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# On the Classification of Real Four-Dimensional Lie Groups

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**Abstract.** A classification of the real four-dimensional connected Lie groups is obtained. Those groups which are linearizable are identified; accompanying matrix Lie groups are exhibited.
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## 1. Introduction

Complex Lie algebras up to dimension four were originally classified by Lie [15]. A few decades later, Bianchi [2] first classified the three-dimensional real Lie algebras. It was only in the 1950s and 1960s that the real four-dimensional case was treated. Several authors, such as Dobrescu [8], Kručkovič [14], Bratzlavsky [6], Mubarakzyanov [17], and Ellis and Sciama [9], independently enumerated the four-dimensional Lie algebras; Mubarakzyanov's scheme (which is complete and nonredundant) seems to be the most popular (cf. [16, 22, 20, 21]). To date, real Lie algebras up to dimension six have been classified; a complete enumeration of both the real and complex Lie algebras up to dimension six can be found in the recent book [23] by Šnobl and Winternitz. The classification of all real Lie algebras of dimensions beyond six seems unrealistic due to the fast growing number of multiparameter families of pairwise nonisomorphic Lie algebras (cf. [23, 18]). For a more detailed account on the history of the classification of lower-dimensional Lie algebras, see e.g., [16, 22, 24].

By Ado's theorem, every finite-dimensional real Lie algebra is (isomorphic to) the Lie algebra of some matrix Lie group. However, not every connected real Lie group is linearizable. Two classic examples of Lie groups that are not linearizable are the universal covering of the three-dimensional special linear group $\mathsf{SL}(2,\mathbb{R})$ and the quotient of the three-dimensional Heisenberg group by a discrete central

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subgroup. In contrast to low-dimensional real Lie algebras, the corresponding connected Lie groups have been fully classified only up to dimension three.

In this paper we classify the four-dimensional connected real Lie groups corresponding to each four-dimensional real Lie algebra. We prefer to use (a modified version of) the enumeration of the four-dimensional Lie algebras due to Mubarakzyanov [17], similar to that used by Patera et al. [20, 21]. Furthermore, we determine exactly which of the groups are linearizable; a matrix representation is supplied for each linearizable group.

The main body of the paper is divided into two sections. Those Lie groups whose Lie algebras are decomposable (as direct sums of lower dimensional Lie algebras) are considered first (Section 3). Most of these Lie algebras are trivial Abelian extensions of three-dimensional Lie algebras. The classification of the corresponding connected Lie groups is thus closely related to the classification of the three-dimensional Lie groups. The Lie groups with indecomposable Lie algebras are considered next (Section 4). For several Lie algebras, the universal covering Lie group has trivial center and so the classification of the corresponding Lie groups is trivial. The remaining five indecomposable Lie algebras are treated individually.

Appendix A.1 contains a classification of the three-dimensional Lie algebras and connected Lie groups. In Appendix A.2 the four-dimensional Lie algebras are enumerated (Tables 2 and 3). Table 4 cross-references the classification scheme used in this paper against some other schemes. Some basic properties of each four-dimensional Lie algebra are tabulated in Table 5. Lastly, the automorphism group of each four-dimensional Lie algebra is given in Tables 6, 7, and 8. Throughout the paper, we use notation for the three- and four-dimensional Lie algebras as specified in Appendix A.

## 2. Preliminaries

**2.1. Universal coverings and discrete central subgroups.** For every Lie algebra $\mathfrak{g}$, there exists a connected simply connected Lie group $\mathsf{G}$ (the universal covering group) with Lie algebra $\mathfrak{g}$. Any other connected Lie group with Lie algebra $\mathfrak{g}$ is isomorphic to a quotient $\mathsf{G}/\mathsf{N}$, where $\mathsf{N}$ is a discrete central subgroup of $\mathsf{G}$. Two such connected Lie groups $\mathsf{G}/\mathsf{N}$ and $\mathsf{G}/\mathsf{N}'$ are isomorphic if and only if there exists an automorphism $\phi \in \mathsf{Aut}(\mathsf{G})$ such that $\phi(\mathsf{N}) = \mathsf{N}'$; in this case we say that $\mathsf{N}$ and $\mathsf{N}'$ are *equivalent*. Accordingly, in order to classify the connected Lie groups with Lie algebra $\mathfrak{g}$, it suffices to classify the discrete central subgroups of $\mathsf{G}$. (For more details, see, e.g., [11, Part I, Chapter 1, Section 4] or [12, Chapter 9, Section 5].)

**Remark 2.1.** $\mathsf{G}/\mathsf{N}$ covers $\mathsf{G}/\mathsf{N}'$ (i.e., there exists a Lie group covering homomorphism $\phi : \mathsf{G}/\mathsf{N} \to \mathsf{G}/\mathsf{N}'$) if and only if there exists $\varphi \in \mathsf{Aut}(\mathsf{G})$ such that $\varphi(\mathsf{N})$ is a subgroup of $\mathsf{N}'$.

We denote the group of restrictions $\phi|_{\mathsf{Z}(\mathsf{G})} : \mathsf{Z}(\mathsf{G}) \to \mathsf{Z}(\mathsf{G})$ of automorphisms $\phi \in \mathsf{Aut}(\mathsf{G})$ to $\mathsf{Z}(\mathsf{G})$ by $\mathsf{Aut}(\mathsf{G})|_{\mathsf{Z}(\mathsf{G})}$. Two discrete central subgroups are equivalent if and only if there exists $\varphi \in \mathsf{Aut}(\mathsf{G})|_{\mathsf{Z}(\mathsf{G})}$ such that $\varphi(\mathsf{N}) = \mathsf{N}'$. Any element $g \in \mathsf{Z}(\mathsf{G})$ can be expressed as an exponential $g = \exp(A)$ of some $A \in \mathfrak{g}$ (see, e.g., [12, Theorem 14.2.1]). As $\mathsf{G}$ is simply connected, the map $d : \mathsf{Aut}(\mathsf{G}) \to \mathsf{Aut}(\mathfrak{g})$ is a bijection. Hence, the group $\mathsf{Aut}(\mathsf{G})|_{\mathsf{Z}(\mathsf{G})}$ consists of maps

$$\phi : \mathsf{Z}(\mathsf{G}) \to \mathsf{Z}(\mathsf{G}), \qquad \exp(A) \mapsto \exp(\psi \cdot A), \qquad \psi \in \mathsf{Aut}(\mathfrak{g}).$$

A standard computation yields the Lie algebra automorphism group $\mathsf{Aut}(\mathfrak{g})$. The automorphism group for each four-dimensional Lie algebra is given in Appendix A.2 (Tables 6, 7, and 8). These explicit automorphisms are used to calculate $\mathsf{Aut}(\mathsf{G})|_{\mathsf{Z}(\mathsf{G})}$, as explained above, in several cases. We note, however, that for many algebras we arrive at our results without depending on the explicit form of $\mathsf{Aut}(\mathfrak{g})$; nonetheless, for the sake of completeness, we tabulate the automorphisms for each Lie algebra in Appendix A.2.

Frequently $\mathsf{Z}(\mathsf{G})$ is isomorphic to $\mathbb{R}^n$ or some subgroup of $\mathbb{R}^n$. Any discrete subgroup $\mathsf{N}$ of $\mathbb{R}^n$ can be expressed as

$$\mathsf{N} = \{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 + \cdots + m_\ell\boldsymbol{v}_\ell \,:\, m_1, m_2, \ldots, m_\ell \in \mathbb{Z}\}$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_\ell$ are linearly independent vectors in $\mathbb{R}^n$ and $\ell \leq n$ (see, e.g, [19, Chapter 2, Theorem 1.1] or [5, Chapter VII, Section 1.1]). We refer to the matrix $B = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_\ell\end{bmatrix} \in \mathbb{R}^{n\times\ell}$ of rank $\ell$ as a basis for $\mathsf{N}$. Two bases $B$ and $B'$ generate the same discrete subgroup $\mathsf{N}$ of $\mathbb{R}^n$ if and only if there exists $U \in \mathsf{GL}\,(\ell, \mathbb{Z})$ such that $B = B'U$ (see, e.g., [5, Chapter VII, Section 1.1]). Here $\mathsf{GL}\,(\ell, \mathbb{Z})$ denotes the group of integer matrices $\{g \in \mathbb{Z}^{\ell\times\ell} \,:\, \det g = \pm 1\}$.

**2.2. Linearizability of real Lie groups.** A Lie group $\mathsf{G}$ is *linearizable* if it admits a faithful finite-dimensional linear representation; for real Lie groups this is equivalent to $\mathsf{G}$ admitting a faithful finite-dimensional linear representation with closed image (see, e.g., [12, Theorem 16.2.10]). In other words, a linearizable (real) Lie group is one that is isomorphic to a matrix Lie group (i.e., a closed subgroup of $\mathsf{GL}(n, \mathbb{R})$). For every Lie algebra $\mathfrak{g}$, there exists at least one connected matrix Lie group which has Lie algebra $\mathfrak{g}$ (some matrix representations for low-dimensional connected Lie groups are given in [10]). We shall make use of the following results to identify exactly which four-dimensional connected Lie groups are linearizable.

First, we consider the linearizability of solvable Lie groups. Note that all but two of the four-dimensional Lie algebras are solvable (Table 5); the two exceptions are the trivial Abelian extensions of the simple three-dimensional Lie algebras $\mathfrak{g}_{3.6} \cong \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{g}_{3.7} \cong \mathfrak{so}(3)$.

**Theorem 2.2** ([12, Chapter 16 and Theorem 11.2.14] or [18, Chapter 2, Section 7 and Chapter 2, Section 3.2, Corollary 2])**.** *If $\mathsf{G}$ is a connected simply connected solvable Lie group, then $\mathsf{G}$ is linearizable and diffeomorphic to $\mathbb{R}^n$.*

When the group is not simply connected, we make use of the following theorem (or rather, its corollary).

**Theorem 2.3** (cf. [12, Theorem 16.2.9])**.** *Let $\mathsf{G}$ be a connected solvable Lie group with Lie algebra $\mathfrak{g}$ and let $\mathsf{T}$ be a maximal torus in $\mathsf{G}$ with Lie algebra*

$\mathfrak{t}$. *Then* $\mathsf{G}$ *is linearizable if and only if the intersection* $\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}]$ *of* $\mathfrak{t}$ *with the commutator subalgebra is trivial.*

**Corollary 2.4.** *Let* $\mathsf{G}$ *be a connected simply connected solvable Lie group with Lie algebra* $\mathfrak{g}$, *let* $\mathsf{N} = \exp(\mathbb{Z}A_1) \cdots \exp(\mathbb{Z}A_\ell)$, $A_1, \ldots, A_\ell \in \mathfrak{g}$ *be a central discrete subgroup, and suppose* $\mathfrak{t} = \operatorname{span}\{A_1, \ldots, A_\ell\}$ *is an Abelian subalgebra of* $\mathfrak{g}$. *Then* $\mathsf{G}/\mathsf{N}$ *is linearizable if and only if* $\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$.

**Proof.** Let $q : \mathsf{G} \to \mathsf{G}/\mathsf{N}$ be the corresponding covering homomorphism and let $\mathsf{T} = \langle \exp \mathfrak{t} \rangle$ be the integral subgroup corresponding to $\mathfrak{t}$. It is not difficult to show that $q(\mathsf{T})$ is a maximal torus in $\mathsf{G}/\mathsf{N}$ and hence the result follows. ■

**Remark 2.5.** For an indecomposable nilpotent Lie algebra, *only* the universal covering Lie group is linearizable (cf. [18, Chapter 2, Theorem 7.3]).

The linearizability of semisimple Lie groups can usually be checked by studying the linearizer of the group or by considering its universal complexification (see, e.g., [18, Chapter 4, Section 3.6] and [12, Chapter 16]). However, there are no semisimple four-dimensional Lie algebras. For the purposes of this paper, it suffices to know that the universal cover $\widetilde{\mathsf{A}}$, as well as the $n$-fold covers $\mathsf{A}_n$, $n \geq 3$, of $\mathsf{SO}(2,1)_0 \cong \mathsf{PSL}(2, \mathbb{R})$ are not linearizable (see, e.g., [12, Example 16.1.8] or [18, Example on p. 152])

Lastly, when a group $\mathsf{G}$ is neither solvable nor semisimple, one considers its Levi decomposition. Let $\mathfrak{g}$ be the Lie algebra of $\mathsf{G}$. There exists a semisimple subalgebra $\mathfrak{s}$ such that $\mathfrak{g}$ decomposes as a semidirect sum

$$\mathfrak{g} = \operatorname{rad}(\mathfrak{g}) \rtimes \mathfrak{s}$$

of the radical $\operatorname{rad}(\mathfrak{g})$ (the maximal solvable ideal) and the Levi subalgebra $\mathfrak{s}$ ([12, Chapter 5, Section 6] or [18, Chapter 1, Section 4]). The integral subgroup $\mathsf{S} = \langle \exp \mathfrak{s} \rangle$ corresponding to $\mathfrak{s}$ is referred to as a Levi subgroup. Let $\operatorname{Rad}(\mathsf{G})$ be the radical of $\mathsf{G}$ (i.e., the maximal connected solvable normal closed subgroup; $\operatorname{Rad}(\mathsf{G})$ has Lie algebra $\operatorname{rad}(\mathfrak{g})$). We have that

$$\mathsf{G} = (\operatorname{Rad}(\mathsf{G}))\mathsf{S}, \quad \dim((\operatorname{Rad}(\mathsf{G})) \cap \mathsf{S}) = 0.$$

When $\mathsf{G}$ is simply connected, then $\operatorname{Rad}(\mathsf{G})$ and $\mathsf{S}$ are simply connected, $\mathsf{S}$ is a closed subgroup, and the above decomposition is a semidirect one ([18, Chapter 1, Section 4]). (Note that in general $\mathsf{S}$ may not be closed in $\mathsf{G}$; also, $\mathsf{S}$ could be dense in $\mathsf{G}$, see, e.g., [18, Example on p. 19]). The linearizability of $\mathsf{G}$ can be reduced to the linearizability of its radical $\operatorname{Rad}(\mathsf{G})$ (which is solvable) and the linearizability of any Levi subgroup $\mathsf{S}$ (which is semisimple).

**Theorem 2.6** ([19, Chapter 1, Theorem 5.2]; cf. [12, Theorem 5.6.13])**.** *A connected Lie group* $\mathsf{G}$ *is linearizable if and only if both* $\operatorname{Rad}(\mathsf{G})$ *and* $\mathsf{S}$ *are linearizable.*

## 3. Groups with decomposable algebras

We consider each of the decomposable four-dimensional Lie algebras (Table 2) and classify the corresponding Lie groups. First, we dispense with the Abelian groups. We then briefly treat those algebras whose universal covering group is a trivial Abelian extension of a simply connected Lie group with trivial center. We then proceed the consider each of the remaining decomposable Lie algebras in turn.

**3.1. Abelian groups.** There are exactly five four-dimensional connected Abelian Lie groups, namely $\mathbb{R}^k \times \mathbb{T}^\ell$, $k+\ell = 4$. Any connected Abelian Lie group is clearly linearizable.

**3.2. Trivial Abelian extensions of simply connected groups with trivial center.** Let $\mathsf{G}$ be the universal covering group for a solvable Lie algebra $\mathfrak{g}$. Suppose $\mathsf{G}$ has a trivial center. The extension $\mathsf{G} \times \mathbb{R}^n$ of $\mathsf{G}$ by the Abelian group $\mathbb{R}^n$ is then the universal covering group for $\mathfrak{g} \oplus n\mathfrak{g}_1$. The center of $\mathsf{G} \times \mathbb{R}^n$ is $\{\mathbf{1}\} \times \mathbb{R}^n$. Moreover, any automorphism of $\mathbb{R}^n$ can be extended to an automorphism of $\mathsf{G} \times \mathbb{R}^n$. Consequently, any connected Lie group with Lie algebra $\mathfrak{g} \oplus n\mathfrak{g}_1$ is isomorphic to a group $\mathsf{G} \times \mathbb{R}^k \times \mathbb{T}^\ell$ where $k+\ell = n$. Moreover, as the Abelian groups and simply connected solvable groups are linearizable, so is their product.

We identify those (decomposable) four-dimensional solvable Lie algebras for which the universal covering group is the trivial Abelian extension of a connected simply connected Lie group with trivial center. By the above argument, we get the following classification of the corresponding Lie groups.

**Theorem 3.1.** *Suppose $\mathsf{G}$ is a connected Lie group with Lie algebra $\mathfrak{g}$.*

1. *If $\mathfrak{g} \cong \mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$, then $\mathsf{G}$ is isomorphic to $\mathsf{G}_{2.1} \times \mathbb{R}^2$, $\mathsf{G}_{2.1} \times \mathbb{R} \times \mathbb{T}$, or $\mathsf{G}_{2.1} \times \mathbb{T}^2$; here*
$$\mathsf{G}_{2.1} = \left\{ \begin{bmatrix} 1 & 0 \\ x & \mathrm{e}^{-y} \end{bmatrix} \; : \; x, y \in \mathbb{R} \right\} \cong \mathsf{Aff}\,(\mathbb{R})_0.$$
2. *If $\mathfrak{g} \cong \mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$, then $\mathsf{G}$ is isomorphic to $\mathsf{G}_{3.2} \times \mathbb{R}$ or $\mathsf{G}_{3.2} \times \mathbb{T}$; here*
$$\mathsf{G}_{3.2} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ y & \mathrm{e}^{z} & 0 \\ x & -z\,\mathrm{e}^{z} & \mathrm{e}^{z} \end{bmatrix} \; : \; x, y, z \in \mathbb{R} \right\}.$$
3. *If $\mathfrak{g} \cong \mathfrak{g}_{3.3} \oplus \mathfrak{g}_1$, then $\mathsf{G}$ is isomorphic to $\mathsf{G}_{3.3} \times \mathbb{R}$ or $\mathsf{G}_{3.3} \times \mathbb{T}$; here*
$$\mathsf{G}_{3.3} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ y & \mathrm{e}^{z} & 0 \\ x & 0 & \mathrm{e}^{z} \end{bmatrix} \; : \; x, y, z \in \mathbb{R} \right\}.$$
4. *If $\mathfrak{g} \cong \mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1$, then $\mathsf{G}$ is isomorphic to $\mathsf{G}_{3.4}^0 \times \mathbb{R}$ or $\mathsf{G}_{3.4}^0 \times \mathbb{T}$; here*
$$\mathsf{G}_{3.4}^0 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh z & -\sinh z \\ y & -\sinh z & \cosh z \end{bmatrix} \; : \; x, y, z \in \mathbb{R} \right\} = \mathsf{SE}\,(1,1).$$

5. *If* $\mathfrak{g} \cong \mathfrak{g}_{3.4}^{\alpha} \oplus \mathfrak{g}_1$, $\alpha > 0$, $\alpha \neq 1$, *then* $\mathsf{G}$ *is isomorphic to* $\mathsf{G}_{3.4}^{\alpha} \times \mathbb{R}$ *or* $\mathsf{G}_{3.4}^{\alpha} \times \mathbb{T}$*; here*

$$\mathsf{G}_{3.4}^{\alpha} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \mathrm{e}^{\alpha z} \cosh z & -\mathrm{e}^{\alpha z} \sinh z \\ y & -\mathrm{e}^{\alpha z} \sinh z & \mathrm{e}^{\alpha z} \cosh z \end{bmatrix} \; : \; x, y, z \in \mathbb{R} \right\}.$$

6. *If* $\mathfrak{g} \cong \mathfrak{g}_{3.5}^{\alpha} \oplus \mathfrak{g}_1$, $\alpha > 0$, *then* $\mathsf{G}$ *is isomorphic to* $\mathsf{G}_{3.5}^{\alpha} \times \mathbb{R}$ *or* $\mathsf{G}_{3.5}^{\alpha} \times \mathbb{T}$*; here*

$$\mathsf{G}_{3.5}^{\alpha} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \mathrm{e}^{\alpha z} \cos z & -\mathrm{e}^{\alpha z} \sin z \\ y & \mathrm{e}^{\alpha z} \sin z & \mathrm{e}^{\alpha z} \cos z \end{bmatrix} \; : \; x, y, z \in \mathbb{R} \right\}.$$

From the list of decomposable algebras (Table 2), only $2\mathfrak{g}_{2.1}$, $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3.5}^{0}$, $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ and $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$ remain; we now proceed to study each of these cases in turn.

**3.3. Groups with algebra $2\mathfrak{g}_{2.1}$.** The universal covering group

$$\mathsf{G}_{2.1} \times \mathsf{G}_{2.1} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ w & \mathrm{e}^{-x} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & y & \mathrm{e}^{-z} \end{bmatrix} \; : \; w, x, y, z \in \mathbb{R} \right\}$$

has trivial center. Thus $\mathsf{G}_{2.1} \times \mathsf{G}_{2.1}$ is the only connected group with algebra $2\mathfrak{g}_{2.1}$.

**3.4. Groups with algebra $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ (trivial extension of the Heisenberg algebra).** There are exactly two connected Lie groups with Lie algebra $\mathfrak{g}_{3.1}$, namely the Heisenberg group

$$\mathsf{H}_3 = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \; : \; x, y, z \in \mathbb{R} \right\}$$

and its quotient $\mathsf{H}_3^* = \mathsf{H}_3 / \mathsf{Z}$ by the discrete central subgroup

$$\mathsf{Z} = \left\{ \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; : \; x \in \mathbb{Z} \right\}.$$

($\mathsf{H}_3^*$ is not linearizable.) The universal covering group for $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ is the trivial Abelian extension

$$\mathsf{H}_3 \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & x & w & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{z} \end{bmatrix} = p(w, x, y, z) \; : \; w, x, y, z \in \mathbb{R} \right\}$$

of $\mathsf{H}_3$. The Lie algebra of $\mathsf{H}_3 \times \mathbb{R}$ is given by

$$\mathfrak{h}_3 \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & x & w & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \; : \; w, x, y, z \in \mathbb{R} \right\}.$$

The following lemma proves useful in classifying the discrete central subgroups.

**Lemma 3.2.** *Suppose* $\alpha, \alpha' \in \mathbb{R}$*. Then there exist* $a_1, a_2, a_3 \in \mathbb{R}$*,* $a_1 a_3 \neq 0$ *such that*

$$\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \left( \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \mathbb{Z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbb{Z} \right) = \begin{bmatrix} 1 \\ \alpha' \end{bmatrix} \mathbb{Z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbb{Z}$$

*if and only if* $\alpha = \dfrac{m_3 + m_1\alpha'}{m_4 + m_2\alpha'}$ *for some* $\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \in \mathsf{GL}(2, \mathbb{Z})$*.*

**Proof.** The discrete subgroups $\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \left( \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \mathbb{Z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbb{Z} \right)$ and $\begin{bmatrix} 1 \\ \alpha' \end{bmatrix} \mathbb{Z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbb{Z}$ are identical if and only if there exists $\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \in \mathsf{GL}(2, \mathbb{Z})$ such that

$$\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha' & 1 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$$

for some $a_1, a_3 \neq 0$ and $a_2 \in \mathbb{R}$. Equivalently,

$$a_2 = m_2, \qquad a_1 = m_1 - m_2\alpha \neq 0, \qquad a_3 = m_4 + m_2\alpha' \neq 0, \quad \text{and} \quad \alpha = \frac{m_3 + m_1\alpha'}{m_4 + m_2\alpha'}$$

for some $\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \in \mathsf{GL}(2, \mathbb{Z})$. It is easy to show that $m_1 - m_2\alpha \neq 0$ provided $m_4 + m_2\alpha' \neq 0$, $\alpha = \frac{m_3 + m_1\alpha'}{m_4 + m_2\alpha'}$, and $\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \in \mathsf{GL}(2, \mathbb{Z})$. ■

**Theorem 3.3.** *Any (nontrivial) discrete central subgroup of* $\mathsf{H}_3 \times \mathbb{R}$ *is equivalent to one of the following discrete subgroups*

$$\begin{aligned} \exp(\mathbb{Z}E_1) &= \{p(w, 0, 0, 0) \,:\, w \in \mathbb{Z}\} \\ \exp(\mathbb{Z}E_4) &= \{p(0, 0, 0, z) \,:\, z \in \mathbb{Z}\} \\ \exp(\mathbb{Z}E_1)\exp(\mathbb{Z}E_4) &= \{p(w, 0, 0, z) \,:\, w, z \in \mathbb{Z}\} \\ \exp(\mathbb{Z}(E_1 + \alpha E_4))\exp(\mathbb{Z}E_4) &= \{p(w, 0, 0, \alpha w + z) \,:\, w, z \in \mathbb{Z}\}. \end{aligned}$$

*Here* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$*,* $0 < \alpha < 1$ *parametrizes a family of discrete subgroups. The subgroups* $\exp(\mathbb{Z}(E_1 + \alpha E_4))\exp(\mathbb{Z}E_4)$ *and* $\exp(\mathbb{Z}(E_1 + \alpha' E_4))\exp(\mathbb{Z}E_4)$ *are equivalent if and only if* $\alpha = \dfrac{m_3 + m_1\alpha'}{m_4 + m_2\alpha'}$ *for some* $\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \in \mathsf{GL}(2, \mathbb{Z})$*; no other pairs of discrete central subgroups are equivalent.*

**Proof.** The center of $\mathsf{H}_3 \times \mathbb{R}$ is $\exp(\mathbb{R}E_1)\exp(\mathbb{R}E_4)$. The group of automorphisms $\mathsf{Aut}(\mathfrak{h}_3 \oplus \mathfrak{g}_1)$ is given by

$$\left\{ \begin{bmatrix} a_2a_7 - a_6a_3 & a_1 & a_5 & a_9 \\ 0 & a_2 & a_6 & 0 \\ 0 & a_3 & a_7 & 0 \\ 0 & a_4 & a_8 & a_{10} \end{bmatrix} \,:\, a_1, \ldots, a_{10} \in \mathbb{R},\ (a_2a_7 - a_3a_6)a_{10} \neq 0 \right\}.$$

Let $\psi \in \mathsf{Aut}(\mathfrak{h}_3 \oplus \mathfrak{g}_1)$ and let $\phi \in \mathsf{Aut}(\mathsf{H}_3 \times \mathbb{R})$ be the unique automorphism such that $T_{\mathbf{1}}\phi = \psi$. We have that

$$\begin{aligned}\phi(\exp(wE_1)\exp(zE_4)) &= \exp(w\psi \cdot E_1)\exp(z\psi \cdot E_4)\\ &= \exp\left(\left((a_2a_7 - a_3a_6)\, w + a_9 z\right) E_1\right)\exp(a_{10}E_4).\end{aligned}$$

We identify an element $\exp(wE_1)\exp(zE_4) \in \mathsf{Z}(\mathsf{H}_3 \times \mathbb{R})$ with the pair $(w, z) \in \mathbb{R}^2$. Hence, the restriction of the automorphism group $\mathsf{Aut}(\mathsf{H}_3 \times \mathbb{R})$ to $\mathsf{Z}(\mathsf{H}_3 \times \mathbb{R})$ is given by

$$\mathsf{Aut}(\mathsf{H}_3 \times \mathbb{R})|_{\mathsf{Z}(\mathsf{H}_3\times\mathbb{R})} = \left\{ \begin{bmatrix} w \\ z \end{bmatrix} \mapsto \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \;:\; a_1, a_2, a_3 \in \mathbb{Z},\; a_1a_3 \neq 0 \right\}.$$

Let $\mathsf{N}$ be a discrete central subgroup. Then $\mathsf{N} = (w, 0)\mathbb{Z}$, $\mathsf{N} = (0, z)\mathbb{Z}$, $\mathsf{N} = (w, z)\mathbb{Z}$, or $\mathsf{N} = (w_1, z_1)\mathbb{Z} + (w_2, z_2)\mathbb{Z}$ for some $w, z, z_1, z_2, w_1, w_2 \in \mathbb{R}$ with $(w_1, z_1)$ and $(w_2, z_2)$ linearly independent, $w, z \neq 0$. If $\mathsf{N} = (w, 0)\mathbb{Z}$, then $\varphi = \operatorname{diag}(\frac{1}{w}, 1) \in \mathsf{Aut}(\mathsf{H}_3 \times \mathbb{R})|_{\mathsf{Z}(\mathsf{H}_3\times\mathbb{R})}$ and $\varphi(\mathsf{N}) = (1, 0)\mathbb{Z} = \exp(\mathbb{Z}E_1)$. Likewise if $\mathsf{N} = (0, z)\mathbb{Z}$, then $\varphi = \operatorname{diag}(1, \frac{1}{z}) \in \mathsf{Aut}(\mathsf{H}_3 \times \mathbb{R})|_{\mathsf{Z}(\mathsf{H}_3\times\mathbb{R})}$ and $\varphi(\mathsf{N}) = (0, 1)\mathbb{Z} = \exp(\mathbb{Z}E_4)$. On the other hand, if $\mathsf{N} = (w, z)\mathbb{Z}$, then

$$\varphi = \begin{bmatrix} z & -w \\ 0 & \frac{1}{z} \end{bmatrix} \in \mathsf{Aut}(\mathsf{H}_3 \times \mathbb{R})|_{\mathsf{Z}(\mathsf{H}_3\times\mathbb{R})}$$

and $\varphi(\mathsf{N}) = (0, 1)\mathbb{Z} = \exp(\mathbb{Z}E_4)$.

Assume $\mathsf{N} = (w_1, z_1)\mathbb{Z} + (w_2, z_2)\mathbb{Z}$. Either $z_1 \neq 0$ or $z_2 \neq 0$; so we may assume $z_1 \neq 0$. Hence

$$\varphi = \begin{bmatrix} \frac{z_1}{w_2z_1 - w_1z_2} & -\frac{w_1}{w_2z_1 - w_1z_2} \\ 0 & \frac{1}{z_1} \end{bmatrix} \in \mathsf{Aut}(\mathsf{H}_3 \times \mathbb{R})|_{\mathsf{Z}(\mathsf{H}_3\times\mathbb{R})}$$

and $\varphi(\mathsf{N}) = (1, \frac{z_2}{z_1})\mathbb{Z} + (0, 1)\mathbb{Z}$. We have $(1, \frac{z_2}{z_1})\mathbb{Z} + (0, 1)\mathbb{Z} = (1, \frac{z_2}{z_1} + \ell)\mathbb{Z} + (0, 1)\mathbb{Z}$ for any $\ell \in \mathbb{Z}$. Therefore $\varphi(\mathsf{N}) = \exp(\mathbb{Z}(E_1 + \alpha E_4))\exp(\mathbb{Z}E_4)$ where $0 \leq \alpha < 1$ and $\alpha = \frac{z_2}{z_1} + \ell$ for some $\ell \in \mathbb{Z}$. Suppose $\alpha \in \mathbb{Q}$. Then $\alpha = \frac{n_1}{n_2}$ where $n_1, n_2 \in \mathbb{Z}$ and $\gcd(n_1, n_2) = 1$. By Bézout's identity, there exists integers $m_1, m_2$ such that $m_1n_1 + m_2n_2 = 1$. Consequently $\alpha = \frac{n_1 - m_2\alpha'}{n_2 + m_1\alpha'}$ where $\alpha' = 0$ and $\begin{bmatrix} -m_2 & m_1 \\ n_1 & n_2 \end{bmatrix} \in \mathsf{GL}\,(2, \mathbb{Z})$. Hence, by Lemma 3.2, there exists $\varphi' \in \mathsf{Aut}(\mathsf{H}_3 \times \mathbb{R})|_{\mathsf{Z}(\mathsf{H}_3\times\mathbb{R})}$ such that $(\varphi' \circ \varphi)(\mathsf{N}) = (1, 0)\mathbb{Z} + (0, 1)\mathbb{Z} = \exp(\mathbb{Z}E_1)\exp(\mathbb{Z}E_4)$.

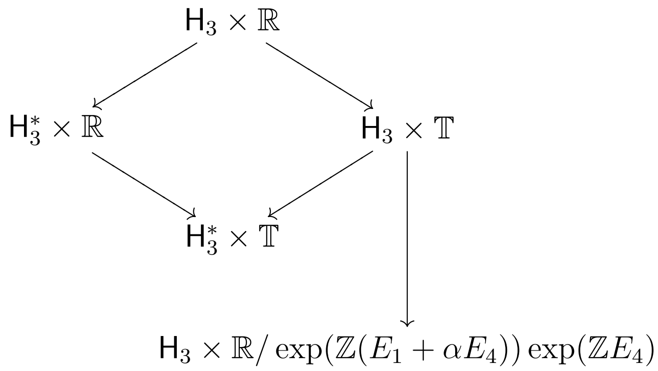
We note that $(1, 0)$ is an eigenvector of the matrix of any automorphism in $\mathsf{Aut}(\mathsf{H}_3 \times \mathbb{R})|_{\mathsf{Z}(\mathsf{H}_3\times\mathbb{R})}$. It follows that $\exp(\mathbb{Z}E_1)$ is not equivalent to any of the other subgroups. Clearly $\exp(\mathbb{Z}E_4)$ is not equivalent to $\exp(\mathbb{Z}(E_1 + \alpha E_4))\exp(\mathbb{Z}E_4)$ for any $0 \leq \alpha < 1$. We claim that $\exp(\mathbb{Z}E_1)\exp(\mathbb{Z}E_4)$ is not equivalent to $\exp(\mathbb{Z}(E_1 + \alpha E_4))\exp(\mathbb{Z}E_4)$ for any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Indeed, if they were equivalent, then by Lemma 3.2 we would have that $\alpha = \frac{m_3}{m_4}$ for some integers $m_3, m_4$, yielding a contradiction. The characterization for when $\exp(\mathbb{Z}(E_1 + \alpha E_4))\exp(\mathbb{Z}E_4)$ and $\exp(\mathbb{Z}(E_1 + \alpha' E_4))\exp(\mathbb{Z}E_4)$ are equivalent follows directly from Lemma 3.2. ■

**Corollary 3.4.** *There are five types of connected Lie groups with Lie algebra* $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$*, namely*

1. $\mathsf{H}_3 \times \mathbb{R}$;
2. $\mathsf{H}_3 \times \mathbb{R}/\exp(\mathbb{Z}E_1) \cong \mathsf{H}_3^* \times \mathbb{R}$;
3. $\mathsf{H}_3 \times \mathbb{R}/\exp(\mathbb{Z}E_4) \cong \mathsf{H}_3 \times \mathbb{T}$;
4. $\mathsf{H}_3 \times \mathbb{R}/\exp(\mathbb{Z}E_1)\exp(\mathbb{Z}E_4) \cong \mathsf{H}_3^* \times \mathbb{T}$;
5. $\mathsf{H}_3 \times \mathbb{R}/\exp(\mathbb{Z}(E_1 + \alpha E_4))\exp(\mathbb{Z}E_4)$.

*Here* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $0 < \alpha < 1$ *parametrizes a family of connected Lie groups;* $\mathsf{H}_3 \times \mathbb{R}/\exp(\mathbb{Z}(E_1 + \alpha E_4))\exp(\mathbb{Z}E_4)$ *and* $\mathsf{H}_3 \times \mathbb{R}/\exp(\mathbb{Z}(E_1 + \alpha' E_4))\exp(\mathbb{Z}E_4)$ *are isomorphic if and only if* $\alpha = \dfrac{m_3 + m_1\alpha'}{m_4 + m_2\alpha'}$ *for some* $\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \in \mathsf{GL}(2, \mathbb{Z})$.

**Remark 3.5.** We have the following diagram of coverings:




**Proposition 3.6.** *Among the connected Lie groups with Lie algebra* $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$, *only* $\mathsf{H}_3 \times \mathbb{R}$ *and* $\mathsf{H}_3 \times \mathbb{R}/\exp(\mathbb{Z}E_4) \cong \mathsf{H}_3 \times \mathbb{T}$ *are linearizable.*

**Proof.** We have that $[\mathfrak{h}_3 \oplus \mathfrak{g}_1, \mathfrak{h}_3 \oplus \mathfrak{g}_1] = \operatorname{span}\{E_1\}$ and so

$$\begin{aligned} \operatorname{span}\{E_1\} \cap [\mathfrak{h}_3 \oplus \mathfrak{g}_1, \mathfrak{h}_3 \oplus \mathfrak{g}_1] &\neq \{0\} \\ \operatorname{span}\{E_4\} \cap [\mathfrak{h}_3 \oplus \mathfrak{g}_1, \mathfrak{h}_3 \oplus \mathfrak{g}_1] &= \{0\} \\ \operatorname{span}\{E_1, E_4\} \cap [\mathfrak{h}_3 \oplus \mathfrak{g}_1, \mathfrak{h}_3 \oplus \mathfrak{g}_1] &\neq \{0\} \\ \operatorname{span}\{E_1 + \alpha E_4, E_4\} \cap [\mathfrak{h}_3 \oplus \mathfrak{g}_1, \mathfrak{h}_3 \oplus \mathfrak{g}_1] &\neq \{0\}. \end{aligned}$$

The result then follows by Corollary 2.4. ∎

### 3.5. Groups with algebra $\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$ (trivial extension of the Euclidean algebra).

Any connected Lie group with Lie algebra $\mathfrak{g}_{3.5}^0$ is isomorphic to either the Euclidean group

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cos z & -\sin z \\ y & \sin z & \cos z \end{bmatrix} \;:\; x, y, z \in \mathbb{R} \right\}$$

its $n$-fold covering

$$\mathsf{SE}_n(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & \cos z & -\sin z & 0 \\ y & \sin z & \cos z & 0 \\ 0 & 0 & 0 & e^{\frac{iz}{n}} \end{bmatrix} \;:\; x, y, z \in \mathbb{R} \right\}$$

or its simply connected universal covering

$$\widetilde{\mathsf{SE}}\,(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & \cos z & -\sin z & 0 \\ y & \sin z & \cos z & 0 \\ 0 & 0 & 0 & \mathrm{e}^z \end{bmatrix} \; : \; x, y, z \in \mathbb{R} \right\}.$$

(Here $\mathsf{SE}_1(2) \cong \mathsf{SE}\,(2)$.) Accordingly, the universal covering for $\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$ is the trivial Abelian extension

$$\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ w & \cos y & -\sin y & 0 & 0 \\ x & \sin y & \cos y & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^y & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^z \end{bmatrix} = p(w,x,y,z) \; : \; w,x,y,z \in \mathbb{R} \right\}$$

of $\widetilde{\mathsf{SE}}\,(2)$. The Lie algebra of $\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R}$ is given by

$$\widetilde{\mathfrak{se}}\,(2) \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ w & 0 & -y & 0 & 0 \\ x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & z \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \; : \; w,z,y,z \in \mathbb{R} \right\}.$$

The following lemma proves useful in classifying the discrete central subgroups.

**Lemma 3.7.** *Suppose* $y_1, y_2 \in \mathbb{Z}$, $z_1, z_2 \in \mathbb{R}$, *and* $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^2$ *are linearly independent. Then there exist* $a_1, a_2 \in \mathbb{R}$, $a_1 > 0$ *such that*

$$\begin{bmatrix} 1 & 0 \\ a_2 & a_1 \end{bmatrix} \left( \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} y_2 \\ z_2 \end{bmatrix} \mathbb{Z} \right) = \begin{bmatrix} \gcd(y_1, y_2) \\ 0 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbb{Z}.$$

**Proof.** By Bézout's identity, there exists integers $m, n \in \mathbb{Z}$ such that $my_1 + ny_2 = \gcd(y_1, y_2) > 0$. Also, as $(y_1, z_1)$ and $(y_2, z_2)$ are linearly independent, we have that $y_1 z_2 - y_2 z_1 \neq 0$. Let $\sigma = \operatorname{sgn}(y_1 z_2 - y_2 z_1)$. We claim that $a_1 = \frac{my_1 + ny_2}{|y_1 z_2 - y_2 z_1|} > 0$ and $a_2 = -\frac{mz_1 + nz_2}{|y_1 z_2 - y_2 z_1|}$ satisfies the conditions of the lemma. Indeed

$$\begin{bmatrix} 1 & 0 \\ -\frac{mz_1 + nz_2}{|y_1 z_2 - y_2 z_1|} & \frac{my_1 + ny_2}{|y_1 z_2 - y_2 z_1|} \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} = \begin{bmatrix} my_1 + ny_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{y_1}{my_1 + ny_2} & \frac{y_2}{my_1 + ny_2} \\ -n\sigma & m\sigma \end{bmatrix}$$

where

$$\begin{bmatrix} \frac{y_1}{my_1 + ny_2} & \frac{y_2}{my_1 + ny_2} \\ -n\sigma & m\sigma \end{bmatrix} \in \mathsf{GL}\,(2, \mathbb{Z}).$$

■

**Theorem 3.8.** *Any (nontrivial) discrete central subgroup of* $\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R}$ *is equivalent to exactly one of the following discrete subgroups*

$$\exp(2n\pi\mathbb{Z}E_3) = \{p(0,0,2n\pi y,0) \; : \; y \in \mathbb{Z}\}$$
$$\exp(\mathbb{Z}E_4) = \{p(0,0,0,z) \; : \; z \in \mathbb{Z}\}$$
$$\exp(2n\pi\mathbb{Z}E_3)\exp(\mathbb{Z}E_4) = \{p(0,0,2n\pi y,z) \; : \; y,z \in \mathbb{Z}\}.$$

*Here* $n \in \mathbb{N}$, $n \geq 1$ *parametrizes families of discrete subgroups, each different value yielding a distinct (nonequivalent) subgroup.*

**Proof.** The center of $\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R}$ is $\exp(2\pi\mathbb{Z}E_3)\exp(\mathbb{R}E_4)$. The group of automorphisms $\mathsf{Aut}(\widetilde{\mathfrak{se}}\,(2) \oplus \mathfrak{g}_1)$ is given by

$$\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ -\sigma a_2 & \sigma a_1 & a_4 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix} : a_1, \ldots, a_6 \in \mathbb{R},\ a_1 a_2 a_6 \neq 0,\ \sigma = \pm 1 \right\}.$$

Let $\psi \in \mathsf{Aut}(\widetilde{\mathfrak{se}}\,(2) \oplus \mathfrak{g}_1)$ and let $\phi \in \mathsf{Aut}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})$ be the unique automorphism such that $T_{\mathbf{1}}\phi = \psi$. We have that

$$\begin{aligned} \phi(\exp(2\pi y E_3)\exp(zE_4)) &= \exp(2\pi y \psi \cdot E_3)\exp(z\psi \cdot E_4) \\ &= \exp\left(2\sigma\pi y E_3\right)\exp((2\pi a_5 y + a_6 z)E_4) \end{aligned}$$

We identify an element $\exp(2\pi y E_3)\exp(zE_4) \in \mathsf{Z}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})$ with the pair $(y,z) \in \mathbb{Z} \times \mathbb{R} \subset \mathbb{R}^2$. Hence, the restriction of the automorphism group $\mathsf{Aut}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})$ to $\mathsf{Z}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})$ is then given by

$$\mathsf{Aut}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})\Big|_{\mathsf{Z}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})} = \left\{ \begin{bmatrix} y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \sigma & 0 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} : a_1, a_2 \in \mathbb{R},\ a_1 \neq 0,\ \sigma = \pm 1 \right\}.$$

Let $\mathsf{N}$ be a discrete central subgroup. Then $\mathsf{N} = (y,0)\mathbb{Z}$, $\mathsf{N} = (0,z)\mathbb{Z}$, $\mathsf{N} = (y,z)\mathbb{Z}$, or $\mathsf{N} = (y_1,z_1)\mathbb{Z} + (y_2,z_2)\mathbb{Z}$ for some $y, y_1, y_2 \in \mathbb{Z}$, $z, z_1, z_2 \in \mathbb{R}$ with $(y_1,z_1)$ and $(y_2,z_2)$ linearly independent, $y \neq 0$ and $z \neq 0$. If $\mathsf{N} = (y,0)\mathbb{Z}$, then $\mathsf{N} = (n,0)\mathbb{Z} = \exp(2n\pi\mathbb{Z}E_3)$ where $n = |y|$. If $\mathsf{N} = (0,z)\mathbb{Z}$, then $\varphi = \operatorname{diag}(1, \frac{1}{z}) \in \mathsf{Aut}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})\Big|_{\mathsf{Z}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})}$ and $\varphi(\mathsf{N}) = (0,1)\mathbb{Z} = \exp(\mathbb{Z}E_4)$. On the other hand, if $\mathsf{N} = (y,z)\mathbb{Z}$, then

$$\varphi = \begin{bmatrix} 1 & 0 \\ -z & y \end{bmatrix} \in \mathsf{Aut}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})\Big|_{\mathsf{Z}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})}$$

and $\varphi(\mathsf{N}) = (y,0)\mathbb{Z} = \exp(2n\pi\mathbb{Z}E_3)$ where $n = |y|$. Lastly, if $\mathsf{N} = (y_1,z_1)\mathbb{Z} + (y_2,z_2)\mathbb{Z}$, then by Lemma 3.7 there exists $\varphi \in \mathsf{Aut}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})\Big|_{\mathsf{Z}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})}$ such that $\varphi(\mathsf{N}) = (n,0)\mathbb{Z} + (0,1)\mathbb{Z} = \exp(2n\pi\mathbb{Z})\exp(\mathbb{Z}E_4)$ where $n = \gcd(y_1, y_2)$.

It remains to be shown that no two of the given discrete subgroups are equivalent. We note that $(0,1)$ is an eigenvector of the matrix of any automorphism in $\mathsf{Aut}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})\Big|_{\mathsf{Z}(\widetilde{\mathsf{SE}}\,(2) \times \mathbb{R})}$. It follows that $\exp(\mathbb{Z}E_4)$ cannot be equivalent to $\exp(2n\pi\mathbb{Z}E_3)$ or $\exp(2n\pi\mathbb{Z}E_3)\exp(\mathbb{Z}E_4)$; likewise $\exp(2n\pi\mathbb{Z}E_3)$ cannot be equivalent to $\exp(2n\pi\mathbb{Z}E_3)\exp(\mathbb{Z}E_4)$. It is straightforward to show that $\exp(2n\pi\mathbb{Z}E_3)$ and $\exp(2n'\pi\mathbb{Z}E_3)$ are equivalent only if $n = n'$ (and similarly for $\exp(2n\pi\mathbb{Z}E_3)\exp(\mathbb{Z}E_4)$). ∎

**Corollary 3.9.** *There are four types of connected Lie groups with Lie algebra* $\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$, *namely*

1. *the universal covering group* $\widetilde{\mathsf{SE}}(2) \times \mathbb{R}$*;*
2. *the* $n$*-fold coverings*

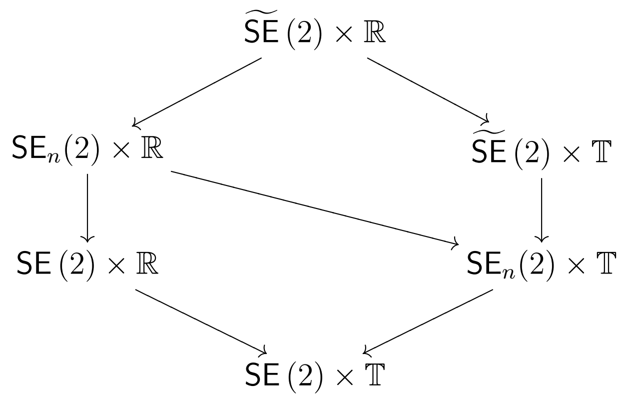
$$\widetilde{\mathsf{SE}}(2) \times \mathbb{R} / \exp(2n\pi\mathbb{Z}E_3) \cong \mathsf{SE}_n(2) \times \mathbb{R}$$

   *of* $\widetilde{\mathsf{SE}}(2) \times \mathbb{R} / \exp(2\pi\mathbb{Z}E_3) \cong \mathsf{SE}(2) \times \mathbb{R}$*;*
3. *the group* $\widetilde{\mathsf{SE}}(2) \times \mathbb{R} / \exp(\mathbb{Z}E_4) \cong \widetilde{\mathsf{SE}}(2) \times \mathbb{T}$*;*
4. *the* $n$*-fold coverings*

$$\widetilde{\mathsf{SE}}(2) \times \mathbb{R} / \exp(2n\pi\mathbb{Z}E_3)\exp(\mathbb{Z}E_4) \cong \mathsf{SE}_n(2) \times \mathbb{T}$$

   *of* $\widetilde{\mathsf{SE}}(2) \times \mathbb{R} / \exp(2\pi\mathbb{Z}E_3)\exp(\mathbb{Z}E_4) \cong \mathsf{SE}(2) \times \mathbb{T}$*.*

**Remark 3.10.** We have the following diagram of coverings:




**Proposition 3.11.** *Any connected Lie group with Lie algebra* $\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$ *is linearizable.*

### 3.6. Groups with algebra $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ (trivial extension of the pseudo-orthogonal algebra).

The pseudo-orthogonal group

$$\mathsf{SO}(2,1) = \{g \in \mathbb{R}^{3\times 3} \,:\, g^\top J g = J, \det g = 1\}, \qquad J = \operatorname{diag}(1,1,-1)$$

has two connected components. The identity component of $\mathsf{SO}(2,1)$ is $\mathsf{SO}(2,1)_0 = \{g \in \mathsf{SO}(2,1) \,:\, g_{33} > 0\}$ where $g = \left[g_{ij}\right]$. Its Lie algebra is given by

$$\mathfrak{so}(2,1) = \{A \in \mathbb{R}^{3\times 3} \,:\, A^\top J + JA = 0\} = \left\{ \begin{bmatrix} 0 & y & x \\ -y & 0 & w \\ x & w & 0 \end{bmatrix} \,:\, w,x,y \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{so}(2,1) \cong \mathfrak{g}_{3.6}$ is the only three-dimensional Lie algebra for which the corresponding universal covering group, which we denote $\widetilde{\mathsf{A}}$, is not linearizable. Any connected Lie group with Lie algebra $\mathfrak{g}_{3.6}$ is isomorphic to either $\widetilde{\mathsf{A}}$ or an $n$-fold covering $\mathsf{A}_n$ of $\mathsf{SO}(2,1)_0$, $n \geq 1$.

**Remark 3.12.** $\mathsf{A}_1$ is isomorphic to both $\mathsf{SO}(2,1)_0$ and the projective special linear group $\mathsf{PSL}(2,\mathbb{R})$. The two-fold cover $\mathsf{A}_2$ is isomorphic to the special linear group $\mathsf{SL}(2,\mathbb{R})$. The four-fold cover $\mathsf{A}_4$ is isomorphic to the metaplectic group $\mathsf{Mp}(2,\mathbb{R})$.

The trivial Abelian extension $\widetilde{\mathsf{A}} \times \mathbb{R}$ of $\widetilde{\mathsf{A}}$ is simply connected and hence the universal covering Lie group for $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$. We note however that this group is not linearizable. Indeed, it turns out that $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ is the only four-dimensional Lie algebra for which the universal covering is not linearizable. Hence we proceed somewhat differently here with respect to the representation of the universal covering group.

Let $(\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3, \widetilde{E}_4)$ be an ordered basis for the Lie algebra $\widetilde{\mathfrak{a}} \oplus \mathfrak{g}_1$ of $\widetilde{\mathsf{A}} \times \mathbb{R}$ such that $\operatorname{span}(\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3) = \widetilde{\mathfrak{a}} \oplus \{0\}$, $\operatorname{span}(\widetilde{E}_4) = \{0\} \oplus \mathfrak{g}_1$ and with nonzero commutator relations $[\widetilde{E}_2, \widetilde{E}_3] = \widetilde{E}_1$, $[\widetilde{E}_3, \widetilde{E}_1] = \widetilde{E}_2$, and $[\widetilde{E}_1, \widetilde{E}_2] = -\widetilde{E}_3$. Now $\widetilde{\mathfrak{a}}$ has Cartan decomposition

$$\widetilde{\mathfrak{a}} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} = \operatorname{span}\{\widetilde{E}_3\}, \ \mathfrak{p} = \operatorname{span}\{\widetilde{E}_1, \widetilde{E}_2\}.$$

Indeed $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, and the Killing form $k(A, B) = \operatorname{tr}(\operatorname{ad} A \operatorname{ad} B) = 2(a_1 b_1 + a_2 b_2 - a_3 b_3)$ is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. (Here $A = \sum_{i=1}^3 a_i \widetilde{E}_i$ and $B = \sum_{i=1}^3 b_i \widetilde{E}_i$.) Hence, the mapping

$$p : \mathbb{R}^3 \to \widetilde{\mathsf{A}}, \qquad (w, x, y) \mapsto \exp(y\widetilde{E}_3) \exp(w\widetilde{E}_1 + x\widetilde{E}_2)$$

is a diffeomorphism (see, e.g., [18, Chapter 4, Section 3.2] or [12, Chapter 13, Section 1]).

The trivial Abelian extension

$$\mathsf{SO}(2,1)_0 \times \mathbb{R} = \left\{ \begin{bmatrix} \multicolumn{2}{c}{g} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & e^z \end{bmatrix} : g \in \mathsf{SO}(2,1)_0, \ z \in \mathbb{R} \right\}$$

of $\mathsf{SO}(2,1)_0$ has Lie algebra

$$\mathfrak{so}(2,1) \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & y & x & 0 \\ -y & 0 & w & 0 \\ x & w & 0 & 0 \\ 0 & 0 & 0 & z \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \ : \ w, x, y, z \in \mathbb{R} \right\}$$

with nonzero commutator relations given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, and $[E_1, E_2] = -E_3$. Accordingly, there exists a Lie group covering homomorphism $q : \widetilde{\mathsf{A}} \times \mathbb{R} \to \mathsf{SO}(2,1)_0 \times \mathbb{R}$ such that

$$q(p(w, x, y), z) = \exp(yE_3) \exp(wE_1 + xE_2) \exp(zE_4)$$

(i.e., $T_{\mathbf{1}} q$ is the identity matrix with respect to the given bases).

**Lemma 3.13.** *The center of* $\widetilde{\mathsf{A}} \times \mathbb{R}$ *is* $p(0, 0, 2\pi\mathbb{Z}) \times \mathbb{R}$.

**Proof.** We have that $Z(\widetilde{\mathsf{A}} \times \mathbb{R}) = q^{-1}(Z(\mathsf{SO}(2,1)_0 \times \mathbb{R}))$. Furthermore, $Z(\mathsf{SO}(2,1)_0 \times \mathbb{R}) = \{\mathbf{1}\} \times \mathbb{R}$ and $\exp(y\, E_3) \exp(wE_1 + xE_2) \exp(zE_4) \in \{\mathbf{1}\} \times \mathbb{R}$ if and only if $w = x = 0$, $y \in 2\pi\mathbb{Z}$ and $z \in \mathbb{R}$. Hence $Z(\widetilde{\mathsf{A}} \times \mathbb{R}) = p(0, 0, 2\pi\mathbb{Z}) \times \mathbb{R}$. ■

We identify an element $(p(0,0,2\pi y), z)$ in the center of $\widetilde{\mathsf{A}} \times \mathbb{R}$ with the pair $(y,z) \in \mathbb{Z} \times \mathbb{R} \subseteq \mathbb{R}^2$.

**Theorem 3.14.** *Any (nontrivial) discrete central subgroup of $\widetilde{\mathsf{A}} \times \mathbb{R}$ is equivalent to exactly one of the following discrete subgroups*

$$\begin{aligned}
\exp(2n\pi\mathbb{Z}\widetilde{E}_3) &= \{(ny, 0) \,:\, y \in \mathbb{Z}\} \\
\exp(\mathbb{Z}\widetilde{E}_4) &= \{(0, z) \,:\, z \in \mathbb{Z}\} \\
\exp(n\mathbb{Z}(2\pi\widetilde{E}_3 + \widetilde{E}_4)) &= \{n(y, y) \,:\, y \in \mathbb{Z}\} \\
\exp(2n\pi\mathbb{Z}\widetilde{E}_3)\exp(\mathbb{Z}\widetilde{E}_4) &= \{(ny, z) \,:\, y, z \in \mathbb{Z}\} \\
\exp(n\mathbb{Z}(2\pi\widetilde{E}_3 + \alpha\widetilde{E}_4))\exp(n\mathbb{Z}\widetilde{E}_4) &= \{n(y, \alpha y + z) \,:\, y, z \in \mathbb{Z}\}.
\end{aligned}$$

*Here $n \in \mathbb{N}$, $n \geq 1$ and $0 < \alpha < 1$ parametrize families of discrete subgroups, each different value yielding a distinct (nonequivalent) subgroup.*

**Proof.** Any automorphism $\phi \in \mathsf{Aut}(\widetilde{\mathsf{A}} \times \mathbb{R})$ decomposes as a direct product $\phi = \phi_1 \times \phi_2$ of automorphisms $\phi_1 \in \mathsf{Aut}(\widetilde{\mathsf{A}})$ and $\phi_2 \in \mathsf{Aut}(\mathbb{R})$. Any automorphism $\phi_1 \in \mathsf{Aut}(\widetilde{\mathsf{A}})$ is the composition of an inner automorphism and the automorphism $\phi' \in \mathsf{Aut}(\widetilde{\mathsf{A}})$ with tangent map $T_{\mathbf{1}}\phi' = \operatorname{diag}(-1, 1, -1)$. Inner automorphisms fix elements in the center of the group. Hence the action of any $\phi' \in \mathsf{Aut}(\widetilde{\mathsf{A}})$ on an element $\exp(2\pi y\widetilde{E}_3) \in \mathsf{Z}(\widetilde{\mathsf{A}})$ is $\exp(2\pi y\widetilde{E}_3) \mapsto \exp(2\sigma\pi y\widetilde{E}_3)$ where $\sigma = \pm 1$. Consequently, the restriction of the automorphism group $\mathsf{Aut}(\widetilde{\mathsf{A}} \times \mathbb{R})$ to $\mathsf{Z}(\widetilde{\mathsf{A}} \times \mathbb{R})$ is given by

$$\mathsf{Aut}(\widetilde{\mathsf{A}} \times \mathbb{R})\big|_{\mathsf{Z}(\widetilde{\mathsf{A}} \times \mathbb{R})} = \left\{ \begin{bmatrix} y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \sigma & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \,:\, a \in \mathbb{R},\, a \neq 0,\, \sigma = \pm 1 \right\}.$$

Let $\mathsf{N}$ be a discrete central subgroup. Then $\mathsf{N} = (y,0)\mathbb{Z}$, $\mathsf{N} = (0,z)\mathbb{Z}$, $\mathsf{N} = (y,z)\mathbb{Z}$, or $\mathsf{N} = (y_1,z_1)\mathbb{Z} + (y_2,z_2)\mathbb{Z}$ for some $y, y_1, y_2 \in \mathbb{Z}$, $z, z_1, z_2 \in \mathbb{R}$ with $(y_1,z_1)$ and $(y_2,z_2)$ linearly independent, $y \neq 0$, and $z \neq 0$. If $\mathsf{N} = (y,0)\mathbb{Z}$, then $\mathsf{N} = (n,0)\mathbb{Z} = \exp(2n\pi\mathbb{Z}\widetilde{E}_3)$ where $n = |y|$. If $\mathsf{N} = (0,z)\mathbb{Z}$, then $\varphi = \operatorname{diag}(1, \frac{1}{z}) \in \mathsf{Aut}(\widetilde{\mathsf{A}} \times \mathbb{R})\big|_{\mathsf{Z}(\widetilde{\mathsf{A}} \times \mathbb{R})}$ and $\varphi(\mathsf{N}) = (0,1)\mathbb{Z} = \exp(\mathbb{Z}\widetilde{E}_4)$. Likewise, if $\mathsf{N} = (y,z)\mathbb{Z}$, then $\varphi = \operatorname{diag}(1, \frac{y}{z}) \in \mathsf{Aut}(\widetilde{\mathsf{A}} \times \mathbb{R})\big|_{\mathsf{Z}(\widetilde{\mathsf{A}} \times \mathbb{R})}$ and $\varphi(\mathsf{N}) = (n,n)\mathbb{Z} = \exp(n\mathbb{Z}(2\pi\widetilde{E}_3 + \widetilde{E}_4))$ where $n = |y|$.

Suppose $\mathsf{N} = (y_1,z_1)\mathbb{Z} + (y_2,z_2)\mathbb{Z}$. Then $\mathsf{N} = (n,z_1')\mathbb{Z} + (0,z_2')\mathbb{Z}$, $n = \gcd(y_1,y_2)$ for some $z_1', z_2' \in \mathbb{R}$, $z_2' \neq 0$. Indeed, by Bézout's identity, there exist integers $m_1, m_2$ such that $m_1y_1 + m_2y_2 = n$ and

$$\begin{bmatrix} y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} = \begin{bmatrix} m_1y_1 + m_2y_2 & 0 \\ m_1z_1 + m_2z_2 & \frac{y_1z_2 - y_2z_1}{m_1y_1 + m_2y_2} \end{bmatrix} \begin{bmatrix} \frac{y_1}{m_1y_1 + m_2y_2} & \frac{y_2}{m_1y_1 + m_2y_2} \\ -m_2 & m_1 \end{bmatrix}$$

where

$$\begin{bmatrix} \frac{y_1}{m_1y_1 + m_2y_2} & \frac{y_2}{m_1y_1 + m_2y_2} \\ -m_2 & m_1 \end{bmatrix} \in \mathsf{GL}\,(2, \mathbb{Z}).$$

Hence $\varphi = \operatorname{diag}(1, \frac{n}{z_2'}) \in \mathsf{Aut}(\widetilde{\mathsf{A}} \times \mathbb{R})\big|_{\mathsf{Z}(\widetilde{\mathsf{A}}\times\mathbb{R})}$ and $\varphi(\mathsf{N}) = (n, n\alpha)\mathbb{Z} + (0,n)\mathbb{Z} = \exp(n\mathbb{Z}(2\pi\widetilde{E}_3 + \alpha\widetilde{E}_4))\exp(n\mathbb{Z}\widetilde{E}_4)$ for some $0 \leq \alpha < 1$. Indeed,

$$\begin{bmatrix} n & 0 \\ n\left(\frac{z_1'}{z_2'} + m\right) & 1 \end{bmatrix} = \begin{bmatrix} n & 0 \\ n\frac{z_1'}{z_2'} & n \end{bmatrix}\begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix} \in \mathsf{GL}\,(2,\mathbb{Z})$$

for any $m \in \mathbb{Z}$. When $\alpha = 0$ it is a simple matter to show that $\exp(n\mathbb{Z}(2\pi\widetilde{E}_3 + \alpha\widetilde{E}_4))\exp(n\mathbb{Z}\widetilde{E}_4)$ is equivalent to $\exp(2n\pi\mathbb{Z}\widetilde{E}_3)\exp(\mathbb{Z}\widetilde{E}_4)$.

It remains to be shown that no two of the discrete subgroups listed in the theorem are equivalent. For any $\phi \in \mathsf{Aut}(\widetilde{\mathsf{A}} \times \mathbb{R})\big|_{\mathsf{Z}(\widetilde{\mathsf{A}}\times\mathbb{R})}$ we have that $\phi(n,0) = (\pm n, 0)$ and $\phi(0,1) = (0,a)$ for some $a \neq 0$. It follows that no two of the subgroups $\exp(2n\pi\mathbb{Z}\widetilde{E}_3)$ $(n \geq 1)$, $\exp(\mathbb{Z}\widetilde{E}_4)$, and $\exp(n\mathbb{Z}(2\pi\widetilde{E}_3 + \widetilde{E}_4))$ $(n \geq 1)$ can be equivalent. Clearly none of these subgroups are equivalent to $\exp(n\mathbb{Z}(2\pi\widetilde{E}_3 + \alpha\widetilde{E}_4))\exp(n\mathbb{Z}\widetilde{E}_4)$ for any $0 < \alpha < 1$.

Suppose that the subgroups

$$\exp(n\mathbb{Z}(2\pi\widetilde{E}_3+\alpha\widetilde{E}_4))\exp(n\mathbb{Z}\widetilde{E}_4) \qquad \text{and} \qquad \exp(n'\mathbb{Z}(2\pi\widetilde{E}_3+\alpha'\widetilde{E}_4)\mathbb{Z})\exp(n'\mathbb{Z}\widetilde{E}_4)$$

are equivalent for some $n, n' \in \mathbb{N}$, $n, n' \geq 1$ and $0 < \alpha, \alpha' < 1$. Then there exists $\begin{bmatrix} \sigma & 0 \\ 0 & a \end{bmatrix} \in \mathsf{Aut}(\widetilde{\mathsf{A}} \times \mathbb{R})\big|_{\mathsf{Z}(\widetilde{\mathsf{A}}\times\mathbb{R})}$ and $\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \in \mathsf{GL}\,(2,\mathbb{R})$ such that

$$\begin{bmatrix} \sigma & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} n & 0 \\ n\alpha & n \end{bmatrix} = \begin{bmatrix} n' & 0 \\ n'\alpha' & n' \end{bmatrix}\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}.$$

That is, $n\sigma = m_1 n'$, $0 = m_2 n'$, $an\alpha = m_3 n' + m_1 n'\alpha'$, and $an = m_4 n' + m_2 n'\alpha'$. As $n' \neq 0$ we have $m_2 = 0$ and so $m_1, m_4 \in \{1, -1\}$. Thus $n = \pm n'$, $an\alpha = n'(m_3 \pm \alpha')$, and $an = \pm n'$. As $n, n' > 0$ we get $n = n'$ and $a = \pm 1$. Hence, as $0 < \alpha, \alpha' < 1$, it follows that $\alpha = \alpha'$. Consequently $\exp(n\mathbb{Z}(2\pi\widetilde{E}_3 + \alpha\widetilde{E}_4))\exp(n\mathbb{Z}\widetilde{E}_4)$ and $\exp(n'\mathbb{Z}(2\pi\widetilde{E}_3 + \alpha'\widetilde{E}_4))\exp(n'\mathbb{Z}\widetilde{E}_4)$ are equivalent only if $n = n'$ and $\alpha = \alpha'$. ∎

**Corollary 3.15.** *There are six types of connected Lie groups with Lie algebra* $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$, *namely*

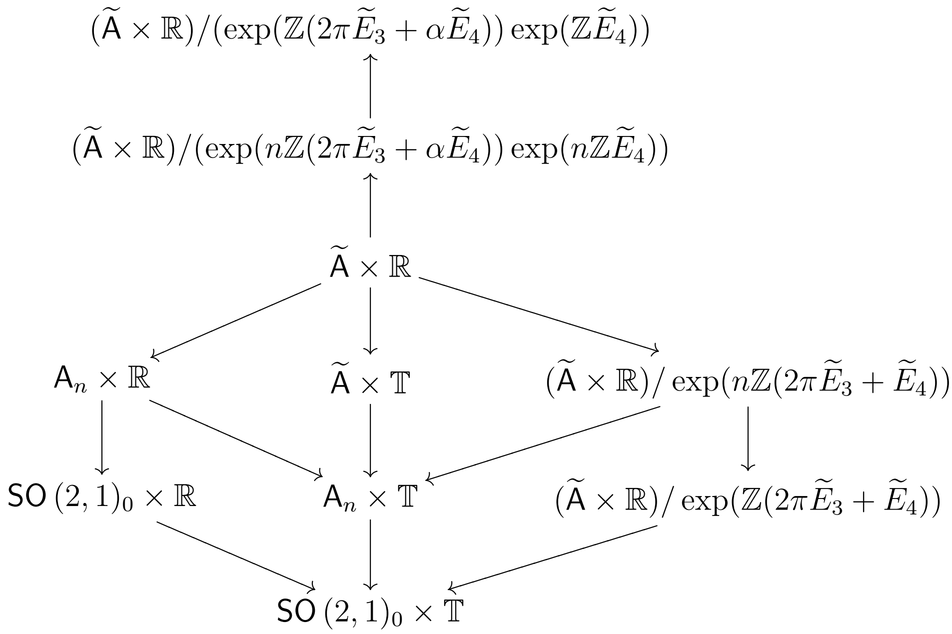
1. *the universal covering* $\widetilde{\mathsf{A}} \times \mathbb{R}$;
2. *the* $n$*-fold covers* $(\widetilde{\mathsf{A}} \times \mathbb{R})/\exp(2n\pi\mathbb{Z}\widetilde{E}_3) \cong \mathsf{A}_n \times \mathbb{R}$ *of* $\mathsf{SO}\,(2,1)_0 \times \mathbb{R}$;
3. *the group* $(\widetilde{\mathsf{A}} \times \mathbb{R})/\exp(\mathbb{Z}\widetilde{E}_4) \cong \widetilde{\mathsf{A}} \times \mathbb{T}$,
4. *the* $n$*-fold covers*
$$(\widetilde{\mathsf{A}} \times \mathbb{R})/\exp(n\mathbb{Z}(2\pi\widetilde{E}_3 + \widetilde{E}_4))$$
*of* $(\widetilde{\mathsf{A}} \times \mathbb{R})/\exp(\mathbb{Z}(2\pi\widetilde{E}_3 + \widetilde{E}_4))$;
5. *the* $n$*-fold covers*
$$(\widetilde{\mathsf{A}} \times \mathbb{R})/(\exp(2n\pi\mathbb{Z}\widetilde{E}_3)\exp(\mathbb{Z}\widetilde{E}_4)) \cong \mathsf{A}_n \times \mathbb{T}$$
*of* $(\widetilde{\mathsf{A}} \times \mathbb{R})/(\exp(\mathbb{Z}\widetilde{E}_3)\exp(\mathbb{Z}\widetilde{E}_4)) \cong \mathsf{SO}\,(2,1)_0 \times \mathbb{T}$;

*6. the $n^2$-fold covers*

$$(\widetilde{\mathsf{A}} \times \mathbb{R})/(\exp(n\mathbb{Z}(2\pi \widetilde{E}_3 + \alpha \widetilde{E}_4)) \exp(n\mathbb{Z}\widetilde{E}_4))$$

*of* $(\widetilde{\mathsf{A}} \times \mathbb{R})/(\exp(\mathbb{Z}(2\pi \widetilde{E}_3 + \alpha \widetilde{E}_4)) \exp(\mathbb{Z}\widetilde{E}_4))$.

**Remark 3.16.** We have the following diagram of coverings:




**Remark 3.17.** The connected component $\mathsf{GL}^+(2,\mathbb{R}) = \{g \in \mathbb{R}^{2\times 2} : \det g > 0\}$ of the general linear group $\mathsf{GL}(2,\mathbb{R})$ is isomorphic to $\mathsf{A}_2 \times \mathbb{R} \cong \mathsf{SL}(2,\mathbb{R}) \times \mathbb{R}$.

**Proposition 3.18.** *Among the connected Lie groups with Lie algebra* $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$, *only* $\mathsf{A}_2 \times \mathbb{R} \cong \mathsf{SL}(2,\mathbb{R}) \times \mathbb{R}$, $\mathsf{A}_2 \times \mathbb{T} \cong \mathsf{SL}(2,\mathbb{R}) \times \mathbb{T}$, $\mathsf{A}_1 \times \mathbb{R} \cong \mathsf{SO}(2,1)_0 \times \mathbb{R}$, *and* $\mathsf{A}_1 \times \mathbb{T} \cong \mathsf{SO}(2,1)_0 \times \mathbb{T}$ *are linearizable.*

**Proof.** Let $\mathsf{N}$ be a discrete central subgroup of $\widetilde{\mathsf{A}} \times \mathbb{R}$ and let $q : \widetilde{\mathsf{A}} \times \mathbb{R} \to \widetilde{\mathsf{A}} \times \mathbb{R}/\mathsf{N}$ be the canonical covering homomorphism. The group $\widetilde{\mathsf{A}} \times \mathbb{R}$ has Levi subgroup $\widetilde{\mathsf{A}} \times \{\mathbf{1}\}$ and radical $\mathrm{Rad}(\widetilde{\mathsf{A}} \times \mathbb{R}) = \{\mathbf{1}\} \times \mathbb{R}$. Accordingly, $\widetilde{\mathsf{A}} \times \mathbb{R}/\mathsf{N}$ has Levi subgroup $q(\widetilde{\mathsf{A}} \times \{\mathbf{1}\})$ and radical $q(\{\mathbf{1}\} \times \mathbb{R})$. We have that $\widetilde{\mathsf{A}} \times \mathbb{R}/\mathsf{N}$ is linearizable if and only if both its radical and the Levi subgroup are linearizable (Theorem 2.6). The one-dimensional connected Abelian Lie group $q(\mathrm{Rad}(\widetilde{\mathsf{A}} \times \mathbb{R}))$ is linearizable. On the other hand, among the connected Lie groups with Lie algebra $\mathfrak{g}_{3.6} \cong \widetilde{\mathfrak{a}}$, only $\mathsf{A}_2 \cong \mathsf{SL}(2,\mathbb{R})$ and $\mathsf{A}_1 \cong \mathsf{SO}(2,1)_0$ are linearizable. Consequently, $q(\widetilde{\mathsf{A}} \times \{\mathbf{1}\}) = (\widetilde{\mathsf{A}} \times \{\mathbf{1}\})/(\mathsf{N} \cap (\widetilde{\mathsf{A}} \times \{\mathbf{1}\}))$ is linearizable if and only if $\mathsf{N} \cap (\widetilde{\mathsf{A}} \times \{\mathbf{1}\}) = \exp(4\pi \widetilde{E}_3 \mathbb{Z})$ or $\mathsf{N} \cap (\widetilde{\mathsf{A}} \times \{\mathbf{1}\}) = \exp(2\pi \widetilde{E}_3 \mathbb{Z})$. ∎

**3.7. Groups with algebra $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$ (trivial extension of the orthogonal algebra).** There are exactly two connected Lie groups with Lie algebra $\mathfrak{g}_{3.7}$: the special orthogonal group $\mathsf{SO}(3)$ and its simply connected double cover, the special unitary group $\mathsf{SU}(2)$. The special unitary group $\mathsf{SU}(2)$ and its Lie algebra $\mathfrak{su}(2)$ are given by

$$\begin{aligned}\mathsf{SU}(2) &= \left\{g \in \mathbb{C}^{2\times 2} \;:\; g^\dagger g = I_2,\, \det g = 1\right\}\\ \mathfrak{su}(2) &= \left\{A \in \mathbb{C}^{2\times 2} \;:\; A^\dagger = -A,\, \operatorname{tr} A = 0\right\}\\ &= \left\{\begin{bmatrix} \frac{i}{2}x & \frac{1}{2}(y+iz)\\ -\frac{1}{2}(y-iz) & -\frac{i}{2}x\end{bmatrix} \;:\; x,y,z \in \mathbb{R}\right\}.\end{aligned}$$

(Here $\dagger$ denotes the conjugate transpose.) The special orthogonal group $\mathsf{SO}(3)$ and its Lie algebra $\mathfrak{so}(3)$ are given by

$$\begin{aligned}\mathsf{SO}(3) &= \left\{g \in \mathbb{R}^{3\times 3} \;:\; g^\top g = I_3,\, \det g = 1\right\}\\ \mathfrak{so}(3) &= \left\{A \in \mathbb{R}^{3\times 3} \;:\; A^\top = -A\right\}\\ &= \left\{\begin{bmatrix} 0 & -z & y\\ z & 0 & -x\\ -y & x & 0\end{bmatrix} \;:\; x,y,z \in \mathbb{R}\right\}.\end{aligned}$$

The trivial Abelian extension

$$\mathsf{SU}(2)\times\mathbb{R} = \left\{\begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$g$}} & 0\\ & & 0\\ 0 & 0 & \mathrm{e}^z\end{bmatrix} \;:\; g \in \mathsf{SU}(2),\; z \in \mathbb{R}\right\}$$

of $\mathsf{SU}(2)$ is the universal covering group for $\mathfrak{g}_{3.7}\oplus\mathfrak{g}_1$. The Lie algebra $\mathfrak{su}(2)\oplus\mathfrak{g}_1$ of $\mathsf{SU}(2)\times\mathbb{R}$ is given by

$$\left\{\begin{bmatrix} \frac{i}{2}w & \frac{1}{2}(x+iy) & 0\\ -\frac{1}{2}(x-iy) & -\frac{i}{2}w & 0\\ 0 & 0 & z\end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \;:\; w,x,y,z \in \mathbb{R}\right\}.$$

**Remark 3.19.** $\mathsf{SU}(2)$ is diffeomorphic to the three-sphere $\mathbb{S}^3$. Moreover, $\mathsf{SU}(2)$ is the only three-dimensional connected simply connected Lie group which is not diffeomorphic to $\mathbb{R}^3$. Likewise, it turns out that $\mathsf{SU}(2)\times\mathbb{R}$ is the only four-dimensional connected simply connected Lie group which is not diffeomorphic to $\mathbb{R}^4$.

We note that the four-dimensional unitary group

$$\mathsf{U}(2) = \{g \in \mathbb{C}^{2\times 2} \;:\; g^\dagger g = I_2\}$$

has Lie algebra

$$\begin{aligned}\mathfrak{u}(2) &= \left\{A \in \mathbb{C}^{2\times 2} \;:\; A^\dagger = -A\right\}\\ &= \left\{\begin{bmatrix} \frac{i}{2}(w+z) & \frac{1}{2}(x+iy)\\ -\frac{1}{2}(x-iy) & \frac{i}{2}(-w+z)\end{bmatrix} \;:\; w,x,y,z \in \mathbb{R}\right\}\end{aligned}$$

isomorphic to $\mathfrak{g}_{3.7}\oplus\mathfrak{g}_1$. However, $\mathsf{U}(2)$ is not simply connected and so it follows that $\mathsf{U}(2)$ is isomorphic to a quotient of $\mathsf{SU}(2)\times\mathbb{R}$ by some discrete central subgroup.

**Theorem 3.20.** *Any (nontrivial) discrete central subgroup of* $\mathsf{SU}(2) \times \mathbb{R}$ *is equivalent to exactly one of the following discrete subgroups*

$$\langle \exp(2\pi E_1) \rangle = \left\{ \begin{bmatrix} \sigma I_2 & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} : \sigma = \pm 1 \right\}$$

$$\langle \exp(E_4) \rangle = \left\{ \begin{bmatrix} I_2 & & 0 \\ & & 0 \\ 0 & 0 & \mathrm{e}^z \end{bmatrix} : z \in \mathbb{Z} \right\}$$

$$\langle \exp(2\pi E_1) \exp(E_4) \rangle = \left\{ \begin{bmatrix} (-1)^z I_2 & & 0 \\ & & 0 \\ 0 & 0 & \mathrm{e}^z \end{bmatrix} : z \in \mathbb{Z} \right\}$$

$$\langle \exp(2\pi E_1) \rangle \langle \exp(E_4) \rangle = \left\{ \begin{bmatrix} \sigma I_2 & & 0 \\ & & 0 \\ 0 & 0 & \mathrm{e}^z \end{bmatrix} : z \in \mathbb{Z}, \sigma = \pm 1 \right\}.$$

*(Here* $\langle g \rangle$ *denotes the subgroup generated by* $g$*.)*

**Proof.** The center of $\mathsf{SU}(2) \times \mathbb{R}$ is

$$\langle \exp(2\pi E_1) \rangle \exp(\mathbb{R} E_4) = \left\{ \begin{bmatrix} \sigma I_2 & & 0 \\ & & 0 \\ 0 & 0 & \mathrm{e}^z \end{bmatrix} : z \in \mathbb{R}, \sigma = \pm 1 \right\}.$$

Any automorphism $\phi$ of $\mathsf{SU}(2) \times \mathbb{R}$ decomposes as a direct product $\phi = \phi_1 \times \phi_2$ of automorphisms $\phi_1 \in \mathsf{Aut}(\mathsf{SU}(2))$ and $\phi_2 \in \mathsf{Aut}(\mathbb{R})$. Furthermore, any automorphism $\phi_1 \in \mathsf{Aut}(\mathsf{SU}(2))$ fixes both elements of the center of $\mathsf{SU}(2)$. Consequently, the action of any automorphism on an element of the center of $\mathsf{SU}(2) \times \mathbb{R}$ is given by

$$\begin{bmatrix} \sigma I_2 & & 0 \\ & & 0 \\ 0 & 0 & \mathrm{e}^z \end{bmatrix} \longmapsto \begin{bmatrix} \sigma I_2 & & 0 \\ & & 0 \\ 0 & 0 & \mathrm{e}^{az} \end{bmatrix}, \qquad a \neq 0.$$

(Here $\sigma = \pm 1$ and $z \in \mathbb{Z}$.) It is a simple matter then to obtain the result (by considering whether or not the intersections of a discrete subgroup $\mathsf{N}$ with $\langle \exp(2\pi E_1) \rangle$ and $\langle \exp(E_4) \rangle$ are trivial or not). ∎

**Corollary 3.21.** *There are five types of connected Lie groups with Lie algebra* $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$*, namely*

1. *the universal covering group* $\mathsf{SU}(2) \times \mathbb{R} \cong \mathsf{SU}(2) \times \mathbb{R}$*;*
2. $\mathsf{SU}(2) \times \mathbb{R} / \langle \exp(2\pi E_1) \rangle \cong \mathsf{SO}(3) \times \mathbb{R}$*;*
3. $\mathsf{SU}(2) \times \mathbb{R} / \langle \exp(E_4) \rangle \cong \mathsf{SU}(2) \times \mathbb{T}$*;*
4. $\mathsf{SU}(2) \times \mathbb{R} / \langle \exp(2\pi E_1) \exp(E_4) \rangle \cong \mathsf{U}(2)$*;*
5. $\mathsf{SU}(2) \times \mathbb{R} / \langle \exp(2\pi E_1) \rangle \langle \exp(E_4) \rangle \cong \mathsf{SO}(3) \times \mathbb{T}$.

**Proof.** We show that $\mathsf{SU}(2) \times \mathbb{R} / \langle \exp(2\pi E_1) \exp(E_4) \rangle$ is indeed isomorphic to $\mathsf{U}(2)$. The mapping $\psi : \mathfrak{su}\,(2) \oplus \mathfrak{g}_1 \to \mathfrak{u}\,(2)$ given by

$$\begin{bmatrix} \frac{i}{2}w & \frac{1}{2}(x+iy) & 0 \\ -\frac{1}{2}(x-iy) & -\frac{i}{2}w & 0 \\ 0 & 0 & z \end{bmatrix} \longmapsto \begin{bmatrix} \frac{iw}{2}+\pi iz & \frac{1}{2}(x+iy) \\ -\frac{1}{2}(x-iy) & -\frac{iw}{2}+\pi iz \end{bmatrix}$$

is a Lie algebra isomorphism. Hence, as $\mathsf{SU}\,(2)\times\mathbb{R}$ is simply connected, there exists a Lie group epimorphism $\phi : \mathsf{SU}\,(2)\times\mathbb{R} \to \mathsf{G}$ such that $T_{\mathbf{1}}\phi = \psi$. Now $\ker\phi \subseteq \mathsf{Z}(\mathsf{SU}\,(2)\times\mathbb{R})$. Let $g \in \mathsf{Z}(\mathsf{SU}\,(2)\times\mathbb{R})$; we have that

$$g = \exp((1-\sigma)\pi E_1)\exp(zE_4), \text{ for some } z\in\mathbb{R} \text{ and } \sigma\in\{-1,1\}.$$

Hence

$$\phi(g) = \exp(\psi\cdot(1-\sigma)\pi E_1)\exp(\psi\cdot zE_4) = \begin{cases} \begin{bmatrix} \mathrm{e}^{i\pi z} & 0 \\ 0 & \mathrm{e}^{i\pi z}\end{bmatrix}, & \sigma = 1 \\[2ex] \begin{bmatrix} -\mathrm{e}^{i\pi z} & 0 \\ 0 & -\mathrm{e}^{i\pi z}\end{bmatrix}, & \sigma = -1. \end{cases}$$

Thus $\phi(g) = \mathbf{1}$ exactly when ($\sigma = 1$ and $z \in 2\mathbb{Z}$) or ($\sigma = -1$ and $z \in 1+2\mathbb{Z}$). Therefore $\ker\phi = \langle \exp(2\pi E_1)\exp(E_4)\rangle$. Consequently, $\mathsf{SU}\,(2)\times\mathbb{R}/\ker q = \mathsf{SU}\,(2)\times\mathbb{R}/\langle\exp(2\pi E_1)\exp(E_4)\rangle \cong \mathsf{U}\,(2)$. ■

**Remark 3.22.** We have the following diagram of coverings:

$$\begin{array}{ccccc} & & \mathsf{SU}\,(2)\times\mathbb{R} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathsf{SO}\,(3)\times\mathbb{R} & & \mathsf{SU}\,(2)\times\mathbb{T} & & \mathsf{U}(2) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathsf{SO}\,(3)\times\mathbb{T} & & \end{array}$$

**Remark 3.23.** $\mathsf{SU}\,(2)\times\mathbb{T}$, $\mathsf{SO}\,(3)\times\mathbb{T}$, $\mathsf{U}\,(2)$ and the Abelian four-torus $\mathbb{T}^4$ are the only compact connected four-dimensional Lie groups.

**Remark 3.24.** The group of nonzero quaternions $\mathbb{H}^\times$ with multiplication is isomorphic to $\mathsf{SU}\,(2)\times\mathbb{R}$.

**Proposition 3.25.** *Any connected Lie group with Lie algebra* $\mathfrak{g}_{3.7}\oplus\mathfrak{g}_1$ *is linearizable.*

## 4. Groups with indecomposable algebras

We consider each of the indecomposable four-dimensional Lie algebras (Table 3) and classify the corresponding Lie groups. First, we dispense with the those algebras whose universal cover has trivial center. We then proceed the consider each of the remaining five indecomposable Lie algebras in turn.

**Remark 4.1.** If a group is linearizable, then so is any subgroup. Hence, any group admitting $\mathsf{H}_3^*$ as a subgroup is not linearizable. It turns out that among the four-dimensional connected Lie groups with indecomposable Lie algebras, a group is not linearizable if and *only if* it admits $\mathsf{H}_3^*$ as a subgroup (cf. [4]). Here $\mathsf{H}_3^*$ is the quotient of the Heisenberg group $\mathsf{H}_3$ as described in Section 3.4.

**4.1. Simply connected groups with trivial center.** For a number of the four-dimensional indecomposable Lie algebras, the associated universal covering Lie group has trivial center. In these cases the classification of the associated connected Lie groups is trivial (in the sense that any connected Lie group with given Lie algebra is isomorphic to the universal covering Lie group).

**Proposition 4.2.** *The universal covering Lie group for each of the Lie algebras* $\mathfrak{g}_{4.2}^{\alpha}$, $\mathfrak{g}_{4.4}$, $\mathfrak{g}_{4.5}^{\alpha,\beta}$, $\mathfrak{g}_{4.6}^{\alpha,\beta}$, $\mathfrak{g}_{4.7}$, $\mathfrak{g}_{4.8}^{\alpha}$, *and* $\mathfrak{g}_{4.9}^{\alpha}$ *has a trivial center; we have the following linear representations of these groups*

$$\mathsf{G}_{4.2}^{\alpha} = \left\{ \begin{bmatrix} e^{-\alpha z} & 0 & 0 & w \\ 0 & e^{-z} & -\alpha z\, e^{-z} & \alpha x \\ 0 & 0 & e^{-z} & y \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}, \quad \alpha \neq 0$$

$$\mathsf{G}_{4.4} = \left\{ \begin{bmatrix} e^{-z} & -z\, e^{-z} & \frac{1}{2} z^2\, e^{-z} & w \\ 0 & e^{-z} & -z\, e^{-z} & x \\ 0 & 0 & e^{-z} & y \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}$$

$$\mathsf{G}_{4.5}^{\alpha,\beta} = \left\{ \begin{bmatrix} e^{-z} & 0 & 0 & w \\ 0 & e^{-\alpha z} & 0 & y \\ 0 & 0 & e^{-\beta z} & x \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}, \quad \begin{array}{c} -1 < \alpha \leq \beta \leq 1,\ \alpha\beta \neq 0 \\ \textit{or}\ \ \alpha = -1,\ 0 < \beta \leq 1 \end{array}$$

$$\mathsf{G}_{4.6}^{\alpha,\beta} = \left\{ \begin{bmatrix} e^{-\alpha z} & 0 & 0 & w \\ 0 & e^{-\beta z} \cos z & -e^{-\beta z} \sin z & -y \\ 0 & e^{-\beta z} \sin z & e^{-\beta z} \cos z & x \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}, \quad \alpha > 0,\ \beta \in \mathbb{R}$$

$$\mathsf{G}_{4.7} = \left\{ \begin{bmatrix} e^{-2z} & -y\, e^{-z} & (x + yz)\, e^{-z} & 2w \\ 0 & e^{-z} & -z\, e^{-z} & x \\ 0 & 0 & e^{-z} & y \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}$$

$$\mathsf{G}_{4.8}^{\alpha} = \left\{ \begin{bmatrix} e^{-(1+\alpha)z} & x & w \\ 0 & e^{-\alpha z} & y \\ 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}, \quad -1 < \alpha \leq 1$$

$$\mathsf{G}_{4.9}^{\alpha} = \left\{ \begin{bmatrix} e^{-2\alpha z} & -e^{-\alpha z}(x \cos z + y \sin z) & e^{-\alpha z}(y \cos z - x \sin z) & -2w \\ 0 & e^{-\alpha z} \cos z & e^{-\alpha z} \sin z & y \\ 0 & -e^{-\alpha z} \sin z & e^{-\alpha z} \cos z & x \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}, \quad \alpha > 0.$$

**Proof.** Standard (though quite tedious) computation. ■

From the list of indecomposable algebras (Table 3), only $\mathfrak{g}_{4.1}$, $\mathfrak{g}_{4.3}$, $\mathfrak{g}_{4.8}^{-1}$, $\mathfrak{g}_{4.9}^{0}$, and $\mathfrak{g}_{4.10}$ remain; we now proceed to study each of these cases in turn.

**4.2. Groups with algebra $\mathfrak{g}_{4.1}$.** The universal covering Lie group

$$\mathsf{G}_{4.1} = \left\{ \begin{bmatrix} 1 & z & \frac{z^2}{2} & w \\ 0 & 1 & z & -x+z \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix} = p(w,x,y,z) \; : \; w,x,y,z \in \mathbb{R} \right\}$$

has Lie algebra

$$\mathfrak{g}_{4.1} = \left\{ \begin{bmatrix} 0 & z & 0 & w \\ 0 & 0 & z & -x+z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \; : \; w,x,y,z \in \mathbb{R} \right\}.$$

**Theorem 4.3.** *Any (nontrivial) discrete central subgroup of $\mathsf{G}_{4.1}$ is equivalent to $\exp(\mathbb{Z}E_1) = \{p(w,0,0,0) \; : \; w \in \mathbb{Z}\}$.*

**Proof.** The center of $\mathsf{G}_{4.1}$ is $\exp(\mathbb{R}E_1)$. Suppose $\mathsf{N}$ is a discrete central subgroup. Then $\mathsf{N} = \exp(a\mathbb{Z}E_1)$ for some $a \neq 0$. Also, there exists a group automorphism $\phi \in \mathsf{Aut}(\mathsf{G}_{4.1})$ such that $T_{\mathbf{1}}\phi = \operatorname{diag}\left(\frac{1}{a}, \frac{1}{a}, \frac{1}{a}, 1\right)$ (see Table 7). Hence $\phi(\mathsf{N}) = \exp(T_{\mathbf{1}}\phi \cdot a\mathbb{Z}E_1) = \exp(\mathbb{Z}E_1)$. ■

**Corollary 4.4.** *There are two types of connected Lie groups with Lie algebra $\mathfrak{g}_{4.1}$, namely $\mathsf{G}_{4.1}$ and $\mathsf{G}_{4.1}/\exp(\mathbb{Z}E_1)$.*

**Proposition 4.5.** *The group $\mathsf{G}_{4.1}/\exp(\mathbb{Z}E_1)$ is not linearizable.*

**Proof.** The commutator subalgebra $[\mathfrak{g}_{4.1}, \mathfrak{g}_{4.1}] = \operatorname{span}\{E_1, E_2\}$ has nontrivial intersection with $\operatorname{span}\{E_1\}$ and hence the result follows by Corollary 2.4. The result also follows by Remark 2.5. ■

**4.3. Groups with algebra $\mathfrak{g}_{4.3}$.** The universal covering Lie group

$$\mathsf{G}_{4.3} = \left\{ \begin{bmatrix} \mathrm{e}^{-z} & 0 & 0 & w \\ 0 & 1 & -z & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix} = p(w,x,y,z) \; : \; w,x,y,z \in \mathbb{R} \right\}$$

has Lie algebra

$$\mathfrak{g}_{4.3} = \left\{ \begin{bmatrix} -z & 0 & 0 & w \\ 0 & 0 & -z & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \; : \; w,x,y,z \in \mathbb{R} \right\}.$$

**Theorem 4.6.** *Any (nontrivial) discrete central subgroup of* $\mathsf{G}_{4.3}$ *is equivalent to* $\exp(\mathbb{Z}E_2) = \{p(0,x,0,0) \,:\, x \in \mathbb{Z}\}$.

**Proof.** The same argument as made for Theorem 4.3 holds. ∎

**Corollary 4.7.** *There are two types of connected Lie groups with Lie algebra* $\mathfrak{g}_{4.3}$*, namely* $\mathsf{G}_{4.3}$ *and* $\mathsf{G}_{4.3}/\exp(\mathbb{Z}E_2)$.

**Proposition 4.8.** *The group* $\mathsf{G}_{4.3}/\exp(\mathbb{Z}E_2)$ *is not linearizable.*

**Proof.** The commutator subalgebra $[\mathfrak{g}_{4.3}, \mathfrak{g}_{4.3}] = \operatorname{span}\{E_1, E_2\}$ has nontrivial intersection with $\operatorname{span}\{E_2\}$ and so the result follows by Corollary 2.4. ∎

**4.4. Groups with algebra $\mathfrak{g}_{4.8}^{-1}$ (central extension of the semi-Euclidean algebra).** The universal covering Lie group

$$\mathsf{G}_{4.8}^{-1} = \left\{ \begin{bmatrix} 1 & x & w \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix} = p(w,x,y,z) \,:\, w,x,y,z \in \mathbb{R} \right\}$$

has Lie algebra

$$\mathfrak{g}_{4.8}^{-1} = \left\{ \begin{bmatrix} 0 & x & w \\ 0 & z & y \\ 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \,:\, w,x,y,z \in \mathbb{R} \right\}.$$

**Theorem 4.9.** *Any (nontrivial) discrete central subgroup of* $\mathsf{G}_{4.1}$ *is equivalent to* $\exp(\mathbb{Z}E_1) = \{p(w,0,0,0) \,:\, w \in \mathbb{Z}\}$.

**Proof.** Again, the same argument as made for Theorem 4.3 holds. ∎

**Corollary 4.10.** *There are two types of connected Lie groups with Lie algebra* $\mathfrak{g}_{4.8}^{-1}$*, namely* $\mathsf{G}_{4.8}^{-1}$ *and* $\mathsf{G}_{4.8}^{-1}/\exp(\mathbb{Z}E_1)$.

**Proposition 4.11.** *The group* $\mathsf{G}_{4.8}^{-1}/\exp(\mathbb{Z}E_1)$ *is not linearizable.*

**Proof.** The commutator subalgebra $[\mathfrak{g}_{4.8}^{-1}, \mathfrak{g}_{4.8}^{-1}] = \operatorname{span}\{E_1, E_2, E_3\}$ has nontrivial intersection with $\operatorname{span}\{E_1\}$ and so the result follows by Corollary 2.4. ∎

**4.5. Groups with algebra $\mathfrak{g}_{4.9}^{0}$ (central extension of the Euclidean algebra).** The universal covering Lie group is given by

$$\mathsf{G}_{4.9}^{0} = \left\{ \begin{bmatrix} 1 & -x\cos z - y\sin z & y\cos z - x\sin z & -2w & 0 \\ 0 & \cos z & \sin z & y & 0 \\ 0 & -\sin z & \cos z & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^z \end{bmatrix} \,:\, w,x,y,z \in \mathbb{R} \right\}.$$

Elements of the group are again denoted by $p(w,x,y,z)$. The corresponding Lie algebra (the so-called oscillator algebra) is given by

$$\mathfrak{g}_{4.9}^0 = \left\{ \begin{bmatrix} 0 & -x & y & -2w & 0 \\ 0 & 0 & z & y & 0 \\ 0 & -z & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \,:\, w,x,y,z \in \mathbb{R} \right\}.$$

**Theorem 4.12.** *Any (nontrivial) discrete central subgroup of* $\mathsf{G}_{4.9}^0$ *is equivalent to exactly one of the following discrete subgroups*

$$\begin{aligned} \exp(2n\pi\mathbb{Z}E_4) &= \{p(0,0,0,2n\pi z) \,:\, z \in \mathbb{Z}\} \\ \exp(\mathbb{Z}E_1) &= \{p(w,0,0,0) \,:\, w \in \mathbb{Z}\} \\ \exp(\mathbb{Z}E_1)\exp(2n\pi\mathbb{Z}E_4) &= \{p(w,0,0,2n\pi z) \,:\, w,z \in \mathbb{Z}\}. \end{aligned}$$

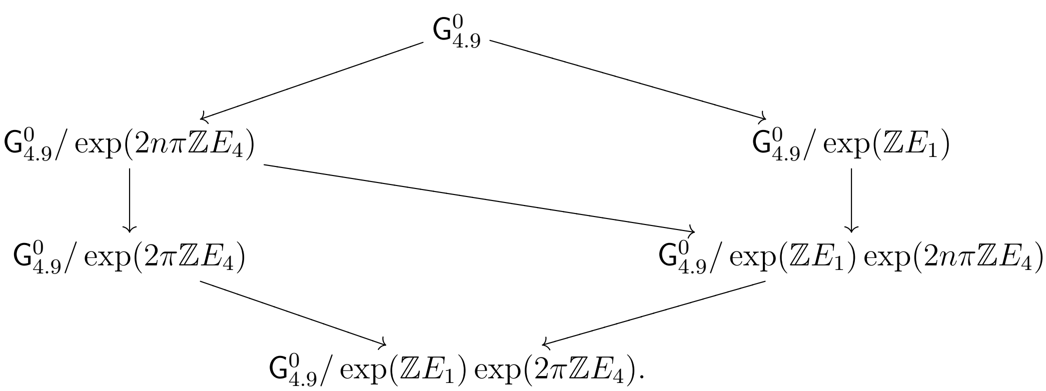
*Here* $n \in \mathbb{N}$, $n \geq 1$ *parametrizes families of discrete subgroups, each different value yielding a distinct (nonequivalent) subgroup.*

**Proof.** A proof of this theorem, very similar to that of Theorem 3.8, appears in [3]. ∎

**Corollary 4.13.** *There are four types of connected Lie groups with Lie algebra* $\mathfrak{g}_{4.9}^0$*, namely*

1. *the universal covering group* $\mathsf{G}_{4.9}^0$*;*
2. *the* $n$*-fold coverings* $\mathsf{G}_{4.9}^0/\exp(2n\pi\mathbb{Z}E_4)$ *of* $\mathsf{G}_{4.9}^0/\exp(2\pi\mathbb{Z}E_4)$*;*
3. *the group* $\mathsf{G}_{4.9}^0/\exp(\mathbb{Z}E_1)$*;*
4. *the* $n$*-fold coverings* $\mathsf{G}_{4.9}^0/\exp(\mathbb{Z}E_1)\exp(2n\pi\mathbb{Z}E_4)$ *of* $\mathsf{G}_{4.9}^0/\exp(\mathbb{Z}E_1)\exp(2\pi\mathbb{Z}E_4)$.

**Remark 4.14.** We have the following diagram of coverings:




**Proposition 4.15** (cf. [3])**.** *The groups* $\mathsf{G}_{4.9}^0$ *and* $\mathsf{G}_{4.9}^0/\exp(2n\pi\mathbb{Z}E_4)$ *are linearizable, whereas* $\mathsf{G}_{4.9}^0/\exp(\mathbb{Z}E_1)$ *and* $\mathsf{G}_{4.9}^0/\exp(\mathbb{Z}E_1)\exp(2n\pi\mathbb{Z}E_4)$ *are not.*

*In particular,* $\mathsf{G}_{4.9}^0/\exp(2n\pi\mathbb{Z}E_4)$ *is isomorphic to the matrix Lie group*

$$\left\{\begin{bmatrix} 1 & -x\cos z - y\sin z & y\cos z - x\sin z & -2w & 0 \\ 0 & \cos z & \sin z & y & 0 \\ 0 & -\sin z & \cos z & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{iz}{n}} \end{bmatrix} : w, x, y, z \in \mathbb{R}\right\}.$$

**Proof.** The commutator subalgebra of $\mathfrak{g}_{4.9}^0$ is $\operatorname{span}\{E_1, E_2, E_3\}$. The result then follows by Corollary 2.4, since $\operatorname{span}\{E_1, E_2, E_3\} \cap \operatorname{span}\{E_4\} = \{0\}$, $\operatorname{span}\{E_1, E_2, E_3\} \cap \operatorname{span}\{E_1\} \neq \{0\}$, $\operatorname{span}\{E_1, E_2, E_3\} \cap \operatorname{span}\{E_1, E_4\} \neq \{0\}$. ■

**4.6. Groups with algebra $\mathfrak{g}_{4.10}$.** The universal covering Lie group

$$\mathsf{G}_{4.10} = \left\{\begin{bmatrix} e^{-y}\cos z & e^{-y}\sin z & x & 0 \\ -e^{-y}\sin z & e^{-y}\cos z & w & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix} = p(w, x, y, z) : w, x, y, z \in \mathbb{R}\right\}$$

has Lie algebra

$$\mathfrak{g}_{4.10} = \left\{\begin{bmatrix} -y & z & x & 0 \\ -z & -y & w & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 : w, x, y, z \in \mathbb{R}\right\}.$$

**Theorem 4.16.** *Any (nontrivial) discrete central subgroup of* $\mathsf{G}_{4.10}$ *is equivalent to exactly one of the subgroups* $\exp(2n\pi\mathbb{Z}E_4) = \{p(0, 0, 0, 2n\pi z) : z \in \mathbb{Z}\}$. *Here* $n \in \mathbb{N}$, $n \geq 1$ *parametrizes a family of discrete subgroups, each different value yielding a distinct (nonequivalent) subgroup.*

**Proof.** The center of $\mathsf{G}_{4.10}$ is $\exp(2\pi\mathbb{Z}E_4)$. Hence, any discrete central subgroup is of the form $\exp(2n\pi\mathbb{Z}E_4)$ for some $n \in \mathbb{N}$. For any automorphism $\phi \in \mathsf{Aut}(\mathsf{G}_{4.10})$ we have that $\phi(\exp(2n\pi\mathbb{Z}E_4)) = \exp(T_{\mathbf{1}}\phi \cdot 2n\pi\mathbb{Z}E_4) = \exp(2n\pi\mathbb{Z}E_4)$ (see Table 8). Hence $\phi(\exp(2n\pi\mathbb{Z}E_4)) = \exp(2m\pi\mathbb{Z}E_4)$ only if $n = m$. ■

**Corollary 4.17.** *There are two types of connected Lie groups with Lie algebra* $\mathfrak{g}_{4.10}$*, namely*

1. *the universal covering Lie group* $\mathsf{G}_{4.10}$*;*
2. *the* $n$*-fold coverings* $\mathsf{G}_{4.10}/\exp(2n\pi\mathbb{Z}E_4)$ *of* $\mathsf{G}_{4.10}/Z(\mathsf{G}_{4.10})$.

**Remark 4.18.** The group $\mathsf{G}_{4.10}/Z(\mathsf{G}_{4.10})$ is isomorphic to the group of affine transformations over $\mathbb{C}$:

$$\mathsf{Aff}(\mathbb{C}) = \left\{\begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} : x, y \in \mathbb{C},\, y \neq 0\right\}.$$

**Proposition 4.19.** *Any connected Lie group with Lie algebra* $\mathfrak{g}_{4.10}$ *is linearizable. In particular,* $\mathsf{G}_{4.10}/\exp(2n\pi\mathbb{Z}E_4)$ *is isomorphic to the matrix Lie group*

$$\left\{\begin{bmatrix} e^{-y}\cos z & e^{-y}\sin z & x & 0 \\ -e^{-y}\sin z & e^{-y}\cos z & w & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\frac{iz}{n}} \end{bmatrix} : w, x, y, z \in \mathbb{R}\right\}.$$

## A. Low-dimensional Lie algebras and groups

The classification of the (real) three- and four-dimensional Lie algebras is well known (see, e.g., [16], [22], [23], and the references therein). We prefer to use (a modified version of) the enumeration of these Lie algebras due to Mubarakzyanov [17], similar to that used by Patera et al. [20, 21]. However, in the three-dimensional case, we use the commutator relations in the Bianchi-Behr form [13].

**A.1. Three-dimensional Lie algebras and groups.** In terms of an (appropriate) ordered basis $(E_1, E_2, E_3)$, the commutator operation is given by

$$[E_2, E_3] = n_1 E_1 - \alpha E_2$$
$$[E_3, E_1] = \alpha E_1 + n_2 E_2$$
$$[E_1, E_2] = n_3 E_3.$$

The structure parameters $\alpha, n_1, n_2, n_3$ for each type are given in Table 1.

| Type | Bianchi | $\alpha$ | $n_1$ | $n_2$ | $n_3$ | Connected Groups |
|---|---|---|---|---|---|---|
| $3\mathfrak{g}_1$ | $I$ | 0 | 0 | 0 | 0 | $\mathbb{R}^3$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times \mathbb{T}^2$, $\mathbb{T}^3$ |
| $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ | $III$ | 1 | 1 | $-1$ | 0 | $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{T}$ |
| $\mathfrak{g}_{3.1}$ | $II$ | 0 | 1 | 0 | 0 | $\mathsf{H}_3$, $\mathsf{H}_3^*$ |
| $\mathfrak{g}_{3.2}$ | $IV$ | 1 | 1 | 0 | 0 | $\mathsf{G}_{3.2}$ |
| $\mathfrak{g}_{3.3}$ | $V$ | 1 | 0 | 0 | 0 | $\mathsf{G}_{3.3}$ |
| $\mathfrak{g}_{3.4}^0$ | $VI_0$ | 0 | 1 | $-1$ | 0 | $\mathsf{SE}(1,1)$ |
| $\mathfrak{g}_{3.4}^\alpha$ | $VI_\alpha$ | $\alpha>0$, $\alpha\neq 1$ | 1 | $-1$ | 0 | $\mathsf{G}_{3.4}^\alpha$ |
| $\mathfrak{g}_{3.5}^0$ | $VII_0$ | 0 | 1 | 1 | 0 | $\widetilde{\mathsf{SE}}(2)$, $\mathsf{SE}_n(2)$, $\mathsf{SE}(2)$ |
| $\mathfrak{g}_{3.5}^\alpha$ | $VII_\alpha$ | $\alpha>0$ | 1 | 1 | 0 | $\mathsf{G}_{3.5}^\alpha$ |
| $\mathfrak{g}_{3.6}$ | $VIII$ | 0 | 1 | 1 | $-1$ | $\widetilde{\mathsf{A}}$, $\mathsf{A}_n$, $\mathsf{SL}(2,\mathbb{R})$, $\mathsf{SO}(2,1)_0$ |
| $\mathfrak{g}_{3.7}$ | $IX$ | 0 | 1 | 1 | 1 | $\mathsf{SU}(2)$, $\mathsf{SO}(3)$ |

Table 1: Bianchi-Behr classification of 3D Lie algebras

A classification of the three-dimensional (real connected) Lie groups can be found in [18, Chapter 7, Section 1]. Let $\mathsf{G}$ be a three-dimensional (real connected) Lie group with Lie algebra $\mathfrak{g}$.

1. If $\mathfrak{g} \cong 3\mathfrak{g}_1$, then $\mathsf{G}$ is isomorphic to $\mathbb{R}^3$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times \mathbb{T}$, or $\mathbb{T}^3$.

2. If $\mathfrak{g} \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$, then $\mathsf{G}$ is isomorphic to $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ or $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{T}$.
3. If $\mathfrak{g} \cong \mathfrak{g}_{3.1}$, then $\mathsf{G}$ is isomorphic to the Heisenberg group $\mathsf{H}_3$ or the Lie group $\mathsf{H}_3^* = \mathsf{H}_3/\, Z(\mathsf{H}_3(\mathbb{Z}))$, where $Z(\mathsf{H}_3(\mathbb{Z}))$ is the group of integer points in the center $Z(\mathsf{H}_3) \cong \mathbb{R}$ of $\mathsf{H}_3$.
4. If $\mathfrak{g} \cong \mathfrak{g}_{3.2}$, $\mathfrak{g}_{3.3}$, $\mathfrak{g}_{3.4}^0$, $\mathfrak{g}_{3.4}^\alpha$, or $\mathfrak{g}_{3.5}^\alpha$, then $\mathsf{G}$ is isomorphic to the simply connected Lie group $\mathsf{G}_{3.2}$, $\mathsf{G}_{3.3}$, $\mathsf{G}_{3.4}^0 = \mathsf{SE}(1,1)$, $\mathsf{G}_{3.4}^\alpha$, or $\mathsf{G}_{3.5}^\alpha$, respectively. (The centers of these groups are trivial.)
5. If $\mathfrak{g} \cong \mathfrak{g}_{3.5}^0$, then $\mathsf{G}$ is isomorphic to the Euclidean group $\mathsf{SE}(2)$, the $n$-fold covering $\mathsf{SE}_n(2)$ of $\mathsf{SE}_1(2) = \mathsf{SE}(2)$, or the universal covering group $\widetilde{\mathsf{SE}}(2)$.
6. If $\mathfrak{g} \cong \mathfrak{g}_{3.6}$, then $\mathsf{G}$ is isomorphic to the pseudo-orthogonal group $\mathsf{SO}(2,1)_0$, the $n$-fold covering $\mathsf{A}_n$ of $\mathsf{SO}(2,1)_0$, or the universal covering group $\widetilde{\mathsf{A}}$. Here $\mathsf{A}_2 \cong \mathsf{SL}(2,\mathbb{R})$.
7. If $\mathfrak{g} \cong \mathfrak{g}_{3.7}$, then $\mathsf{G}$ is isomorphic to either the special unitary group $\mathsf{SU}(2)$ or the special orthogonal group $\mathsf{SO}(3)$.

Among these Lie groups, only $\mathsf{H}_3^*$, $\mathsf{A}_n$, $n \geq 3$, and $\widetilde{\mathsf{A}}$ are not linearizable.

**A.2. Four-dimensional Lie algebras.** In terms of an (appropriate) ordered basis $(E_1, E_2, E_3, E_4)$, the nonzero commutator relations for each four-dimensional Lie algebra are given in Tables 2 and 3. In Table 4, we cross-reference the classification scheme used in this paper against the schemes of Mubarakzyanov [17], Patera et al. [20, 21], and Šnobl and Winternitz [23]. We note that in the scheme of Mubarakzyanov, resp. Patera et al, the parameters in the family $g_{4.5}$, resp. $A_{4.5}^{ab}$, should be restricted slightly further to ensure nonredundancy. Cross-referencing to further schemes (such as those of Bratzlavsky [6] and Kruchkovich [14]) can be found in MacCallum's paper [16] (see also [1]).

We collect some basic properties for the four-dimensional Lie algebras in Table 5. For each algebra $\mathfrak{g}$, its nilradical is identified; the quotient $\mathfrak{g}/\,Z(\mathfrak{g})$ is displayed whenever $Z(\mathfrak{g})$ is nontrivial. We indicate which Lie algebras are unimodular, which are nilpotent, which are completely solvable (or triangular, see, e.g., [18, Chapter 2, Section 2]), which are exponential (see, e.g., [18, Chapter 2, Section 6.4]), and which are solvable. Most of these classes of Lie algebras can be characterized in terms of the adjoint operator $\mathrm{ad}_A = [A, \cdot]$; a Lie algebra $\mathfrak{g}$ is

- nilpotent if and only if the eigenvalues of $\mathrm{ad}_A$ are all zero for every $A \in \mathfrak{g}$.
- completely solvable if and only if the eigenvalues of $\mathrm{ad}_A$ are all real for every $A \in \mathfrak{g}$.
- exponential if and only if $\mathrm{ad}_A$ does not have any purely imaginary eigenvalues for any $A \in \mathfrak{g}$.
- unimodular if and only if $\mathrm{tr}(\mathrm{ad}_A) = 0$ for every $A \in \mathfrak{g}$.

We note that nilpotent algebras are completely solvable, completely solvable algebras are exponential, and exponential algebras are solvable. Furthermore, we indicate those algebras that admit an invariant scalar product (abbreviated ISP), i.e., a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ satisfying $\langle A, [B, C] \rangle = \langle [A, B], C \rangle$ for all $A, B, C \in \mathfrak{g}$.

A standard computation yields the automorphism group for each Lie algebra (cf. [22, 7]). In Tables 6, 7 and 8 the automorphisms of each Lie algebra, with respect to the ordered basis $(E_1, E_2, E_3, E_4)$, are displayed (as a parametrized matrix Lie group). For the parametrized families of Lie algebras, the automorphism group may depend on the parameter(s); in these cases the automorphism group along with the appropriate restriction(s) on the parameter(s) is displayed. The parameters $a_1, a_2, \ldots$ (which parametrize $\mathsf{Aut}(\mathfrak{g})$) are real numbers with the restriction that the determinant of the matrix is nonzero. In a few cases an additional parameter $\sigma = \pm 1$ is also used.

| Type | Nonzero commutators | | Parameters |
|---|---|---|---|
| $4\mathfrak{g}_1$ | | | |
| $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$ | $[E_1, E_2] = E_1$ | | |
| $2\mathfrak{g}_{2.1}$ | $[E_1, E_2] = E_1$ | $[E_3, E_4] = E_3$ | |
| $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ | $[E_2, E_3] = E_1$ | | |
| $\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$ | $[E_2, E_3] = E_1 - E_2$ | $[E_3, E_1] = E_1$ | |
| $\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1$ | $[E_2, E_3] = -E_2$ | $[E_3, E_1] = E_1$ | |
| $\mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1$ | $[E_2, E_3] = E_1$ | $[E_3, E_1] = -E_2$ | |
| $\mathfrak{g}_{3.4}^\alpha \oplus \mathfrak{g}_1$ | $[E_2, E_3] = E_1 - \alpha E_2$ | $[E_3, E_1] = \alpha E_1 - E_2$ | $\alpha > 0$, $\alpha \neq 1$ |
| $\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$ | $[E_2, E_3] = E_1$ | $[E_3, E_1] = E_2$ | |
| $\mathfrak{g}_{3.5}^\alpha \oplus \mathfrak{g}_1$ | $[E_2, E_3] = E_1 - \alpha E_2$ | $[E_3, E_1] = \alpha E_1 + E_2$ | $\alpha > 0$ |
| $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ | $[E_2, E_3] = E_1$<br>$[E_1, E_2] = -E_3$ | $[E_3, E_1] = E_2$ | |
| $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$ | $[E_2, E_3] = E_1$<br>$[E_1, E_2] = E_3$ | $[E_3, E_1] = E_2$ | |

Table 2: Decomposable four-dimensional Lie algebras

| Type | Nonzero commutators | | Parameters |
|---|---|---|---|
| $\mathfrak{g}_{4.1}$ | $[E_2, E_4] = E_1$ | $[E_3, E_4] = E_2$ | |
| $\mathfrak{g}_{4.2}^{\alpha}$ | $[E_1, E_4] = \alpha\, E_1$ | $[E_2, E_4] = E_2$ | $\alpha \neq 0$ |
| | $[E_3, E_4] = E_2 + E_3$ | | |
| $\mathfrak{g}_{4.3}$ | $[E_1, E_4] = E_1$ | $[E_3, E_4] = E_2$ | |
| $\mathfrak{g}_{4.4}$ | $[E_1, E_4] = E_1$ | $[E_2, E_4] = E_1 + E_2$ | |
| | $[E_3, E_4] = E_2 + E_3$ | | |
| $\mathfrak{g}_{4.5}^{\alpha,\beta}$ | $[E_1, E_4] = E_1$ | $[E_2, E_4] = \beta\, E_2$ | $-1<\alpha\leq\beta\leq 1, \alpha\beta\neq 0$ or $\alpha=-1, 0<\beta\leq 1$ |
| | $[E_3, E_4] = \alpha\, E_3$ | | |
| $\mathfrak{g}_{4.6}^{\alpha,\beta}$ | $[E_1, E_4] = \alpha\, E_1$ | $[E_2, E_4] = \beta\, E_2 - E_3$ | $\alpha > 0,\ \beta \in \mathbb{R}$ |
| | $[E_3, E_4] = E_2 + \beta\, E_3$ | | |
| $\mathfrak{g}_{4.7}$ | $[E_1, E_4] = 2E_1$ | $[E_2, E_4] = E_2$ | |
| | $[E_3, E_4] = E_2 + E_3$ | $[E_2, E_3] = E_1$ | |
| $\mathfrak{g}_{4.8}^{-1}$ | $[E_2, E_3] = E_1$ | $[E_2, E_4] = E_2$ | |
| | $[E_3, E_4] = -E_3$ | | |
| $\mathfrak{g}_{4.8}^{\alpha}$ | $[E_1, E_4] = (1+\alpha)\, E_1$ | $[E_2, E_4] = E_2$ | $-1 < \alpha \leq 1$ |
| | $[E_3, E_4] = \alpha\, E_3$ | $[E_2, E_3] = E_1$ | |
| $\mathfrak{g}_{4.9}^{0}$ | $[E_2, E_3] = E_1$ | $[E_2, E_4] = -E_3$ | |
| | $[E_3, E_4] = E_2$ | | |
| $\mathfrak{g}_{4.9}^{\alpha}$ | $[E_1, E_4] = 2\alpha\, E_1$ | $[E_2, E_4] = \alpha\, E_2 - E_3$ | $\alpha > 0$ |
| | $[E_3, E_4] = E_2 + \alpha\, E_3$ | $[E_2, E_3] = E_1$ | |
| $\mathfrak{g}_{4.10}$ | $[E_1, E_3] = E_1$ | $[E_2, E_3] = E_2$ | |
| | $[E_1, E_4] = -E_2$ | $[E_2, E_4] = E_1$ | |

Table 3: Indecomposable four-dimensional Lie algebras

| | Type | Mubarakzyanov [17] | Patera et al. [20, 21] | Šnobl and Winternitz [23] | Other |
|---|---|---|---|---|---|
| 1D | $\mathfrak{g}_1$ | $g_1$ | $A_1$ | $\mathfrak{n}_{1,1}$ | $\mathbb{R}$ |
| 2D | $\mathfrak{g}_{2.1}$ | $g_2$ | $A_2$ | $\mathfrak{s}_{2,1}$ | $\mathfrak{aff}(\mathbb{R})$ |
| 3D | $\mathfrak{g}_{3.1}$ | $g_{3,1}$ | $A_{3,1}$ | $\mathfrak{n}_{3,1}$ | $\mathfrak{h}_3$ |
| | $\mathfrak{g}_{3.2}$ | $g_{3,2}$ | $A_{3,2}$ | $\mathfrak{s}_{3,2}$ | |
| | $\mathfrak{g}_{3.3}$ | $g_{3,3}$ | $A_{3,3}$ | $\mathfrak{s}_{3,1},\ a{=}1$ | |
| | $\mathfrak{g}_{3.4}^0$ | $g_{3,4},\ h{=}-1$ | $A_{3,4}$ | $\mathfrak{s}_{3,1},\ a{=}-1$ | $\mathfrak{se}(1,1)$ |
| | $\mathfrak{g}_{3.4}^\alpha$ | $g_{3,4},\ 0{<}\lvert h\rvert{<}1$ | $A_{3,5}^a$ | $\mathfrak{s}_{3,1},\ 0{<}\lvert a\rvert{<}1$ | |
| | $\mathfrak{g}_{3.5}^0$ | $g_{3,5},\ p{=}0$ | $A_{3,6}$ | $\mathfrak{s}_{3,3},\ \alpha{=}0$ | $\mathfrak{se}(2)$ |
| | $\mathfrak{g}_{3.5}^\alpha$ | $g_{3,5},\ p{>}0$ | $A_{3,7}^a$ | $\mathfrak{s}_{3,3},\ \alpha{>}0$ | |
| | $\mathfrak{g}_{3.6}$ | $g_{3,6}$ | $A_{3,8}$ | $\mathfrak{sl}(2,\mathbb{R})$ | $\mathfrak{so}(2,1)$ |
| | $\mathfrak{g}_{3.7}$ | $g_{3,7}$ | $A_{3,9}$ | $\mathfrak{so}(3,\mathbb{R})$ | $\mathfrak{su}(2)$ |
| 4D | $\mathfrak{g}_{4.1}$ | $g_{4,1}$ | $A_{4,1}$ | $\mathfrak{n}_{4,1}$ | |
| | $\mathfrak{g}_{4.2}^\alpha$ | $g_{4,2}$ | $A_{4,2}^a$ | $\mathfrak{s}_{4,4}$ | |
| | $\mathfrak{g}_{4.3}$ | $g_{4,3}$ | $A_{4,3}$ | $\mathfrak{s}_{4,1}$ | |
| | $\mathfrak{g}_{4.4}$ | $g_{4,4}$ | $A_{4,4}$ | $\mathfrak{s}_{4,2}$ | |
| | $\mathfrak{g}_{4.5}^{\alpha,\beta}$ | $g_{4,5}$ | $A_{4,5}^{ab}$ | $\mathfrak{s}_{4,3}$ | |
| | $\mathfrak{g}_{4.6}^{\alpha,\beta}$ | $g_{4,6}$ | $A_{4,6}^{ab}$ | $\mathfrak{s}_{4,5}$ | |
| | $\mathfrak{g}_{4.7}$ | $g_{4,7}$ | $A_{4,7}$ | $\mathfrak{s}_{4,10}$ | |
| | $\mathfrak{g}_{4.8}^{-1}$ | $g_{4,8},\ h{=}-1$ | $A_{4,8}$ | $\mathfrak{s}_{4,6}$ | |
| | $\mathfrak{g}_{4.8}^\alpha$ | $g_{4,8},\ -1{<}h{\leq}1$ | $A_{4,9}^b$ | $\mathfrak{s}_{4,8}$ if $\alpha{\neq}0$ | |
| | | | | $\mathfrak{s}_{4,11}$ if $\alpha{=}0$ | |
| | $\mathfrak{g}_{4.9}^0$ | $g_{4,9},\ p{=}0$ | $A_{4,10}$ | $\mathfrak{s}_{4,7}$ | oscillator |
| | $\mathfrak{g}_{4.9}^\alpha$ | $g_{4,9},\ p{>}0$ | $A_{4,11}^a$ | $\mathfrak{s}_{4,9}$ | |
| | $\mathfrak{g}_{4.10}$ | $g_{4,10}$ | $A_{4,12}$ | $\mathfrak{s}_{4,12}$ | $\mathfrak{aff}(\mathbb{C})$ |

Table 4: Cross-reference of indecomposable lower-dimensional Lie algebras

| Type | Nilradical | $\mathfrak{g}/Z(\mathfrak{g})$ | Unimodular | Nilpotent | Compl. Solv. | Exponential | Solvable | Admits ISP |
|---|---|---|---|---|---|---|---|---|
| $4\mathfrak{g}_1$ | $4\mathfrak{g}_1$ | $\{0\}$ | • | • | • | • | • | • |
| $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$ | $3\mathfrak{g}_1$ | $\mathfrak{g}_{2.1}$ | | | • | • | • | |
| $2\mathfrak{g}_{2.1}$ | $2\mathfrak{g}_1$ | — | | | • | • | • | |
| $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ | $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ | $2\mathfrak{g}_1$ | • | • | • | • | • | |
| $\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$ | $3\mathfrak{g}_1$ | $\mathfrak{g}_{3.2}$ | | | • | • | • | |
| $\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1$ | $3\mathfrak{g}_1$ | $\mathfrak{g}_{3.3}$ | | | • | • | • | |
| $\mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1$ | $3\mathfrak{g}_1$ | $\mathfrak{g}_{3.4}^0$ | • | | • | • | • | |
| $\mathfrak{g}_{3.4}^\alpha \oplus \mathfrak{g}_1$ | $3\mathfrak{g}_1$ | $\mathfrak{g}_{3.4}^\alpha$ | | | • | • | • | |
| $\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$ | $3\mathfrak{g}_1$ | $\mathfrak{g}_{3.5}^0$ | • | | | | • | |
| $\mathfrak{g}_{3.5}^\alpha \oplus \mathfrak{g}_1$ | $3\mathfrak{g}_1$ | $\mathfrak{g}_{3.5}^\alpha$ | | | | • | • | |
| $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ | $\mathfrak{g}_1$ | $\mathfrak{g}_{3.6}$ | • | | | | | • |
| $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$ | $\mathfrak{g}_1$ | $\mathfrak{g}_{3.7}$ | • | | | | | • |
| $\mathfrak{g}_{4.1}$ | $\mathfrak{g}_{4.1}$ | $\mathfrak{g}_{3.1}$ | • | • | • | • | • | |
| $\mathfrak{g}_{4.2}^\alpha$ | $3\mathfrak{g}_1$ | — | $\alpha=-2$ | | • | • | • | |
| $\mathfrak{g}_{4.3}$ | $3\mathfrak{g}_1$ | $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ | | | • | • | • | |
| $\mathfrak{g}_{4.4}$ | $3\mathfrak{g}_1$ | — | | | • | • | • | |
| $\mathfrak{g}_{4.5}^{\alpha,\beta}$ | $3\mathfrak{g}_1$ | — | $\alpha+\beta=-1$ | | • | • | • | |
| $\mathfrak{g}_{4.6}^{\alpha,\beta}$ | $3\mathfrak{g}_1$ | — | $\alpha=-2\beta$ | | | $\beta\neq0$ | • | |
| $\mathfrak{g}_{4.7}$ | $\mathfrak{g}_{3.1}$ | — | | | • | • | • | |
| $\mathfrak{g}_{4.8}^{-1}$ | $\mathfrak{g}_{3.1}$ | $\mathfrak{g}_{3.4}^0$ | • | | • | • | • | • |
| $\mathfrak{g}_{4.8}^\alpha$ | $\mathfrak{g}_{3.1}$ | — | | | • | • | • | |
| $\mathfrak{g}_{4.9}^0$ | $\mathfrak{g}_{3.1}$ | $\mathfrak{g}_{3.5}^0$ | • | | | | • | • |
| $\mathfrak{g}_{4.9}^\alpha$ | $\mathfrak{g}_{3.1}$ | — | | | | • | • | |
| $\mathfrak{g}_{4.10}$ | $2\mathfrak{g}_1$ | — | | | | | • | |

Table 5: Four-dimensional Lie algebras (properties)

| Type | Automorphisms |
|---|---|
| $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$ | $\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_3 & a_4 & a_5 \\ 0 & a_6 & a_7 & a_8 \end{bmatrix}$ |
| $2\mathfrak{g}_{2.1}$ | $\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & a_3 & a_4 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ |
| $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ | $\begin{bmatrix} a_2a_7 - a_6a_3 & a_1 & a_5 & a_9 \\ 0 & a_2 & a_6 & 0 \\ 0 & a_3 & a_7 & 0 \\ 0 & a_4 & a_8 & a_{10} \end{bmatrix}$ |
| $\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$ | $\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ 0 & a_1 & a_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix}$ |
| $\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1$ | $\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ a_4 & a_5 & a_6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_7 & a_8 \end{bmatrix}$ |
| $\mathfrak{g}^0_{3.4} \oplus \mathfrak{g}_1$ | $\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ \sigma\, a_2 & \sigma\, a_1 & a_4 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix}, \ \sigma = \pm 1$ |
| $\mathfrak{g}^\alpha_{3.4} \oplus \mathfrak{g}_1$ $\alpha>0,\ \alpha\neq 1$ | $\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & a_1 & a_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix}$ |
| $\mathfrak{g}^0_{3.5} \oplus \mathfrak{g}_1$ | $\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ -\sigma a_2 & \sigma a_1 & a_4 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix}, \ \sigma = \pm 1$ |
| $\mathfrak{g}^\alpha_{3.5} \oplus \mathfrak{g}_1$ $\alpha>0$ | $\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ -a_2 & a_1 & a_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix}$ |
| $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$ | $\begin{bmatrix} \mathsf{SO}(2,1) & \mathbf{0} \\ \mathbf{0} & a_4 \end{bmatrix}, \quad \mathsf{SO}(2,1) = \{g \in \mathbb{R}^{3\times 3} \, : \, g^\top J g = J, \ \det g = 1\}$ $J = \operatorname{diag}(1,1,-1)$ |
| $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$ | $\begin{bmatrix} \mathsf{SO}(3) & \mathbf{0} \\ \mathbf{0} & a_4 \end{bmatrix}, \quad \mathsf{SO}(3) = \{g \in \mathbb{R}^{3\times 3} \, : \, g^\top g = I, \ \det g = 1\}$ $I = \operatorname{diag}(1,1,1)$ |

Table 6: Automorphisms of four-dimensional decomposable Lie algebras

| Type | Automorphisms |
|---|---|
| $\mathfrak{g}_{4.1}$ | $\begin{bmatrix} a_1a_2^2 & a_2a_3 & a_4 & a_5 \\ 0 & a_1a_2 & a_3 & a_6 \\ 0 & 0 & a_1 & a_7 \\ 0 & 0 & 0 & a_2 \end{bmatrix}$ |
| $\mathfrak{g}_{4.2}^{\alpha}$ $\alpha\neq 0$ | $\begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & a_3 & a_5 \\ 0 & 0 & a_2 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \alpha \neq 1$ $\begin{bmatrix} a_1 & 0 & a_4 & a_6 \\ a_2 & a_3 & a_5 & a_7 \\ 0 & 0 & a_3 & a_8 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \alpha = 1$ |
| $\mathfrak{g}_{4.3}$ | $\begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & a_3 & a_5 \\ 0 & 0 & a_2 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\mathfrak{g}_{4.4}$ | $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_2 & a_5 \\ 0 & 0 & a_1 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\mathfrak{g}_{4.5}^{\alpha,\beta}$ $-1<\alpha\leq\beta\leq 1, \alpha\beta\neq 0$ or $\alpha=-1, 0<\beta\leq 1$ | $\begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & 0 & a_5 \\ 0 & 0 & a_3 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \begin{matrix} \alpha \neq 1 \\ \beta \neq 1 \\ \alpha \neq \beta \end{matrix}$ $\begin{bmatrix} a_1 & 0 & 0 & a_6 \\ 0 & a_2 & a_4 & a_7 \\ 0 & a_3 & a_5 & a_8 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \begin{matrix} \alpha \neq 1 \\ \alpha = \beta \end{matrix}$ $\begin{bmatrix} a_1 & a_3 & 0 & a_6 \\ a_2 & a_4 & 0 & a_7 \\ 0 & 0 & a_5 & a_8 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \begin{matrix} \alpha \neq 1 \\ \beta = 1 \end{matrix}$ $\begin{bmatrix} a_1 & a_4 & a_7 & a_{10} \\ a_2 & a_5 & a_8 & a_{11} \\ a_3 & a_6 & a_9 & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \begin{matrix} \alpha = 1 \\ \beta = 1 \end{matrix}$ |
| $\mathfrak{g}_{4.6}^{\alpha,\beta}$ $\alpha>0, \beta\in\mathbb{R}$ | $\begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & a_3 & a_5 \\ 0 & -a_3 & a_2 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |

Table 7: Automorphisms of four-dimensional indecomposable Lie algebras

| Type | Automorphisms |
| --- | --- |
| $\mathfrak{g}_{4.7}$ | $\begin{bmatrix} a_1^2 & -a_1a_3 & a_1a_4-(a_1+a_2)\,a_3 & a_5 \\ 0 & a_1 & a_2 & a_4 \\ 0 & 0 & a_1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\mathfrak{g}_{4.8}^{-1}$ | $\begin{bmatrix} a_1a_2 & a_1a_3 & a_2a_4 & a_5 \\ 0 & a_1 & 0 & a_4 \\ 0 & 0 & a_2 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -a_1a_2 & -a_2a_4 & -a_1a_3 & a_5 \\ 0 & 0 & a_1 & a_4 \\ 0 & a_2 & 0 & a_3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ |
| $\mathfrak{g}_{4.8}^{\alpha}$ $_{-1<\alpha\leq 1}$ | $\begin{bmatrix} a_1a_2 & -a_1a_3 & a_2a_4 & a_5 \\ 0 & a_1 & 0 & a_4 \\ 0 & 0 & a_2 & \alpha\, a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \alpha \neq 0, \ \alpha \neq 1 \quad \begin{bmatrix} a_1a_2 & a_3 & a_2a_4 & a_5 \\ 0 & a_1 & 0 & a_4 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \alpha = 0$ <br> $\begin{bmatrix} a_1a_2 - a_6a_7 & -a_1a_3 + a_4a_6 & a_2a_4 - a_3a_7 & a_5 \\ 0 & a_1 & a_7 & a_4 \\ 0 & a_6 & a_2 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \alpha = 1$ |
| $\mathfrak{g}_{4.9}^{0}$ | $\begin{bmatrix} \sigma\left(a_1^2+a_2^2\right) & -\sigma a_1a_4 + a_2a_5 & -a_1a_5 - \sigma a_2a_4 & a_3 \\ 0 & a_1 & a_2 & a_4 \\ 0 & -\sigma a_2 & \sigma a_1 & a_5 \\ 0 & 0 & 0 & \sigma \end{bmatrix}, \ \sigma = \pm 1$ |
| $\mathfrak{g}_{4.9}^{\alpha}$ $_{\alpha>0}$ | $\begin{bmatrix} a_1^2+a_2^2 & \frac{-a_2(\alpha\, a_4-a_5)-a_1(a_4+\alpha\, a_5)}{1+\alpha^2} & \frac{a_1(\alpha\, a_4-a_5)-a_2(a_4+\alpha\, a_5)}{1+\alpha^2} & a_3 \\ 0 & a_1 & a_2 & a_4 \\ 0 & -a_2 & a_1 & a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\mathfrak{g}_{4.10}$ | $\begin{bmatrix} a_1 & \sigma\, a_2 & a_3 & \sigma\, a_4 \\ -a_2 & \sigma\, a_1 & a_4 & -\sigma\, a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix}, \ \sigma = \pm 1$ |

Table 8: Automorphisms of four-dimensional indecomposable Lie algebras (cont.)

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