

A Class of Lie Conformal Superalgebras in Higher Dimensions

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Communicated by A. Fialowski

Abstract. Fix a positive integer number r . A class of Lie conformal superalgebras in r dimensions called r -dim i -linear Lie conformal superalgebras is studied for $1 \leq i \leq r$. It is shown that an r -dim i -linear Lie conformal superalgebra is equivalent to an $(r - 1)$ -dim super Gel'fand-Dorfman conformal bialgebra, which is a generalization of a super-Gel'fand-Dorfman bialgebra in the conformal sense. In particular, a special Lie conformal superalgebra named r -dim linear Lie conformal superalgebra can be characterized by a generalized super Gel'fand-Dorfman algebra which has a Lie superalgebra structure and r Novikov superalgebra structures adjoint with some compatibility conditions. Moreover, by these equivalent characterizations, several constructions and examples of Lie conformal superalgebras in higher dimensions are given.

Mathematics Subject Classification 2010: 17B60, 17B63, 17B67, 17B69, 17D99.

Key Words and Phrases: Lie conformal superalgebra, Gel'fand-Dorfman bialgebra, Novikov-Poisson superalgebra, Novikov conformal superalgebra.

1. Introduction

Throughout this paper, set r a positive integer number and denote by \mathbb{C} the field of complex numbers; \mathbb{Z} the set of integer numbers; \mathbb{Z}_+ the set of non-negative integer numbers. In addition, all vectors are defined over \mathbb{C} and tensor products over \mathbb{C} are denoted by \otimes .

Lie conformal superalgebra, introduced by Kac in [14], gives an axiomatic description of the singular part of the operator product expansion of chiral fields in conformal field theory. It is an useful tool to study vertex superalgebras (see [14]) and has many applications in the theory of infinite-dimensional Lie superalgebras and integrable systems. In particular, the category of Lie conformal superalgebras is equivalent to the category of some infinite-dimensional Lie superalgebras named formal distribution Lie superalgebras in some sense (see [14]). The main examples of Lie conformal superalgebras are Virasoro conformal algebra, current conformal

* This project is supported by the National Natural Science Foundation of China (No.11501515 and No.11401533) and the Zhejiang Provincial Natural Science Foundation of China (No.LQ16A010011).

superalgebras, Neveu-Schwarz conformal superalgebra and so on. It is known that Lie conformal superalgebra is a $\mathbb{C}[T]$ -module. Therefore, there is a natural generalization, i.e. we change $\mathbb{C}[T]$ by $\mathbb{C}[T_1, \dots, T_n]$. This is just the n -dim Lie conformal superalgebra (see [2]). More generally, $\mathbb{C}[T]$ can be generalized to a cocommutative Hopf algebra. This is the case of Lie pseudoalgebras (see [1]). The structure theory [11], representation theory [9, 10] and cohomology theory [2] of finite Lie conformal algebras has been developed.

A class of Lie conformal superalgebras named “quadratic” Lie conformal superalgebras are also studied in [12, 17]. It was essentially stated in [12] that a “quadratic” Lie conformal superalgebra is equivalent to a bialgebra structure. One is a Lie superalgebra structure, the other is a Novikov superalgebra structure and they satisfy a compatibility condition. The above bialgebra structure is called a super Gel’fand-Dorfman bialgebra by Xu in [17]. In fact, a quadratic Lie conformal algebra corresponds to a Hamiltonian pair in [12], which plays fundamental roles in completely integrable systems. Then, Xu in [17] presented several constructions of super Gel’fand-Dorfman bialgebras and classified the Gel’fand-Dorfman bialgebra structures on simple Novikov algebras. Therefore, there are some natural questions whether there exists a similar characterization of Lie conformal superalgebras in higher dimensions and if there exists, what is the corresponding algebraic structure. In this paper, we study this question. We prove that an r -dim i -linear Lie conformal superalgebra (see Definition 2.6) is equivalent to one $(r - 1)$ -dim conformal superalgebra with a Lie conformal superalgebra structure and a Novikov conformal superalgebra structure adjoint with some compatibility conditions. We call such an algebraic structure an $(r - 1)$ -dim super Gel’fand-Dorfman conformal bialgebra. In particular, we also show that an r -dim linear Lie conformal superalgebra (see Definition 2.6) is equivalent to a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space with a Lie superalgebra structure and r Novikov superalgebra structures adjoint with some compatibility conditions. Here, any one of r Novikov superalgebra structures and the Lie superalgebra structure construct a super Gel’fand-Dorfman bialgebra structure. We call such an algebraic structure a generalized super Gel’fand-Dorfman algebra. Moreover, we present several constructions and examples of super Gel’fand-Dorfman conformal bialgebras in r dimension and generalized super Gel’fand-Dorfman algebras. Since an r -dim Lie conformal superalgebra corresponds to an infinite-dimensional Lie superalgebra, we can give many interesting examples of infinite-dimensional Lie superalgebras using super Gel’fand-Dorfman conformal bialgebras in r dimension and generalized super Gel’fand-Dorfman algebras.

This paper is organized as follows. In Section 2, we introduce some basic definitions and some facts about r -dim Lie conformal superalgebras and Novikov conformal superalgebras. The definitions of r -dim i -linear Lie conformal superalgebras and linear Lie conformal superalgebras are presented for $1 \leq i \leq r$. In Section 3, we mainly present two equivalent characterizations of r -dim i -linear Lie conformal superalgebras and linear Lie conformal superalgebras. Several constructions and examples of Lie conformal superalgebras in higher dimensions are given.

2. Preliminaries

In this section, we will introduce some basic definitions and some facts about r -dim Lie conformal superalgebras and r -dim Novikov conformal superalgebras. These facts can be found in [2, 14].

First, we introduce the definitions of Lie superalgebra (see [15]) and Lie-Poisson superalgebra.

Definition 2.1. A Lie superalgebra \mathfrak{g} is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with an operation $[\cdot, \cdot]$ satisfying the following axioms:

$$[a, b] = -(-1)^{\alpha\beta}[b, a], \quad [a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]],$$

for $a \in \mathfrak{g}_\alpha$, $b \in \mathfrak{g}_\beta$ and $c \in \mathfrak{g}$.

A Lie-Poisson superalgebra $(A, [\cdot, \cdot], \cdot)$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with two operations $[\cdot, \cdot]$ and \cdot such that $(A, [\cdot, \cdot])$ forms a Lie superalgebra and (A, \cdot) forms a commutative associative superalgebra with the compatibility condition:

$$[a, b \cdot c] = [a, b] \cdot c + (-1)^{\alpha\beta} b \cdot [a, c], \quad a \in A_\alpha, \quad b \in A_\beta, \quad c \in A.$$

If $A_{\bar{1}} = \{0\}$, it is called a Lie-Poisson algebra.

Set $\mathbf{T} = (T_1, \dots, T_r)$ and $\lambda = (\lambda_1, \dots, \lambda_r)$. Then, \mathbf{T} and λ are r dimensional vector variables. Assume that $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$ and $e_i = \underbrace{(0, \dots, 1, \dots, 0)}_i \in \mathbb{Z}^r$ for $1 \leq i \leq r$. Set $\mathbb{C}[\mathbf{T}] = \mathbb{C}[T_1, \dots, T_r]$.

Next, we introduce the definition of r -dim Lie conformal superalgebra.

Definition 2.2. An r -dim Lie conformal superalgebra $(R, [\cdot, \cdot])$ is a left $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{C}[\mathbf{T}]$ -module $R = R_{\bar{0}} \oplus R_{\bar{1}}$ endowed with a λ -bracket $[a_\lambda b]$ which defines a \mathbb{C} -linear map $R \otimes R \rightarrow R[\lambda]$, where $R[\lambda] = \mathbb{C}[\lambda] \otimes R = \mathbb{C}[\lambda_1, \dots, \lambda_r] \otimes R$, subject to the following axioms:

$$[T_i a_\lambda b] = -\lambda_i [a_\lambda b], \quad [a_\lambda T_i b] = (T_i + \lambda_i) [a_\lambda b], \tag{2.1}$$

$$[a_\lambda b] = -(-1)^{\alpha\beta} [b_{-(\lambda+\mathbf{T})} a], \tag{2.2}$$

$$[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{\alpha\beta} [b_\mu [a_\lambda c]], \tag{2.3}$$

for $a \in R_\alpha$, $b \in R_\beta$ and $c \in R$. When $R_{\bar{1}} = \{0\}$, R is called an r -dim Lie conformal algebra. If $r = 1$, R is called a Lie conformal superalgebra.

If $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mathbf{m} = (m_1, \dots, m_r)$, we set $\lambda^{(\mathbf{m})} = \lambda_1^{(m_1)} \dots \lambda_r^{(m_r)} = \frac{\lambda_1^{m_1}}{(m_1)!} \dots \frac{\lambda_r^{m_r}}{(m_r)!}$. We can rewrite

$$[a_\lambda b] = \sum_{\mathbf{m} \in \mathbb{Z}_+^r} \lambda^{(\mathbf{m})} a_{(\mathbf{m})} b.$$

Given an r -dim Lie conformal superalgebra R , we can associate a Lie superalgebra as follows. Let $\text{Lie}R$ be the quotient of the $\mathbb{Z}/2\mathbb{Z}$ -graded vector

space with the basis $a_{\mathbf{n}}$ ($a \in R, \mathbf{n} \in \mathbb{Z}^r$) by the subspace spanned over \mathbb{C} by elements:

$$(\alpha a)_{\mathbf{n}} - \alpha a_{\mathbf{n}}, (a+b)_{\mathbf{n}} - a_{\mathbf{n}} - b_{\mathbf{n}}, (T_i a)_{\mathbf{n} + n_i \mathbf{e}_i}, \text{ where } a, b \in R, \alpha \in \mathbb{C}, \mathbf{n} \in \mathbb{Z}^r.$$

Here, if $a \in R_{\alpha}, a_{\mathbf{n}} \in (\text{Lie}R)_{\alpha}$. The operation on $\text{Lie}R$ is defined as follows:

$$[a_{\mathbf{m}}, b_{\mathbf{n}}] = \sum_{\mathbf{j} \in \mathbb{Z}_+^r} \binom{\mathbf{m}}{\mathbf{j}} (a_{(\mathbf{j})} b)_{\mathbf{m} + \mathbf{n} - \mathbf{j}}, \tag{2.4}$$

where $\binom{\mathbf{m}}{\mathbf{j}} = \binom{m_1}{j_1} \cdots \binom{m_r}{j_r}$, if $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{j} = (j_1, \dots, j_r)$.

Example 2.3. Let \mathfrak{g} be a Lie superalgebra. Then $\text{Curg} \mathfrak{g} = \mathbb{C}[\mathbf{T}] \otimes \mathfrak{g}$ can be endowed with the following λ -bracket ($a, b \in \mathfrak{g}$):

$$[a_{\lambda} b] = [a, b].$$

$\tilde{\mathfrak{g}} = \text{Lie}(\text{Curg} \mathfrak{g})$ is just the current Lie superalgebra in r variables, i.e. $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$ with the following Lie bracket:

$$[a \otimes x_1^{m_1} \cdots x_r^{m_r}, b \otimes x_1^{n_1} \cdots x_r^{n_r}] = [a, b] \otimes x_1^{m_1+n_1} \cdots x_r^{m_r+n_r},$$

where $a, b \in \mathfrak{g}$, and $m_1, \dots, m_r, n_1, \dots, n_r \in \mathbb{Z}$.

Example 2.4. Let $R = \bigoplus_{i=1}^r \mathbb{C}[\mathbf{T}]L^i$. It can be endowed with the following λ -bracket:

$$[L_{\lambda}^i L^j] = T_i L^j + \lambda_i L^j + \lambda_j L^i.$$

$\text{Lie}R$ is just the Lie algebra $W_r = \text{Der}\mathbb{C}[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$. For $r = 1$, R is the Virasoro Lie conformal algebra.

Example 2.5. Let $r = 2s$ and $R = \mathbb{C}[\mathbf{T}]L$. The following λ -bracket endows R a Lie conformal algebra structure:

$$[L_{\lambda} L] = \sum_{i=1}^s (\lambda_{s+i} T_i L - \lambda_i T_{s+i} L).$$

$\text{Lie}R$ is just the subalgebra H_r of Hamiltonian derivations of W_r .

In the following of this paper, we mainly study a special class of r -dim Lie conformal superalgebras.

Let $\bar{\lambda}_i = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_r)$ and $\bar{\mathbf{T}}_i = (T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_r)$ for $i \in \{1, \dots, r\}$.

Definition 2.6. Suppose that R is a free $\mathbb{C}[\mathbf{T}]$ -module over a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space V . R is called an r -dim i -linear Lie conformal superalgebra if for any $a, b \in V$, its λ -bracket is of the following form:

$$[a_{\lambda} b] = T_i f(\bar{\lambda}_i, \bar{\mathbf{T}}_i) u + \lambda_i g(\bar{\lambda}_i, \bar{\mathbf{T}}_i) v + h(\bar{\lambda}_i, \bar{\mathbf{T}}_i) w, \tag{2.5}$$

where $u, v, w \in V, f(\bar{\lambda}_i, \bar{\mathbf{T}}_i), g(\bar{\lambda}_i, \bar{\mathbf{T}}_i), h(\bar{\lambda}_i, \bar{\mathbf{T}}_i) \in \mathbb{C}[\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_r, T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_r], i \in \{1, \dots, r\}$. In particular, R is called an r -dim linear Lie conformal superalgebra if for any $a, b \in V$, its λ -bracket is of the following form:

$$[a_\lambda b] = \sum_{i=1}^r \lambda_i w_i + \sum_{j=1}^r T_j u_j + v, \quad u_i, w_i, v \in V. \tag{2.6}$$

When $r = 1$, for simplicity, it is also called a linear Lie conformal superalgebra.

Remark 2.7. Since the right side of (2.5) can be seen a linear function with two variables λ_i, T_i , we take the name “ r -dim i -linear Lie conformal superalgebra”. Similarly, since the right side of (2.6) can be seen a linear function with variables $\lambda_1, \dots, \lambda_r, T_1, \dots, T_r$, we call it an “ r -dim linear Lie conformal superalgebra”. When $r = 1$, it is called a “quadratic Lie conformal superalgebra” in [17], since the definition of Lie conformal superalgebra is given by the language of formal distribution (see [14]).

Remark 2.8. Obviously, Examples 2.3 and 2.4 are r -dim linear Lie conformal superalgebras. Example 2.5 is an r -dim r -linear Lie conformal superalgebra.

Next, we introduce the definitions of Novikov superalgebra and Novikov-Poisson superalgebra.

Definition 2.9. A (left) Novikov superalgebra (A, \circ) is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with an operation “ \circ ” satisfying the following axioms: for $a \in A_\alpha, b \in A_\beta, c \in A_\gamma$,

$$(a \circ b) \circ c = (-1)^{\beta\gamma}(a \circ c) \circ b, \\ (a \circ b) \circ c - a \circ (b \circ c) = (-1)^{\alpha\beta}[(b \circ a) \circ c - b \circ (a \circ c)].$$

When $A_{\bar{1}} = 0$, we call A a (left) Novikov algebra.

A Novikov-Poisson superalgebra (A, \circ, \cdot) is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with two operations “ \circ ” and “ \cdot ”, where (A, \circ) is a Novikov superalgebra, (A, \cdot) is a commutative associative superalgebra, and they satisfy the following axioms:

$$(a \circ b) \cdot c - a \circ (b \cdot c) = (-1)^{\alpha\beta}((b \circ a) \cdot c - b \circ (a \cdot c)), \\ (a \cdot b) \circ c = a \cdot (b \circ c),$$

for $a \in A_\alpha, b \in A_\beta$ and $c \in A$. When $A_{\bar{1}} = \{0\}$, it is called a Novikov-Poisson algebra.

Remark 2.10. Novikov algebra was essentially stated in [12] that it corresponds to a certain Hamiltonian operator. Such an algebraic structure appeared in [3] from the point of view of Poisson structures of hydrodynamic type. The name “Novikov algebra” was given by Osborn [16]. Moreover, Novikov-Poisson algebra is introduced by Xu in [18] to study Novikov algebra, and the author also studied the constructions of Novikov-Poisson algebra in [19].

Definition 2.11. An r -dim right Novikov conformal superalgebra is a left $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{C}[\mathbf{T}]$ -module $R = R_{\bar{0}} \oplus R_{\bar{1}}$ endowed with a λ -product $a_\lambda b$ which defines a \mathbb{C} -linear map $R \otimes R \rightarrow R[\lambda]$, where $R[\lambda] = \mathbb{C}[\lambda] \otimes R = \mathbb{C}[\lambda_1, \dots, \lambda_r] \otimes R$, subject to the following axioms:

$$(T_i a_\lambda b) = -\lambda_i(a_\lambda b), \quad (a_\lambda T_i b) = (T_i + \lambda_i)(a_\lambda b), \tag{2.7}$$

$$a_\lambda(b_\mu c) - (a_\lambda b)_{\lambda+\mu} c = (-1)^{\beta\gamma}(a_\lambda(c_{-\mu-\mathbf{T}} b) - (a_\lambda c)_{-\mu-\mathbf{T}} b), \tag{2.8}$$

$$a_\lambda(b_\mu c) = (-1)^{\alpha\beta} b_\mu(a_\lambda c). \tag{2.9}$$

for $a \in R_\alpha$, $b \in R_\beta$ and $c \in R_\gamma$. When $R_{\bar{1}} = \{0\}$, R is called an r -dim right Novikov conformal algebra. If $r = 1$, it is called a right Novikov conformal superalgebra.

An r -dim (left) Novikov conformal superalgebra is a left $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{C}[\mathbf{T}]$ -module $R = R_{\bar{0}} \oplus R_{\bar{1}}$ endowed with a λ -product $a_\lambda b$ which defines a \mathbb{C} -linear map $R \otimes R \rightarrow R[\lambda]$, where $R[\lambda] = \mathbb{C}[\lambda] \otimes R = \mathbb{C}[\lambda_1, \dots, \lambda_r] \otimes R$, subject to the following axioms:

$$(T_i a_\lambda b) = -\lambda_i(a_\lambda b), \quad (a_\lambda T_i b) = (T_i + \lambda_i)(a_\lambda b), \tag{2.10}$$

$$(a_\lambda b)_{\lambda+\mu} c - a_\lambda(b_\mu c) = (-1)^{\alpha\beta}((b_\mu a)_{\lambda+\mu} c - b_\mu(a_\lambda c)), \tag{2.11}$$

$$(a_\lambda b)_{\lambda+\mu} c = (-1)^{\beta\gamma}(a_\lambda c)_{-\mu-\mathbf{T}} b, \tag{2.12}$$

for $a \in R_\alpha$, $b \in R_\beta$ and $c \in R_\gamma$. When $R_{\bar{1}} = \{0\}$, R is called an r -dim (left) Novikov conformal algebra. If $r = 1$, it is called a (left) Novikov conformal superalgebra.

Remark 2.12. The definition of (left) Novikov conformal algebra is introduced in [13]. Obviously, if $(R, \cdot_\lambda \cdot)$ is a left (resp. right) Novikov conformal superalgebra, then (R, \circ_λ) is a right (resp. left) Novikov conformal superalgebra with $a \circ_\lambda b = (-1)^{\alpha\beta} b_{-\lambda-\mathbf{T}} a$ for any $a \in R_\alpha$, $b \in R_\beta$.

Example 2.13. If (A, \circ) is a Novikov superalgebra, then $R = \mathbb{C}[T]A$ is a Novikov conformal superalgebra with the following λ -product:

$$a_\lambda b = a \circ b, \quad a, b \in A.$$

Let $R = R_{\bar{0}} \oplus R_{\bar{1}} = \mathbb{C}[T]a \oplus \mathbb{C}[T]b$ be a free $\mathbb{C}[T]$ -module generated by a , b . Then, R is a Novikov conformal superalgebra with the following λ -product:

$$a_\lambda a = (\lambda + T + C_1)a, \quad a_\lambda b = (\lambda + T + C_1)b, \quad b_\lambda a = (\lambda + T + C_1)b, \quad b_\lambda b = 0,$$

where $C_1 \in \mathbb{C}$.

Next, we present a theorem about the classification of torsion-free Novikov conformal algebras of rank 1 in [13].

Proposition 2.14. Let $R = \mathbb{C}[T]x$ be a Novikov conformal algebra that is free of rank one as a $\mathbb{C}[T]$ -module. Then R is isomorphic to the Novikov conformal algebra with λ -product which is one of three cases as follows:

- (i) $x_\lambda x = 0$;
- (ii) $x_\lambda x = x$, i.e. R is associative;
- (iii) $x_\lambda x = (\lambda + T + a)x$, for any $a \in \mathbb{C}$.

Next, we also introduce a construction of Novikov conformal superalgebras from Novikov-Poisson superalgebras (see [13]).

Proposition 2.15. *If (V, \circ, \cdot) is a Novikov-Poisson superalgebra, then $R = \mathbb{C}[T]V$ is a Novikov conformal superalgebra with the following λ -product:*

$$a_\lambda b = (\lambda + T)(a \cdot b) + a \circ b, \quad a \in V_\alpha, \quad b \in V_\beta. \tag{2.13}$$

In addition, it is easy to check the following proposition.

Proposition 2.16. *If $(R, \cdot_\lambda \cdot)$ is an r -dim left (resp. right) Novikov conformal superalgebra, then $(R, [\cdot_\lambda \cdot])$ is an r -dim Lie conformal superalgebra with the following λ -bracket:*

$$[a_\lambda b] = a_\lambda b - (-1)^{\alpha\beta}(b_{-\lambda-T}a), \quad a \in R_\alpha, \quad b \in R_\beta.$$

Definition 2.17. (see [17]) A *super Gel'fand-Dorfman bialgebra* $(\mathfrak{A}, [\cdot, \cdot], \circ)$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $\mathfrak{A} = \mathfrak{A}_{\bar{0}} \oplus \mathfrak{A}_{\bar{1}}$ with two algebraic operations $[\cdot, \cdot]$ and \circ such that $(\mathfrak{A}, [\cdot, \cdot])$ forms a Lie superalgebra, (\mathfrak{A}, \circ) forms a Novikov superalgebra and the following compatibility condition holds:

$$\begin{aligned} [a \circ b, c] + [a, b] \circ c - a \circ [b, c] - (-1)^{\beta\gamma}[a \circ c, b] \\ - (-1)^{\beta\gamma}[a, c] \circ b = 0, \end{aligned} \tag{2.14}$$

for $a \in \mathfrak{A}$, $b \in \mathfrak{A}_\beta$, and $c \in \mathfrak{A}_\gamma$. When $\mathfrak{A}_{\bar{1}} = 0$, it is called a *Gel'fand-Dorfman bialgebra* (see also [17]).

A characterization of linear Lie conformal superalgebra is as follows.

Theorem 2.18. (see [17],[12]) *A linear Lie conformal superalgebra $R = \mathbb{C}[T]V$ is equivalent to that V is a super Gel'fand-Dorfman bialgebra.*

3. A characterization of i -linear Lie conformal superalgebra in r dimension

In this section, we will give characterizations of r -dim i -linear Lie conformal superalgebra and r -dim linear Lie conformal superalgebra for $i \in \{1, \dots, r\}$.

Theorem 3.1. *An r -dim i -linear Lie conformal superalgebra $R = \mathbb{C}[\mathbf{T}]V$ is equivalent to that $\bar{R}_i = \mathbb{C}[\bar{\mathbf{T}}_i]V$ is equipped with an $(r - 1)$ -dim Novikov conformal superalgebra structure $(\bar{R}_i, \cdot_{\bar{\lambda}} \cdot)$, an $(r - 1)$ -dim Lie conformal superalgebra structure $(\bar{R}_i, [\cdot_{\bar{\lambda}} \cdot])$ and they satisfying the following condition:*

$$[(a_{-\bar{\mu}_i - \bar{\mathbf{T}}_i} b)_{-\bar{\lambda}_i - \bar{\mathbf{T}}_i} c] + [a_{-\bar{\mu}_i - \bar{\mathbf{T}}_i} b]_{-\bar{\lambda}_i - \bar{\mathbf{T}}_i} c - a_{-\bar{\lambda}_i - \bar{\mu}_i - \bar{\mathbf{T}}_i} [b_{-\bar{\lambda}_i - \bar{\mathbf{T}}_i} c] \tag{3.15}$$

$$-(-1)^{\beta\gamma}[(a_{-\bar{\lambda}_i-\bar{\mathbf{T}}_i}c)_{-\bar{\mu}_i-\bar{\mathbf{T}}_i}b] - (-1)^{\beta\gamma}[a_{-\bar{\lambda}_i-\bar{\mathbf{T}}_i}c]_{-\bar{\mu}_i-\bar{\mathbf{T}}_i}b = 0,$$

for $a \in V$, $b \in V_\beta$, $c \in V_\gamma$. We call such $\bar{R}_i = \mathbb{C}[\bar{\mathbf{T}}_i]V$ an $(r - 1)$ -dim super Gel'fand-Dorfman conformal bialgebra. Denote it by $(\bar{R}_i, [\cdot_{\bar{\lambda}\cdot}], \cdot_{\bar{\lambda}\cdot})$.

Proof. Obviously, we only need to prove this theorem for one special case, i.e. $i = r$. Other cases can be obtained similarly.

For convenience, set $\bar{\lambda} = \bar{\lambda}_r$, $\bar{\mathbf{T}} = \bar{\mathbf{T}}_r$, $\bar{\mu} = \bar{\mu}_r$ and $\bar{R} = \bar{R}_r = \mathbb{C}[\bar{\mathbf{T}}_r]V$.

By (2.5), we can assume that

$$[a_\lambda b] = T_r(a \circ_{\bar{\lambda}} b) + \lambda_r(a *_{\bar{\lambda}} b) + [a_{\bar{\lambda}}b], \quad a, b \in V. \tag{3.16}$$

Here, $\circ_{\bar{\lambda}}$, $*_{\bar{\lambda}}$ and $[\cdot_{\bar{\lambda}\cdot}]$ are three \mathbb{C} -bilinear maps from $\bar{R} \otimes \bar{R} \rightarrow \mathbb{C}[\bar{\lambda}] \otimes \bar{R}$ where $\bar{R} = \mathbb{C}[\bar{\mathbf{T}}]V$.

Since $[T_i a_\lambda b] = -\lambda_i[a_\lambda b]$ and $[a_\lambda T_i b] = (\lambda_i + T_i)[a_\lambda b]$ for $1 \leq i \leq r - 1$, by comparing the coefficients of T_r , λ_r and $T_r^0 \lambda_r^0$, we can obtain that

$$(T_i a) \circ_{\bar{\lambda}} b = -\lambda_i(a \circ_{\bar{\lambda}} b), \quad a \circ_{\bar{\lambda}} (T_i b) = (\lambda_i + T_i)(a \circ_{\bar{\lambda}} b), \tag{3.17}$$

$$(T_i a) *_{\bar{\lambda}} b = -\lambda_i(a *_{\bar{\lambda}} b), \quad a *_{\bar{\lambda}} (T_i b) = (\lambda_i + T_i)(a *_{\bar{\lambda}} b), \tag{3.18}$$

$$[(T_i a)_{\bar{\lambda}} b] = -\lambda_i[a_{\bar{\lambda}} b], \quad [a_{\bar{\lambda}} T_i b] = (\lambda_i + T_i)[a_{\bar{\lambda}} b], \tag{3.19}$$

for $1 \leq i \leq r - 1$.

For any $a \in V_\alpha$, $b \in V_\beta$, using (2.5), (2.2) becomes:

$$\begin{aligned} & T_r(a \circ_{\bar{\lambda}} b) + \lambda_r(a *_{\bar{\lambda}} b) + [a_{\bar{\lambda}} b] \\ &= -(-1)^{\alpha\beta}(T_r(b \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) - (\lambda_r + T_r)(b *_{-\bar{\lambda}-\bar{\mathbf{T}}} a) + [b_{-\bar{\lambda}-\bar{\mathbf{T}}} a]). \end{aligned}$$

Since R is a free $\mathbb{C}[\mathbf{T}]$ -module, by comparing the coefficients of T_r , λ_r and $T_r^0 \lambda_r^0$, we get

$$a *_{\bar{\lambda}} b = (-1)^{\alpha\beta} b *_{-\bar{\lambda}-\bar{\mathbf{T}}} a, \tag{3.20}$$

$$b *_{-\bar{\lambda}-\bar{\mathbf{T}}} a = b \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a + (-1)^{\alpha\beta} a \circ_{\bar{\lambda}} b, \tag{3.21}$$

$$[a_{\bar{\lambda}} b] = -(-1)^{\alpha\beta} [b_{-\bar{\lambda}-\bar{\mathbf{T}}} a]. \tag{3.22}$$

By (3.20) and (3.21), we obtain

$$a *_{\bar{\lambda}} b = a \circ_{\bar{\lambda}} b + (-1)^{\alpha\beta} b \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a. \tag{3.23}$$

Obviously, (3.20) can be obtained from (3.23). Moreover, (3.18) follows from (3.17) and (3.23).

Next, we consider (2.3). For any $a \in V_\alpha$, $b \in V_\beta$, $c \in V_\gamma$, we have

$$\begin{aligned}
 & [a_\lambda[b_\mu c]] \\
 = & [a_\lambda(T_r(b \circ_{\bar{\mu}} c) + \mu_r(b *_{\bar{\mu}} c) + [b_{\bar{\mu}}c])] \\
 = & (\lambda_r + T_r)[a_\lambda(b \circ_{\bar{\mu}} c)] + \mu_r[a_\lambda(b *_{\bar{\mu}} c)] + [a_\lambda[b_{\bar{\mu}}c]] \\
 = & (\lambda_r + T_r)(T_r(a \circ_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)) + \lambda_r(a *_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)) + [a_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)]) \\
 & + \mu_r(T_r(a \circ_{\bar{\lambda}}(b *_{\bar{\mu}} c)) + \lambda_r(a *_{\bar{\lambda}}(b *_{\bar{\mu}} c)) + [a_{\bar{\lambda}}(b *_{\bar{\mu}} c)]) \\
 & + T_r(a \circ_{\bar{\lambda}}[b_{\bar{\mu}}c]) + \lambda_r(a *_{\bar{\lambda}}[b_{\bar{\mu}}c]) + [a_{\bar{\lambda}}[b_{\bar{\mu}}c]] \\
 = & T_r^2(a \circ_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)) + \lambda_r T_r(a \circ_{\bar{\lambda}}(b \circ_{\bar{\mu}} c) + a *_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)) \\
 & + \lambda_r \mu_r(a *_{\bar{\lambda}}(b *_{\bar{\mu}} c)) + \mu_r T_r(a \circ_{\bar{\lambda}}(b *_{\bar{\mu}} c)) + \lambda_r^2(a *_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)) \\
 & + \lambda_r([a_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)] + a *_{\bar{\lambda}}[b_{\bar{\mu}}c]) + T_r([a_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)] + a \circ_{\bar{\lambda}}[b_{\bar{\mu}}c]) \\
 & + \mu_r[a_{\bar{\lambda}}(b *_{\bar{\mu}} c)] + [a_{\bar{\lambda}}[b_{\bar{\mu}}c]].
 \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
 & [b_\mu[a_\lambda c]] \\
 = & T_r^2(b \circ_{\bar{\mu}}(a \circ_{\bar{\lambda}} c)) + \mu_r T_r(b \circ_{\bar{\mu}}(a \circ_{\bar{\lambda}} c) + b *_{\bar{\mu}}(a \circ_{\bar{\lambda}} c)) \\
 & + \lambda_r \mu_r(b *_{\bar{\mu}}(a *_{\bar{\lambda}} c)) + \lambda_r T_r(b \circ_{\bar{\mu}}(a *_{\bar{\lambda}} c)) + \mu_r^2(b *_{\bar{\mu}}(a \circ_{\bar{\lambda}} c)) \\
 & + \mu_r([b_{\bar{\mu}}(a \circ_{\bar{\lambda}} c)] + b *_{\bar{\mu}}[a_{\bar{\lambda}}c]) + T_r([b_{\bar{\mu}}(a \circ_{\bar{\lambda}} c)] + b \circ_{\bar{\mu}}[a_{\bar{\lambda}}c]) \\
 & + \lambda_r[b_{\bar{\mu}}(a *_{\bar{\lambda}} c)] + [b_{\bar{\mu}}[a_{\bar{\lambda}}c]].
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 & [[a_\lambda b]_{\lambda+\mu} c] \\
 = & [(T_r(a \circ_{\bar{\lambda}} b) + \lambda_r(a *_{\bar{\lambda}} b) + [a_{\bar{\lambda}}b])_{\lambda+\mu} c] \\
 = & (-\lambda_r - \mu_r)[(a \circ_{\bar{\lambda}} b)_{\lambda+\mu} c] + \lambda_r[(a *_{\bar{\lambda}} b)_{\lambda+\mu} c] + [[a_{\bar{\lambda}}b]_{\lambda+\mu} c] \\
 = & (-\lambda_r - \mu_r)(T_r((a \circ_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c) + (\lambda_r + \mu_r)((a \circ_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c) + [(a \circ_{\bar{\lambda}} b)_{\bar{\lambda}+\bar{\mu}} c]) \\
 & + \lambda_r(T_r((a *_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c) + (\lambda_r + \mu_r)((a *_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c) + [(a *_{\bar{\lambda}} b)_{\bar{\lambda}+\bar{\mu}} c]) \\
 & + T_r([a_{\bar{\lambda}}b] \circ_{\bar{\lambda}+\bar{\mu}} c) + (\lambda_r + \mu_r)([a_{\bar{\lambda}}b] *_{\bar{\lambda}+\bar{\mu}} c) + [[a_{\bar{\lambda}}b]_{\bar{\lambda}+\bar{\mu}} c]) \\
 = & \lambda_r^2((a *_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c - (a \circ_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c) + \lambda_r T_r((a *_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c - (a \circ_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c) \\
 & + \lambda_r \mu_r((a *_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c - 2(a \circ_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c) - \mu_r T_r(a \circ_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c \\
 & - \mu_r^2(a \circ_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c + \lambda_r([(a *_{\bar{\lambda}} b)_{\bar{\lambda}+\bar{\mu}} c] - [(a \circ_{\bar{\lambda}} b)_{\bar{\lambda}+\bar{\mu}} c] + [a_{\bar{\lambda}}b] *_{\bar{\lambda}+\bar{\mu}} c) \\
 & + T_r([a_{\bar{\lambda}}b] \circ_{\bar{\lambda}+\bar{\mu}} c) + \mu_r([a_{\bar{\lambda}}b] *_{\bar{\lambda}+\bar{\mu}} c - [(a \circ_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c]) + [[a_{\bar{\lambda}}b]_{\bar{\lambda}+\bar{\mu}} c].
 \end{aligned}$$

By (2.3), comparing the coefficients of T_r^2 , $\lambda_r T_r$, $\mu_r T_r$, $\lambda_r \mu_r$, λ_r^2 , μ_r^2 , T_r , λ_r , μ_r and 1, we obtain the following equalities:

$$a \circ_{\bar{\lambda}}(b \circ_{\bar{\mu}} c) - (-1)^{\alpha\beta} b \circ_{\bar{\mu}}(a \circ_{\bar{\lambda}} c) = 0, \tag{3.24}$$

$$a \circ_{\bar{\lambda}}(b \circ_{\bar{\mu}} c) + a *_{\bar{\lambda}}(b \circ_{\bar{\mu}} c) + (a \circ_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c - (a *_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c \tag{3.25}$$

$$-(-1)^{\alpha\beta} b \circ_{\bar{\mu}}(a *_{\bar{\lambda}} c) = 0,$$

$$a \circ_{\bar{\lambda}}(b *_{\bar{\mu}} c) + (a \circ_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c - (-1)^{\alpha\beta}(b \circ_{\bar{\mu}}(a \circ_{\bar{\lambda}} c)) \tag{3.26}$$

$$+b *_{\bar{\mu}} (a \circ_{\bar{\lambda}} c) = 0,$$

$$a *_{\bar{\lambda}} (b *_{\bar{\mu}} c) + 2(a \circ_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c - (a *_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c \quad (3.27)$$

$$-(-1)^{\alpha\beta} b *_{\bar{\mu}} (a *_{\bar{\lambda}} c) = 0,$$

$$a *_{\bar{\lambda}} (b \circ_{\bar{\mu}} c) - (a *_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c + (a \circ_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c = 0, \quad (3.28)$$

$$(a \circ_{\bar{\lambda}} b) *_{\bar{\lambda}+\bar{\mu}} c - (-1)^{\alpha\beta} b *_{\bar{\mu}} (a \circ_{\bar{\lambda}} c) = 0, \quad (3.29)$$

$$[a_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)] + a \circ_{\bar{\lambda}} [b_{\bar{\mu}}c] - [a_{\bar{\lambda}}b] \circ_{\bar{\lambda}+\bar{\mu}} c \quad (3.30)$$

$$-(-1)^{\alpha\beta} ([b_{\bar{\mu}}(a \circ_{\bar{\lambda}} c)] + b \circ_{\bar{\mu}} [a_{\bar{\lambda}}c]) = 0,$$

$$[a_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)] + a *_{\bar{\lambda}} [b_{\bar{\mu}}c] + [(a \circ_{\bar{\lambda}} b)_{\bar{\lambda}+\bar{\mu}}c] - [(a *_{\bar{\lambda}} b)_{\bar{\lambda}+\bar{\mu}}c] \quad (3.31)$$

$$-[a_{\bar{\lambda}}b] *_{\bar{\lambda}+\bar{\mu}} c - (-1)^{\alpha\beta} [b_{\bar{\mu}}(a *_{\bar{\lambda}} c)] = 0,$$

$$[a_{\bar{\lambda}}(b *_{\bar{\mu}} c)] + [(a \circ_{\bar{\lambda}} b)_{\bar{\lambda}+\bar{\mu}}c] - [a_{\bar{\lambda}}b] *_{\bar{\lambda}+\bar{\mu}} c \quad (3.32)$$

$$-(-1)^{\alpha\beta} (b *_{\bar{\mu}} [a_{\bar{\lambda}}c] + [b_{\bar{\mu}}(a \circ_{\bar{\lambda}} c)]) = 0,$$

$$[a_{\bar{\lambda}}[b_{\bar{\mu}}c]] - [[a_{\bar{\lambda}}b]_{\bar{\lambda}+\bar{\mu}}c] - (-1)^{\alpha\beta} [b_{\bar{\mu}}[a_{\bar{\lambda}}c]] = 0. \quad (3.33)$$

By (3.19), (3.22) and (3.33), we know that $(\bar{R}, [\cdot]_{\bar{\lambda}})$ is an $(r-1)$ -dim Lie conformal superalgebra.

We first consider (3.26). Taking (3.23) into (3.26) and according to (3.24), we can get

$$a \circ_{\bar{\lambda}} (b_{\bar{\mu}}c) - (a \circ_{\bar{\lambda}} b)_{\bar{\lambda}+\bar{\mu}}c = (-1)^{\beta\gamma} (a \circ_{\bar{\lambda}} (c_{-\bar{\mu}-\bar{\mathbf{T}}}b) - (a_{\bar{\lambda}}c)_{-\bar{\mu}-\bar{\mathbf{T}}}b). \quad (3.34)$$

By (3.17), (3.24) and (3.34), we know that $(\bar{R}, \cdot \circ_{\bar{\lambda}} \cdot)$ is an $(r-1)$ -dim right Novikov conformal superalgebra.

Next, we show that when (3.24) and (3.34) are satisfied, (3.25), (3.27), (3.28) and (3.29) holds. Here, we only prove that (3.27) holds. Other equalities

can be checked similarly. The right side of (3.27) is equal to the following:

$$\begin{aligned}
 & a \circ_{\bar{\lambda}} (b *_{\bar{\mu}} c) + (-1)^{\alpha(\beta+\gamma)} (b *_{\bar{\mu}} c) \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a + 2(a \circ_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c \\
 & + 2(-1)^{(\alpha+\beta)\gamma} c \circ_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}} (a \circ_{\bar{\lambda}} b) - (a *_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c \\
 & - (-1)^{\gamma(\alpha+\beta)} c \circ_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}} (a *_{\bar{\lambda}} b) - (-1)^{\alpha\beta} b \circ_{\bar{\mu}} (a *_{\bar{\lambda}} c) \\
 & - (-1)^{\beta\gamma} (a *_{\bar{\lambda}} c) \circ_{-\bar{\mu}-\bar{\mathbf{T}}} b \\
 & = a \circ_{\bar{\lambda}} (b \circ_{\bar{\mu}} c) + (-1)^{\beta\gamma} a \circ_{\bar{\lambda}} (c \circ_{-\bar{\mu}-\bar{\mathbf{T}}} b) + (-1)^{\alpha(\beta+\gamma)} (b \circ_{\bar{\mu}} c) \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a \\
 & + (-1)^{\alpha(\beta+\gamma)+\beta\gamma} (c \circ_{-\bar{\mu}-\bar{\mathbf{T}}} b) \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a + 2(a \circ_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c \\
 & + 2(-1)^{(\alpha+\beta)\gamma} c \circ_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}} (a \circ_{\bar{\lambda}} b) - (a \circ_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c - (-1)^{\alpha\beta} (b \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) \circ_{\bar{\lambda}+\bar{\mu}} c \\
 & - (-1)^{\gamma(\alpha+\beta)} c \circ_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}} (a \circ_{\bar{\lambda}} b) - (-1)^{\gamma(\alpha+\beta)+\alpha\beta} c \circ_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}} (b \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) \\
 & - (-1)^{\alpha\beta} b \circ_{\bar{\mu}} (a \circ_{\bar{\lambda}} c) - (-1)^{\alpha(\beta+\gamma)} b \circ_{\bar{\mu}} (c \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) \\
 & - (-1)^{\beta\gamma} (a \circ_{\bar{\lambda}} c) \circ_{-\bar{\mu}-\bar{\mathbf{T}}} b - (-1)^{\gamma(\alpha+\beta)} (c \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) \circ_{-\bar{\mu}-\bar{\mathbf{T}}} b \\
 & = (-1)^{(\alpha+\beta)\gamma} (c \circ_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}} (a \circ_{\bar{\lambda}} b) - (c \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) \circ_{-\bar{\mu}-\bar{\mathbf{T}}} b) \\
 & - (-1)^{\alpha\beta} c \circ_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}} (b \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) + (-1)^{\alpha\beta} (c \circ_{-\bar{\mu}-\bar{\mathbf{T}}} b) \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) \\
 & - (a \circ_{\bar{\lambda}} (b \circ_{\bar{\mu}} c) - (a \circ_{\bar{\lambda}} b) \circ_{\bar{\lambda}+\bar{\mu}} c - (-1)^{\beta\gamma} a \circ_{\bar{\lambda}} (c \circ_{-\bar{\mu}-\bar{\mathbf{T}}} b) \\
 & + (-1)^{\beta\gamma} (a \circ_{\bar{\lambda}} c) \circ_{-\bar{\mu}-\bar{\mathbf{T}}} b) + (-1)^{\alpha\beta} (b \circ_{\bar{\mu}} (a \circ_{\bar{\lambda}} c) - (b \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) \circ_{\bar{\lambda}+\bar{\mu}} c \\
 & - (-1)^{\alpha\gamma} b \circ_{\bar{\mu}} (c \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) + (-1)^{\alpha\gamma} (b \circ_{\bar{\mu}} c) \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) \\
 & = 0.
 \end{aligned}$$

Finally, we show that (3.31) and (3.32) can be obtained from (3.30). Here, we only prove that (3.31) holds. (3.32) can be checked similarly. Using (3.22) and (3.30), the right side of (3.31) is as follows:

$$\begin{aligned}
 & [a_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)] + a \circ_{\bar{\lambda}} [b_{\bar{\mu}}c] + (-1)^{\alpha(\beta+\gamma)} [b_{\bar{\mu}}c] \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a + [(a \circ_{\bar{\lambda}} b)_{\bar{\lambda}+\bar{\mu}}c] \\
 & - [(a \circ_{\bar{\lambda}} b)_{\bar{\lambda}+\bar{\mu}}c] - (-1)^{\alpha\beta} [(b \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a)_{\bar{\lambda}+\bar{\mu}}c] - [a_{\bar{\lambda}}b] \circ_{\bar{\lambda}+\bar{\mu}} c \\
 & - (-1)^{(\alpha+\beta)\gamma} c \circ_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}} [a_{\bar{\lambda}}b] - (-1)^{\alpha\beta} [b_{\bar{\mu}}(a \circ_{\bar{\lambda}} c)] - (-1)^{\alpha(\beta+\gamma)} [b_{\bar{\mu}}(c \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a)] \\
 & = ([a_{\bar{\lambda}}(b \circ_{\bar{\mu}} c)] + a \circ_{\bar{\lambda}} [b_{\bar{\mu}}c] - [a_{\bar{\lambda}}b] \circ_{\bar{\lambda}+\bar{\mu}} c - (-1)^{\alpha\beta} [b_{\bar{\mu}}(a \circ_{\bar{\lambda}} c)]) \\
 & + (-1)^{\alpha(\beta+\gamma)} [b_{\bar{\mu}}c] \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a + (-1)^{\alpha\beta+(\alpha+\beta)\gamma} [c_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}}(b \circ_{\bar{\lambda}-\bar{\mathbf{T}}} a)] \\
 & + (-1)^{(\alpha+\beta)\gamma+\alpha\beta} c \circ_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}} [b_{-\bar{\lambda}-\bar{\mathbf{T}}}a] - (-1)^{\alpha(\beta+\gamma)} [b_{\bar{\mu}}(c \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a)] \\
 & = -(-1)^{\alpha(\beta+\gamma)} ([b_{\bar{\mu}}(c \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a)] + b \circ_{\bar{\mu}} [c_{-\bar{\lambda}-\bar{\mathbf{T}}}a] - [b_{\bar{\mu}}c] \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) \\
 & - (-1)^{\beta\gamma} [c_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}}(b \circ_{\bar{\lambda}-\bar{\mathbf{T}}} a)] - (-1)^{\beta\gamma} c \circ_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}} [b_{-\bar{\lambda}-\bar{\mathbf{T}}}a] \\
 & = 0.
 \end{aligned}$$

Therefore, from the above discussion, the Lie conformal superalgebra structure of R is determined by $(\bar{R}, [\cdot_{\bar{\lambda}}], \cdot \circ_{\bar{\lambda}} \cdot)$ where $(\bar{R}, [\cdot_{\bar{\lambda}}])$ is an $(r - 1)$ -dim Lie conformal superalgebra, $(\bar{R}, \cdot \circ_{\bar{\lambda}} \cdot)$ is an $(r - 1)$ -dim right Novikov conformal superalgebra, and they satisfy (3.30).

For the unity with Theorem 2.18, set

$$a_{\bar{\lambda}}b = (-1)^{\alpha\beta} (b \circ_{-\bar{\lambda}-\bar{\mathbf{T}}} a) \tag{3.35}$$

for $a \in \bar{R}_\alpha, b \in \bar{R}_\beta$. Therefore, $(\bar{R}, \cdot_{\bar{\lambda}})$ is an $(r - 1)$ -dim Novikov conformal algebra. By (3.30), using (3.35), we can obtain the following equality:

$$\begin{aligned}
 & [(c_{-\bar{\mu}-\bar{\mathbf{T}}}b)_{-\bar{\lambda}-\bar{\mathbf{T}}}a] + [c_{-\bar{\mu}-\bar{\mathbf{T}}}b]_{-\bar{\lambda}-\bar{\mathbf{T}}}a - c_{-\bar{\lambda}-\bar{\mu}-\bar{\mathbf{T}}}[b_{-\bar{\lambda}-\bar{\mathbf{T}}}a] \\
 & - (-1)^{\alpha\beta}[(c_{-\bar{\lambda}-\bar{\mathbf{T}}}a)_{-\bar{\mu}-\bar{\mathbf{T}}}b] - (-1)^{\alpha\beta}[c_{-\bar{\lambda}-\bar{\mathbf{T}}}a]_{-\bar{\mu}-\bar{\mathbf{T}}}b = 0,
 \end{aligned}
 \tag{3.36}$$

for $a \in \bar{R}_\alpha, b \in \bar{R}_\beta, c \in \bar{R}$. Therefore, by (3.36), we obtain that $(\bar{R}, [\cdot_{\bar{\lambda}}], \cdot_{\bar{\lambda}})$ is an $(r - 1)$ -dim super Gel'fand-Dorfman conformal bialgebra.

Thus, the theorem holds when $i = r$. So, the proof is finished. ■

By Theorem 3.1, we know that one $(r - 1)$ -dim super Gel'fand-Dorfman conformal bialgebra can correspond to r Lie conformal superalgebras in r dimension. In the following, we only consider the r -dim r -linear Lie conformal superalgebras.

Next, we give two remarks about r -dim super Gel'fand-Dorfman conformal bialgebras.

Remark 3.2. In fact, there are some relations between r -dim super Gel'fand-Dorfman conformal bialgebra and super Gel'fand-Dorfman bialgebra. If $(R = \mathbb{C}[\mathbf{T}]V, [\cdot_{\lambda}], \cdot_{\lambda})$ is an r -dim super Gel'fand-Dorfman conformal bialgebra, then $(\text{Lie}R, [\cdot, \cdot], \circ)$ is a super Gel'fand-Dorfman bialgebra with the Lie bracket and the Novikov algebraic operation as follows:

$$[a_{\mathbf{m}}, b_{\mathbf{n}}] = \sum_{\mathbf{j} \in \mathbb{Z}_+^r} \binom{\mathbf{m}}{\mathbf{j}} (a_{(\mathbf{j})}b)_{\mathbf{m}+\mathbf{n}-\mathbf{j}},
 \tag{3.37}$$

$$a_{\mathbf{m}} \circ b_{\mathbf{n}} = \sum_{\mathbf{j} \in \mathbb{Z}_+^r} \binom{\mathbf{m}}{\mathbf{j}} (a_{\mathbf{j}}b)_{\mathbf{m}+\mathbf{n}-\mathbf{j}},
 \tag{3.38}$$

where $[a_{\lambda}b] = \sum_{\mathbf{m} \in \mathbb{Z}_+^r} \lambda^{(\mathbf{m})} a_{(\mathbf{m})}b$ and $a_{\lambda}b = \sum_{\mathbf{m} \in \mathbb{Z}_+^r} \lambda^{(\mathbf{m})} a_{\mathbf{m}}b$. The proof is similar to that in Section 2.7 in [14].

Remark 3.3. Obviously, any r -dim Novikov conformal superalgebra $(R = \mathbb{C}[\mathbf{T}]V, \cdot_{\lambda})$ with the trivial r -dim Lie conformal superalgebra can construct a super Gel'fand-Dorfman conformal bialgebra. Therefore, by Theorem 3.1, given an r -dim Novikov conformal superalgebra $(R = \mathbb{C}[\mathbf{T}]V, \cdot_{\lambda})$, we can obtain an $(r + 1)$ -dim Lie conformal superalgebra $(\tilde{R} = \mathbb{C}[T_1, \dots, T_r, T_{r+1}]V, [\cdot_{\tilde{\lambda}}])$ with the λ -bracket as follows:

$$[a_{\tilde{\lambda}}b] = T_{r+1}((-1)^{\alpha\beta}b_{-\lambda-\mathbf{T}}a) + \lambda_{r+1}((-1)^{\alpha\beta}b_{-\lambda-\mathbf{T}}a + a_{\lambda}b),
 \tag{3.39}$$

where $\tilde{\lambda} = (\lambda_1, \dots, \lambda_r, \lambda_{r+1}), a \in V_\alpha, b \in V_\beta$.

Proposition 3.4. Let (R, \cdot_{λ}) be an r -dim Novikov conformal superalgebra. Then, $(R, [\cdot_{\lambda}], \cdot_{\lambda})$ is an r -dim super Gel'fand-Dorfman conformal bialgebra with the λ -bracket given as follows:

$$[a_{\lambda}b] = a_{\lambda}b - (-1)^{\alpha\beta}b_{-\lambda-\mathbf{T}}a, \quad a \in R_\alpha, \quad b \in R_\beta.
 \tag{3.40}$$

Proof. By Proposition 2.16, we only need to check that (3.15) holds. By (3.40), the right side of (3.15) is the following:

$$\begin{aligned}
 & (a_{-\mu-\mathbf{T}b})_{-\lambda-\mathbf{T}c} - (-1)^{(\alpha+\beta)\gamma}c_\lambda(a_{-\mu-\mathbf{T}b}) + (a_{-\mu-\mathbf{T}b})_{-\lambda-\mathbf{T}c} \\
 & - (-1)^{\alpha\beta}(b_\mu a)_{-\lambda-\mathbf{T}c} - a_{-\lambda-\mu-\mathbf{T}}(b_{-\lambda-\mathbf{T}c}) + (-1)^{\beta\gamma}a_{-\lambda-\mu-\mathbf{T}}(c_\lambda b) \\
 & - (-1)^{\beta\gamma}(a_{-\lambda-\mathbf{T}c})_{-\mu-\mathbf{T}b} + (-1)^{\alpha\beta}b_\mu(a_{-\lambda-\mathbf{T}c}) \\
 & - (-1)^{\beta\gamma}(a_{-\lambda-\mathbf{T}c})_{-\mu-\mathbf{T}b} + (-1)^{(\alpha+\beta)\gamma}(c_\lambda a)_{-\mu-\mathbf{T}b} \\
 = & ((a_{-\mu-\mathbf{T}b})_{-\lambda-\mathbf{T}c} - a_{-\lambda-\mu-\mathbf{T}}(b_{-\lambda-\mathbf{T}c}) - (-1)^{\alpha\beta}(b_\mu a)_{-\lambda-\mathbf{T}c} \\
 & + (-1)^{\alpha\beta}b_\mu(a_{-\lambda-\mathbf{T}c})) - (-1)^{\beta\gamma}((a_{-\lambda-\mathbf{T}c})_{-\mu-\mathbf{T}b} - a_{-\lambda-\mu-\mathbf{T}}(c_\lambda b) \\
 & + (-1)^{\alpha\gamma}c_\lambda(a_{-\mu-\mathbf{T}b}) - (-1)^{\alpha\gamma}(c_\lambda a)_{-\mu-\mathbf{T}b}) \\
 & + ((a_{-\mu-\mathbf{T}b})_{-\lambda-\mathbf{T}c} - (-1)^{\beta\gamma}(a_{-\lambda-\mathbf{T}c})_{-\mu-\mathbf{T}b}) \\
 = & 0.
 \end{aligned}$$

Therefore, $(R, [\cdot, \lambda \cdot], \cdot, \lambda \cdot)$ is an r -dim super Gel'fand-Dorfman conformal bialgebra. ■

By Propositions 3.4 and 2.15, we can easily obtain the following corollary.

Corollary 3.5. *If (V, \circ, \cdot) is a Novikov-Poisson superalgebra, then $R = \mathbb{C}[T]V$ is a super Gel'fand-Dorfman conformal bialgebra with the following λ -product and λ -bracket ($a \in V_\alpha, b \in V_\beta$):*

$$a_\lambda b = (\lambda + T)(a \cdot b) + a \circ b, \tag{3.41}$$

$$[a_\lambda b] = (T + 2\lambda)(a \cdot b) + (a \circ b - (-1)^{\alpha\beta}b \circ a). \tag{3.42}$$

Remark 3.6. In fact, by Corollary 3.5 and Remark 3.2, we can obtain a super Gel'fand-Dorfman bialgebra from a Novikov-Poisson superalgebra. The construction is as follows.

Let (V, \circ, \cdot) is a Novikov-Poisson superalgebra, then $(A = V \otimes \mathbb{C}[t, t^{-1}], [\cdot, \cdot], *)$ is a super Gel'fand-Dorfman bialgebra with the following Novikov superalgebra structure and the Lie superalgebra structure:

$$\begin{aligned}
 (a \otimes t^m) * (b \otimes t^n) &= (a \circ b) \otimes t^{m+n} - n(a \cdot b) \otimes t^{m+n-1}, \\
 [a \otimes t^m, b \otimes t^n] &= (m - n)(a \cdot b) \otimes t^{m+n-1} + (a \circ b - (-1)^{\alpha\beta}b \circ a) \otimes t^{m+n},
 \end{aligned}$$

where $a \in V_\alpha, b \in V_\beta, m, n \in \mathbb{Z}$.

Next, we present several examples of super Gel'fand-Dorfman conformal bialgebras.

Example 3.7. By Proposition 2.14 and Proposition 3.4, we have the following Gel'fand-Dorfman conformal bialgebra $R = \mathbb{C}[T]x$ with the λ -product and λ -bracket as follows ($a \in \mathbb{C}$):

$$x_\lambda x = (\lambda + T + a)x, \quad [x_\lambda x] = (T + 2\lambda)x.$$

This Gel'fand-Dorfman conformal bialgebra corresponds to a 2-dim 2-linear Lie conformal algebra $\tilde{R} = \mathbb{C}[T_1, T_2]x$ with the following λ -bracket:

$$[x_\lambda x] = T_2(-\lambda_1 + a)x + \lambda_2(T_1 + 2a)x + (T_1 + 2\lambda_1)x.$$

$\text{Lie}\tilde{R} = \mathbb{C}x \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ is a Lie algebra with the following Lie bracket:

$$\begin{aligned} & [x \otimes t_1^{m_1}t_2^{n_1}, x \otimes t_1^{m_2}t_2^{n_2}] \\ &= a(n_1 - n_2)x \otimes t_1^{m_1+m_2}t_2^{n_1+n_2-1} + (m_1 - m_2)x \otimes t_1^{m_1+m_2-1}t_2^{n_1+n_2} \\ &+ (m_1n_2 - n_1m_2)x \otimes t_1^{m_1+m_2-1}t_2^{n_1+n_2-1}, \end{aligned}$$

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$.

Example 3.8. Let $R = R_0 \oplus R_{\bar{1}} = \mathbb{C}[T]a \oplus \mathbb{C}[T]b$ be the Novikov conformal superalgebra given in Example 2.13. Then, by Proposition 3.4, we obtain a super Gel'fand-Dorfman bialgebra $(R, \cdot_\lambda, [\cdot_\lambda \cdot])$ given by as follows:

$$\begin{aligned} a_\lambda a &= (\lambda + T + C_1)a, & a_\lambda b &= (\lambda + T + C_1)b, & b_\lambda a &= (\lambda + T + C_1)b, & b_\lambda b &= 0, \\ [a_\lambda a] &= (T + 2\lambda)a, & [a_\lambda b] &= (T + 2\lambda)b, & [b_\lambda b] &= 0, \end{aligned}$$

for $C_1 \in \mathbb{C}$.

This Gel'fand-Dorfman conformal bialgebra corresponds to a 2-dim 2-linear Lie conformal algebra $\tilde{R} = \mathbb{C}[T_1, T_2]a \oplus \mathbb{C}[T_1, T_2]b$ with the following λ -bracket:

$$\begin{aligned} [a_\lambda a] &= T_2(-\lambda_1 + C_1)a + \lambda_2(T_1 + 2C_1)a + (T_1 + 2\lambda_1)a, \\ [a_\lambda b] &= T_2(-\lambda_1 + C_1)b + \lambda_2(T_1 + 2C_1)b + (T_1 + 2\lambda_1)b, \\ [b_\lambda b] &= 0, \end{aligned}$$

for $C_1 \in \mathbb{C}$.

Example 3.9. If (A, \cdot) is a commutative associative algebra and $D : A \rightarrow A$ is a derivation of A , then (A, \circ, \cdot) is a Novikov-Poisson algebra with

$$a \circ b = a \cdot Db.$$

Then, by Corollary 3.5, we obtain a 2-dim 2-linear Lie conformal algebra $R = \mathbb{C}[T_1, T_2]A$ with the following λ -product and λ -bracket:

$$\begin{aligned} [a_\lambda b] &= T_2(-\lambda_1(a \cdot b) + b \cdot Da) + \lambda_2(T_1(a \cdot b) + b \cdot Da + a \cdot Db) \\ &+ (T_1 + 2\lambda_1)(a \cdot b) + (a \cdot Db - b \cdot Da), \quad a, b \in A. \end{aligned}$$

Next, we present another construction of Gel'fand-Dorfman conformal bialgebras.

Proposition 3.10. *If $(V, [\cdot, \cdot], \circ)$ is a Gel'fand-Dorfman bialgebra, (V, \circ, \cdot) is Novikov-Poisson algebra and $(V, [\cdot, \cdot], \cdot)$ is a Lie-Poisson algebra, then $R = \mathbb{C}[T]V$ is a Gel'fand-Dorfman conformal bialgebra with the following λ -product and λ -bracket ($a \in V_\alpha, b \in V_\beta$):*

$$\begin{aligned} a_\lambda b &= (\lambda + T)(a \cdot b) + a \circ b, \\ [a_\lambda b] &= T(b \circ a) + \lambda(a \circ b + b \circ a) + [b, a]. \end{aligned}$$

The quadruple $(V, [\cdot, \cdot], \circ, \cdot)$ is called a Gel'fand-Dorfman Novikov-Poisson algebra.

Proof. It can be check directly. ■

Remark 3.11. If the operation “ \cdot ” in the above proposition is trival, then we can obtain a construction of Gel’fand-Dorfman conformal bialgebras from Gel’fand-Dorfman bialgebras.

If $(V, [\cdot, \cdot], \circ)$ is a Gel’fand-Dorfman bialgebra, then $R = \mathbb{C}[T]V$ is a Gel’fand-Dorfman conformal bialgebra with the following λ -product and λ -bracket $(a \in V_\alpha, b \in V_\beta)$:

$$\begin{aligned} a_\lambda b &= a \circ b, \\ [a_\lambda b] &= T(b \circ a) + \lambda(a \circ b + b \circ a) + [b, a]. \end{aligned}$$

In fact, by Remark 3.2, we can also obtain a construction of Gel’fand-Dorfman bialgebras as follows.

If $(V, [\cdot, \cdot], \circ)$ is a Gel’fand-Dorfman bialgebra, then $(\text{Lie}R = V \otimes \mathbb{C}[t, t^{-1}], *, [\cdot, \cdot])$ is a Gel’fand-Dorfman bialgebra with the Novikov operation and Lie algebra operation as follows $(a \in V_\alpha, b \in V_\beta, m, n \in \mathbb{C})$:

$$\begin{aligned} (a \otimes t^m) * (b \otimes t^n) &= (a \circ b) \otimes t^{m+n}, \\ [a \otimes t^m, b \otimes t^n] &= (ma \circ b - nb \circ a) \otimes t^{m+n-1} + [b, a] \otimes t^{m+n}. \end{aligned}$$

We give a construction of Gel’fand-Dorfman Novikov-Poisson algebras in the following.

Example 3.12. Let $(A, [\cdot, \cdot], \cdot)$ be a Lie-Poisson algebra and let d be a derivation of the algebra (A, \cdot) such that

$$d[u, v] = [d(u), v] + [u, d(v)] + \xi[u, v], \quad \text{for } u, v \in A, \tag{3.43}$$

where $\xi \in \mathbb{C}$ is a constant. We define another algebraic operation \circ on A by

$$u \circ v = u \cdot d(v) + \xi u \cdot v \quad \text{for } u, v \in A. \tag{3.44}$$

By Corollary 2.6 in [19], (A, \circ, \cdot) is a Novikov-Poisson algebra. And, by Theorem 3.2 in [17], $(A, [\cdot, \cdot], \circ)$ forms a Gel’fand-Dorfman bialgebra. Therefore, $(V, [\cdot, \cdot], \circ, \cdot)$ is a Gel’fand-Dorfman Novikov-Poisson algebra.

Finally, we will present a theorem about an equivalent characterization of r -dim linear Lie conformal superalgebra.

Theorem 3.13. *An r -dim linear Lie conformal superalgebra $R = \mathbb{C}[\mathbf{T}]V$ is equivalent to that $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is equipped with one Lie superalgebraic operation $[\cdot, \cdot]$, and r Novikov superalgebraic operations \circ_i such that $(V, [\cdot, \cdot], \circ_i)$ is a super Gel’fand-Dorfman bialgebra for each $1 \leq i \leq r$ and \circ_i and \circ_j satisfying the following conditions:*

$$a \circ_i (b \circ_j c) - (a \circ_i b) \circ_j c = (-1)^{\alpha\beta} (b \circ_j (a \circ_i c) - (b \circ_j a) \circ_i c), \tag{3.45}$$

$$(c \circ_i a) \circ_j b + (c \circ_j a) \circ_i b = (-1)^{\alpha\beta} ((c \circ_i b) \circ_j a + (c \circ_j b) \circ_i a), \tag{3.46}$$

for $a \in V_\alpha, b \in V_\beta, c \in V$ and $i \neq j$. We call such $(V, [\cdot, \cdot], \circ_1, \dots, \circ_r)$ a generalized super Gel’fand-Dorfman algebra.

Proof. By (2.6), we can assume that:

$$[a_\lambda b] = \sum_{i=1}^r T_i(b \circ_i a) + \sum_{j=1}^r \lambda_j(b *_j a) + [b, a] \quad \text{for } a, b \in V, \tag{3.47}$$

where $\circ_i, *_j, [\cdot, \cdot]$ are algebraic operations on V . Then, the proof is similar to that in Theorem 3.1. Here, $b *_i a = (b \circ_i a) + (-1)^{\alpha\beta} a \circ_i b$, where $a \in V_\alpha, b \in V_\beta$. ■

Remark 3.14. If $(V, [\cdot, \cdot], \circ_i)$ is a super Gel'fand-Dorfman bialgebra for one of $1 \leq i \leq r$, then $(V, [\cdot, \cdot], \circ_1, \dots, \circ_r)$ with \circ_j trivial for each $1 \leq j \leq r$ and $j \neq i$ is a generalized super Gel'fand-Dorfman algebra, and its corresponding r -dim Lie conformal superalgebra $R = \mathbb{C}[\mathbf{T}]V$ is determined by the following λ -bracket:

$$[a_\lambda b] = T_i(b \circ_i a) + \lambda_i(b \circ_i a + (-1)^{\alpha\beta} a \circ_i b) + [b, a],$$

for $a \in V_\alpha$ and $b \in V_\beta$.

In addition, by Theorem 3.13, we can classify r -dim linear Lie conformal superalgebras by classifying generalized super Gel'fand-Dorfman algebras. One useful method to classify generalized super Gel'fand-Dorfman algebras is that given a Lie superalgebra, we first classify all Novikov superalgebra structures compatible with this Lie superalgebra, i.e. constructing a super Gel'fand-Dorfman bialgebra and then study which Novikov superalgebra structures satisfy (3.45) and (3.46).

Example 3.15. Let $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. Then, $(V, [\cdot, \cdot])$ is a Lie algebra with the Lie bracket $[e_1, e_2] = e_2$.

Set $e_i \circ e_j = \sum_{l=1}^2 c_{ij}^l e_l$ for any $i, j \in \{1, 2\}$. If $[\cdot, \cdot]$ and \circ satisfy (2.14), by some computations, we can obtain

$$c_{11}^1 = c_{21}^2, c_{12}^1 + c_{21}^1 = 0, c_{22}^2 = 0, c_{21}^1 = c_{22}^2.$$

For simplicity, set $a = c_{11}^1 = c_{21}^2, b = c_{11}^2, c = c_{12}^2, d = c_{21}^1 = c_{22}^2 = -c_{12}^1$. Then, if (V, \circ) is a Novikov algebra, we get $cd = bd = 0$.

Therefore, $(V, [\cdot, \cdot], \circ)$ is a Gel'fand-Dorfman bialgebra if and only if the Novikov algebraic operation is defined by

$$e_1 \circ e_1 = ae_1 + be_2, \quad e_1 \circ e_2 = -de_1 + ce_2, \quad e_2 \circ e_1 = de_1 + ae_2, \quad e_2 \circ e_2 = de_2,$$

with $cd = bd = 0$.

Assume that $(V, [\cdot, \cdot], \circ_1)$ and $(V, [\cdot, \cdot], \circ_2)$ are two Gel'fand-Dorfman bialgebras with the Novikov algebraic operations

$$e_1 \circ_i e_1 = a_i e_1 + b_i e_2, \quad e_1 \circ_i e_2 = -d_i e_1 + c_i e_2, \tag{3.48}$$

$$e_2 \circ_i e_1 = d_i e_1 + a_i e_2, \quad e_2 \circ_i e_2 = d_i e_2, \tag{3.49}$$

with

$$c_i d_i = b_i d_i = 0, \quad i \in \{1, 2\}. \tag{3.50}$$

For \circ_1 and \circ_2 satisfying (3.45) and (3.46), by some computations, we have

$$a_1c_2 = a_2c_1, \quad b_2(c_1 - a_1) = b_1(c_2 - a_2). \quad (3.51)$$

Therefore, $(V, [\cdot, \cdot], \circ_1, \circ_2)$ is a generalized Gel'fand-Dorfman algebra with the Novikov algebraic operations defined by (3.48),(3.49) if and only if (3.50) and (3.51) hold.

For example, $(V = \mathbb{C}e_1 \oplus \mathbb{C}e_2, [\cdot, \cdot], \circ_1, \circ_2)$ is a generalized Gel'fand-Dorfman algebra with the Lie algebra structure and Novikov algebra structures as follows

$$\begin{aligned} [e_1, e_2] &= e_2, \\ e_1 \circ_1 e_1 &= e_2, \quad e_1 \circ_1 e_2 = e_2 \circ_1 e_1 = e_2 \circ_1 e_2 = 0, \\ e_1 \circ_2 e_1 &= e_1, \quad e_1 \circ_2 e_2 = e_2, \quad e_2 \circ_2 e_1 = e_2, \quad e_2 \circ_2 e_2 = 0. \end{aligned}$$

It corresponds to a 2-dim linear Lie conformal algebra $R = \mathbb{C}[T_1, T_2]V$ with the following λ -bracket

$$\begin{aligned} [e_{1\lambda}e_1] &= (T_1 + 2\lambda_1)e_2 + (T_2 + 2\lambda_2)e_1, \\ [e_{1\lambda}e_2] &= (T_2 + 2\lambda_2)e_2 - e_2, \\ [e_{2\lambda}e_2] &= 0. \end{aligned}$$

Proposition 3.16. *Let $(V, [\cdot, \cdot], \circ_i)$ be a super Gel'fand-Dorfman bialgebra for some i . Assume that other operations \circ_j for $1 \leq j \leq r$ and $j \neq i$ are defined as follows:*

$$a \circ_j b = k_j a \circ_i b, \quad \text{for some } k_j \in \mathbb{C}. \quad (3.52)$$

Then, $(V, [\cdot, \cdot], \circ_1, \dots, \circ_r)$ is a generalized super Gel'fand-Dorfman algebra.

Proof. It is easy to check. ■

Acknowledgments. *We would like to thank the referees for a careful reading of this paper and for many useful comments.*

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Received October 1, 2014
and in final form May 7, 2016