

On the Integral Representations for Dunkl Kernels of Type A_2

Bécher Amri*

Communicated by B. Ørsted

Abstract. We give an explicit integral formula for the Dunkl kernel associated to root system of type A_2 and parameter $k > 0$, by exploiting recent results in B. Amri, *Note on Bessel functions of type A_{N-1}* , *Integral Transforms and Special Functions* **25** (2014), 448–461.

Mathematics Subject Classification 2010: Primary 33E30; Secondary 17B22, 20F55.

Key Words and Phrases: Dunkl operators, root systems, reflection groups.

1. Introduction and statement of the result.

As originally constructed by Dunkl [2], the investigation of intertwining operator between Dunkl operators and ordinary derivatives is still ambiguous. Except for a few cases the finding of explicit form is an open problem. One of the most important contributions is due to Rösler [7] who proved that attached with finite reflection group and nonnegative multiplicity function the Dunkl intertwining operator is positive and can be expressed as an integral transform with positive kernel. Dunkl kernels $E_k(\cdot, y)$, first defined by Dunkl by means of intertwining operator [3], are the joint eigenfunctions of the Dunkl operators and so they are considered as the generalization of the usual exponential functions $e^{\langle \cdot, y \rangle}$. The basic fact, derived from the main result of [7] is the validity of the representation of E_k by an integral of Laplace type,

$$E_k(x, y) = \int e^{\langle x, y \rangle} d\mu_x(y) \quad (1)$$

where μ_x is a compactly supported probability measure. This representation is useful to determine the behavior of these functions and so making explicit computation for μ_x becomes increasingly important.

In this paper we mainly focus on Dunkl kernels associated to root systems of type A , for the purpose of finding an explicit integral representation of type

* The author was partially supported by the DGRST research project LR11ES11 and the CMCU Research project 15G 1504

(1), following our recent work on the symmetric case. We outline here a simple method that leads us to such formulas for the A_2 root system and provide an elementary proof of Dunkl’s formula for the intertwining operator established in [4] for parameter $k > 1/2$. The key idea here is that a Dunkl kernel can be obtained by differentiating its symmetric counterpart, namely the Dunkl-Bessel function (Lemma 2.1). The proof is then achieved by using the integral representation of the Dunkl-Bessel function as already established in [1] and is concentrated on finding similar representations for products of these functions and its derivatives by certain polynomials. It would be interesting to generalize Theorem 1.1 to the higher dimensional but our approach is far from being applied to other cases of A_n , $n \geq 3$, because it becomes more complicated as calculus when using products of Dunkl operators.

We begin by summarizing some facts from [1], general references are [2, 3, 4, 5, 7, 8, 9]. Let \mathbb{V} be the hyperplane in the Euclidean space \mathbb{R}^3 given by

$$\mathbb{V} = \{(x, y, z) \in \mathbb{R}^3; x + y + z = 0\}$$

and $R = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3)\}$ be the root system of type A_2 in \mathbb{V} , where (e_1, e_2, e_3) is the standard basis of \mathbb{R}^3 . Fix $(e_1 - e_2, e_2 - e_3)$ as the basis of simple roots and C the corresponding fundamental Weyl chamber,

$$C = \{\lambda = (\lambda_1, \lambda_2, \lambda_3); \lambda_3 < \lambda_2 < \lambda_1\}.$$

The Weyl group is isomorphic to the symmetric group S_3 . The Dunkl operators are given by

$$T_i = \frac{\partial}{\partial x_i} + k \sum_{1 \leq i < j \leq 3} \frac{1 - s_{i,j}}{x_i - x_j}, \quad i = 1, 2, 3$$

where k is a positive real parameter and $s_{i,j}$ acts on functions of variables (x_1, x_2, x_3) by interchanging the variables x_i and x_j . The Dunkl kernel $E_k(\cdot, y)$, $y \in \mathbb{R}^3$ is characterized by being the unique solution of the following eigenvalue problem

$$T_i(E_k(\cdot, y))(x) = y_i E_k(x, y); \quad E_k(0, y) = 0, \quad i = 1, 2, 3.$$

Let J_k the generalized Bessel function associated with R and k , given by

$$J_k(x, y) = \frac{1}{6} \sum_{\sigma \in G} E_k(\sigma.x, y). \tag{2}$$

The functions J_k are related to the ordinary modified Bessel functions $\mathcal{J}_{k-\frac{1}{2}}$, by (see [1]):

$$J_k(\mu, \lambda) = \frac{\Gamma(3k)}{V(\lambda)^{2k-1} \Gamma(k)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2}\right) (\nu_1 - \nu_2) W_k(\mu, \lambda) d\nu_1 d\nu_2, \tag{3}$$

for all $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{V} \cap C$ and $\mu \in \mathbb{R}^3$, where

$$V(\lambda) = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3),$$

$$W_k(\nu, \lambda) = \left((\lambda_1 - \nu_1)(\lambda_1 - \nu_2)(\lambda_2 - \nu_2)(\nu_1 - \lambda_2)(\nu_1 - \lambda_3)(\nu_2 - \lambda_3) \right)^{k-1}.$$

Recall that the modified Bessel functions $\mathcal{J}_{k-\frac{1}{2}}$ is given by

$$\mathcal{J}_{k-\frac{1}{2}}(x) = \Gamma(k + 1/2) \sum_{n=0}^{+\infty} \frac{(x/2)^{2n}}{n! \Gamma(n + k + 1/2)}$$

and has the integral representation

$$\mathcal{J}_{k-\frac{1}{2}}(z) = \frac{\Gamma(2k)}{2^{2k-1} \Gamma(k)^2} \int_{-1}^1 e^{zt} (1-t^2)^{k-1} dt, \quad z \in \mathbb{R}.$$

In the next section we shall use this fact to construct an integral formula for E_k . The following theorem is the main result of this article.

Theorem 1.1. *The Dunkl kernel of type A_2 has the following integral formula*

$$\begin{aligned} E_k(\mu, \lambda) = & \frac{\Gamma(3k)}{V(\lambda)^{2k} \Gamma(k)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} \left\{ 3(\lambda_1 - \lambda_2)(\nu_1 - \nu_2) \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \right. \\ & \left. - 6 \left(\nu_1 \nu_2 + \frac{\lambda_3}{2} (\nu_1 + \nu_2) + \lambda_1 \lambda_2 \right) \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \right\} \\ & (\lambda_3 - \nu_1)(\lambda_3 - \nu_2) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} W_k(\nu, \lambda) d\nu_1 d\nu_2, \end{aligned} \tag{4}$$

for all $\lambda \in \mathbb{V} \cap C$ and $\mu \in \mathbb{R}^3$.

2. Outline of the proof

An interesting relation between J_k and J_{k+1} is given in ([6], p.369) by the following functional equation

$$T_V(J_{k+1}(\cdot, y)V(\cdot))(x) = \gamma_k J_k(x, y) \tag{5}$$

where $T_V = (T_1 - T_2)(T_2 - T_3)(T_1 - T_3)$ and $\gamma_k = T_V(V(\cdot))(0) = \left((2k + 1)(3k + 1)(3k + 2) \right)^{-1}$. This together with Proposition 1.4 of [4] implies

$$\sum_{\sigma \in G} \det(\sigma) E_k(\sigma \cdot \mu, \lambda) = \gamma_k V(\mu)V(\lambda) J_{k+1}(\mu, \lambda). \tag{6}$$

Combining (6) with (2) yields for all $\mu \in \mathbb{R}^3$ and $\lambda \in \mathbb{V}$

$$E_k(\mu, \lambda) + E_k(\mu, \sigma \cdot \lambda) + E_k(\mu, \sigma^2 \cdot \lambda) = \frac{1}{2} \left(\gamma_k V(\lambda)V(\mu) J_{k+1}(\mu, \lambda) + 6J_k(\mu, \lambda) \right) \tag{7}$$

where $\sigma = s_{1,3}s_{1,2}$. This is a starting point from which we have the following

Lemma 2.1. *Let $\lambda \in \mathbb{V}$ and T^λ be the operator*

$$T^\lambda = \frac{2\lambda_1 + \lambda_2}{\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2} T_1 + \frac{2\lambda_2 + \lambda_1}{\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2} T_2 + 1 = \alpha(\lambda) T_1 + \beta(\lambda) T_2 + 1.$$

Then we have

$$E_k(\mu, \lambda) = T^\lambda \left(\frac{\gamma_k}{6} V(\lambda) V(\cdot) J_{k+1}(\cdot, \lambda) + J_k(\cdot, \lambda) \right) (\mu), \quad \mu \in \mathbb{R}^3.$$

The proof is a straightforward calculation which we shall omit. However, to obtain our integral formula for E_k , it therefore comes down to express the following terms with suitable integrals

- (i) $V(\mu)J_{k+1}(\mu, \lambda)$
- (ii) $(\mu_1 - \mu_2)(\mu_2 - \mu_3)J_{k+1}(\mu, \lambda)$
- (iii) $(\mu_1 - \mu_2)(\mu_1 - \mu_3)J_{k+1}(\mu, \lambda)$
- (iv) $T_1(V(\cdot)J_{k+1}(\cdot, \lambda))(\mu) = V(\mu)\frac{\partial J_{k+1}}{\partial \mu_1}(\mu, \lambda) + (2k + 1)\frac{\partial V(\mu)}{\partial \mu_1}J_{k+1}(\mu, \lambda)$
- (v) $T_2(V(\cdot)J_{k+1}(\cdot, \lambda))(\mu) = V(\mu)\frac{\partial J_{k+1}}{\partial \mu_2}(\mu, \lambda) + (2k + 1)\frac{\partial V(\mu)}{\partial \mu_2}J_{k+1}(\mu, \lambda)$

We will need to use the following classical equations of the modified Bessel function \mathcal{J}_α , $\alpha > -\frac{1}{2}$,

$$z\mathcal{J}_{\alpha+1}(z) = 2(\alpha + 1)\mathcal{J}'_\alpha(z) \tag{8}$$

$$\mathcal{J}_\alpha(z) = \mathcal{J}''_\alpha(z) + \frac{2\alpha + 1}{z}\mathcal{J}'_\alpha(z) \tag{9}$$

and the following facts:

$$(\mu_1 - \mu_2)(\mu_1 - \mu_3) = \frac{(\mu_1 - \mu_2)(\mu_1 + \mu_2 - 2\mu_3) + (\mu_1 - \mu_2)^2}{2} \tag{10}$$

$$(\mu_1 - \mu_2)(\mu_2 - \mu_3) = \frac{(\mu_1 - \mu_2)(\mu_1 + \mu_2 - 2\mu_3) - (\mu_1 - \mu_2)^2}{2} \tag{11}$$

$$(\mu_1 - \mu_3)(\mu_2 - \mu_3) = \frac{(\mu_1 + \mu_2 - 2\mu_3)^2 - (\mu_1 - \mu_2)^2}{4} \tag{12}$$

$$V(\mu) = \frac{(\mu_1 + \mu_2 - 2\mu_3)^2(\mu_1 - \mu_2) - (\mu_1 - \mu_2)^3}{4} . \tag{13}$$

Further, the rule of integration by parts will be used repeatedly with the derivatives $\partial_{\nu_1} + \partial_{\nu_2}$ and $\partial_{\nu_1} - \partial_{\nu_2}$, since by looking the integrant of (3) the machine works well with the obvious facts: $(\partial_{\nu_1} + \partial_{\nu_2})h(\nu_1 + \nu_2) = 2h'(\nu_1 + \nu_2)$, $(\partial_{\nu_1} + \partial_{\nu_2})h(\nu_1 - \nu_2) = 0$, $(\partial_{\nu_1} - \partial_{\nu_2})h(\nu_1 - \nu_2) = 2h'(\nu_1 - \nu_2)$ and $(\partial_{\nu_1} - \partial_{\nu_2})h(\nu_1 + \nu_2) = 0$, for a C^1 -function h .

We begin by treating (i). From (8) we have

$$\begin{aligned} & (\mu_1 - \mu_2)J_{k+1}(\mu, \lambda) \\ &= \frac{(4k + 2)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2}\right) \\ & \quad W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2 \end{aligned}$$

and by using integration by parts,

$$\begin{aligned} & (\mu_1 - \mu_2)^2 J_{k+1}(\mu, \lambda) \\ &= \frac{(4k + 2)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2}\right) \\ & \quad (\partial_{\nu_1} - \partial_{\nu_2})W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2 . \end{aligned}$$

Making use of (9) we have

$$\begin{aligned}
 &(\mu_1 - \mu_2)^3 J_{k+1}(\mu, \lambda) = \\
 &-\frac{(4k + 2)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} (\mu_1 - \mu_2) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}}'' \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 &\quad (\partial_{\nu_1} - \partial_{\nu_2}) W_{k+1}(\nu, \lambda) \, d\nu_1 d\nu_2 \\
 &-\frac{4k(4k + 2)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}}' \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 &\quad \frac{(\partial_{\nu_1} - \partial_{\nu_2}) W_{k+1}(\nu, \lambda)}{\nu_1 - \nu_2} \, d\nu_1 d\nu_2
 \end{aligned}$$

and by integration by parts one-time,

$$\begin{aligned}
 &(\mu_1 + \mu_2 - 2\mu_3)^2 (\mu_1 - \mu_2) J_{k+1}(\mu, \lambda) = \\
 &-\frac{(4k + 2)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} (\mu_1 + \mu_2 - 2\mu_3) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \\
 &\quad \mathcal{J}_{k-\frac{1}{2}}' \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) (\partial_{\nu_1} + \partial_{\nu_2}) W_{k+1}(\nu, \lambda) \, d\nu_1 d\nu_2.
 \end{aligned}$$

Note that the condition $k > 0$ is not sufficient to make an integration by parts again using the derivative operators $\partial_{\nu_1} + \partial_{\nu_2}$ or $\partial_{\nu_1} - \partial_{\nu_2}$, because the appearance of $\partial_{\nu_1}^2 W_{k+1}$ and $\partial_{\nu_2}^2 W_{k+1}$. However, we see that

$$\begin{aligned}
 &-(\mu_1 + \mu_2 - 2\mu_3) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}}' \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) (\partial_{\nu_1} + \partial_{\nu_2}) W_{k+1}(\nu, \lambda) \\
 &+(\mu_1 - \mu_2) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}}'' \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) (\partial_{\nu_1} - \partial_{\nu_2}) W_{k+1}(\nu, \lambda) \\
 &= -2\partial_{\nu_1} \left\{ e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}}' \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \right\} \partial_{\nu_2} W_{k+1}(\nu, \lambda) \\
 &\quad -2\partial_{\nu_2} \left\{ e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}}' \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \right\} \partial_{\nu_1} W_{k+1}(\nu, \lambda).
 \end{aligned}$$

Thus from (13) and integration by parts we obtain

$$\begin{aligned}
 &V(\mu) J_{k+1}(\mu, \lambda) \\
 &= \frac{(4k + 2)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}}' \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 &\quad \left(\partial_{\nu_1} \partial_{\nu_2} + k \frac{\partial_{\nu_1} - \partial_{\nu_2}}{\nu_1 - \nu_2} \right) W_{k+1}(\nu, \lambda) \, d\nu_1 d\nu_2,
 \end{aligned}$$

which is a nice integral formula for (i).

Next, using (10) and (11) with integration by parts,

$$\begin{aligned}
 (\mu_1 - \mu_2)(\mu_1 - \mu_3)J_{k+1}(\mu, \lambda) = & \\
 & - \frac{(4k + 2)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & (\partial_{\nu_1} - \partial_{\nu_2})W_{k+1}(\nu, \lambda)d\nu_1d\nu_2 \\
 & - \frac{(4k + 2)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & (\partial_{\nu_1} + \partial_{\nu_2})W_{k+1}(\nu, \lambda)d\nu_1d\nu_2
 \end{aligned}$$

and

$$\begin{aligned}
 & (\mu_1 - \mu_2)(\mu_2 - \mu_3)J_{k+1}(\mu, \lambda) \\
 = & \frac{(2k + 1)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & (\partial_{\nu_1} - \partial_{\nu_2})W_{k+1}(\lambda, \mu)d\nu_1d\nu_2 \\
 & - \frac{(2k + 1)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & (\partial_{\nu_1} + \partial_{\nu_2})W_{k+1}(\nu, \lambda)d\nu_1d\nu_2.
 \end{aligned}$$

For (iv) we make use of the fact that

$$z\mathcal{J}'_{\alpha+1}(z) = 2(\alpha + 1)\left(\mathcal{J}_\alpha(z) - \mathcal{J}_{\alpha+1}(z)\right),$$

and write

$$\begin{aligned}
 & V(\mu)\frac{\partial J_{k+1}}{\partial \mu_1}(\mu) \\
 = & \frac{\Gamma(3k + 3)}{2V(\lambda)^{2k+1}\Gamma(k + 1)^3}V(\mu) \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k+\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & (\nu_1 - \nu_2)(\nu_1 + \nu_2)W_{k+1}(\nu, \lambda)d\nu_1d\nu_2 \\
 + & \frac{(2k + 1)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3}(\mu_1 - \mu_3)(\mu_2 - \mu_3) \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \\
 & \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) (\nu_1 - \nu_2)W_{k+1}(\nu, \lambda)d\nu_1d\nu_2 \\
 & - (2k + 1)(\mu_1 - \mu_3)(\mu_2 - \mu_3)J_{k+1}.
 \end{aligned}$$

Proceeding as for the integral representation of (i), we have

$$\begin{aligned}
 & \frac{\Gamma(3k + 3)}{2V(\lambda)^{2k+1}\Gamma(k + 1)^3}V(\mu) \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k+\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & (\nu_1 - \nu_2)(\nu_1 + \nu_2)W_{k+1}(\nu, \lambda)d\nu_1d\nu_2 \\
 = & \frac{(2k + 1)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & \left\{ \partial_{\nu_1}\partial_{\nu_2} \left((\nu_1 + \nu_2)W_{k+1}(\nu, \lambda) \right) + k \frac{(\partial_{\nu_1} - \partial_{\nu_2}) \left((\nu_1 + \nu_2)W_{k+1}(\nu, \lambda) \right)}{\nu_1 - \nu_2} \right\} d\nu_1d\nu_2.
 \end{aligned}$$

On the other hand, by using (9) and (12) with integration by parts,

$$\begin{aligned}
 & \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3} (\mu_1 - \mu_3)(\mu_2 - \mu_3) \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \\
 & \quad \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) (\nu_1 - \nu_2) W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2 \\
 = & -\frac{(2k+1)\Gamma(3k+3)}{4V(\lambda)^{2k+1}\Gamma(k+1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} (\mu_1 + \mu_2 - 2\mu_3) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \\
 & \quad \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) (\partial\nu_1 + \partial\nu_2) (\nu_1 - \nu_2) W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2 \\
 + & \frac{(2k+1)\Gamma(3k+3)}{4V(\lambda)^{2k+1}\Gamma(k+1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} (\mu_1 - \mu_2) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \\
 & \quad \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) (\partial\nu_1 - \partial\nu_2) (\nu_1 - \nu_2) W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2 \\
 + & \frac{k(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & \quad (\partial\nu_1 - \partial\nu_2) W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2.
 \end{aligned}$$

As we noted above for the use of integration by parts a second time, we can do it by the following

$$\begin{aligned}
 & -(\mu_1 + \mu_2 - 2\mu_3) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & \quad (\partial\nu_1 + \partial\nu_2) \left((\nu_1 - \nu_2) W_{k+1}(\nu, \lambda) \right) \\
 + & (\mu_1 - \mu_2) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & \quad (\partial\nu_1 - \partial\nu_2) \left((\nu_1 - \nu_2) W_{k+1}(\nu, \lambda) \right) \\
 = & -2\partial\nu_1 \left\{ e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \right\} \partial\nu_2 \left((\nu_1 - \nu_2) W_{k+1}(\nu, \lambda) \right) \\
 & -2\partial\nu_2 \left\{ e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \right\} \partial\nu_1 \left((\nu_1 - \nu_2) W_{k+1}(\nu, \lambda) \right).
 \end{aligned}$$

Thus

$$\begin{aligned}
 & \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3} (\mu_1 - \mu_3)(\mu_2 - \mu_3) \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \\
 & \quad \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) (\nu_1 - \nu_2) W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2 \\
 = & \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & \quad \left\{ \partial\nu_1 \partial\nu_2 \left((\nu_1 - \nu_2) W_{k+1}(\nu, \lambda) \right) + k(\partial\nu_1 - \partial\nu_2) W_{k+1}(\nu, \lambda) \right\} d\nu_1 d\nu_2.
 \end{aligned}$$

From these calculations it follows that

$$\begin{aligned}
 & T_1(V(\cdot)J_{k+1}(\cdot, \lambda))(\mu) \\
 &= V(\mu) \frac{\partial J_{k+1}}{\partial \mu_1}(\mu) + (2k + 1) \left((\mu_1 - \mu_3)(\mu_2 - \mu_3) + (\mu_1 - \mu_2)(\mu_2 - \mu_3) \right) J_{k+1}(\mu) \\
 &= \frac{(2k + 1)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 &\quad \left\{ (\nu_1 + \nu_2) \left(\partial\nu_1 \partial\nu_2 + k \frac{\partial\nu_1 - \partial\nu_2}{\nu_1 - \nu_2} \right) - 2k(\partial\nu_1 + \partial\nu_2) \right\} W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2 \\
 + & \frac{(2k + 1)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 &\quad (\nu_1 - \nu_2) \left(\partial\nu_1 \partial\nu_2 + 3k \frac{(\partial\nu_1 - \partial\nu_2)}{\nu_1 - \nu_2} \right) W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2.
 \end{aligned}$$

By the fact that

$$T_2(V(\cdot)J_{k+1}(\cdot, \lambda))(\mu_1, \mu_2, \mu_3) = -T_1(V(\cdot)J_{k+1}(\cdot, \lambda))(\mu_2, \mu_1, \mu_3),$$

we also have

$$\begin{aligned}
 & T_2(V(\cdot)J_{k+1}(\cdot, \lambda))(\mu) \\
 &= \frac{(2k + 1)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 &\quad \left\{ (\nu_1 + \nu_2) \left(\partial\nu_1 \partial\nu_2 + k \frac{\partial\nu_1 - \partial\nu_2}{\nu_1 - \nu_2} \right) - 2k(\partial\nu_1 + \partial\nu_2) \right\} W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2 \\
 - & \frac{(2k + 1)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 & (\nu_1 - \nu_2) \left(\partial\nu_1 \partial\nu_2 + 3k \frac{(\partial\nu_1 - \partial\nu_2)}{\nu_1 - \nu_2} \right) W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2.
 \end{aligned}$$

By virtue of these integral formulas we obtain

$$\begin{aligned}
 & T^\lambda(V(\cdot)J_{k+1}(\cdot, \lambda))(\mu) \\
 &= \frac{(2k + 1)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 &\quad \left\{ ((\alpha + \beta)(\nu_1 + \nu_2) + 2) \left(\partial\nu_1 \partial\nu_2 + k \frac{\partial\nu_1 - \partial\nu_2}{\nu_1 - \nu_2} \right) - 2k(\alpha + \beta)(\partial\nu_1 + \partial\nu_2) \right\} \\
 &\quad W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2 \\
 + & \frac{(2k + 1)\Gamma(3k + 3)}{V(\lambda)^{2k+1}\Gamma(k + 1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 &\quad (\nu_1 - \nu_2)(\alpha - \beta) \left(\partial\nu_1 \partial\nu_2 + 3k \frac{\partial\nu_1 - \partial\nu_2}{\nu_1 - \nu_2} \right) W_{k+1}(\nu, \lambda) d\nu_1 d\nu_2.
 \end{aligned}$$

Put $a(\lambda) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ and $b(\lambda) = -\lambda_1\lambda_2\lambda_3$, we have

$$\begin{aligned} & \left(\partial\nu_1\partial\nu_2 + k\frac{\partial\nu_1 - \partial\nu_2}{\nu_1 - \nu_2} \right) W_{k+1}(\nu, \lambda) \\ & \quad = -k^2 \left(6\nu_1^2\nu_2^2 + 2a(\nu_1^2 + \nu_2^2 + \nu_1\nu_2) + 3b(\nu_1 + \nu_2) \right) W_k(\nu, \lambda), \\ & \left\{ (\nu_1 + \nu_2) \left(\partial\nu_1\partial\nu_2 + k\frac{\partial\nu_1 - \partial\nu_2}{\nu_1 - \nu_2} \right) - 2k(\partial\nu_1 + \partial\nu_2) \right\} W_{k+1}(\nu, \lambda), \\ & \quad = k^2 \left(2a\nu_1\nu_2(\nu_1 + \nu_2) + 3b(\nu_1 - \nu_2)^2 + 2a^2(\nu_1 + \nu_2) + 4ab \right) W_k(\nu, \lambda) \\ & \left(\partial\nu_1\partial\nu_2 + 3k\frac{(\partial\nu_1 - \partial\nu_2)}{\nu_1 - \nu_2} \right) W_{k+1}(\nu, \lambda) \\ & \quad = k^2 \left(-6a\nu_1\nu_2 - 9b(\nu_1 + \nu_2) + 2a^2 \right) W_k(\nu, \lambda), \\ & \left\{ ((\alpha + \beta)(\nu_1 + \nu_2) + 2) \left(\partial\nu_1\partial\nu_2 + k\frac{\partial\nu_1 - \partial\nu_2}{\nu_1 - \nu_2} \right) \right. \\ & \quad \quad \left. - 2k(\alpha + \beta)(\partial\nu_1 + \partial\nu_2) \right\} W_{k+1}(\nu, \lambda) \\ & = -k^2 \left(12\nu_1^2\nu_2^2 + 4a(\nu_1^2 + \nu_2^2 + \nu_1\nu_2) + 6b(\nu_1 + \nu_2) \right) W_k(\nu, \lambda) \\ & \quad + (\alpha + \beta)k^2 \left(2a\nu_1\nu_2(\nu_1 + \nu_2) + 3b(\nu_1 - \nu_2)^2 + 2a^2(\nu_1 + \nu_2) + 4ab \right) W_k(\nu, \lambda). \end{aligned}$$

Now by the fact that

$$\begin{aligned} & T^\lambda(J_k(\cdot, \lambda))(\mu) \\ & = \frac{\Gamma(3k)}{V(\lambda)^{2k-1}\Gamma(k)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} \left(\frac{\alpha + \beta}{2} \right) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\ & \quad (\nu_1 - \nu_2)(\nu_1 + \nu_2) W_k(\mu, \lambda) d\nu_1 d\nu_2, \\ & + \frac{\Gamma(3k)}{V(\lambda)^{2k-1}\Gamma(k)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} \left(\frac{\alpha - \beta}{2} \right) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\ & \quad (\nu_1 - \nu_2)^2 W_k(\mu, \lambda) d\nu_1 d\nu_2 \end{aligned}$$

and

$$\frac{(2k + 1)\gamma_k\Gamma(3k + 3)}{6\Gamma(k + 1)^3} = \frac{\Gamma(3k)}{2\Gamma(k)^3},$$

it follows from Lemma 2.1 that

$$\begin{aligned}
 E_k(\mu, \lambda) &= \frac{\Gamma(3k)}{V(\lambda)^{2k}\Gamma(k)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} (\nu_1 - \nu_2) e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \\
 &\quad \mathcal{J}_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \left\{ \frac{\alpha - \beta}{2} \left(-6a\nu_1\nu_2 - 9b(\nu_1 + \nu_2) + 2a^2 \right) \right. \\
 &\quad \left. + \left(\frac{\alpha + \beta}{2}(\nu_1 + \nu_2) + 1 \right) V(\lambda) \right\} W_k(\nu, \lambda) d\nu_1 d\nu_2 \\
 + \frac{\Gamma(3k)}{V(\lambda)^{2k}\Gamma(k)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1 + \mu_2 - 2\mu_3)(\nu_1 + \nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}} \left(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2} \right) \\
 &\quad \left\{ \frac{(\alpha + \beta)}{2} \left(2a\nu_1\nu_2(\nu_1 + \nu_2) + 3b(\nu_1 - \nu_2)^2 + 2a^2(\nu_1 + \nu_2) + 4ab \right) \right. \\
 &\quad \left. - \left(6\nu_1^2\nu_2^2 + 2a(\nu_1^2 + \nu_2^2 + \nu_1\nu_2) + 3b(\nu_1 + \nu_2) \right) + \frac{\alpha - \beta}{2}(\nu_1 - \nu_2)^2 V(\lambda) \right\} \\
 &\quad W_k(\nu, \lambda) d\nu_1 d\nu_2.
 \end{aligned}$$

Now, a simple computation gives the following

$$\begin{aligned}
 &\frac{\alpha - \beta}{2} \left(-6a\nu_1\nu_2 - 9b(\nu_1 + \nu_2) + 2a^2 \right) + \left(\frac{\alpha + \beta}{2}(\nu_1 + \nu_2) + 1 \right) V(\lambda) \\
 &= 3(\lambda_1 - \lambda_2)\nu_1\nu_2 + 3(\lambda_1^2 - \lambda_2^2)(\nu_1 + \nu_2) + 3\lambda_3^2(\lambda_1 - \lambda_2) \\
 &= 3(\lambda_1 - \lambda_2)(\lambda_3 - \nu_1)(\lambda_3 - \nu_2)
 \end{aligned}$$

and

$$\begin{aligned}
 &\frac{(\alpha + \beta)}{2} \left(2a\nu_1\nu_2(\nu_1 + \nu_2) + 3b(\nu_1 - \nu_2)^2 + 2a^2(\nu_1 + \nu_2) + 4ab \right) \\
 &\quad - \left(6\nu_1^2\nu_2^2 + 2a(\nu_1^2 + \nu_2^2 + \nu_1\nu_2) + 3b(\nu_1 + \nu_2) \right) + \frac{\alpha - \beta}{2}(\nu_1 - \nu_2)^2 V(\lambda) \\
 &= -6\nu_1^2\nu_2^2 + 3\lambda_3\nu_1\nu_2(\nu_1 + \nu_2) - 3\lambda_3(\lambda_1^2 + \lambda_2^2)(\nu_1 + \nu_2) - 6\lambda_1\lambda_2\lambda_3^2 \\
 &\quad - 2(\nu_1^2 + \nu_2^2 + \nu_1\nu_2)(\lambda_1\lambda_2 - \lambda_3^2) + (2\lambda_1\lambda_2 + \lambda_3^2)(\nu_1 - \nu_2)^2 \\
 &= -6(\lambda_3 - \nu_1)(\lambda_3 - \nu_2) \left(\nu_1\nu_2 + \frac{\lambda_3}{2}(\nu_1 + \nu_2) + \lambda_1\lambda_2 \right).
 \end{aligned}$$

This conclude the proof of the theorem.

Now if we equipped the space \mathbb{V} with the basis $(e_1 - e_3, e_2 - e_3)$ and with the Lebesgue measure $d\nu = d\nu_1 d\nu_2$, then we can state

Corollary 2.2. *The Dunkl kernel E_k connected with the exponential function by*

$$E_k(\mu, \lambda) = \int_{co(\lambda)} e^{\langle \mu, \nu \rangle} F_k \left(\frac{\nu_1 + \nu_2}{2}, \frac{\nu_1 - \nu_2}{2}, \lambda \right) d\nu \tag{14}$$

where $\mu \in \mathbb{V}$, $\lambda \in \mathbb{V} \cap C$, $co(\lambda) = \{ \nu \in \mathbb{V}, \lambda_3 \leq \nu_1, \nu_2, \nu_3 \leq \lambda_1 \}$, the convex

hull of the orbit $G.\lambda$ and the function F_k is given by

$$\begin{aligned}
 &F_k(x, y, \lambda) \\
 &= \frac{6\Gamma(2k)\Gamma(3k)}{2^{2k-2}\Gamma(k)^5V(\lambda)^{2k}} \int_{\max(|y|, |x-\lambda_2|)}^{\min(x-\lambda_3, \lambda_1-x)} \left(z^2(y + \lambda_1 - \lambda_2) + y(\lambda_1 - x)(x - \lambda_2) \right) \\
 &\quad \left(\frac{(\lambda_3 - x)^2 - z^2}{z^2} \right)^k \left((z^2 - y^2)((\lambda_1 - x)^2 - z^2)(z^2 - (\lambda_2 - x)^2) \right)^{k-1} dz, \quad (15)
 \end{aligned}$$

if $\max(|y|, |x - \lambda_2|) \leq \min(x - \lambda_3, \lambda_1 - x)$ and equal 0 otherwise. Moreover F_k is a nonnegative function.

Proof. Recall that

$$\begin{aligned}
 \mathcal{J}_{k-\frac{1}{2}}((\mu_1 - \mu_2)z) &= \frac{\Gamma(2k)}{2^{2k-1}\Gamma(k)^2} \int_{\mathbb{R}} e^{(\mu_1-\mu_2)y} \left(1 - \frac{y^2}{z^2}\right)^{k-1} \chi_{[-1,1]} \left(\frac{y}{z}\right) z^{-1} dy, \\
 \mathcal{J}'_{k-\frac{1}{2}}((\mu_1 - \mu_2)z) &= \frac{\Gamma(2k)}{2^{2k-1}\Gamma(k)^2} \int_{\mathbb{R}} e^{(\mu_1-\mu_2)y} \left(1 - \frac{y^2}{z^2}\right)^{k-1} \frac{y}{z^2} \chi_{[-1,1]} \left(\frac{y}{z}\right) dy.
 \end{aligned}$$

Inserting these into (4) and making use the change of variables:

$$x = \frac{\nu_1 + \nu_2}{2}, \quad z = \frac{\nu_1 - \nu_2}{2},$$

with Fubini's Theorem, we obtain

$$E_k(\mu, \lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{(\mu_1+\mu_2-2\mu_3)x+(\mu_1-\mu_2)y} F_k(x, y, \lambda) dx dy$$

where

$$\begin{aligned}
 &F_k(x, y, \lambda) = \\
 &\quad \frac{6\Gamma(2k)\Gamma(3k)}{2^{2k-2}\Gamma(k)^5V(\lambda)^{2k}} \int_{\mathbb{R}} \left(z^2(\lambda_1 - \lambda_2) - y(x^2 - z^2 + \lambda_3x + \lambda_1\lambda_2) \right) \\
 &\quad \left(\frac{(\lambda_3 - x)^2 - z^2}{z^2} \right)^k \left((z^2 - y^2)(\lambda_1 - x)^2 - z^2 \right) (z^2 - (\lambda_2 - x)^2) \right)^{k-1} \\
 &\quad \chi_{[-1,1]} \left(\frac{y}{z}\right) \chi_{[\lambda_2, \lambda_1]}(x + z) \chi_{[\lambda_3, \lambda_2]}(x - z) dz \\
 &= \frac{6\Gamma(2k)\Gamma(3k)}{2^{2k-2}\Gamma(k)^5V(\lambda)^{2k}} \int_{\max(|y|, |x-\lambda_2|)}^{\min(x-\lambda_3, \lambda_1-x)} \left(z^2(y + \lambda_1 - \lambda_2) + y(\lambda_1 - x)(x - \lambda_2) \right) \\
 &\quad \left(\frac{(\lambda_3 - x)^2 - z^2}{z^2} \right)^k \left((z^2 - y^2)((\lambda_1 - x)^2 - z^2)(z^2 - (\lambda_2 - x)^2) \right)^{k-1} dz
 \end{aligned}$$

where we used the fact that

$$\chi_{[-1,1]} \left(\frac{y}{z}\right) \chi_{[\lambda_2, \lambda_1]}(x + z) \chi_{[\lambda_3, \lambda_2]}(x - z) = \chi_{\max(|y|, |x-\lambda_2|) \leq z \leq \min(x-\lambda_3, \lambda_1-x)}.$$

The change of variables

$$x = \frac{\nu_1 + \nu_2}{2}, \quad y = \frac{\nu_1 - \nu_2}{2},$$

gives

$$E_k(\mu, \lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{(\mu, \nu)} F_k \left(\frac{\nu_1 + \nu_2}{2}, \frac{\nu_1 - \nu_2}{2}, \lambda \right) d\nu_1 \nu_2.$$

Noting here that the integral is over the set $co(\lambda)$, since

$$\left\{ \nu \in \mathbb{V}; \quad \max \left(\frac{|\nu_1 - \nu_2|}{2}, \left| \frac{\nu_1 + \nu_2}{2} - \lambda_2 \right| \right) \leq \min \left(\frac{\nu_1 + \nu_2}{2} - \lambda_3, \lambda_1 - \frac{\nu_1 + \nu_2}{2} \right) \right\} = co(\lambda).$$

We shall now prove that the integrand in (15) is positive. Indeed, under the condition

$$\max(|y|, |x - \lambda_2|) \leq z \leq \min(x - \lambda_3, \lambda_1 - x), \quad (16)$$

we argue as follows. Suppose first $y + \lambda_1 - \lambda_2 \geq 0$. Observe that

$$\begin{aligned} (x - \lambda_2)^2(y + \lambda_1 - \lambda_2) + y(\lambda_1 - x)(x - \lambda_2) &= (x - \lambda_2)(\lambda_1 - \lambda_2)(y + x - \lambda_2), \\ y^2(y + \lambda_1 - \lambda_2) + y(\lambda_1 - x)(x - \lambda_2) &= y(y + \lambda_1 - x)(y + x - \lambda_2) \end{aligned}$$

and from (16)

$$\begin{aligned} &z^2(y + \lambda_1 - \lambda_2) + y(\lambda_1 - x)(x - \lambda_2) \\ &\geq \max \left((x - \lambda_2)(\lambda_1 - \lambda_2)(y + x - \lambda_2), y(y + \lambda_1 - x)(y + x - \lambda_2) \right). \end{aligned}$$

So if $x - \lambda_2 \geq 0$ and $y + x - \lambda_2 \geq 0$ then $(x - \lambda_2)(\lambda_1 - \lambda_2)(y + x - \lambda_2) \geq 0$. But if $x - \lambda_2 \geq 0$ and $y + x - \lambda_2 \leq 0$ then we must have $y \leq 0$ and $y(y + \lambda_1 - x)(y + x - \lambda_2) \geq 0$, since $y + \lambda_1 - x \geq 0$.

Similarly, if $x - \lambda_2 \leq 0$ and $y + x - \lambda_2 \leq 0$ then $(x - \lambda_2)(\lambda_1 - \lambda_2)(y + x - \lambda_2) \geq 0$ and if $x - \lambda_2 \leq 0$ and $y + x - \lambda_2 \geq 0$ then we must have $y \geq 0$ and $y(y + \lambda_1 - x)(y + x - \lambda_2) \geq 0$. This yields that in the case $y + \lambda_1 - \lambda_2 \geq 0$,

$$z^2(y + \lambda_1 - \lambda_2) + y(\lambda_1 - x)(x - \lambda_2) \geq 0.$$

However, when $y + \lambda_1 - \lambda_2 \leq 0$, we get from (16)

$$\begin{aligned} z^2(y + \lambda_1 - \lambda_2) + y(\lambda_1 - x)(x - \lambda_2) &\geq (x - \lambda_1)^2(y + \lambda_1 - \lambda_2) + y(\lambda_1 - x)(x - \lambda_2) \\ &= (\lambda_1 - x)(\lambda_1 - \lambda_2)(y + \lambda_1 - x) \geq 0. \end{aligned}$$

This achieves the proof of the positivity of F_k and so, the proof of Corollary 2.2. ■

Acknowledgments. We are grateful to the referee, whose suggestions improved the presentation of the paper.

References

- [1] Amri, B., *Note on Bessel functions of type A_{N-1}* , Integral Transforms and Special Functions **25** (2014), 448–461.
- [2] Dunkl, C. F., *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.
- [3] —, *Integral kernels with reflection group invariance*, Canadian J. Math. **43** (1991), 1213–1227.
- [4] —, *Intertwining operators associated to the group S_3* , Trans. Amer. Math. Soc. **347** (1995), 3347–3374.
- [5] de Jeu, M. F. E., *The Dunkl transform*, Invent. Math. **113** (1993), 147–162.
- [6] Opdam, E. M., *Dunkl operators, Bessel functions, and the discriminant of a finite Coxeter group*, Compos. Math. **85** (1993), 333–373.
- [7] Rösler, M., *Positivity of Dunkl’s intertwining operator*, Duke Math. J. **98** (1999), 445–463.
- [8] —, *A positive radial product formula for the Dunkl kernel*, Trans. Amer. Math. Soc. **355** (2003), 2413–2438.
- [9] —, *Dunkl operators: Theory and applications*, in: Orthogonal Polynomials and Special Functions (Leuven, 2002), Lecture Notes in Math. **1817**, Springer, Berlin, 2003, 93–136,

Bechir Amri
Université de Tunis El Manar
Faculté des Sciences de Tunis
Laboratoire d’Analyse
Mathématiques et Applications
LR11ES11, 2092-El Manar I, Tunisia
bechir.amri@ipeit.rnu.tn

Received April 8, 2015
and in final form May 11, 2016