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Isomorphisms and rigidity of arithmetic Kac–Moody groups

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Abstract. In this article we solve the isomorphism problem for subgroups of integral points of two-spherical Kac–Moody groups over the rational numbers. Along the way we establish versions of Mostow–Margulis strong rigidity and Margulis superrigidity with target in two-spherical split Kac–Moody groups over the rational numbers for arithmetically defined subgroups.
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1. Introduction

Fundamental work by Caprace and Caprace–Mühlherr [CM05], [CM06], [Cap09] provided a solution to the isomorphism problem of split Kac–Moody groups over fields: among other things they prove that two Kac–Moody functors $\mathcal{G}$, $\mathcal{G}'$ are identical over fields if and only if there exists a sufficiently large field $\mathbb{K}$ such that the groups $\mathcal{G}(\mathbb{K})$ and $\mathcal{G}'(\mathbb{K})$ of $\mathbb{K}$-rational points are isomorphic.

The purpose of our article is to generalize this result to the ring $\mathbb{Z}$ of integers:

Theorem 4.9. *Let $\mathcal{G}$, $\mathcal{G}'$ be simply connected irreducible two-spherical split Kac–Moody functors. If there exists a group isomorphism $\mathcal{G}(\mathbb{Z}) \cong \mathcal{G}'(\mathbb{Z})$, then $\mathcal{G} = \mathcal{G}'$ over fields.*

In other words, a simply connected irreducible two-spherical split Kac–Moody functor

$$\mathcal{G} : \text{CommRings} \to \text{Grps}$$

is uniquely determined by its evaluation on the initial object $\mathbb{Z}$ of CommRings.

Our overall strategy of proof is similar to the one by Caprace and Caprace–Mühlherr: we first use a fixed-point theorem by Caprace–Monod [CM09] in order to establish that an isomorphism $\mathcal{G}(\mathbb{Z}) \cong \mathcal{G}'(\mathbb{Z})$ necessarily has to preserve bounded subgroups and then conduct a local analysis. While the concept of regular and coregular diagonalizable subgroups ([Cap09, Section 4.2]) is not applicable (the ring

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$\mathbb{Z}$ only has two invertible elements), the classical theory of rigidity of arithmetic groups by Margulis and Mostow–Margulis ([Mar91]) comes to our rescue. A positive side effect of our variant of the overall strategy is that next to the solution of the isomorphism problem we also establish versions of the Mostow–Margulis and Margulis rigidity results for arithmetic Kac–Moody groups (with target in the $\mathbb{Q}$-rational points).

While in this article we focus exclusively on the ring $\mathbb{Z}$ of integers, we firmly believe Theorem 4.9 to be true for arbitrary rings of S-integers in characteristic zero; in fact, the concept of regular and coregular diagonalizable subgroups should be applicable to nearly all rings of S-integers, as should our approach via classical rigidity. Moreover, there certainly is room for generalization of our Margulis superrigidity result (see Proposition 4.4) to targets in the $\mathbb{R}$-rational points instead of the $\mathbb{Q}$-rational points: Finite generation of arithmetic Kac–Moody groups in fact implies that it is enough to consider targets in the $\mathbb{K}$-rational points where $\mathbb{K}$ is an algebraic number field so that by Galois descent this naturally leads to the study of rigidity properties of arithmetic Kac–Moody groups in the quasi-split situation.

Note that our proof of the Caprace–Monod fixed-point theorem (see Theorem A.1) fails in positive characteristic, as the theory of U-elements requires characteristic 0 (see [LMR00, Theorem 2.15]).

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2. Twin buildings and Kac–Moody groups

Buildings. Buildings can be studied from the point of view of simplicial complexes (as done in [Tit74]) or, equivalently, from the point of view of chamber systems (as introduced in [Tit81]). The book [AB08] is a comprehensive introduction into the theory of buildings that explains both concepts in detail, also including the theory of twin buildings. For the necessary background information on reflection groups, Coxeter systems, and their associated root systems we refer to [AB08, Sections 1.5, 3.4] or to [Hum90].

Definition 2.1. Let (W, S) be a Coxeter system. A building of type (W, S) is a pair (Δ, δ) consisting of a set Δ, whose elements are called chambers, together with a distance function $\delta : \Delta \times \Delta \to W$ satisfying the following axioms, where $x, y \in \Delta$ and $\delta(x, y) = w$:

(Bu1) $w = 1$ if and only if $x = y$.

(Bu2) If $z \in \Delta$ such that $\delta(y, z) = s \in S$, then $\delta(x, z) \in \{ws, w\}$. If additionally $l(ws) > l(w)$, then $\delta(x, z) = ws$.

(Bu3) If $s \in S$, there exists $z \in \Delta$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

In fact, (W, S) itself is a building:

Example 2.2. Let (W, S) be a Coxeter system. Then $\Delta := W$ and $\delta : \Delta \times \Delta \to W : (x, y) \mapsto x^{-1}y$ yields a building of type (W, S), denoted by $\Delta(W, S)$. For any three chambers $x, y, z \in \Delta$ one has $\delta(x, z) = x^{-1}z = x^{-1}yy^{-1}z = \delta(x, y)\delta(y, z)$; see also [AB08, Lemma 5.55].

Definition 2.3. For every $c \in \Delta$ and every subset $S' \subseteq S$ we define the S'-residue $R_{S'}(c)$ to be

$$R_{S'}(c) := \{d \in \Delta \mid \delta(c, d) \in W_{S'} = \langle s \mid s \in S' \rangle\};$$

the collection of all S'-residues in Δ will be denoted $\mathrm{Res}_{S'}(\Delta)$.

Lemma 2.4 ([AB08, Lemma 5.16 and Corollary 5.30])**.** *Any residue of a building Δ is again a building. For any $S' \subseteq S$ the elements of $\mathrm{Res}_{S'}(\Delta)$ partition Δ.*

A building is called spherical if W is a finite reflection group, i.e., if the Coxeter complex of (W, S) has a geometric realization on a sphere. A building is called k-spherical if all residues of rank $\leq k$ are spherical buildings. Every building is one-spherical, and a building of type (W, S) is $|S|$-spherical if and only if it is spherical. For our purposes the class of two-spherical buildings and their spherical residues will play a key role.

Another key observation that we will make use of in this article is that buildings can be considered as complete CAT(0) spaces. Recall that a complete CAT(0) space is a complete metric space (X, d_X) such that for $x, y, z, p \in X$ with $d_X(x, p) = td_X(x, y)$ and $d_X(p, y) = (1 - t)d_X(x, y)$ for some $t \in [0, 1]$ one has

$$(d_X(z, p))^2 \leq (1 - t)(d_X(z, x))^2 + t(d_X(z, y))^2 - t(1 - t)(d_X(x, y))^2.$$

Theorem 2.5 ([Dav08, Theorem 18.3.1])**.** *Let Δ be a building. Then there exists a complete* CAT(0) *space (X, d_X), called the Davis realization of Δ, such that* $\mathrm{Aut}(\Delta) = \mathrm{Aut}(X, d_X)$.

For euclidean or hyperbolic buildings, i.e., buildings with a euclidean or hyperbolic Coxeter system (W, S), the Davis realization is straightforward to construct from the canonical euclidean, resp. hyperbolic geometric realization of the Coxeter complex (see [BT72, Section 2.5], also [AB08, Section 11.2]), whereas in

general for buildings of arbitrary type this construction can be rather involved (see [Dav98], also [Dav08, Chapter 12 and Sections 18.2, 18.3] and [AB08, Chapter 12]).
Twin buildings.

Definition 2.6. A twin building of type (W, S) is a triple $((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ consisting of two buildings (Δ_+, δ_+) and (Δ_-, δ_-) of type (W, S) and a codistance function $\delta^* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \to W$ subject to the following conditions, where $x \in \Delta_\pm$, $y \in \Delta_\mp$ and $\delta^*(x, y) = w$:

(Tw1) $\delta^*(y, x) = w^{-1}$,

(Tw2) if $z \in \Delta_\mp$ such that $\delta_\mp(y, z) = s \in S$, and $l(ws) < l(w)$, then $\delta^*(x, z) = ws$, and

(Tw3) if $s \in S$, then there exists $z \in \Delta_\mp$ such that $\delta_\mp(y, z) = s$ and $\delta^*(x, z) = ws$.

A twin building is called spherical, resp. k-spherical if both of its halves have the corresponding property.

Example 2.7. Let (W, S) be any Coxeter system and let $\Delta_\pm := W$ and $\delta_\pm$ as in Example 2.2. Moreover, define $\delta^* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \to W$ by $\delta^*(v, w) := v^{-1}w$. Then $((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ is a twin building. In this case the distance and the codistance function are related by the formula

$$\delta^*(x, z) = x^{-1}z = x^{-1}yy^{-1}z = \delta^*(x, y)\delta_\mp(y, z), \quad x \in \Delta_\pm, y, z \in \Delta_\mp;$$

see also [AB08, Lemma 5.173(4)].

RGD systems. A group G acts by isometries on a twin building

$$\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$$

if it acts on each half and preserves the distances and the codistance. Here we describe a class of twin buildings endowed with a natural group action coming from an RGD system. For more details on RGD systems we strongly recommend to consult [AB08, Chapters 7, 8] or [CR09]

Definition 2.8. Let G be a group and let $\{U_\alpha\}_{\alpha \in \Phi}$ be a family of subgroups of G, indexed by some root system Φ of type (W, S), let $\Pi = \{\alpha_s \in \Phi \mid s \in S\}$ be a system of simple roots, with $\Phi^+ \subset \Phi$ the corresponding set of positive roots, and let T be a subgroup of G. The triple $(G, \{U_\alpha\}_{\alpha \in \Phi}, T)$ is called an RGD system of type (W, S) if it satisfies the following assertions.

(RGD0) For each root $\alpha \in \Phi$, one has $U_\alpha \neq \{1\}$.

(RGD1) For each prenilpotent pair $\{\alpha, \beta\} \subseteq \Phi$ of distinct roots, one has $[U_\alpha, U_\beta] \subseteq \langle U_\gamma \mid \gamma \in]\alpha, \beta[\rangle$.
(Cf. [AB08, Sections 8.5.2, 8.5.3] for a definition of a prenilpotent pair, the "closed" interval $[\alpha, \beta]$ and the "open" interval $]\alpha, \beta[$.)

(RGD2) For each $s \in S$ there exists a function $\mu_s : U_{\alpha_s} \backslash \{1\} \to G$ such that for all $u \in U_{\alpha_s} \backslash \{1\}$ and $\alpha \in \Phi$ one has $\mu_s(u) \in U_{-\alpha_s} u U_{-\alpha_s}$ and $\mu_s(u) U_\alpha \mu_s(u)^{-1} = U_{s(\alpha)}$.

(RGD3) For each $s \in S$ one has $U_{-\alpha_s} \not\subseteq U_+ := \langle U_\alpha \mid \alpha \in \Phi^+ \rangle$.

(RGD4) $G = T.\langle U_\alpha \mid \alpha \in \Phi \rangle$.

(RGD5) The group T normalises every U_α.

The tuple $(\{U_\alpha\}_{\alpha \in \Phi}, T)$ is called a root group datum, the U_α are called the root subgroups, and the $G_\alpha := \langle U_{\pm\alpha} \rangle$ are called the rank one subgroups. Occasionally, for pairwise distinct simple roots $\alpha_1, \ldots, \alpha_r$, we use the notation $G_{\alpha_1,\ldots,\alpha_r}$ for the group generated by $G_{\alpha_1} \cup \cdots \cup G_{\alpha_r}$. These groups are then referred to as fundamental rank r subgroups.

The RGD system is called centered if G is generated by its root subgroups, i.e., if $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$.

Twin BN-pairs from RGD systems. Root group data give rise to twin BN-pairs in the sense of the following definition:

Definition 2.9. Let G be a group and let B, N be subgroups of G. The pair (B, N) is called a BN-pair for G, if G is generated by B and N, the intersection $T := B \cap N$ is normal in N, and the quotient group $W := N/T$ admits a set S of generating involutions such that

(BN1) for all $w \in W$ and $s \in S$ one has $wBs \subseteq BwsB \cup BwB$, and

(BN2) $sBs \not\subseteq B$ for each $s \in S$.

Two BN-pairs (B_+, N) and (B_-, N) of the same group G satisfying $B_+ \cap N = B_- \cap N$ yield a twin BN-pair (B_+, B_-, N), if the following additional assertions hold:

(TBN1) for $\epsilon \in \{+, -\}$ and all $w \in W$, $s \in S$ such that $l(sw) < l(w)$, one has $B_\epsilon s B_\epsilon w B_{-\epsilon} = B_\epsilon sw B_{-\epsilon}$, and

(TBN2) for each $s \in S$ one has $B_+ s \cap B_- = \emptyset$.

If (B, N) is a BN-pair for G and S is as above then the quadruple (G, B, N, S) is called a Tits system with Weyl group W. The notion of a twin Tits system (G, B_+, B_-, N, S) is defined accordingly. We remark that the pair (W, S) is a Coxeter system; cf. [AB08, Theorem 6.56(1)].

A group G with a BN-pair admits a Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$, cf. [AB08, Theorems 6.17 and 6.56(1)], and a group G with a twin BN-pair admits a Birkhoff decomposition $G = \bigsqcup_{w \in W} B_\epsilon w B_{-\epsilon}$, cf. [AB08, Proposition 6.81].

Important examples arise from root group data:

Proposition 2.10 ([AB08, Theorem 8.80]). *Let G be a group with a root group datum $(\{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W,S) and, for each $s \in S$, let $\mu_s : U_{\alpha_s}\backslash\{1\} \to U_{-\alpha_s}U_{\alpha_s}U_{-\alpha_s}$ be the map provided by* (RGD2). *Then the groups*

$$\begin{aligned} N &:= T.\langle \mu_s(u) \mid u \in U_\alpha\backslash\{1\}, s \in S\rangle, \\ B_+ &:= T.U_+, \\ B_- &:= T.U_- \end{aligned}$$

yield a twin BN-pair (B_+, B_-, N) of the group G.

Twin buildings from twin BN-pairs. For a twin Tits system (G, B_+, B_-, N, S) with Weyl group W define $\Delta_\pm := G/B_\pm$. Given $gB_\pm, hB_\pm \in \Delta_\pm$ using the Bruhat decomposition let

$$\delta_\pm(gB_\pm, hB_\pm) := w \in W \quad \text{if and only if} \quad B_\pm g^{-1}hB_\pm = B_\pm wB_\pm.$$

Similarly using the Birkhoff decomposition instead, given $gB_\pm \in \Delta_\pm$ and $hB_\mp \in \Delta_\mp$ let

$$\delta^*(gB_\pm, hB_\mp) := w \in W \quad \text{if and only if} \quad B_\pm g^{-1}hB_\mp = B_\pm wB_\mp.$$

Then $((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ is a twin building of type (W,S), see [AB08, Theorem 6.56 and Definition 6.82].

Definition 2.11. The above twin building is referred to as the twin building associated to the twin Tits system (G, B_+, B_-, N, S). Similarly, if the twin Tits system arises from an RGD system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$, we refer to it as the twin building associated to the RGD system.

Definition 2.12. If $\Delta(G, B_+, B_-, N, S)$ is a twin building coming from a twin BN-pair as above, a subgroup of G is called bounded, if it stabilizes spherical residues R_+ of G/B_+ and R_- of G/B_-.

In other words, a subgroup of G is bounded if it is contained in the intersection $A := P_+ \cap P_-$ for $P_+ = \mathrm{Stab}_G(R_+)$ and $P_- = \mathrm{Stab}_G(R_-)$. Considering the Davis realizations $X_\pm$ from Theorem 2.5 of the two halves $G/B_\pm$ of the twin building, a subgroup of G is bounded if and only if there exist $x_+ \in X_+$ and $x_- \in X_-$ such that G fixes both x_+ and x_- (see e.g. [AB08, Corollary 12.67]).

Kac–Moody groups.

Definition 2.13. A generalized Cartan matrix is a matrix $A = (a_{ij})_{1\le i,j\le n} \in \mathbb{Z}^{n\times n}$ satisfying $a_{ii} = 2$, $a_{ij} \le 0$ for $i \ne j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Let $I = \{1, \ldots, n\}$ and let $A = (a_{ij})_{1\le i,j\le n}$ be a generalized Cartan matrix. A quintuple $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ is called a Kac–Moody root datum if Λ is a free $\mathbb{Z}$-module, each c_i is an element of Λ and each h_i is in the $\mathbb{Z}$-dual $\Lambda^\vee$ of Λ such that for all $i, j \in I$ one has $h_i(c_j) = a_{ij}$.

Following [Tit87, 3.6] to a Kac–Moody root datum $\mathcal{D}$ one associates a triple $(\mathcal{G}, \{\varphi_i\}_{i\in I}, \eta)$, where $\mathcal{G}$ is a group functor on the category of commutative unital rings, the φ_i are maps $\mathrm{SL}_2(R) \to \mathcal{G}(R)$, and η is a natural transformation $\mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], -) \to \mathcal{G}$ such that the following assertions hold:

(KMG1) If $\mathbb{K}$ is a field, then the group $\mathcal{G}(\mathbb{K})$ is generated by the images of the φ_i and $\eta(\mathbb{K})$.

(KMG2) For all rings R the homomorphism $\eta(R) : \mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], R) \to \mathcal{G}(R)$ is injective.

(KMG3) Given a ring R, $i \in I$ and $u \in R^\times$, one has

$$\varphi_i \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \eta\left(\lambda \mapsto u^{h_i(\lambda)}\right).$$

(KMG4) If R is a ring, $\mathbb{K}$ is a field and $\iota : R \to \mathbb{K}$ is a monomorphism, then $\mathcal{G}(\iota) : \mathcal{G}(R) \to \mathcal{G}(\mathbb{K})$ is a monomorphism as well.

(KMG5) If $\mathfrak{g}$ is the complex Kac–Moody algebra of type A, then there exists a homomorphism $\mathbf{Ad} : \mathcal{G}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g})$ such that $\ker(\mathbf{Ad}) \subseteq \eta(\mathbb{C})(\mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], \mathbb{C}))$ and for a given $z \in \mathbb{C}$ one has

$$\mathbf{Ad}\left(\varphi_i \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \exp(\mathbf{Ad}_{ze_i}),$$
$$\mathbf{Ad}\left(\varphi_i \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) = \exp(\mathbf{Ad}_{-zf_i});$$

where $\{e_i, f_i\}$ are part of a standard $\mathfrak{sl}_2$-triple for the fundamental Kac–Moody sub-Lie algebra corresponding to the simple root α_i; furthermore, for every homomorphism $\gamma \in \mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], \mathbb{C})$ one has

$$\mathbf{Ad}(\eta(\mathbb{C})(\gamma))(e_i) = \gamma(c_i) \cdot e_i, \quad \mathbf{Ad}(\eta(\mathbb{C})(\gamma))(f_i) = \gamma(-c_i) \cdot f_i.$$

The Kac–Moody root datum $\mathcal{D}$ is called simply connected if the h_i form a basis of $\Lambda^\vee$. It is called centered if the following stronger version of (KMG1) is satisfied: If $\mathbb{K}$ is a field, then the group $\mathcal{G}(\mathbb{K})$ is generated by the images of the φ_i. Note that simply connected implies centered.

For a given Kac–Moody root datum $\mathcal{D}$ the group $G_{\mathcal{D}}(R) := \mathcal{G}(R)$ is called a split Kac–Moody group of type $\mathcal{D}$ over R.

A split Kac–Moody group over a field is an example of a group with an RGD system by the following result. In particular, a split Kac–Moody group naturally admits a twin building that it acts on. Note that if the Kac–Moody datum is centered, the corresponding RGD system is centered.

Proposition 2.14 ([Rém02, Proposition 8.4.1], [Cap09, Lemma 1.4]). *Let $\mathbb{K}$ be a field, let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac–Moody root datum, and let $G_{\mathcal{D}}(\mathbb{K}) := \mathcal{G}(\mathbb{K})$ be the corresponding split Kac–Moody group of type $\mathcal{D}$ over $\mathbb{K}$.*

Then $G_{\mathcal{D}}(\mathbb{K})$ admits an RGD system as follows. Let $M(A)$ be the associated Coxeter matrix of type (W,S) and choose a set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$ such that the reflection associated to α_i is $s_i \in S$. Define the set of real roots as $\Phi^{re} := W.\Pi$. Given $i \in I$, let U_{α_i} and $U_{-\alpha_i}$ be the respective images of the subgroups of strictly upper, resp. strictly lower triangular matrices of the matrix group $\mathrm{SL}_2(\mathbb{K})$ under the map φ_i, and denote by T the image of $\eta(\mathbb{F})$ in $G_{\mathcal{D}}(\mathbb{K})$.

Then there is an isomorphism $W \to N_{G_{\mathcal{D}}(\mathbb{K})}(T)/T : w \mapsto \overline{w}T$ for suitably defined $\overline{w} \in N_{G_{\mathcal{D}}(\mathbb{K})}(T)$. Moreover, T normalizes each $U_{\pm\alpha_i}$, $i \in I$; for a real root $\alpha = w.\alpha_i$ ($w \in W$, $i \in I$), one defines the root group $U_\alpha := \overline{w}U_{\alpha_i}\overline{w}^{-1}$ and obtains an RGD system $(G_{\mathcal{D}}(\mathbb{K}), \{U_\alpha\}_{\alpha\in\Phi^{re}}, T)$ and $T = \bigcap_{\alpha\in\Phi^{re}} N_{G_{\mathcal{D}}(\mathbb{K})}(U_\alpha)$.

Bounded subgroups of Kac–Moody groups are related to algebraic groups as follows:

Proposition 2.15 ([Rém02, §10.3], [Cap09, Proposition 3.6])**.** *Let $\mathbb{K}$ be a field, let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac–Moody root datum, and let $G_{\mathcal{D}}(\mathbb{K}) := \mathcal{G}(\mathbb{K})$ be the corresponding split Kac–Moody group of type $\mathcal{D}$ over $\mathbb{K}$. Let $\Delta_\pm$ be the twin building associated to the RGD system described in Proposition 2.14, let $R_+ \subseteq \Delta_+$ and $R_- \subseteq \Delta_-$ be spherical residues, and let $A := \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{K})}(R_+) \cap \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{K})}(R_-)$, a bounded subgroup.*

Let H be a conjugate of T contained in A and let $A = L \ltimes U$ be a Levi decomposition (cf. [Rém02, §6.3], [Cap09, Proposition 3.2]) such that $H \leq L$.

Then there exists a finite-dimensional subspace W of the universal enveloping algebra $\mathcal{U}_{\mathcal{D}}(\mathbb{K})$ over $\mathbb{K}$ of the corresponding Kac–Moody algebra such that $A = \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{K})}(W)$ and, moreover, the following assertions hold:

(a) *The Zariski closure $\overline{A}$ (resp. $\overline{L}$, $\overline{U}$, $\overline{H}$) of $\mathbf{Ad}_{\mathbb{K}}(A)_{|W}$ (resp. $\mathbf{Ad}_{\mathbb{K}}(L)_{|W}$, $\mathbf{Ad}_{\mathbb{K}}(U)_{|W}$, $\mathbf{Ad}_{\mathbb{K}}(H)_{|W}$) in $\mathrm{GL}(W_{\overline{\mathbb{K}}})$ is a connected algebraic $\mathbb{K}$-group, where $W_{\overline{\mathbb{K}}} := W \otimes_{\mathbb{K}} \overline{\mathbb{K}} \subset (\mathcal{U}_{\mathcal{D}})_{\overline{\mathbb{K}}}$.*

(b) *$\overline{L}$ is reductive, $\overline{H}$ is a maximal torus of $\overline{L}$, the group $\overline{U}$ is unipotent, and $\overline{A} = \overline{L} \ltimes \overline{U}$ is a Levi decomposition. Moreover, $\mathbf{Ad}_{\mathbb{K}}$ maps root subgroups of L (in the sense of Definition 2.8) to root subgroups of $\overline{L}$ (in the sense of algebraic groups).*

(c) *The kernel of the restriction $\mathbf{Ad}_{\mathbb{K}} : A \to \mathrm{GL}(W)$ is the center of A and is contained in the center of L, which is $\mathbf{Ad}_{\mathbb{K}}$-diagonalizable.*

Remark 2.16. The proof of [HKM13, Lemma 7.5] implies that the center of a simply connected Kac–Moody group $\mathcal{G}$ over an arbitrary field $\mathbb{F}$ is determined by the roots of unity contained in $\mathbb{F}$. In particular, $Z(\mathcal{G}(\mathbb{Q})) = Z(\mathcal{G}(\mathbb{R}))$, as $\mathbb{Q}$ and $\mathbb{R}$ contain the same roots of unity 1 and -1.

Topologies on Kac–Moody groups.

Theorem 2.17. *Let $\mathbb{K}$ be a local field and let $G_{\mathcal{D}}(\mathbb{K})$ be a split Kac–Moody group of type $\mathcal{D}$ over $\mathbb{K}$. Then there exists a topology τ on $G_{\mathcal{D}}(\mathbb{K})$, called the Kac–Peterson topology, with the following properties:*

(a) τ *is Hausdorff,*

(b) τ *is a group topology,*

(c) τ *restricts to the Lie group topology on any bounded subgroup,*

(d) τ *is the finest group topology satisfying these properties.*

Proof. (a) and (b) are [HKM13, Proposition 7.10], (c) is [HKM13, Corollary 7.30], (d) is [HKM13, Proposition 7.21]. See also [KP83, Theorem 4] for a statement without proof of a similar result. ■

Let $\mathbb{K}$ be a local field and $G_{\mathcal{D}}(\mathbb{K})$ a centered Kac–Moody group. For each simple root α, choose a parametrization $x_{\pm\alpha}\colon \mathbb{K} \to U_{\pm\alpha}$ of the root groups. For each finite sequence of positive or negative simple roots $\overline{\beta} = (\beta_1, \dots, \beta_k)$ denote by

$$x_{\overline{\beta}} : \mathbb{K}^k \to G_{\mathcal{D}}(\mathbb{K}) : (t_1, \dots, t_k) \mapsto x_{\beta_1}(t_1) \cdots x_{\beta_k}(t_k)$$

the composition of the chosen parametrizations with the product map of $G_{\mathcal{D}}(\mathbb{K})$. As, by the Gauss algorithm/Bruhat decomposition, for each simple root α one has $G_\alpha = U_\alpha U_{-\alpha} U_\alpha U_{-\alpha}$, the final topology on $G_{\mathcal{D}}(\mathbb{K})$ with respect to the maps $x_{\overline{\beta}}$ coincides with the Kac–Peterson topology.

An important tool in the study of the Kac–Peterson topology, which goes back to the original work of Kac and Peterson (see [KP83, Section 2A]), are weakly regular functions in the sense of the following definition:

Definition 2.18. A function $f : G_{\mathcal{D}}(\mathbb{K}) \to \mathbb{K}$ is called weakly regular, if $f \circ x_{\overline{\beta}} : \mathbb{F}^k \to \mathbb{F}$ is a polynomial function for each finite sequence $\overline{\beta}$ of positive or negative simple roots and all $k \in \mathbb{N}$. Let $\mathbb{K}[G_{\mathcal{D}}(\mathbb{K})]_{w.r.}$ be the $\mathbb{K}$-algebra of weakly regular functions. The weak Zariski topology on $G_{\mathcal{D}}(\mathbb{K})$ is the topology arising from the algebra $\mathbb{K}[G_{\mathcal{D}}(\mathbb{K})]_{w.r.}$: a weakly Zariski-closed subset of $G_{\mathcal{D}}(\mathbb{K})$ is the set of zeros of an ideal of $\mathbb{K}[G_{\mathcal{D}}(\mathbb{K})]_{w.r.}$.

The link between weakly regular functions and the Kac–Peterson topology is provided by the following lemma:

Lemma 2.19 ([HKM13, Lemma 7.14])**.** *Every weakly regular function is continuous with respect to the Kac–Peterson topology. In particular, any weakly Zariski-closed set is Kac–Peterson closed.*

Theorem 2.20 (Topological Curtis–Tits Theorem)**.** *Let $\mathbb{K}$ be a local field and let $G_{\mathcal{D}}(\mathbb{K})$ be a two-spherical simply connected split Kac–Moody group. Let Φ^{re} be the set of real roots and let Π be a basis of simple roots for Φ^{re}. For $\alpha, \beta \in \Pi$, set $G_\alpha := \varphi_\alpha(\mathrm{SL}_2(\mathbb{K}))$ and $G_{\alpha\beta} := \langle G_\alpha \cup G_\beta \rangle$. Moreover, let $\iota_{\alpha\beta}\colon G_\alpha \hookrightarrow G_{\alpha\beta}$ be the canonical inclusion morphisms.*

Then the topological group $(G_{\mathcal{D}}(\mathbb{K}), \tau)$ (endowed with the Kac–Peterson topology from Theorem 2.17) is a universal enveloping group of the amalgam $\{G_\alpha, G_{\alpha\beta}; \iota_{\alpha\beta}\}$ in the categories of

(a) *abstract groups,*

(b) *Hausdorff topological groups and*

(c) k_ω *groups.*

Proof. (a) is the main result of [AM97]. (b) and (c) follow from [HKM13, Theorem 7.22]. ∎

Remark 2.21. Within the setting of the preceding theorem, if α, β are non-orthogonal, the subgroup $G_{\alpha,\beta}$ is in fact a copy of the simply connected Chevalley group of type determined by the root system generated by α and β. Indeed, a centered Kac–Moody/Chevalley group has simply connected isogeny type if and only if the torus functor is the direct product of k copies of the functor GL_1 that assigns to a ring the group of its invertible elements, where k is the rank of the Kac–Moody group.

This means that there is a subfunctor $\mathcal{G}_{\alpha,\beta}$ of $\mathcal{G}_{\mathcal{D}}$ which is in fact a simply connected Chevalley group functor. More precisely, there is a natural transformation $\eta_{\alpha,\beta} : \mathcal{G}_{\alpha,\beta} \to \mathcal{G}$ such that for every commutative unital ring R, the morphism $\eta_{\alpha,\beta}(R) : \mathcal{G}_{\alpha,\beta}(R) \to \mathcal{G}(R)$ is a monomorphism with image $G_{\alpha,\beta}(R) := \langle \varphi_\alpha(\mathrm{SL}_2(R)), \varphi_\beta(\mathrm{SL}_2(R))\rangle$. This observation will be crucial for many subsequent definitions and the proof of our result.

3. Arithmetically defined subgroups of Kac–Moody groups

Throughout this article we assume the reader has some basic familiarity with the concepts of Chevalley groups, their arithmetic subgroups, their rigidity properties, split Kac–Moody groups, and their (twin) buildings realised as (pairs of) CAT(0) polyhedral complexes as discussed in [Cap09], [Dav08], [Mar91], [Rém02]. We will use concepts and definitions freely throughout the whole article and will refer to the relevant results from the above literature in the course of our proofs. For the reader's convenience we have collected a quick reminder of the key concepts used in this article in Section 2 and the appendices.

Within this section, let $\mathcal{G}$ be a simply connected two-spherical split Kac–Moody functor, let Φ^{re} be the set of real roots for $\mathcal{G}$, and let Π be a system of simple roots for Φ^{re}. As is the case for simply connected Chevalley groups, a simply connected Kac–Moody functor $\mathcal{G}$ (see [Rém02, Section 7.1.2]) is centered, i.e., $\mathcal{G}(\mathbb{K}) = \langle U_\alpha(\mathbb{K}) \mid \alpha \in \Phi^{re}\rangle$ for any field $\mathbb{K}$.

An element $\alpha \in \Pi$ is called isolated, if it is orthogonal to each $\beta \in \Pi\backslash\{\alpha\}$. For $\alpha, \beta \in \Pi$, we denote by $\mathcal{G}_\alpha := \mathcal{G}_{\alpha,\alpha}$ and $\mathcal{G}_{\alpha,\beta}$ the simply connected Chevalley group subfunctors described in Remark 2.21. By slight abuse of notation, for any commutative unital ring R, we identify $\mathcal{G}_\alpha(R)$ resp. $\mathcal{G}_{\alpha,\beta}(R)$ with the corresponding isomorphic fundamental subgroups of $\mathcal{G}(R)$. Note that

$$\mathcal{G}_\alpha(\mathbb{K}) = \langle U_\alpha(\mathbb{K}), U_{-\alpha}(\mathbb{K})\rangle \quad \text{and} \quad \mathcal{G}_{\alpha,\beta}(\mathbb{K}) = \langle U_\alpha(\mathbb{K}), U_{-\alpha}(\mathbb{K}), U_\beta(\mathbb{K}), U_{-\beta}(\mathbb{K})\rangle$$

and

$$\mathcal{G}_{\alpha,\beta}(\mathbb{Z}) = \langle U_\gamma(\mathbb{Z}) \mid \gamma \in (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi^{re}\rangle.$$

In another slight abuse of notation, and in view of (KMG4), we will treat $\mathcal{G}(\mathbb{Z})$ as a subgroup of $\mathcal{G}(\mathbb{Q})$; in fact, we will pretend $\mathcal{G}(\mathbb{Z}) \leq \mathcal{G}(\mathbb{Q}) \leq \mathcal{G}(\mathbb{R})$.

The purpose of this section is to study some basic properties of the following arithmetically defined subgroups of a split Kac–Moody group over the rational numbers.

Definition 3.1. Let $n \in \mathbb{N}$ and $\alpha, \beta \in \Pi$.

- $U_\alpha(n\mathbb{Z})$ is the subgroup of $n\mathbb{Z}$-points of the root subgroup U_α.
- $\Lambda_{\alpha,\beta}(n) := \langle U_{\pm\alpha}(n\mathbb{Z}), U_{\pm\beta}(n\mathbb{Z})\rangle$; in case $\alpha = \beta$ let $\Lambda_\alpha(n) := \Lambda_{\alpha,\alpha}(n)$.
- $\Lambda_\Pi(n) := \langle U_\alpha(n\mathbb{Z}) \mid \alpha \in \pm\Pi\rangle = \langle \Lambda_\alpha(n) \mid \alpha \in \Pi\rangle$.
- $\Gamma_{\alpha,\beta}(n)$ is the kernel of the homomorphism $\mathcal{G}_{\alpha,\beta}(\mathbb{Z}) \to \mathcal{G}_{\alpha,\beta}(\mathbb{Z}/n\mathbb{Z})$.
- $\Gamma(\mathbb{Z}) := \langle U_\alpha(\mathbb{Z}) \mid \alpha \in \Phi^{re}\rangle \leq \mathcal{G}(\mathbb{Z}) \hookrightarrow \mathcal{G}(\mathbb{Q})$.
- $\Gamma_\Pi(n) := \langle \Gamma_{\alpha,\beta}(n) \mid \alpha, \beta \in \Pi\rangle \leq \Gamma(\mathbb{Z})$.

Remark 3.2.

(a) In the preceding definition one has $\Gamma(\mathbb{Z}) = \Gamma_\Pi(1) = \Lambda_\Pi(1)$. Moreover, for $n \in \mathbb{N}$,

$$\Lambda_{\alpha,\beta}(n) \leq \Gamma_{\alpha,\beta}(n) \leq \mathcal{G}_{\alpha,\beta}(\mathbb{Z}) \leq \Gamma(\mathbb{Z}) \qquad \text{and} \qquad \Lambda_\Pi(n) \leq \Gamma_\Pi(n).$$

(b) In the classical rigidity theory for arithmetic groups the extensions of homomorphisms are always virtual. For instance, if $\varphi : \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_\ell(\mathbb{Q})$ is a group homomorphism, then there exist a finite-index subgroup $\Gamma \leq \mathrm{SL}_n(\mathbb{Z})$ and a $\mathbb{Q}$-morphism $\varphi_\mathbb{Q} : \mathrm{SL}_n \to \mathrm{SL}_\ell$ such that the restriction of $\varphi_\mathbb{Q}$ to Γ coincides with the restriction of φ to Γ; cf. Theorem C.4.

Unfortunately, in the non-spherical case, the kernel of $\mathcal{G}(\mathbb{Z}) \to \mathcal{G}(\mathbb{Z}/n\mathbb{Z})$ does not have finite index. Instead, the groups $\Gamma_\Pi(n)$ will serve as a substitute for these finite-index subgroups. The following are the key properties of these groups that we are going to use in case Π does not contain any isolated elements:

- if the root system Φ^{re} is spherical, then for each $n \in \mathbb{N}$ the group $\Gamma_\Pi(n)$ is of finite index in $\mathcal{G}(\mathbb{Z})$;
- for each $g \in \mathcal{G}(\mathbb{Q})$ and for each $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $\Gamma_\Pi(k) \leq \Gamma_\Pi^g(n)$.

The first statement is a consequence of Corollary B.3, as $\Lambda_\Pi(n) \leq \Gamma_\Pi(n) \leq \mathcal{G}(\mathbb{Z})$; the second statement is Lemma 3.6.

(c) Not much is known to us about the difference between $\Gamma(\mathbb{Z})$ and $\mathcal{G}(\mathbb{Z})$. Since both groups admit $\mathcal{G}(\mathbb{F}_p)$ as a quotient for any prime p, any potential difference actually can be seen in their principal congruence subgroups.

Lemma 3.3. *For all distinct non-orthogonal $\alpha, \beta \in \Pi$ and $n \in \mathbb{N}$ there exists $m_{\alpha,\beta} \in \mathbb{N}$ such that $\Gamma_{\alpha,\beta}(m_{\alpha,\beta}) \le \Lambda_{\alpha,\beta}(n)$.*

Proof. The group $\Lambda_{\alpha,\beta}(n)$ has finite index in $\mathcal{G}_{\alpha,\beta}(\mathbb{Z})$ by Corollary B.3. By the congruence subgroup property (see Theorem B.1) there exists $m_{\alpha,\beta}$ such that $\Gamma_{\alpha,\beta}(m_{\alpha,\beta}) \le \Lambda_{\alpha,\beta}(n)$. ■

Lemma 3.4. *Assume that Π does not contain any isolated elements. Then for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\Gamma_\Pi(m) \le \Lambda_\Pi(n)$.*

Proof. The claim follows from Lemma 3.3, by setting

$$m := \operatorname{lcm}\{m_{\alpha,\beta} \mid \alpha, \beta \in \Pi \text{ distinct and non-orthogonal}\}.$$ ■

Lemma 3.5. *Let $g \in \mathcal{G}(\mathbb{Q})$, let $n \in \mathbb{N}$ and assume that Π does not contain any isolated elements. Then there exists $m \in \mathbb{N}$ such that $\Lambda_\Pi(m) \le \Lambda_\Pi^g(n)$*

Proof. Suppose $g \in U_\alpha(\mathbb{Q})$ with $\alpha \in \pm\Pi$. Since Π contains no isolated elements, there is $\beta \in \Pi \setminus \{\alpha\}$ which is non-orthogonal to $\pm\alpha$. By Lemma 3.3, there is $m_{\alpha,\beta} \in \mathbb{N}$ such that $\Gamma_{\alpha,\beta}(m_{\alpha,\beta}) \le \Lambda_{\alpha,\beta}(n)$. Thus both groups have finite index in $\mathcal{G}_{\alpha,\beta}(\mathbb{Z})$, hence are commensurable to it. This implies that $\Lambda^g_{\alpha,\beta}(n)$ is commensurable to $\mathcal{G}_{\alpha,\beta}(\mathbb{Z})$, because

$$g \in U_\alpha(\mathbb{Q}) \le \mathcal{G}_{\alpha,\beta}(\mathbb{Q}) = \operatorname{Comm}_{\mathcal{G}_{\alpha,\beta}(\mathbb{Q})}(\mathcal{G}_{\alpha,\beta}(\mathbb{Z})) = \operatorname{Comm}_{\mathcal{G}_{\alpha,\beta}(\mathbb{Q})}(\Lambda_{\alpha,\beta}(n)).$$

Consequently, by the congruence subgroup property (see Theorem B.1), there is $m'_{\alpha,\beta} \in \mathbb{N}$ such that

$$\Lambda_{\alpha,\beta}(m'_{\alpha,\beta}) \le \Gamma_{\alpha,\beta}(m'_{\alpha,\beta}) \le \Lambda^g_{\alpha,\beta}(n) \le \Lambda^g_\Pi(n).$$

Hence, for

$$m_\alpha := \operatorname{lcm}\{m'_{\alpha,\beta} \mid \beta \in \Pi \text{ distinct from and non-orthogonal to } \alpha\},$$

we have $\Lambda_{\alpha,\beta}(m_\alpha) \le \Lambda^g_\Pi(n)$ for all $\beta \in \Pi$ distinct from and non-orthogonal to α. Furthermore, as $g \in U_\alpha(\mathbb{Q})$ centralizes each $\mathcal{G}_\beta(\mathbb{Q}) \ge \Lambda_\beta(m_\alpha)$ for $\beta \in \Pi$ orthogonal to α, certainly

$$\Lambda_\beta(m_\alpha) = \Lambda^g_\beta(m_\alpha) \le \Lambda^g_\Pi(n).$$

Since $\Lambda_\Pi(m_\alpha)$ is generated by the union of the sets $\Lambda_{\alpha,\beta}(m_\alpha)$, $\beta \in \Pi$ distinct from and non-orthogonal to α, and $\Lambda_\beta(m_\alpha)$, $\beta \in \Pi$ orthogonal to α, one has $\Lambda_\Pi(m_\alpha) \le \Lambda^g_\Pi(n)$. For general $g \in \mathcal{G}(\mathbb{Q})$ the claim now follows by using that $\mathcal{G}(\mathbb{Q})$ is generated by the $U_\alpha(\mathbb{Q})$ with $\alpha \in \pm\Pi$. ■

Lemma 3.6. *Let $g \in \mathcal{G}(\mathbb{Q})$, let $n \in \mathbb{N}$ and assume that Π does not contain any isolated elements. Then there exists $k \in \mathbb{N}$ such that $\Gamma_\Pi(k) \le \Gamma^g_\Pi(n)$.*

Proof. There exist $k, m \in \mathbb{N}$ such that

$$\Gamma_\Pi(k) \overset{3.4}{\le} \Lambda_\Pi(m) \overset{3.5}{\le} \Lambda^g_\Pi(n) \overset{3.2}{\le} \Gamma^g_\Pi(n).$$ ■

4. Rigidity

In this section we study rigidity properties of the arithmetically defined subgroups introduced before and prove our main results. We start with investigating homomorphisms from arithmetic subgroups of Chevalley groups into Kac–Moody groups in order to establish a boundedness result, then homomorphisms from arithmetic subgroups of Kac–Moody groups into Kac–Moody groups in order to prove a version of Margulis superrigidity with target in the $\mathbb{Q}$-rational points, and finally isomorphisms between arithmetic subgroups of Kac–Moody groups in order to prove a weak version of Mostow–Margulis strong rigidity.

Lemma 4.1 (Boundedness)**.** *Let M be an irreducible simply connected Chevalley group functor of rank at least two, and let Γ_n denote the principal congruence subgroup of $M(\mathbb{Z})$ of level n, i.e., the kernel of the epimorphism $M(\mathbb{Z}) \to M(\mathbb{Z}/n\mathbb{Z})$. Let $\mathcal{G}$ be a centered two-spherical split Kac–Moody functor and let $\varphi : M(\mathbb{Z}) \to \mathcal{G}(\mathbb{Q})$ be a group homomorphism with infinite image.*

(a) *Then there exist*

- *a uniquely determined (up to central isogeny) semisimple connected linear algebraic $\mathbb{Q}$-group H,*
- *a uniquely determined central $\mathbb{Q}$-isogeny $\psi : M \to H$,*
- *a bounded subgroup $B_{\mathbb{Q}}$ of $\mathcal{G}(\mathbb{Q})$ that is Zariski-closed in any Levi factor of a parabolic of spherical type that it is contained in,*
- *a group homomorphism $\varphi_{\mathbb{Q}} : M(\mathbb{Q}) \to B_{\mathbb{Q}}$ defined over $\mathbb{Q}$,*
- *a group homomorphism $\iota : B_{\mathbb{Q}} \to H(\mathbb{Q})$ defined over $\mathbb{Q}$,*

satisfying, for a suitable $n \in \mathbb{N}$, that

$$\varphi_{\mathbb{Q}|\Gamma_n} = \varphi_{|\Gamma_n},$$

and making the following diagram commute:

$$\begin{array}{ccccc} M(\mathbb{Q}) & \xrightarrow{\varphi_{\mathbb{Q}}} & B_{\mathbb{Q}} & \hookrightarrow & \mathcal{G}(\mathbb{Q}) . \\ & \searrow^{\psi} & \downarrow{\scriptstyle \iota} & & \\ & & H(\mathbb{Q}) & & \end{array}$$

Moreover, the map

$$M(\mathbb{Q}) \to \mathcal{G}(\mathbb{Q})$$

is continuous with respect to the Zariski topology on $M(\mathbb{Q})$ and the weak Zariski topology on $\mathcal{G}(\mathbb{Q})$ (cf. Definition 2.18).

(b) *Furthermore, there exist*

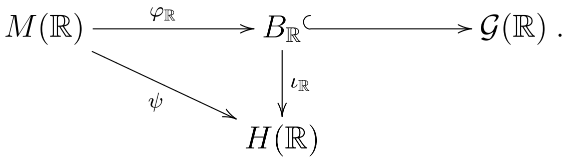
- *a bounded subgroup $B_{\mathbb{R}}$ of $\mathcal{G}(\mathbb{R})$ which, with respect to the induced topology from the Kac–Peterson topology on $\mathcal{G}(\mathbb{R})$, is a semisimple Lie group (cf. Theorem 2.17) and which satisfies $B_{\mathbb{R}} \cap \mathcal{G}(\mathbb{Q}) = B_{\mathbb{Q}}$,*

- *a uniquely determined continuous group homomorphism* $\varphi_{\mathbb{R}} : M(\mathbb{R}) \to B_{\mathbb{R}}$,
- *a uniquely determined continuous group homomorphism* $\iota_{\mathbb{R}} : B_{\mathbb{R}} \to H(\mathbb{R})$,

satisfying

$$\varphi_{\mathbb{R}|G(\mathbb{Q})} = \varphi_{\mathbb{Q}} \qquad \text{and} \qquad \iota_{\mathbb{R}|B_{\mathbb{Q}}} = \iota$$

and making the following diagram commute:



Proof. By Corollary A.2 the image $\varphi(M(\mathbb{Z}))$ fixes points $x_+ \in X_+$ and $x_- \in X_-$ in the CAT(0) Davis realizations $X_\pm$ of the positive, resp. negative halves of the twin building of $\mathcal{G}(\mathbb{Q})$ (cf. Theorem 2.5). Hence $\varphi(M(\mathbb{Z}))$ is contained in the bounded subgroup $A := P_+ \cap P_-$ of $\mathcal{G}(\mathbb{Q})$, where $P_\pm$ denote the respective stabilizers in $\mathcal{G}(\mathbb{Q})$ of $x_\pm$.

Next, we apply Proposition 2.15 to the bounded subgroup A. Let $\mathcal{U}_{\mathcal{G}}(\mathbb{Q})$ be the universal enveloping algebra over $\mathbb{Q}$ of the Kac–Moody algebra corresponding to $\mathcal{G}$, and let $\mathbf{Ad}_{\mathbb{Q}} : \mathcal{G}(\mathbb{Q}) \to \mathcal{U}_{\mathcal{G}}(\mathbb{Q})$ be the adjoint representation of $\mathcal{G}(\mathbb{Q})$.

Then by Proposition 2.15 there exists a finite-dimensional subspace $W \leq \mathcal{U}_{\mathcal{G}}(\mathbb{Q})$ such that $A = \mathrm{Stab}_{\mathcal{G}(\mathbb{Q})}(W)$. Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$, let $W_{\overline{\mathbb{Q}}} := W \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, and let $\overline{A}$ be the linear algebraic $\mathbb{Q}$-group defined as the Zariski closure of $\mathbf{Ad}_{\mathbb{Q}}(A)$ in $\mathrm{GL}(W_{\overline{\mathbb{Q}}})$.

Given a Levi decomposition $A = L \ltimes U$ in the sense of Kac–Moody theory, by Proposition 2.15(b) passage to Zariski closures in $\mathrm{GL}(W_{\overline{\mathbb{Q}}})$ of the images under $\mathbf{Ad}_{\mathbb{Q}}$ provides a Levi decomposition $\overline{A} = \overline{L} \ltimes \overline{U}$ in the sense of the theory of algebraic groups; conversely, any Levi decomposition of $\overline{A}$ is induced in that way by a Levi decomposition of A, as can be seen by taking the $\mathbf{Ad}_{\mathbb{Q}}$-preimage of the $\mathbb{Q}$-points of the two factors of the given Levi decomposition of $\overline{A}$. Moreover, $\mathbf{Ad}_{\mathbb{Q}}$ maps root subgroups of L in the sense of the theory of RGD systems to root subgroups of $\overline{L}$ in the sense of the theory of algebraic groups.

By construction, $\varphi(M(\mathbb{Z})) \leq A$. Let H_1 be the linear algebraic $\mathbb{Q}$-group defined as the Zariski closure of $\mathbf{Ad}_{\mathbb{Q}}(\varphi(M(\mathbb{Z})))$ in $\overline{A}$. By Theorem C.5 (applied to the connected semisimple Lie group $M(\mathbb{R})$, its irreducible lattice $M(\mathbb{Z})$ and the target $\overline{A}$) the linear algebraic $\mathbb{Q}$-group H_1 is semisimple. We conclude that H_1 is a (semisimple) $\mathbb{Q}$-subgroup of a (reductive) Levi $\mathbb{Q}$-subgroup $\overline{L}$ of $\overline{A}$. In fact H_1 is contained in the (semisimple) derived $\mathbb{Q}$-group $\overline{L}'$.

Superrigidity of arithmetic groups Theorem C.4 (applied to the simply connected algebraic $\mathbb{Q}$-group M, its arithmetic subgroup $M(\mathbb{Z})$ and the target H_1) implies the unique existence of a morphism of algebraic $\mathbb{Q}$-groups $\psi_1 : M \to H_1$ that virtually extends the group homomorphism $\mathbf{Ad}_{\mathbb{Q}} \circ \varphi : M(\mathbb{Z}) \to H_1(\mathbb{Q})$; i.e., there furthermore exists a group homomorphism $\nu : M(\mathbb{Z}) \to H_1(\mathbb{Q})$ with finite image such that for all $g \in M(\mathbb{Z})$ one has $(\mathbf{Ad}_{\mathbb{Q}} \circ \varphi)(g) = \psi_1(g)\nu(g)$.

Since M is connected, the morphism of $\mathbb{Q}$-groups $\psi_1 : M \to H_1$ corestricts to a morphism of $\mathbb{Q}$-groups $\psi : M \to H := (H_1)^0$ with connected semisimple target H. Let $B_{\mathbb{Q}} := (\mathbf{Ad}_{\mathbb{Q}})^{-1}(H(\mathbb{Q}))$ and define $\iota : B_{\mathbb{Q}} \to H(\mathbb{Q}) : g \mapsto \mathbf{Ad}_{\mathbb{Q}}(g)$. By Theorem B.1 the finite-index subgroup $\ker(\nu)$ of $M(\mathbb{Z})$ contains a principal congruence subgroup Γ_n; for all $g \in \Gamma_n$ one has $(\mathbf{Ad}_{\mathbb{Q}} \circ \varphi)(g) = \psi_1(g)\nu(g) = \psi(g) \cdot 1 = \psi(g) \in H(\mathbb{Q})$.

By construction, the morphism of $\mathbb{Q}$-groups $\psi : M \to H$ is surjective. Moreover, the group M is irreducible, whence absolutely almost simple. Also $\varphi(M(\mathbb{Z}))$ is infinite, hence $\psi(M(\mathbb{Z}))$ is infinite. Altogether it follows that ψ is a central isogeny.

For each root α of M the root subgroup $U_\alpha(n\mathbb{Z}) \leq \Gamma_n$ of M is mapped by the central isogeny ψ into a root subgroup of H. Moreover, using the notation introduced above, since $\mathbf{Ad}_{\mathbb{Q}}$ maps root subgroups of L to root subgroups of $\overline{L}$ (cf. Proposition 2.15(b)), the image $\varphi(U_\alpha(n\mathbb{Z}))$ lies in a root subgroup of $A = L \ltimes U$. Let x_α be a generator of $U_\alpha(\mathbb{Z})$. Then x_α^n generates $U_\alpha(n\mathbb{Z})$. The unique divisibility of $(\mathbb{Q}, +)$ and the injectivity of the restriction of $\mathbf{Ad}_{\mathbb{Q}}$ to root subgroups yield

$$\sqrt[n]{\varphi(x_\alpha^n)} = ({\mathbf{Ad}_{\mathbb{Q}}}^{-1} \circ \psi)(x_\alpha)$$

and, more generally, for $p, q \in \mathbb{Z},\ q \neq 0$,

$$\sqrt[qn]{\varphi(x_\alpha^{pn})} = ({\mathbf{Ad}_{\mathbb{Q}}}^{-1} \circ \psi)({x_\alpha}^{\frac{p}{q}}).$$

Thus, for each root α of G one obtains a group homomorphism

$$\varphi_{\mathbb{Q}}^{\alpha} : U_\alpha(\mathbb{Q}) \to B_{\mathbb{Q}},\ {x_\alpha}^{\frac{p}{q}} \mapsto ({\mathbf{Ad}_{\mathbb{Q}}}^{-1} \circ \psi)({x_\alpha}^{\frac{p}{q}})$$

such that ${\varphi_{\mathbb{Q}}^{\alpha}}_{|U_\alpha(n\mathbb{Z})} = \varphi_{|U_\alpha(n\mathbb{Z})}$ and such that the following diagram commutes (as $\iota(g) = \mathbf{Ad}_{\mathbb{Q}}(g)$ for $g \in B_{\mathbb{Q}}$):

$$\begin{array}{ccc} U_\alpha(\mathbb{Q}) & \xrightarrow{\varphi_{\mathbb{Q}}^{\alpha}} & B_{\mathbb{Q}} \\ & \searrow^{\psi_{|U_\alpha(\mathbb{Q})}} & \downarrow \iota \\ & & H(\mathbb{Q}) \end{array}$$

Since $M(\mathbb{Q}) = \langle U_{\pm\alpha}(\mathbb{Q}) \mid \alpha \text{ simple root of } M \rangle$, since M is simply connected and since $\ker(\iota)$ is contained in the center of L (cf. Proposition 2.15(c)), the $\varphi_{\mathbb{Q}}^{\alpha}$ together induce a group homomorphism

$$\varphi_{\mathbb{Q}} : M(\mathbb{Q}) \to B_{\mathbb{Q}}$$

that yields the desired commutative diagram:

$$\begin{array}{ccc} M(\mathbb{Q}) & \xrightarrow{\varphi_{\mathbb{Q}}} & B_{\mathbb{Q}} \\ & \searrow^{\psi} & \downarrow \iota \\ & & H(\mathbb{Q}) \end{array}$$

The weak Zariski topology on $\mathcal{G}(\mathbb{Q})$ induces the usual Zariski topology on any Levi factor of any of its parabolic subgroup of spherical type. By definition the group $B_{\mathbb{Q}}$ is Zariski-closed in any Levi factor of a parabolic subgroup of $\mathcal{G}(\mathbb{Q})$ of spherical type that it is contained in and the group homomorphism $\iota_{\mathbb{Q}} : B_{\mathbb{Q}} \to H(\mathbb{Q})$ is defined over $\mathbb{Q}$. Theorem C.1 applied to the homomorphism $\varphi_{\mathbb{Q}} : M(\mathbb{Q}) \to B_{\mathbb{Q}}$ furthermore implies that $\varphi_{\mathbb{Q}}$ is also defined over $\mathbb{Q}$. Continuity of $M(\mathbb{Q}) \to \mathcal{G}(\mathbb{Q})$ with respect to the (weak) Zariski topologies is immediate. Assertion (a) has been proved.

For Assertion (b) consider the embedding $\mathcal{G}(\mathbb{Q}) \hookrightarrow \mathcal{G}(\mathbb{R})$. Let $B_{\mathbb{R}}$ be the closure of $B_{\mathbb{Q}}$ in $\mathcal{G}(\mathbb{R})$ with respect to the Kac–Peterson topology. Then $B_{\mathbb{R}} = (\mathbf{Ad}_{\mathbb{R}})^{-1}(H(\mathbb{R}))$. Since H is semisimple as an algebraic group, the group $H(\mathbb{R})$ of $\mathbb{R}$-points is semisimple as a Lie group. Since $\iota_{\mathbb{R}} : B_{\mathbb{R}} \to H(\mathbb{R})$ has finite kernel and semisimplicity of Lie groups is a commensurability invariant, also $B_{\mathbb{R}}$ is a semisimple Lie group. Extension of the group homomorphisms $\varphi^{\alpha}_{\mathbb{Q}} : U_\alpha(\mathbb{Q}) \to B_{\mathbb{Q}}$ provides continuous group homomorphisms $\varphi^{\alpha}_{\mathbb{R}} : U_\alpha(\mathbb{R}) \to B_{\mathbb{R}}$, which as above yields the desired commutative diagram

$$\begin{array}{ccc} M(\mathbb{R}) & \xrightarrow{\varphi_{\mathbb{R}}} & B_{\mathbb{R}} \\ & \searrow^{\psi} & \downarrow{\scriptstyle \iota_{\mathbb{R}}} \\ & & H(\mathbb{R}) \end{array}$$

■

Remark 4.2. Recall that $Z(\mathcal{G}(\mathbb{Q})) = Z(\mathcal{G}(\mathbb{R}))$ by Lemma 2.16. The center of $\mathcal{G}(\mathbb{Q})$ and $\mathcal{G}(\mathbb{R})$ is the kernel of their adjoint representations, as well as the kernel of the action on their associated buildings. Therefore Lemma 4.1 above and Lemma 4.3 below remain correct if we replace $\mathcal{G}(\mathbb{Q})$ and $\mathcal{G}(\mathbb{R})$ by central quotients $\mathcal{G}(\mathbb{Q})/N$ and $\mathcal{G}(\mathbb{R})/N$.

Lemma 4.3 (Local commensurability)**.** *Let M be an irreducible simply connected Chevalley group functor of rank at least two, let $M_{\mathbb{Z}}$ be a subgroup of $\Gamma(\mathbb{Z})$ isomorphic to a central quotient of $M(\mathbb{Z})$ and assume that Π does not contain any isolated elements. Then for each $n \in \mathbb{N}$ the intersection $M_{\mathbb{Z}} \cap \Gamma_\Pi(n)$ has finite index in $M_{\mathbb{Z}}$*

Proof. We apply the preceding result to the homomorphism $M(\mathbb{Z}) \to M_{\mathbb{Z}} \hookrightarrow \Gamma(\mathbb{Z}) \hookrightarrow \mathcal{G}(\mathbb{Q})$ and obtain the commutative diagram

$$\begin{array}{ccc} M(\mathbb{Q}) & \xrightarrow{\varphi_{\mathbb{Q}}} & B_{\mathbb{Q}} \\ & \searrow^{\psi} & \downarrow{\scriptstyle \iota} \\ & & H(\mathbb{Q}) \end{array}$$

as in assertion (a) of Lemma 4.1. There exists $g \in \mathcal{G}(\mathbb{Q})$ and a spherical subset $\Pi_{\mathrm{sph}} \subset \Pi$ such that $B^g_{\mathbb{Q}}$ is contained in $\mathcal{G}_{\Pi_{\mathrm{sph}}}(\mathbb{Q}) := \langle \mathcal{G}_\alpha(\mathbb{Q}) \mid \alpha \in \Pi_{\mathrm{sph}} \rangle$. By definition the intersection $\Gamma(\mathbb{Z}) \cap \mathcal{G}_{\Pi_{\mathrm{sph}}}(\mathbb{Q})$ is an arithmetic subgroup of $\mathcal{G}_{\Pi_{\mathrm{sph}}}(\mathbb{Q})$. The embedding of rational points of Chevalley groups $B^g_{\mathbb{Q}} \leq \mathcal{G}_{\Pi_{\mathrm{sph}}}(\mathbb{Q})$ is defined over $\mathbb{Q}$, as $g \in \mathcal{G}(\mathbb{Q})$ and by Lemma 4.1 the group $B_{\mathbb{Q}}$ is Zariski-closed in the Levi factor $\mathcal{G}_{\Pi_{\mathrm{sph}}}(\mathbb{Q})^{g^{-1}}$. Therefore the intersection $\Gamma(\mathbb{Z}) \cap B^g_{\mathbb{Q}} = \Gamma(\mathbb{Z}) \cap \mathcal{G}_{\Pi_{\mathrm{sph}}}(\mathbb{Q}) \cap B^g_{\mathbb{Q}}$ is commensurable to the arithmetic subgroup $M^g_{\mathbb{Z}} \leq B^g_{\mathbb{Q}}$. Hence the intersections $\Gamma(\mathbb{Z}) \cap M^g_{\mathbb{Z}}$ and $\Gamma_\Pi(m) \cap M^g_{\mathbb{Z}}$, $m \in \mathbb{N}$, have finite index in $M^g_{\mathbb{Z}}$ (for the latter intersection see the first bullet item in Remark 3.2). We conclude that for each $m \in \mathbb{N}$ the intersection $\Gamma_\Pi(m)^{g^{-1}} \cap M_{\mathbb{Z}}$ has finite index in $M_{\mathbb{Z}}$. By Lemma 3.6 there exists $m \in \mathbb{N}$ such that $\Gamma_\Pi(m) \leq \Gamma_\Pi(n)^g$, whence

$$\Gamma_\Pi(m)^{g^{-1}} \cap M_{\mathbb{Z}} \leq \Gamma_\Pi(n) \cap M_{\mathbb{Z}}.$$

That is, also $\Gamma_\Pi(n) \cap M_{\mathbb{Z}}$ has finite index in $M_{\mathbb{Z}}$. ∎

We now establish a superrigidity result for group homomorphisms $\mathcal{G}(\mathbb{Z}) \to \mathcal{G}'(\mathbb{Q})$. In view of Remark 3.2 and Lemma 4.3 the groups $\Gamma_\Pi(n)$ seem to be reasonable replacement for the finite-index subgroups of the arithmetic groups occurring in the virtual extensions in the classical theory.

Proposition 4.4 (Superrigidity into $\mathbb{Q}$-points)**.** *Let $\mathcal{G}$ be a simply connected irreducible two-spherical split Kac–Moody functor, let $\mathcal{G}'$ be a centered two-spherical split Kac–Moody functor, let Φ^{re} be a set of real roots for $\mathcal{G}$, let Π be a system of simple roots for Φ^{re}, and let $\varphi : \Gamma(\mathbb{Z}) \to \mathcal{G}'(\mathbb{Q})$ be a group homomorphism such that each simple root subgroup has infinite image. Then there exists a uniquely determined group homomorphism*

$$\varphi_{\mathbb{Q}} : \mathcal{G}(\mathbb{Q}) \to \mathcal{G}'(\mathbb{Q}) \qquad \textit{satisfying} \qquad \varphi_{\mathbb{Q}|\Gamma_\Pi(n)} = \varphi_{|\Gamma_\Pi(n)}$$

for some $n \in \mathbb{N}$. This group homomorphism is continuous with respect to the weak Zariski topologies on $\mathcal{G}(\mathbb{Q})$ and $\mathcal{G}'(\mathbb{Q})$. Furthermore, there exists a uniquely determined continuous (with respect to the Kac–Peterson topology) group homomorphism

$$\varphi_{\mathbb{R}} : \mathcal{G}(\mathbb{R}) \to \mathcal{G}'(\mathbb{R}) \qquad \textit{satisfying} \qquad \varphi_{\mathbb{R}|\mathcal{G}(\mathbb{Q})} = \varphi_{\mathbb{Q}}.$$

Proof. For each distinct pair of non-orthogonal simple roots $\alpha, \beta \in \Pi$, the fundamental subgroup $\mathcal{G}_{\alpha,\beta}$ of $\mathcal{G}$ is an irreducible simply connected Chevalley group of rank two. Hence Lemma 4.1 applies to the group homomorphism

$$\varphi^{\alpha,\beta} : \mathcal{G}_{\alpha,\beta}(\mathbb{Z}) \to \mathcal{G}'(\mathbb{Q}) : g \mapsto \varphi(g).$$

In particular, for such a pair of simple roots α, β there exist a bounded subgroup $B^{\alpha,\beta}_{\mathbb{Q}} \leq \mathcal{G}'(\mathbb{Q})$ and a group homomorphism $\varphi^{\alpha,\beta}_{\mathbb{Q}} : \mathcal{G}_{\alpha,\beta}(\mathbb{Q}) \to B^{\alpha,\beta}_{\mathbb{Q}}$ which coincides with $\varphi^{\alpha,\beta}$ on a finite index subgroup of $\mathcal{G}_{\alpha,\beta}(\mathbb{Z})$. The strategy of proof is to show that different maps $\varphi^{\alpha,\beta}_{\mathbb{Q}}$ coincide on the intersections of their respective domains of definition. This will allow one to then use the universality of two-spherical simply connected Kac–Moody groups with respect to the amalgam of their fundamental subgroups of ranks one and two provided by the Curtis–Tits theorem (see Theorem 2.20) in order to conclude the existence of the desired group homomorphism $\varphi_{\mathbb{Q}} : \mathcal{G}(\mathbb{Q}) \to \mathcal{G}'(\mathbb{Q})$.

To this end let $\alpha_1, \alpha_2, \alpha_3 \in \Pi$ be pairwise distinct simple roots such that α_i and α_{i+1} are non-orthogonal. Since $\mathcal{G}$ is simply connected and two-spherical, one has

$$\mathcal{G}_{\alpha_1,\alpha_2}(\mathbb{Q}) \cap \mathcal{G}_{\alpha_2,\alpha_3}(\mathbb{Q}) = \mathcal{G}_{\alpha_2}(\mathbb{Q}) \cong \mathrm{SL}_2(\mathbb{Q}).$$

Consider the commutative diagrams

$$\begin{array}{ccc} \mathcal{G}_{\alpha_1,\alpha_2}(\mathbb{Q}) & \xrightarrow{\varphi_{\mathbb{Q}}^{\alpha_1,\alpha_2}} & B_{\mathbb{Q}}^{\alpha_1,\alpha_2} \\ & \searrow^{\psi^{\alpha_1,\alpha_2}} & \downarrow \iota^{\alpha_1,\alpha_2} \\ & & H_{\alpha_1,\alpha_2}(\mathbb{Q}) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{G}_{\alpha_2,\alpha_3}(\mathbb{Q}) & \xrightarrow{\varphi_{\mathbb{Q}}^{\alpha_2,\alpha_3}} & B_{\mathbb{Q}}^{\alpha_2,\alpha_3} \\ & \searrow^{\psi^{\alpha_2,\alpha_3}} & \downarrow \iota^{\alpha_2,\alpha_3} \\ & & H_{\alpha_2,\alpha_3}(\mathbb{Q}) \end{array}$$

whose existence is guaranteed by Lemma 4.1. A lifting argument for root subgroups as in the proof of Lemma 4.1 guarantees $\varphi_{\mathbb{Q}}^{\alpha_1,\alpha_2}(\mathcal{G}_{\alpha_2}(\mathbb{Q})) \subset B_{\mathbb{Q}}^{\alpha_2,\alpha_3} \subset \mathcal{G}'(\mathbb{Q})$ and $\varphi_{\mathbb{Q}}^{\alpha_2,\alpha_3}(\mathcal{G}_{\alpha_2}(\mathbb{Q})) \subset B_{\mathbb{Q}}^{\alpha_1,\alpha_2} \subset \mathcal{G}'(\mathbb{Q})$ and ${\varphi_{\mathbb{Q}}^{\alpha_1,\alpha_2}}_{|\mathcal{G}_{\alpha_2}(\mathbb{Q})} = {\varphi_{\mathbb{Q}}^{\alpha_2,\alpha_3}}_{|\mathcal{G}_{\alpha_2}(\mathbb{Q})}$.

Since by the Curtis–Tits theorem (see Theorem 2.20) the group $\mathcal{G}(\mathbb{Q})$ is the universal enveloping group of the amalgam consisting of its fundamental rank one and two subgroups $(\mathcal{G}_\alpha(\mathbb{Q}))_{\alpha\in\Pi}$ and $(\mathcal{G}_{\alpha,\beta}(\mathbb{Q}))_{\alpha,\beta\in\Pi}$, there exists a group homomorphism $\varphi_{\mathbb{Q}} : \mathcal{G}(\mathbb{Q}) \to \mathcal{G}'(\mathbb{Q})$ whose restriction to $\mathcal{G}_{\alpha,\beta}(\mathbb{Q})$ with α, β non-orthogonal coincides with $\mathcal{G}_{\alpha,\beta}(\mathbb{Q}) \to \mathcal{G}'(\mathbb{Q}) : g \mapsto \varphi_{\mathbb{Q}}^{\alpha,\beta}(g)$.

By Lemma 4.1 for each pair α, β of non-orthogonal simple roots there exist $n_{\alpha,\beta} \in \mathbb{N}$ such that $\varphi_{\mathbb{Q}|\Gamma_{\alpha,\beta}(n_{\alpha,\beta})} = {\varphi_{\mathbb{Q}}^{\alpha,\beta}}_{|\Gamma_{\alpha,\beta}(n_{\alpha,\beta})} = \varphi_{|\Gamma_{\alpha,\beta}(n_{\alpha,\beta})}$. Defining n as the least common multiple of the $n_{\alpha,\beta}$ therefore provides $\varphi_{\mathbb{Q}|\Gamma_\Pi(n)} = \varphi_{|\Gamma_\Pi(n)}$ as claimed, because by definition $\Gamma_\Pi(n)$ is the subgroup of $\mathcal{G}(\mathbb{Q})$ generated by the family of principal congruence subgroups $(\Gamma_{\alpha,\beta}(n))_{\alpha,\beta\in\Pi}$.

Uniqueness of $\varphi_{\mathbb{Q}}$ is implied by the uniqueness of the $\mathbb{Q}$-isogenies $\psi^{\alpha,\beta} : \mathcal{G}_{\alpha,\beta} \to H_{\alpha,\beta}$, cf. Lemma 4.1. Continuity of $\varphi_{\mathbb{Q}}$ also follows from Lemma 4.1, as the weak Zariski topology on $\mathcal{G}(\mathbb{Q})$ is defined by using its root subgroups, which are contained in the conjugates of the $\mathcal{G}_{\alpha,\beta}(\mathbb{Q})$; cf. Definition 2.18.

The existence of $\varphi_{\mathbb{R}}$ follows from a similar combination of Lemma 4.1 and the Curtis–Tits theorem (see Theorem 2.20). ■

A combination of the preceding three results allows one to establish a weak version of the Mostow–Margulis strong rigidity result, cf. Theorem C.3. We refer to it as "weak", as the unique "extension" of the lattice isomorphism we construct only is known to coincide with the original isomorphism on a subgroup, whereas for a strong rigidity result, the extension should extend the original homomorphism on the whole lattice. Whether it actually does so or not in our situation is an open problem.

Definition 4.5. For a Kac–Moody functor $\mathcal{G}$, let

- $\overline{\mathcal{G}}(\mathbb{R}) := \mathcal{G}(\mathbb{R})/Z(\mathcal{G}(\mathbb{R}))$ and $\pi : \mathcal{G}(\mathbb{R}) \to \overline{\mathcal{G}}(\mathbb{R})$ the natural quotient map,
- $\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{R}) := \pi(\mathcal{G}_{\alpha,\beta}(\mathbb{R}))$,
- $\overline{\mathcal{G}}(\mathbb{Q}) := \pi(\mathcal{G}(\mathbb{Q}))$,

- $\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Q}) := \pi(\mathcal{G}_{\alpha,\beta}(\mathbb{Q}))$,
- $\overline{\mathcal{G}}(\mathbb{Z}) := \pi(\mathcal{G}(\mathbb{Z}))$,
- $\overline{\Gamma}(\mathbb{Z}) := \pi(\Gamma(\mathbb{Z}))$,
- $\overline{\Gamma}_\Pi(n) := \pi(\Gamma_\Pi(n))$.

Remark 4.6. (a) Recall that if $\mathcal{G}$ is a simply connected split Kac–Moody functor, then $\overline{\mathcal{G}}(\mathbb{R})$ has trivial center, as can be seen from the action of $\overline{\mathcal{G}}(\mathbb{R})$ on the twin building of $\mathcal{G}(\mathbb{R})$.

(b) If $\mathcal{G}$ is a simply connected irreducible split Kac–Moody functor, then $Z(\mathcal{G}(\mathbb{R})) \cap \mathcal{G}(\mathbb{Q}) = Z(\mathcal{G}(\mathbb{Q}))$ by Remark 2.16. Hence

$$\overline{\mathcal{G}}(\mathbb{Q}) = \mathcal{G}(\mathbb{Q})/Z(\mathcal{G}(\mathbb{Q})).$$

(c) For each $n \in \mathbb{N}$ one has

$$\Gamma_\Pi(n) \leq \Gamma(\mathbb{Z}) \leq \mathcal{G}(\mathbb{Z}) \leq \mathcal{G}(\mathbb{Q}),$$

hence

$$\overline{\Gamma}_\Pi(n) \leq \overline{\Gamma}(\mathbb{Z}) \leq \overline{\mathcal{G}}(\mathbb{Z}) \leq \overline{\mathcal{G}}(\mathbb{Q}).$$

Proposition 4.7 (Weak rigidity). *Let $\mathcal{G}$, $\mathcal{G}'$ be simply connected irreducible two-spherical split Kac–Moody functors of rank at least two, let Φ be a set of real roots for $\mathcal{G}$, let Π be a system of simple roots for Φ, and let* **either** $\varphi : \overline{\Gamma}(\mathbb{Z}) \to \overline{\Gamma}'(\mathbb{Z})$ **or** $\varphi : \overline{\mathcal{G}}(\mathbb{Z}) \to \overline{\mathcal{G}}'(\mathbb{Z})$ *be a group isomorphism. Then there exists a uniquely determined group isomorphism*

$$\overline{\varphi}_{\mathbb{Q}} : \overline{\mathcal{G}}(\mathbb{Q}) \to \overline{\mathcal{G}}'(\mathbb{Q}) \qquad \textit{satisfying} \qquad \overline{\varphi}_{\mathbb{Q}|\overline{\Gamma}_\Pi(n)} = \varphi_{|\overline{\Gamma}_\Pi(n)}$$

for some $n \in \mathbb{N}$. This group isomorphism is continuous with respect to the weak Zariski topologies on $\overline{\mathcal{G}}(\mathbb{Q})$ and $\overline{\mathcal{G}}'(\mathbb{Q})$. Furthermore, there exists a uniquely determined continuous group isomorphism with respect to the Kac–Peterson topology

$$\overline{\varphi}_{\mathbb{R}} : \mathcal{G}(\mathbb{R})/Z(\mathcal{G}(\mathbb{R})) \to \mathcal{G}'(\mathbb{R})/Z(\mathcal{G}'(\mathbb{R})) \qquad \textit{satisfying} \qquad \overline{\varphi}_{\mathbb{R}|\overline{\mathcal{G}}(\mathbb{Q})} = \overline{\varphi}_{\mathbb{Q}}.$$

Proof. Let Φ' be a set of real roots for $\mathcal{G}'$ and let Π' be a system of simple roots for Φ'. Define π', $\overline{\Gamma}'(\mathbb{Z})$, etc. analogously to Definition 4.5. Now consider the group homomorphisms

$$\varphi \circ \pi : \Gamma(\mathbb{Z}) \to \overline{\mathcal{G}}'(\mathbb{Q}) \qquad \text{and} \qquad \varphi^{-1} \circ \pi' : \Gamma'(\mathbb{Z}) \to \overline{\mathcal{G}}(\mathbb{Q}).$$

By Proposition 4.4 (applied twice and after passing to the least common multiple of the two natural numbers obtained) there exist $m \in \mathbb{N}$ and uniquely determined group homomorphisms

$$\varphi^1_{\mathbb{Q}} : \mathcal{G}(\mathbb{Q}) \to \overline{\mathcal{G}}'(\mathbb{Q}) \qquad \text{and} \qquad \varphi^2_{\mathbb{Q}} : \mathcal{G}'(\mathbb{Q}) \to \overline{\mathcal{G}}(\mathbb{Q})$$

satisfying

$$\varphi^1_{\mathbb{Q}|\Gamma_\Pi(m)} = (\varphi \circ \pi)_{|\Gamma_\Pi(m)} \qquad \text{and} \qquad \varphi^2_{\mathbb{Q}|\Gamma_{\Pi'}(m)} = (\varphi^{-1} \circ \pi')_{|\Gamma_{\Pi'}(m)}.$$

Applying the Borel density Theorem C.2 to the arithmetic subgroups $\mathcal{G}'_{\alpha,\beta}(\mathbb{Z})$ of $\mathcal{G}'_{\alpha,\beta}(\mathbb{Q})$ yields that the former are dense in the latter. But

$$\langle \mathcal{G}'_{\alpha,\beta}(\mathbb{Z}) \mid \alpha, \beta \in \Pi' \rangle \leq \Gamma'(\mathbb{Z}) \leq \mathcal{G}'(\mathbb{Z}) \leq \mathcal{G}'(\mathbb{Q}) = \langle \mathcal{G}'_{\alpha,\beta}(\mathbb{Q}) \mid \alpha, \beta \in \Pi' \rangle,$$

hence both $\overline{\Gamma}'(\mathbb{Z})$ and $\overline{\mathcal{G}}'(\mathbb{Z})$ are Zariski-dense in $\overline{\mathcal{G}}'(\mathbb{Q})$. By Lemma 4.3 we have

$$[\varphi^{-1}(\overline{\mathcal{G}}'_{\alpha,\beta}(\mathbb{Z})) : \varphi^{-1}(\overline{\mathcal{G}}'_{\alpha,\beta}(\mathbb{Z})) \cap \overline{\Gamma}_{\Pi}(m)] < \infty$$

and thus

$$[\overline{\mathcal{G}}'_{\alpha,\beta}(\mathbb{Z}) : \overline{\mathcal{G}}'_{\alpha,\beta}(\mathbb{Z}) \cap \varphi(\overline{\Gamma}_{\Pi}(m))] < \infty.$$

Consequently, $\varphi(\overline{\Gamma}_{\Pi}(m)) = (\varphi \circ \pi)(\Gamma_{\Pi}(m))$ also is Zariski-dense in $\overline{\mathcal{G}}'(\mathbb{Q})$, i.e., the image of $\varphi^1_{\mathbb{Q}}$ is Zariski dense in $\overline{\mathcal{G}}'(\mathbb{Q})$. In addition, by Proposition 4.4 the map $\varphi^1_{\mathbb{Q}}$ is continuous with respect to the weak Zariski topologies. Combining these two observations implies that $\varphi^1_{\mathbb{Q}}$ is surjective. Thus $\varphi^1_{\mathbb{Q}}$ maps $Z(\mathcal{G}(\mathbb{Q}))$ into the center of $\overline{\mathcal{G}}'(\mathbb{Q})$, which is trivial. Hence $\varphi^1_{\mathbb{Q}}$ factors through $\overline{\mathcal{G}}(\mathbb{Q})$ and yields a group homomorphism

$$\overline{\varphi}^1_{\mathbb{Q}} : \overline{\mathcal{G}}(\mathbb{Q}) \to \overline{\mathcal{G}}'(\mathbb{Q}) \qquad \text{satisfying} \qquad \overline{\varphi}^1_{\mathbb{Q}|\overline{\Gamma}_{\Pi}(m)} = \overline{\varphi}_{|\overline{\Gamma}_{\Pi}(m)}.$$

Similarly, there exists a group homomorphism

$$\overline{\varphi}^2_{\mathbb{Q}} : \overline{\mathcal{G}}'(\mathbb{Q}) \to \overline{\mathcal{G}}(\mathbb{Q}) \qquad \text{satisfying} \qquad \overline{\varphi}^2_{\mathbb{Q}|\overline{\Gamma}'_{\Pi}(m)} = \overline{\varphi}^{-1}{}_{|\overline{\Gamma}'_{\Pi}(m)}.$$

We will prove that $\overline{\varphi}^1_{\mathbb{Q}}$ and $\overline{\varphi}^2_{\mathbb{Q}}$ are mutually inverse maps. Since the Kac-Moody groups $\mathcal{G}$ and $\mathcal{G}'$ are simply connected, they are generated by their fundamental rank two subgroups; thus the same holds for $\overline{\mathcal{G}}(\mathbb{Q})$ and $\overline{\mathcal{G}}'(\mathbb{Q})$. By this and by symmetry of the argument it suffices to prove for each non-orthogonal pair of simple roots $\alpha, \beta \in \Pi$ that

$$(\overline{\varphi}^2_{\mathbb{Q}} \circ \overline{\varphi}^1_{\mathbb{Q}})_{|\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Q})} = \mathrm{id}\,.$$

Let $B^{\alpha,\beta}_{\mathbb{Q}}$ be the image of $\overline{\varphi}^1_{\mathbb{Q}}$. It is in fact a bounded subgroup of $\overline{\mathcal{G}}'(\mathbb{Q})$ as a consequence of Lemma 4.1 applied to the homomorphism

$$\overline{\varphi}^1_{\mathbb{Q}} \circ \pi : \mathcal{G}_{\alpha,\beta}(\mathbb{Z}) \to \overline{\mathcal{G}}'(\mathbb{Q}).$$

Restricting and co-restricting $\overline{\varphi}^1_{\mathbb{Q}}$ suitably, one obtains an epimorphism

$$\overline{\varphi}^{1\,\alpha,\beta}_{\mathbb{Q}} := \overline{\varphi}^1_{\mathbb{Q}|\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Q})} : \overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Q}) \to B^{\alpha,\beta}_{\mathbb{Q}}$$

satisfying $\overline{\varphi}^{1\,\alpha,\beta}_{\mathbb{Q}}{}_{|\overline{\Gamma}_{\alpha,\beta}(n)} = \varphi_{|\overline{\Gamma}_{\alpha,\beta}(n)}$. Applying Lemma 4.1 again, this time to

$$\overline{\varphi}^{1\,\alpha,\beta}_{\mathbb{Q}} \circ \pi : \mathcal{G}_{\alpha,\beta}(\mathbb{Q}) \to B^{\alpha,\beta}_{\mathbb{Q}}$$

and using that $\mathcal{G}_{\alpha,\beta}(\mathbb{R}) \twoheadrightarrow \overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{R})$ has finite kernel, whence is a covering map of topological groups, in particular open, there exists a continuous surjective extension

$$\overline{\varphi}^{1\,\alpha,\beta}_{\mathbb{R}} : \overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{R}) \to B^{\alpha,\beta}_{\mathbb{R}}$$

with the semisimple Lie group $B^{\alpha,\beta}_{\mathbb{R}} \leq \mathcal{G}'(\mathbb{R})$ as bounded target. Analysis of the commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{R}) & \xrightarrow{\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\beta}} & B^{\alpha,\beta}_{\mathbb{R}} \\ & \searrow^{\psi^{\alpha,\beta}} & \downarrow{\iota^{\alpha,\beta}_{\mathbb{R}}} \\ & & H_{\alpha,\beta}(\mathbb{R}) \end{array}$$

shows that $\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\beta}(\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Z}))$ is a lattice in $B^{\alpha,\beta}_{\mathbb{R}}$: Indeed, $\psi^{\alpha,\beta}$ is a central $\mathbb{Q}$-isogeny by Lemma 4.1, whence $\psi^{\alpha,\beta}(\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Z}))$ is an arithmetic lattice in $H_{\alpha,\beta}(\mathbb{R})$, so that this assertion follows from the fact that $\iota^{\alpha,\beta}_{\mathbb{R}}$ has finite (central) kernel. A case-by-case analysis in fact shows that $\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\beta}$ necessarily is injective:

- For types A_2 or G_2, the Lie group $\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{R})$ is simple, hence center-free.
- For type C_2 we may have $\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{R}) \cong \mathrm{Sp}(4,\mathbb{R})$ or $\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{R}) \cong \mathrm{PSp}(4,\mathbb{R})$. In the latter case there again is nothing to show as $\mathrm{PSp}(4,\mathbb{R})$ is center-free. In the former case, we claim that the kernel of $\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\beta}$ necessarily has to lie in the center of $\overline{\mathcal{G}}(\mathbb{R})$, which is trivial: indeed, suppose α is the long root in the root system generated by α, β. Then $Z(\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{R})) = Z(\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Q}))$ is in fact equal to the center $Z(\mathcal{G}_{\alpha}(\mathbb{R})) = Z(\mathcal{G}_{\alpha}(\mathbb{Q}))$ of its rank one group $\mathcal{G}_{\alpha}$. If it were contained in the kernel of $\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\beta}$, then it would also be contained in the kernel of $\overline{\varphi}^{1}_{\mathbb{Q}}{}^{\alpha,\beta}$, hence of $\overline{\varphi}^{1}_{\mathbb{Q}}$. Going in the reverse direction, we would then have for every simple root $\gamma \in \Pi \setminus \{\alpha\}$ that $Z(\mathcal{G}_{\alpha}(\mathbb{R}))$ is contained in the kernel of $\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\gamma}$. This would imply that the root system generated by α, γ has type $A_1 \oplus A_1$ or C_2, and in the latter case α would always have to be the long root. But then $Z(\mathcal{G}_{\alpha}(\mathbb{R}))$ would be contained in $Z(\overline{\mathcal{G}}(\mathbb{R}))$, which is trivial, a contradiction. We conclude that $\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\beta} : \overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{R}) \to B^{\alpha,\beta}_{\mathbb{R}}$ and, hence, $\overline{\varphi}^{1}_{\mathbb{Q}}{}^{\alpha,\beta} : \overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Q}) \to B^{\alpha,\beta}_{\mathbb{Q}}$ are group isomorphisms.

Since $\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\beta}$ is an extension of $\overline{\varphi}^{1}_{\mathbb{Q}}{}^{\alpha,\beta} : \overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Q}) \to B^{\alpha,\beta}_{\mathbb{Q}}$, the lattice $\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\beta}(\mathcal{G}_{\alpha,\beta}(\mathbb{Z}))$ is an arithmetic subgroup of $B^{\alpha,\beta}_{\mathbb{Q}}$. By Lemma 4.3 the intersection $\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\beta}(\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Z})) \cap \overline{\Gamma}_{\Pi'}(m)$ has finite index in $\overline{\varphi}^{1}_{\mathbb{R}}{}^{\alpha,\beta}(\overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Z}))$. By Theorem B.1 this finite-index subgroup contains a principal congruence subgroup Γ_0 of the arithmetic subgroup of $B^{\alpha,\beta}_{\mathbb{Q}}$. The maps

$$\left(\overline{\varphi}^{1}_{\mathbb{Q}}{}^{\alpha,\beta}\right)^{-1} : B^{\alpha,\beta}_{\mathbb{Q}} \to \overline{\mathcal{G}}_{\alpha,\beta}(\mathbb{Q}) \leq \overline{\mathcal{G}}(\mathbb{Q}) \qquad \text{and} \qquad \overline{\varphi}^{2}_{\mathbb{Q}|B^{\alpha,\beta}_{\mathbb{Q}}} : B^{\alpha,\beta}_{\mathbb{Q}} \to \overline{\mathcal{G}}(\mathbb{Q})$$

both coincide on Γ_0 with φ^{-1}. By Lemma 4.1 one concludes equality, i.e., $(\overline{\varphi}^{2}_{\mathbb{Q}} \circ \overline{\varphi}^{1}_{\mathbb{Q}})_{|\overline{\mathcal{G}}_{\alpha,\beta}} = \mathrm{id}$. ■

Remark 4.8. Observe that, if $\varphi, \varphi' : \overline{\Gamma}(\mathbb{Z}) \to \overline{\Gamma}'(\mathbb{Z})$ are group isomorphisms that induce the same $\overline{\varphi}_{\mathbb{R}}$, then the automorphism $\varphi^{-1} \circ \varphi' : \overline{\Gamma}(\mathbb{Z}) \to \overline{\Gamma}(\mathbb{Z})$ induces the identity on $\mathcal{G}(\mathbb{R})/Z(\mathcal{G}(\mathbb{R}))$. In this remark we argue that this in fact is enough to conclude $\varphi^{-1} \circ \varphi' = \mathrm{id}$, i.e., $\varphi = \varphi'$.

By Proposition 4.7 there exists $m \in \mathbb{N}$ such that $(\varphi^{-1} \circ \varphi')_{|\overline{\Gamma}_{\Pi}(m)} = \mathrm{id}$. In particular, for each pair of non-orthogonal simple roots $\alpha, \beta \in \Pi$, one has

$(\varphi^{-1}\circ\varphi')_{|\Gamma_{\alpha,\beta}(m)} = \mathrm{id}$. We conclude from Theorem C.3 that $(\varphi^{-1}\circ\varphi')_{|\mathcal{G}_{\alpha,\beta}(\mathbb{Z})} = \mathrm{id}$: Indeed, by Theorem C.3 both $(\varphi^{-1}\circ\varphi')_{|\Gamma_{\alpha,\beta}(m)}$ and $(\varphi^{-1}\circ\varphi')_{|\mathcal{G}_{\alpha,\beta}(\mathbb{Z})}$ extend uniquely to a continuous automorphism of $\mathcal{G}_{\alpha,\beta}(\mathbb{R})$; the former necessarily extends to the identity, whence also the latter has to extend to the identity, as it is an extension of both $(\varphi^{-1}\circ\varphi')_{|\Gamma_{\alpha,\beta}(m)}$ and $(\varphi^{-1}\circ\varphi')_{|\mathcal{G}_{\alpha,\beta}(\mathbb{Z})}$. The claim follows as $\overline{\Gamma}(\mathbb{Z})$ is generated by the $\mathcal{G}_{\alpha,\beta}(\mathbb{Z})$ for $\alpha,\beta\in\Pi$ distinct and non-orthogonal. (Note that formally we can only apply Theorem C.3 in the adjoint situation. That is, as in the proof of Proposition 4.7 the case $\mathcal{G}_{\alpha,\beta} = \mathrm{Sp}_4$ requires special attention. A simple lifting argument from the adjoint situation works in this case.)

Altogether we have shown our main result:

Theorem 4.9 (Solution of the isomorphism problem). *Let $\mathcal{G}$, $\mathcal{G}'$ be simply connected irreducible two-spherical split Kac–Moody functors. If there exists a group isomorphism $\Gamma(\mathbb{Z})\cong\Gamma'(\mathbb{Z})$ or $\mathcal{G}(\mathbb{Z})\cong\mathcal{G}'(\mathbb{Z})$, then $\mathcal{G}=\mathcal{G}'$ over fields.*

Proof. $\Gamma(\mathbb{Z})\cong\Gamma'(\mathbb{Z})$ implies $\overline{\Gamma}(\mathbb{Z})\cong\Gamma(\mathbb{Z})/Z(\Gamma(\mathbb{Z}))\cong\Gamma'(\mathbb{Z})/Z(\Gamma'(\mathbb{Z}))\cong\overline{\Gamma}'(\mathbb{Z})$, whereas $\mathcal{G}(\mathbb{Z})\cong\mathcal{G}'(\mathbb{Z})$ implies $\overline{\mathcal{G}}(\mathbb{Z})\cong\mathcal{G}(\mathbb{Z})/Z(\mathcal{G}(\mathbb{Z}))\cong\mathcal{G}'(\mathbb{Z})/Z(\mathcal{G}'(\mathbb{Z}))\cong\overline{\mathcal{G}}'(\mathbb{Z})$. By Proposition 4.7 there exists a group isomorphism $\overline{\mathcal{G}}(\mathbb{Q})\cong\overline{\mathcal{G}}'(\mathbb{Q})$. The claim therefore follows from [Cap09, Theorem A]. ■

A. A fixed point theorem

The purpose of the first part of the appendix is to recall the statement and proof of a fixed point theorem by Caprace and Monod.

Theorem A.1 ([CM09, Lemma 8.1 and the preceding and subsequent paragraphs]). *Let (X,d_X) be a complete $\mathrm{CAT}(0)$ space (cf. Theorem 2.5) such that isometries of zero translation length are elliptic (i.e., fix a point) and let G be an irreducible simply connected Chevalley group functor of rank at least two. Then any action of $G(\mathbb{Z})$ on X by isometries admits a fixed point.*

Proof. Let Φ be the set of roots of G, let $(U_\alpha)_{\alpha\in\Phi}$ be the root subgroups, and let $x_\alpha:\mathbb{Z}\to U_\alpha(\mathbb{Z})$ be group isomorphisms. Then the group $G(\mathbb{Z})$ is generated by the finite set $\Sigma := \{x_\alpha(1)\mid\alpha\in\Phi\}$. We compute the translation length

$$\begin{aligned}|x_\alpha(1)| &:= \inf\{d_X(x,x_\alpha(1).x)\mid x\in X\}\\ &= \lim_{n\to\infty}\frac{d_X(x,x_\alpha(1)^n.x)}{n} \quad \text{for any } x\in X \qquad \text{by [BH99, Exercise II.6.6(1)]}\\ &\le \lim_{n\to\infty}\frac{l_\Sigma(x_\alpha(1)^n)\cdot\max\{d_X(x,s.x)\mid s\in\Sigma\}}{n}\\ &= \max\{d_X(x,s.x)\mid s\in\Sigma\}\cdot\lim_{n\to\infty}\frac{O(\log(n))}{n} \qquad \text{by [LMR00, Theorem 2.15]}\\ &= 0.\end{aligned}$$

Hence the isometry $x_\alpha(1)$ has a fixed point by hypothesis and so does the group $U_\alpha(\mathbb{Z})$ generated by it. By [Tav90] the group $G(\mathbb{Z})$ is boundedly generated by the family $(U_\alpha(\mathbb{Z}))_{\alpha\in\Phi}$, whence also has a fixed point by the Bruhat–Tits fixed point theorem (cf. [Cap09, Corollary 2.5]). ■

Corollary A.2. *Let X be the* CAT(0) *Davis realization of a building (cf. Theorem 2.5) and let H be an irreducible simply connected Chevalley group functor of rank at least two. Then any action of $H(\mathbb{Z})$ on X by cellular isometries admits a fixed point.*

Proof. The geometric realization X is a polyhedral complex admitting finitely many shapes. Therefore by [Bri99, Theorem A] there do not exist parabolic cellular isometries, i.e., any cellular isometry of zero translation length is elliptic. ■

B. The congruence subgroup property

In the second part of the appendix we recall some results concerning congruence subgroups.

Theorem B.1 (The congruence subgroup property; [Mat69, Corollary 12.6], also [Sur03, Section 6-7.5])**.** *Let G be an irreducible simply connected Chevalley group functor of rank at least two and Γ an arithmetic subgroup of $G(\mathbb{Q})$, i.e., a subgroup of $G(\mathbb{Q})$ commensurable to $G(\mathbb{Z})$. Then Γ is a congruence subgroup, i.e., there exists $n \in \mathbb{N}$ such that Γ contains the kernel of the homomorphism $G(\mathbb{Z}) \to G(\mathbb{Z}/n\mathbb{Z})$, called principal congruence subgroup of level n.*

Proposition B.2 ([Tit76, Proposition 4])**.** *Let G be an irreducible Chevalley group functor of rank at least two, let Φ be its root system, let $(U_\alpha)_{\alpha\in\Phi}$ be the root subgroups of G, and let $n \in \mathbb{N}$. Then the group $\langle U_\alpha(n\mathbb{Z}) \mid \alpha \in \Phi\rangle$ has finite index in $G(\mathbb{Z})$.*

Corollary B.3. *Let G be an irreducible Chevalley group functor of rank at least two, let Φ be its root system, let Π be a system of simple roots, let $(U_\alpha)_{\alpha\in\Phi}$ be the root subgroups of G, and let $n \in \mathbb{N}$. Then the group $\langle U_\alpha(n\mathbb{Z}) \mid \alpha \in \Pi\rangle$ has finite index in $G(\mathbb{Z})$.*

Proof. By the commutator relations (see [Che55, p. 27]), for each root $\beta \in \Phi$ there exists $m_\beta \in \mathbb{Z}$ such that the group $\langle U_\alpha(n\mathbb{Z}) \mid \alpha \in \Pi\rangle$ contains the group $U_\beta(m_\beta\mathbb{Z})$. Therefore, for $m := \mathrm{lcm}\{m_\beta \mid \beta \in \Phi\}$, one has $\langle U_\alpha(m\mathbb{Z}) \mid \alpha \in \Phi\rangle \leq \langle U_\alpha(n\mathbb{Z}) \mid \alpha \in \Pi\rangle \leq G(\mathbb{Z})$. By Proposition B.2 the former group has finite index in $G(\mathbb{Z})$ and hence so has the latter group. ■

C. Homomorphisms of algebraic and arithmetic groups into algebraic groups

In the third part of the appendix we recall the classical results concerning the Borel–Tits theorem, Mostow–Margulis strong rigidity, Margulis superrigidity, and Borel density that we are using in this article. Furthermore, we describe the manner in which we apply these classical results in the main text.

Theorem C.1 (Abstract homomorphisms of algebraic groups; [BT73, Theorem A], also [Mar91, Theorem I.1.8.1])**.** *Let G be connected simply connected non-commutative reductive almost $\mathbb{Q}$-simple $\mathbb{Q}$-isotropic $\mathbb{Q}$-group, let H be a subgroup of $G(\mathbb{Q})$ containing the subgroup $G(\mathbb{Q})^+$ generated by the $\mathbb{Q}$-points of the root subgroups of G, let G' be a connected non-commutative absolutely almost simple $\mathbb{Q}$-group, and let $\delta : H \to G'(\mathbb{Q})$ be a group homomorphism with Zariski-dense image. Then there exist a $\mathbb{Q}$-epimorphism $\beta : G \to G'$ and a group homomorphism $\tau : H \to Z(G')(\mathbb{Q})$ such that*

$$\delta(g) = \tau(g)\beta(g)$$

for each $g \in H$.

Application in the main text. This result is applied in Lemma 4.1 in the following way: *Let M be an irreducible simply connected Chevalley group functor of rank at least two, let B be an irreducible Chevalley group functor, and let $\delta : M(\mathbb{Q}) \to B(\mathbb{Q})$ be a group homomorphism with Zariski-dense image. Then there exists a $\mathbb{Q}$-epimorphism $\beta : M \to B$ such that*

$$\delta(g) = \beta(g)$$

for each $g \in M(\mathbb{Q})$.

Proof. Any irreducible Chevalley group is connected, non-commutative, reductive, defined over $\mathbb{Z}$ and almost simple and isotropic over any field of characteristic zero. Since M is simply connected it is generated by its root subgroups and, hence, perfect and semisimple. Theorem C.1 applies and provides a $\mathbb{Q}$-epimorphism $M \to B$. Since $M(\mathbb{Q})$ is perfect, it cannot have non-trivial abelian quotients, so that the group homomorphism τ from Theorem C.1 is trivial, i.e., $\delta(g) = \beta(g)$ for each $g \in M(\mathbb{Q})$. ■

Theorem C.2 (Borel density of arithmetic subgroups; [Bor60], also [Mar91, Proposition I.3.2.11])**.** *Let G be a connected semisimple $\mathbb{Q}$-group without $\mathbb{Q}$-anisotropic factors. Then every arithmetic subgroup of $G(\mathbb{Q})$ is Zariski dense in G.*

Application in the main text. This result is applied in Proposition 4.7 and implicitly in our variant of Theorem C.5 below in the following way: *Let G be a simply connected Chevalley group functor. Then $G(\mathbb{Z})$ is Zariski-dense in $G(\mathbb{Q})$.*

Proof. The key observation is that G is semisimple, as it is generated by its root subgroups. ■

Theorem C.3 (Strong rigidity of lattices; [Mar91, Theorem VII.7.1])**.** *Let G, G' be connected semisimple adjoint $\mathbb{R}$-simple algebraic $\mathbb{R}$-groups of $\mathbb{R}$-rank at least two, let $\Gamma < G(\mathbb{R})$ and $\Gamma' < G'(\mathbb{R})$ be lattices, and let $f : \Gamma \to \Gamma'$ be an isomorphism. Then f extends uniquely to a continuous group homomorphism $\tilde{f} : G(\mathbb{R}) \to G'(\mathbb{R})$ which is an isomorphism of topological groups and of algebraic $\mathbb{R}$-groups.*

Application in the main text. This result only plays a minor role for our proof. It is exclusively used as a motivation for the name of Proposition 4.7 and for a discussion in Remark 4.8. It is applied in the following way: *Let G be an irreducible adjoint Chevalley group functor of rank two. Then any automorphism of $G(\mathbb{Z})$ extends uniquely to a continuous endomorphism of $G(\mathbb{R})$ that is an automorphism of topological groups and of algebraic $\mathbb{R}$-groups.*

Proof. Borel–Harish-Chandra reduction theory (see [BHC62, Theorem 12.3], also [Mar91, Theorem I.3.2.7]) implies that $G(\mathbb{Z})$ is a lattice in $G(\mathbb{R})$. ■

Theorem C.4 (Superrigidity of arithmetic groups; [Mar91, Theorem VIII.3.12])**.** *Let G be a connected simply connected non-commutative almost $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group of $\mathbb{Q}$-rank at least two, let Λ be an arithmetic subgroup of $G(\mathbb{Q})$, let H be an algebraic $\mathbb{Q}$-group, and let $\delta : \Lambda \to H(\mathbb{Q})$ be a group homomorphism. Then there exist a (uniquely determined) $\mathbb{Q}$-morphism $\varphi : G \to H$ of algebraic groups and a (uniquely determined) group homomorphism $\nu : \Lambda \to H(\mathbb{Q})$ with finite image $\nu(\Lambda)$ commuting with $\varphi(G)$ and such that $\delta(\lambda) = \nu(\lambda)\varphi(\lambda)$ for all $\lambda \in \Lambda$.*

Application in the main text. This result is applied in Remark 3.2 and Lemma 4.1 in the following way: *Let M be an irreducible simply connected Chevalley group functor of rank at least two, let H be an algebraic $\mathbb{Q}$-group, and let $\delta : M(\mathbb{Z}) \to H(\mathbb{Q})$ be a group homomorphism. Then there exist a (uniquely determined) $\mathbb{Q}$-morphism $\varphi : M \to H$ of algebraic groups and a (uniquely determined) group homomorphism $\nu : M(\mathbb{Z}) \to H(\mathbb{Q})$ with finite image $\nu(M(\mathbb{Z}))$ commuting with $\varphi(M)$ and such that $\delta(\lambda) = \nu(\lambda)\varphi(\lambda)$ for all $\lambda \in M(\mathbb{Z})$. In particular, there exists a finite-index subgroup $\Gamma \leq G(\mathbb{Z})$ such that $\delta(\lambda) = \varphi(\lambda)$ for all $\lambda \in \Gamma$.*

Proof. Only the final statement requires a clarification. It follows by defining $\Gamma := \ker(\nu)$, which by the homomorphism theorem of groups has finite index in $G(\mathbb{Z})$. ■

Theorem C.5 (Semisimple Zariski closure of image of lattice; [Mar91, Theorem IX.6.15])**.** *Let H be a quasi-simple split connected semisimple Lie group of rank at least two with finite center, let Γ be a Zariski-dense lattice of H, let F be an algebraic $\mathbb{Q}$-group, and let $\delta : \Gamma \to F(\mathbb{Q})$ be a group homomorphism. Then the Zariski closure $\overline{\delta(\Gamma)}$ in F is a semisimple $\mathbb{Q}$-group.*

Application in the main text. This result is applied in Lemma 4.1 in the following way: *Let M be an irreducible simply connected Chevalley group functor of rank at least two, let A be an algebraic $\mathbb{Q}$-group, and let $\delta : M(\mathbb{Z}) \to A(\mathbb{Q})$ be a group homomorphism. Then the Zariski closure $\overline{\delta(M(\mathbb{Z}))}$ in A is a semisimple $\mathbb{Q}$-group.*

Proof. By Borel density C.2 the arithmetic group $M(\mathbb{Z})$ is Zariski dense. It is a lattice in $M(\mathbb{R})$ by Borel–Harish-Chandra reduction theory (see [BHC62, Theorem 12.3], also [Mar91, Theorem I.3.2.7]). The center of an irreducible simply connected Chevalley group is finite. ■

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