1$ and the type of $\mathfrak{g}$ is $(2, 1, \ldots, 1)$.

One of the most classical example of filiform Lie algebras is the following.

Example 2.1. Let $L = L(n)$ be the n-dimensional vector space spanned by $\{e_1 \ldots, e_n\}$ over $\mathbb{K}$. Define a Lie bracket $[\ ,\ ]_L$ on L by

$$[e_1, e_i]_L = e_{i+1}$$

for any $i = 2, \ldots, n-1$ and the undefined brackets are zero. Then L is a filiform Lie algebra. We call L the standard filiform Lie algebra.

Let L be the standard filiform Lie algebra and $Z^2(L, L)$ be the space of 2-cocycles of the adjoint module L. Let I_n be an index set defined by

$$I_n^0 = \{(k, s) \in \mathbb{N} \times \mathbb{N} \mid 2 \leq k \leq [n/2], 2k+1 \leq s \leq n\} \text{ and}$$
$$I_n = \begin{cases} I_n^0 & \text{if } n \text{ is odd,} \\ I_n^0 \cup \{(n/2, n)\} & \text{if } n \text{ is even.} \end{cases}$$

Then for any element $(k, s) \in I_n$, an associated 2-cocycle $\psi_{k,s} \in Z^2(L, L)$ is defined by

$$\psi_{k,s}(e_i \wedge e_j) = \begin{cases} (-1)^{k-i} \dbinom{j-k-1}{k-i} e_{s+i+j-2k-1} & \text{if } 2 \leq i \leq k < j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi = \displaystyle\sum_{(k,s) \in I_n} \alpha_{k,s} \psi_{k,s}$ be a 2-cocycle such that

$$\psi(X \wedge \psi(Y \wedge X)) + \psi(Y \wedge \psi(Z \wedge X)) + \psi(Z \wedge \psi(X \wedge Y)) = 0$$

for any $X, Y, Z \in L$, where $\alpha_{k,s} \in \mathbb{K}$. Then we can define a new structure of Lie algebras $[\ ,\ ]$ on $L = \langle e_1, \ldots, e_n \rangle_{\mathbb{K}}$ by the following:

$$[e_i, e_j] = [e_i, e_j]_L + \psi(e_i \wedge e_j).$$

Definition 2.2. We call the Lie algebra L with the above Lie bracket the Lie algebra associated with ψ and denote by L_ψ.

It is known that any n-dimensional filiform Lie algebra over $\mathbb{C}$ is isomorphic to L_ψ for some $\psi \in Z^2(L,L)$ (cf. [5], [10]).

By a computation, we can see that the brackets of L_ψ are given by

$$[e_1, e_j] = e_{j+1}, \quad 2 \le j \le n-1,$$
$$[e_i, e_j] = \sum_{r=1}^{n} \left(\sum_{l=0}^{[(j-i-1)/2]} (-1)^l \binom{j-i-l-1}{l} \alpha_{i+l, r-j+i+2l+1} \right) e_r, \quad 2 \le i < j \le n,$$

where $\alpha_{k,s} = 0$ if $(k,s) \notin I_n$. In fact, the above brackets of L_ψ are determined by the following brackets:

$$[e_1, e_j] = e_{j+1}, \quad 2 \le j \le n-1.$$
$$[e_k, e_{k+1}] = \alpha_{k,2k+1} e_{2k+1} + \cdots + \alpha_{k,n} e_n, \quad 2 \le k \le [(n-1)/2].$$
$$[e_{\frac{n}{2}}, e_{\frac{n+2}{2}}] = \alpha_{\frac{n}{2},n} e_n \quad \text{if } n \text{ is even.}$$

By the above equations, we have the following proposition.

Proposition 2.3. *Let* $\psi = \sum_{(k,s)\in I_n} \alpha_{k,s}\psi_{k,s}$ *be a* 2*-cocycle and* $L_\psi = \langle e_1, \ldots, e_n \rangle_{\mathbb{K}}$ *be the Lie algebra associated with* ψ*. Then* L_ψ *satisfies the following properties:*

(i) $L_\psi^{(k)} = \langle e_{k+1}, \ldots, e_n \rangle_{\mathbb{K}}$ *for any* $k = 2, \ldots, n-1$.

(ii) If n *is odd, then* $[e_i, e_j]_\psi \in \langle e_{i+j}, \ldots, e_n \rangle_{\mathbb{K}}$ *for any* i *and* j.

(iii) If n *is even and* $\alpha_{\frac{n}{2},n} = 0$*, then* $[e_i, e_j]_\psi \in \langle e_{i+j}, \ldots, e_n \rangle_{\mathbb{K}}$ *for any* i *and* j.

(iv) The centralizer $\mathfrak{z}(L_\psi^{(n-2)})$ *of* $L_\psi^{(n-2)}$ *is spanned by* $\{e_2, \ldots, e_n\}$.

3. Main Theorem

Let $L = L(n) = \langle e_1, \ldots, e_n \rangle_{\mathbb{K}}$ be the standard filiform Lie algebra of dimension $n \ge 7$, $\psi = \sum_{(k,s)\in I_n} \alpha_{k,s}\psi_{k,s}$ a 2-cocycle, and L_ψ the Lie algebra associated with ψ. Let $\mathfrak{h}$ be a t-dimensional r-step nilpotent Lie algebra over $\mathbb{K}$, where $0 \le r \le n-2$ and 0-step means $\mathfrak{h}$ is the 0-dimensional Lie algebra. Let $\mathfrak{g} = L_\psi \times \mathfrak{h}$ be the product Lie algebra of L_ψ and $\mathfrak{h}$.

Fix a basis $\{v_1, \ldots, v_t\}$ of $\mathfrak{h}$ such that $\{v_1, \ldots, v_{s_k}\}$ span the ideal $\mathfrak{h}^{(k)}$ for each $k = 2, \ldots, r$. If $\mathfrak{h}$ is 0-step, we put $t = 0$. Then $\mathfrak{g}$ is spanned by $\{e_1, \ldots, e_n, v_1, \ldots, v_t\}$. By Proposition 2.3, $\mathfrak{g}^{(k)}$ satisfies the following properties:

$$\mathfrak{g}^{(k)} = \langle e_{k+1}, \ldots, e_n, v_1, \ldots, v_{s_k} \rangle_{\mathbb{K}} \qquad \text{for any } k = 2, \ldots, r.$$
$$\mathfrak{g}^{(k)} = \langle e_{k+1}, \ldots, e_n \rangle_{\mathbb{K}} \qquad \text{for any } k = r+1, \ldots, n-1.$$
$$\mathfrak{z}(\mathfrak{g}^{(n-2)}) = \langle e_2, \ldots, e_n, v_1, \ldots, v_t \rangle_{\mathbb{K}}.$$

Let a_{ij}^k be the structure constants of L_ψ with respect to the basis $\{e_1, \dots, e_n\}$. Suppose that n is odd or n is even and $\alpha_{\frac{n}{2},n} = 0$. Then we have

$$[e_i, e_j] = \sum_{k=i+j}^{n} a_{ij}^k e_k. \tag{3.1}$$

For each $j = 2, \dots, n-1$, we have $a_{1,j}^{j+1} = 1$ since $[e_1, e_j] = e_{j+1}$.

Lemma 3.1. *Let p, q be integers satisfying $p > 1, q > 1$, $p \neq q$ and $1+p+q = n$. Suppose that r satisfies $p \geq r+1$ and $q \geq r+1$. If $a_{p,q}^{p+q}$ and $a_{q,p+1}^n$ are non-zero, and $\mathfrak{g}$ has a rational structure, then $a_{q,p+1}^n / a_{p,q}^{p+q}$ is a rational number.*

Proof. Suppose that $\mathfrak{g}$ has a rational structure. By Corollary 5.2.2 of [2] ideals $\mathfrak{g}^{(k)}$ are rational for each $k = 2, \dots, n-1$. Since $\mathfrak{g}^{(n-2)}$ is rational, by Proposition 5 of [3], the centralizer $\mathfrak{z}(\mathfrak{g}^{(n-2)})$ is also rational. Therefore there exists a strong Mal'cev basis $\{\epsilon'_1, \dots, \epsilon'_{n+t}\}$ over $\mathbb{Q}$ through $\mathfrak{z}(\mathfrak{g}^{(n-2)}), \mathfrak{g}^{(2)}, \dots, \mathfrak{g}^{(n-1)}$ (see [2]), that is, there exists a basis $\{\epsilon'_1, \dots, \epsilon'_{n+t}\}$ which satisfies the following conditions:

(a) For each $k = 1, \dots, n+t$, $\langle \epsilon'_k, \dots, \epsilon'_{n+t} \rangle_{\mathbb{K}}$ is an ideal of $\mathfrak{g}$.

(b) $\mathfrak{z}(\mathfrak{g}^{(n-2)}) = \langle \epsilon'_2, \dots, \epsilon'_{t+n} \rangle_{\mathbb{K}}$,
$\mathfrak{g}^{(k)} = \langle \epsilon'_{t-s_k+k+1}, \dots, \epsilon'_{t+n} \rangle_{\mathbb{K}}$ for each $k = 2, \dots, r$, and
$\mathfrak{g}^{(k)} = \langle \epsilon'_{t+k+1}, \dots, \epsilon'_{t+n} \rangle_{\mathbb{K}}$ for each $k = r+1, \dots, n-1$.

(c) The structure constants of $\mathfrak{g}$ with respect to the basis $\{\epsilon'_1, \dots, \epsilon'_{t+n}\}$ are rational.

We define a new basis $\{\epsilon_1, \dots, \epsilon_{n+t}\}$ by

$$\begin{aligned}
\epsilon_1 &= \epsilon'_1 \\
\epsilon_2 &= \epsilon'_2, \\
\epsilon_{k+1} &= \epsilon'_{t-s_k+k+1}, \quad k = 2, \dots, r, \\
\epsilon_{k+1} &= \epsilon'_{t+k+1}, \quad k = r+1, \dots, n-1, \text{ and} \\
\epsilon_{n+s_{k+1}+j} &= \epsilon'_{t-s_k+k+1+j}, \quad k = 1, \dots, m, \ j = 1, \dots, s_k - s_{k+1},
\end{aligned}$$

where $s_1 = t$ and $s_{r+1} = 0$. Then the basis $\{\epsilon_1, \dots, \epsilon_{n+t}\}$ satisfies the following conditions:

(i) $\mathfrak{z}(\mathfrak{g}^{(n-2)}) = \langle \epsilon_2, \dots, \epsilon_{n+t} \rangle_{\mathbb{K}}$,
$\mathfrak{g}^{(k)} = \langle \epsilon_{k+1}, \dots, \epsilon_{n+s_k} \rangle_{\mathbb{K}}$ for each $k = 2, \dots, r$, and
$\mathfrak{g}^{(k)} = \langle \epsilon_{k+1}, \dots, \epsilon_n \rangle_{\mathbb{K}}$ for each $k = r+1, \dots, n-1$.

(ii) The structure constants c_{ij}^k of $\mathfrak{g}$ with respect to the basis $\{\epsilon_1, \dots, \epsilon_{n+t}\}$ are rational.

Let (b^i_j) be the matrix for changing the basis $\{e_1, \dots, e_n, v_1, \dots, v_t\}$ to the basis $\{\epsilon_1, \dots, \epsilon_{n+t}\}$. By the condition (i) above, we obtain the following equations:

$$\begin{aligned}
\epsilon_1 &= \sum_{k=1}^{n} b^k_1 e_k + \sum_{k=1}^{t} b^{k+n}_1 v_k. \\
\epsilon_j &= \sum_{k=j}^{n} b^k_j e_k + \sum_{k=1}^{s_{j-1}} b^{k+n}_j v_k, \quad j = 2, \dots, r+1. \\
\epsilon_j &= \sum_{k=j}^{n} b^k_j e_k, \quad j = r+2, \dots, n. \\
\epsilon_j &= \sum_{k=l+1}^{n} b^k_j e_k + \sum_{k=1}^{s_l} b^{k+n}_j v_k, \quad l = 1, \dots, r,\ j = n + s_{l+1} + 1, \dots, n + s_l.
\end{aligned} \tag{3.2}$$

Since (b^i_j) is regular, $b^j_j \neq 0$ for each $j = 1, r+2, \dots, n$. For each $j = 2, \dots, r+1$, by changing ϵ_j to ϵ_{n+l} for a suitable $l = s_j + 1, \dots, s_{j-1}$ if necessary, we may assume that $b^j_j \neq 0$ for each $j = 2, \dots, r+1$.

By equations (3.1) and (3.2) and the fact that $[\mathfrak{h}^{(i)}, \mathfrak{h}^{(j)}] \subset \mathfrak{h}^{(i+j)}$, we have the following:

$$\begin{aligned}
&[\epsilon_1, \epsilon_j] \in \langle e_{1+j}, \dots, e_n, v_1, \dots, v_{s_j} \rangle_{\mathbb{K}} = \mathfrak{g}^{(j)}, \quad j = 2, \dots, n-1. \\
&[\epsilon_i, \epsilon_j] \in \langle e_{i+j}, \dots, e_n, v_1, \dots, v_{s_{i+j-2}} \rangle_{\mathbb{K}} \subset \mathfrak{g}^{(i+j-2)}, \quad 2 \leq i, j \leq n.
\end{aligned}$$

In particular, we have

$$[\epsilon_1, \epsilon_j] \in \langle e_{1+j}, \dots, e_n \rangle_{\mathbb{K}} = \langle \epsilon_{j+1}, \dots, \epsilon_n \rangle_{\mathbb{K}}$$

if j satisfies $r+1 \leq j \leq n-1$ and

$$[\epsilon_i, \epsilon_j] \in \langle e_{i+j}, \dots, e_n \rangle_{\mathbb{K}} = \langle \epsilon_{i+j}, \dots, \epsilon_n \rangle_{\mathbb{K}}$$

if i and j satisfy $2 \leq i, j \leq n$ and $i + j - 2 \geq r + 1$.

By the assumption of p, q, and r, these integers satisfy $1 + p + q = n$, $p \geq r+1$, $q \geq r+1$, and $p + q - 2 \geq r + 1$. Therefore, by equations (3.1) and (3.2), we obtain the equations

$$\begin{aligned}
&[\epsilon_q, [\epsilon_1, \epsilon_p]] = b^q_q b^p_p b^1_1 a^n_{q\,1+p} a^{1+p}_{1p} e_n \quad \text{and} \\
&[\epsilon_1, [\epsilon_p, \epsilon_q]] = b^1_1 b^p_p b^q_q a^n_{1\,p+q} a^{p+q}_{pq} e_n.
\end{aligned} \tag{3.3}$$

On the other hand, we have the following equations:

$$\begin{aligned}
[\epsilon_q,[\epsilon_1,\epsilon_p]] &= [\epsilon_q, \sum_{k=p+1}^{n} c_{1,p}^k \epsilon_k] \\
&= \sum_{k=p+1}^{n} \sum_{l=q+k}^{n} c_{1,p}^k c_{q,k}^l \epsilon_l \\
&= c_{q1+p}^n c_{1p}^{1+p} b_n^n e_n. \\
[\epsilon_1,[\epsilon_p,\epsilon_q]] &= [\epsilon_1, \sum_{k=p+q}^{n} c_{p,q}^k \epsilon_k] \\
&= \sum_{k=p+q}^{n} \sum_{l=1+k}^{n} c_{p,q}^k c_{1,k}^l \epsilon_l \\
&= c_{p,q}^{p+q} c_{1,p+q}^n b_n^n e_n.
\end{aligned} \tag{3.4}$$

Since $a_{1j}^{1+j} = 1$ for each $j = 2, \ldots, n-1$, $a_{pq}^{p+q} \neq 0$, and $a_{q1+p}^n \neq 0$, by equations (3.3) and (3.4), we obtain

$$\begin{aligned}
\frac{b_1^1 b_p^p b_q^q}{b_n^n} &= \frac{c_{q1+p}^n c_{1p}^{1+p}}{a_{q1+p}^n} \quad \text{and} \\
\frac{b_1^1 b_p^p b_q^q}{b_n^n} &= \frac{c_{1p+q}^n c_{pq}^{p+q}}{a_{pq}^{p+q}}.
\end{aligned} \tag{3.5}$$

If i and j satisfy either $i = 1$ and $r+1 \leq j \leq n-1$ or $2 \leq i, j \leq n-1$ and $i+j-2 \geq r+1$, then we have

$$\begin{aligned}
[\epsilon_i, \epsilon_j] &= \sum_{k=i+j}^{n} \sum_{l=k}^{n} c_{ij}^k b_k^l e_l \quad \text{and} \\
[\epsilon_i, \epsilon_j] &= \sum_{k=i}^{n} \sum_{l=j}^{n} \sum_{m=k+l}^{n} b_i^k b_j^l a_{k,l}^m e_m.
\end{aligned}$$

By comparing the $(i+j)$-part of the right-hand sides of these equations, we obtain the equation

$$c_{ij}^{i+j} b_{i+j}^{i+j} = b_i^i b_j^j a_{ij}^{i+j}.$$

Therefore, since $a_{pq}^{p+q} \neq 0$, $a_{1j}^{1+j} = 1$ for each $j = 2, \ldots, n-1$, and $b_k^k \neq 0$ for each $k = 1, \ldots, n$ we have $c_{pq}^{p+q} \neq 0$ and $c_{1j}^{1+j} \neq 0$ for each $j = 2, \ldots, n-1$.

Hence, by the equations (3.5), we have

$$\frac{a_{q1+p}^n}{a_{pq}^{p+q}} = \frac{c_{q1+p}^n c_{1p}^{1+p}}{c_{1p+q}^n c_{pq}^{p+q}}.$$

Since each c_{ij}^k is rational, we have $a_{q1+p}^n / a_{pq}^{p+q} \in \mathbb{Q}$. ∎

Theorem 3.2. *Let* $\psi = \sum_{(k,s)\in I_n} \alpha_{k,s}\psi_{k,s}$ *be a* 2*-cocycle and* L_ψ *be the Lie algebra associated with* ψ*. Let* $\mathfrak{h}$ *be an arbitrary* r*-step nilpotent Lie algebra.*

(i) Suppose that $n = 2m+1$ *and* $r \le m-2$*. If* $\alpha_{m,2m+1}/\alpha_{m-1,2m-1}$ *is irrational, then* $L_\psi \times \mathfrak{h}$ *has no rational structures.*

(ii) Suppose that $n = 2m$*,* $r \le m-3$*, and* $\alpha_{m,n} = 0$*. If* $\alpha_{m-2,2m-3}/\alpha_{m-1,2m-1}$ *is irrational, then* $L_\psi \times \mathfrak{h}$ *has no rational structures.*

Proof. First, we assume that $n = 2m+1$. Let $p = m-1$ and $q = m+1$. Then we have

$$\begin{aligned}
[e_{m-1}, e_{m+1}] &= \sum_{r=1}^{n} \left(\sum_{l=0}^{[1/2]} (-1)^l \binom{1-l}{l} \alpha_{m-1+l,r+2l-1} \right) e_r \\
&= \sum_{r=1}^{n} \alpha_{m-1,r-1} e_r \\
&= \alpha_{m-1,2m-1} e_{2m} + \alpha_{m-1,2m} e_{2m+1} \qquad \text{and} \\
[e_m, e_{m+1}] &= \alpha_{m,2m+1} e_{2m+1}.
\end{aligned}$$

Hence we obtain $a^{2m}_{m-1,m+1} = \alpha_{m-1,2m-1}$. By the assumption,

$$\begin{aligned}
\alpha_{m,2m+1}/\alpha_{m-1,2m-1} &= a^{2m+1}_{m,m+1}/a^{2m}_{m-1,m+1} \\
&= -a^{2m+1}_{m+1,m}/a^{2m}_{m-1,m+1}
\end{aligned}$$

is irrational. Therefore, by Lemma 3.1, $L_\psi \times \mathfrak{h}$ has no rational structures.

Next, we assume that $n = 2m$ and $\alpha_{m,n} = 0$. Let $p = m-2$ and $q = m+1$. Then we have the following equations:

$$\begin{aligned}
[e_{m-2}, e_{m+1}] &= \sum_{r=1}^{n} \left(\sum_{l=0}^{1} (-1)^l \binom{2-l}{l} \alpha_{m-2+l,r+2l-2} \right) e_r \\
&= \sum_{r=1}^{n} (\alpha_{m-2,r-2} - \alpha_{m-1,r}) e_r \\
&= (\alpha_{m-2,2m-3} - \alpha_{m-1,2m-1}) e_{2m-1} + (\alpha_{m-2,2m-2} - \alpha_{m-1,2m}) e_{2m}. \\
[e_{m-1}, e_{m+1}] &= \alpha_{m-1,2m-1} e_{2m}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
a^{2m-1}_{m-2,m+1} &= \alpha_{m-2,2m-3} - \alpha_{m-1,2m-1} \qquad \text{and} \\
a^{2m}_{m-1,m+1} &= \alpha_{m-1,2m-1}.
\end{aligned}$$

Therefore we have

$$a^{2m}_{m+1,m-1}/a^{2m-1}_{m-2,m+1} = -\alpha_{m-1,2m-1}/(\alpha_{m-2,2m-3} - \alpha_{m-1,2m-1}).$$

By the assumption, $\alpha_{m-2,2m-3}/\alpha_{m-1,2m-1}$ is irrational. Hence $a^{2m}_{m+1,m-1}/a^{2m-1}_{m-2,m+1}$ is also irrational. Therefore, by Lemma 3.1, $L_\psi \times \mathfrak{h}$ has no rational structures. ■

4. Examples of Lie algebras without rational structures

By using Theorem 3.2, we give concrete examples of nilpotent Lie algebras without rational structures.

Example 4.1. Let $\psi = \alpha\psi_{2,5} + \psi_{3,7}$ be a 2-cocycle and $\mathfrak{g}_\alpha$ be the 7-dimensional Lie algebra associated with ψ. Then $\mathfrak{g}_\alpha$ is spanned by $\{e_1, \dots e_7\}$ and the Lie bracket is defined by

$$\begin{aligned}
&[e_1, e_j] = e_{j+1} \quad \text{for any } j = 2, \dots, 6, \\
&[e_2, e_3] = \alpha e_5, \\
&[e_2, e_4] = \alpha e_6, \\
&[e_2, e_5] = (\alpha - 1)e_7, \text{ and} \\
&[e_3, e_4] = e_7,
\end{aligned}$$

where undefined brackets are zero. Since $\alpha_{3,7}/\alpha_{2,5} = 1/\alpha$, $\mathfrak{g}_\alpha$ has no rational structures if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

This Lie algebra $\mathfrak{g}_\alpha$ is isomorphic to the example constructed by Hamrouni and Souissi in [6].

Example 4.2. Let $\psi = \alpha\psi_{2,5} + \psi_{3,7}$ be a 2-cocycle and $\mathfrak{g}_\alpha$ be the 8-dimensional Lie algebra associated with ψ. Then $\mathfrak{g}_\alpha$ is spanned by $\{e_1, \dots, e_8\}$ and the Lie bracket is defined by

$$\begin{aligned}
&[e_1, e_j] = e_{j+1} \quad \text{for any } j = 2, \dots, 7, \\
&[e_2, e_3] = \alpha e_5, \\
&[e_2, e_4] = \alpha e_6, \\
&[e_2, e_5] = (\alpha - 1)e_7, \\
&[e_2, 2_6] = (\alpha - 2)e_8, \\
&[e_3, e_4] = e_7, \text{ and} \\
&[e_3, e_5] = e_8,
\end{aligned}$$

where the undefined brackets are zero. Since $\alpha_{2,5}/\alpha_{3,7} = \alpha$, $\mathfrak{g}_\alpha$ is no rational structures if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

We remark that this 8-dimensional Lie algebra $\mathfrak{g}_\alpha$ is not isomorphic to the Lie algebra constructed by Scheuneman in [9].

Example 4.3. Let $\psi = \psi_{2,5} + \alpha\psi_{3,7} + \frac{3\alpha^2}{\alpha+2}\psi_{4,9}$ be a 2-cocycle, where $\alpha \neq -2$, and $\mathfrak{g}_\alpha$ be the 9-dimensional Lie algebra associated with ψ. Then $\mathfrak{g}_\alpha$ is spanned by $\{e_1, \ldots, e_9\}$ and the Lie bracket is defined by

$$
\begin{aligned}
[e_1, e_j] &= e_{j+1} \quad \text{for any } j = 2, \ldots, 8, \\
[e_2, e_3] &= e_5, \\
[e_2, e_4] &= e_6, \\
[e_2, e_5] &= (1-\alpha)e_7, \\
[e_2, e_6] &= (1-2\alpha)e_8, \\
[e_2, e_7] &= \frac{-5\alpha+2}{\alpha+2}e_9, \\
[e_3, e_4] &= \alpha e_7, \\
[e_3, e_5] &= \alpha e_8, \\
[e_3, e_6] &= \frac{2\alpha(-\alpha+1)}{\alpha+2}e_9, \text{ and} \\
[e_4, e_5] &= \frac{3\alpha^2}{\alpha+2}e_9,
\end{aligned}
$$

where the undefined brackets are zero. Since $\alpha_{4,9}/\alpha_{3,7} = \frac{3\alpha}{\alpha+2}$, $\mathfrak{g}_\alpha$ is no rational structures if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Example 4.4. Let $\psi = \psi_{2,5} + \alpha\psi_{3,7} + \frac{3\alpha^2}{\alpha+2}\psi_{4,9}$ be a 2-cocycle, where $\alpha \neq -2$, and $\mathfrak{g}_\alpha$ be the 10-dimensional Lie algebra associated with ψ. Since $\alpha_{3,7}/\alpha_{4,9} = \frac{\alpha+2}{3\alpha}$, $\mathfrak{g}_\alpha$ is no rational structures if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Example 4.5. Let $\mathfrak{g}_\alpha$ be a nilpotent Lie algebra defined in Example 4.1, 4.2, 4.3, and 4.4. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then $\mathfrak{g}_\alpha \times \mathbb{R}^n$ has no rational structures for any n.

This example is also obtained by using a theorem of Ghorbel and Hamrouni (see theorem 5 in [3]).

Example 4.6. Let $\mathfrak{g}_\alpha$ be a nilpotent Lie algebra defined in Example 4.3 and 4.4. Let $\mathfrak{h}$ be an arbitrary dimensional 2-step nilpotent Lie algebra. Then $\mathfrak{g}_\alpha \times \mathfrak{h}$ has no rational structures if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

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