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Manifolds Admitting a Continuous Cancellative Binary Operation are Orientable

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Abstract. Generalizing the well-known result on the orientability of Lie groups, we prove that a topological manifold (possibly with boundary) admitting a continuous cancellative binary operation is orientable. This implies that the Möbius band admits no cancellative continuous binary operation and answers a question posed by the second author in 2010.
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Introduction

It is known that the presence of a compatible algebraic structure on a topological space imposes some restrictions on its topological structure. This phenomenon can be illustrated by the famous Adams' Theorem [1] saying that a sphere S^n of finite dimension $n \notin \{0, 1, 3, 7\}$ admits no continuous binary operation with two-sided unit. Each connected locally path connected space admitting a continuous semilattice operation has trivial homotopy groups (this follows from Lemma 2 in [6]). Another result of the same flavor says that a topological manifold admitting a continuous (quasi)group operation is orientable, see [17, 15.19], [9, 15.10], [19, 1.45]. This implies that underlying manifolds of Lie groups are orientable.

In this paper we shall generalize the latter result proving that a topological manifold M (possibly with boundary) carrying a continuous cancellative binary operation is (isotopically) orientable. The notion of isotopically orientable topological space is introduced in Section 1 where we prove that for topological manifolds the isotopical orientability is equivalent to the orientability in the classical sense. In Section 2 we prove that a locally compact locally connected metrizable topological space is isotopically orientable provided it admits a continuous binary operation with open injective shifts and some extra property expressing the continuous cancellativity of the operation. Applying this criterion to topological manifolds, we obtain our main result which says that a topological manifold admitting a continuous cancellative binary operation is orientable.

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1. Isotopically orientable manifolds

In this section we introduce the notion of isotopical orientability of manifolds.

Let E be a fixed topological space called a *model space*. By an *E-manifold* we understand a paracompact topological space M such that each point $x \in M$ has an open neighborhood U_x that admits an open topological embedding $\varphi_x \colon U_x \to E$, called an *open chart*. The family of pairs $\{(U_x, \varphi_x) : x \in M\}$ is called an *atlas* on M. For points $x, y \in X$ the function

$$\varphi_{x,y} = \varphi_y \circ \varphi_x^{-1} | \varphi_x(U_x \cap U_y) : \varphi_x(U_x \cap U_y) \to E$$

is called the *transition function*. This function is an open topological embedding.

An E-manifold M is defined to be *isotopically E-orientable* if it admits an atlas $\{(U_x, \varphi_x) : x \in M\}$ such that for any points $x, y \in M$, every point $z \in \varphi_x(U_x) \cap \varphi_y(U_y)$ has an open neighborhood $U_z \subset \varphi_x(U_x \cap U_y) \subset E$ such that the transition map $\varphi_{x,y}|U_z : U_z \to \varphi_y(U_y) \subset E$ is open-isotopic to the identity embedding $\mathrm{id} : U_z \to E$. The latter means that there is a homotopy $(h_t)_{t \in [0,1]} : U_z \to E$ such that $h_0 = \varphi_{x,y}|U_z$, $h_1 = \mathrm{id}$ and each map $h_t : U_z \to E$ is an open embedding. Such homotopies will be called *open-isotopies*.

Each model space E is an isotopically E-oriented E-manifold with the single chart $\mathrm{id} : E \to E$. In particular, the open Möbius band $\mathbb{M}$ is an isotopically $\mathbb{M}$-oriented $\mathbb{M}$-manifold. However, considered as an $\mathbb{R}^2$-manifold, the Möbius band $\mathbb{M}$ is not isotopically $\mathbb{R}^2$-oriented.

In order to kill such an ambiguity, we introduce the following

Definition 1. A paracompact space X is called *isotopically orientable* if there is a point $e \in X$ such that for any open neighborhood $E \subset X$ of e the space X is an E-orientable E-manifold.

If a topological space X is an E-orientable E-manifold for some open set $E \subset X$, then it is also an E'-orientable E'-manifold for any open subset $E' \subset X$ containing the set E. Consequently, a paracompact space X is isotopically orientable if and only if X contains a point $e \in X$ that has a neighborhood base $\mathcal{B}_e$ such that for any neighborhood $E \in \mathcal{B}_e$ the space X is an E-orientable E-manifold.

Now our aim to show that for topological manifolds the isotopical orientability is equivalent to the usual orientability (defined via homologies). By a *topological manifold* we understand an $\mathbb{R}^n_+$-manifold modeled on the half-space

$$\mathbb{R}^n_+ = \{(x_i)_{i=1}^n \in \mathbb{R}^n : x_n \geq 0\}$$

for some $n \in \mathbb{N}$. The *boundary* ∂M of an $\mathbb{R}^n_+$-manifold M consists of all points $x \in M$ that have no open neighborhoods homeomorphic to $\mathbb{R}^n$.
An $\mathbb{R}^n$-manifold M is called *orientable* if it admits an atlas $\{(U_x, \varphi_x) : x \in M\}$ such that for any points $x, y \in M$ the transition function

$$\varphi_{x,y} : \varphi_x(U_x \cap U_y) \to \mathbb{R}^n, \quad \varphi_{x,y} : z \mapsto \varphi_y \circ \varphi_x^{-1}(z),$$

is orientation-preserving. An embedding $f : U \to \mathbb{R}^n$ of an open subset $U \subset \mathbb{R}^n$ is called *orientation-preserving* if each point $z \in U$ has an open neighborhood $U_z \subset U$ such that the homomorphism in n-th homologies

$$H_n f : H_n(U_z, U_z \setminus \{z\}; \mathbb{Z}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{f(z)\}; \mathbb{Z})$$

induced by the map $f|U_z$ coincides with the homomorphism

$$H_n s : H_n(U_z, U_z \setminus \{z\}; \mathbb{Z}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{f(z)\}; \mathbb{Z})$$

induced by the shift $s : U_z \to \mathbb{R}^n, \quad s : u \mapsto f(z) - z + u$.

Theorem 1. *An $\mathbb{R}^n$-manifold M is orientable if and only if it is isotopically orientable.*

Proof. To prove the "if" part, assume that the $\mathbb{R}^n$-manifold M is isotopically orientable. Then there is a point $e \in M$ such that for every open neighborhood $E \subset M$ of e the space M is an E-orientable E-manifold. Choose an open neighborhood $E \subset M$ of e homeomorphic to $\mathbb{R}^n$. To simplify notation we identify E with the Euclidean space $\mathbb{R}^n$.

Let $\{(U_x, \varphi_x) : x \in M\}$ be an atlas witnessing that M is an E-orientable E-manifold. Given any points $x, y \in M$, we should check that the transition map $\varphi_{x,y} : \varphi_x(U_x \cap U_y) \to \varphi_y(U_x \cap U_y)$ is orientation-preserving. Given any point $z \in \varphi_x(U_x \cap U_y)$ the E-orientability of M yields an open neighborhood $U_z \subset \varphi_x(U_x \cap U_y)$ and an open-isotopy $(i_t)_{t\in[0,1]} : U_z \to E$ connecting the map $i_0 = \varphi_{x,y}|U_z : U_z \to \varphi_y(U_y) \subset E$ with the identity inclusion $i_1 : U_z \to \varphi_x(U_x) \subset E$ of U_z. The isotopy (i_t) induces the isotopy $(j_t) : U_z \to \mathbb{R}^n$ defined by the formula $j_t(u) = i_t(u) + i_0(z) - i_t(z)$ for $t \in [0, 1]$ and $u \in U_z$. In fact, the isotopy (j_t) is a homotopy of the pairs $(j_t) : (U_z, U_z \setminus \{z\}) \to (E, E \setminus \{\varphi_{x,y}(z)\})$. By the homotopical invariance of homology, the maps $\varphi_{x,y} = i_0 = j_0$ and $j_1 : u \mapsto u - z + \varphi_{x,y}(u)$ induce the same homomorphism in n-th homologies

$$H_n j_0 = H_n j_1 : H_n(U_z, U_z \setminus \{x\}) \to (E, E \setminus \{\varphi_{x,y}(z)\}),$$

witnessing that $\varphi_{x,y}$ is orientation-preserving and M is orientable.

To prove the "only if"part, assume that the $\mathbb{R}^n$-manifold M is orientable. Fix any point $e \in M$. Given any open neighborhood $E \subset M$ of x we shall prove that M is an E-orientable E-manifold. Replacing E by a smaller open neighborhood, we can assume that E is homeomorphic to $\mathbb{R}^n$. To simplify notation, we shall identify E with $\mathbb{R}^n$. Since M is an orientable $\mathbb{R}^n$-manifold, there exists an atlas $\{(U_x, \varphi_x) : x \in M\}$ such that for any points $x, y \in M$ the transition map $\varphi_{x,y} : \varphi_x(U_x \cap U_y) \to \varphi_y(U_x \cap U_y)$ is orientation preserving. This atlas witnesses that M is an E-manifold. To show that this E-manifold is E-orientable, for any points $x, y \in M$ and $z \in \varphi_x(U_x \cap U_y)$ we should find a neighborhood $U_z \subset \varphi_x(U_x \cap U_y)$ such that the map $\varphi_{x,y}|U_z : U_z \to \varphi_y(U_x \cap U_y) \subset E$ is open-isotopic to the identity embedding $U_z \subset \varphi_x(U_x \cap U_y) \subset E$. Choose two open neighborhoods $U_z \subset W_z$ of z in $\varphi_x(U_x \cap U_y)$ such that the pair $(W_z, \overline{U}_z)$ is homeomorphic to the pair of balls $(B(2), \overline{B}(1))$ in $\mathbb{R}^n$. Here for $r > 0$ by $B(r) = \{a \in \mathbb{R}^n : \|a\| < r\}$ and $\overline{B}(r) = \{a \in \mathbb{R}^n : \|a\| \le r\}$ we denote the open and closed r-balls centered at the origin. It follows that the boundaries ∂U_z and

$\varphi_{x,y}(\partial U_z)$ are bicollared topological $(n-1)$-spheres in $\mathbb{R}^n$. Choose a positive real number R so large that $\overline{U_z} \cup \varphi_{x,y}(\overline{U_z}) \subset B(R)$. The Annulus Theorem (proved by combined efforts of Radó [20], Moise [18], Kirby [16] and Quinn [21]), allows us to extend the homeomorphism $\varphi_{x,y}|\overline{U}_z$ to a homeomorphism h of the closed ball $\overline{B}(R)$. Since the embedding $\varphi_{x,y}|U_z : U_z \to B(R)$ preserves the orientation, the homeomorphism h preserves the orientation of $\overline{B}(R)$ and the restriction $h|S(R)$ preserves the orientation of the sphere $S(R) = \overline{B}(R) \setminus B(R)$. By Brown-Gluck Theorem [8], the Annulus Theorem implies the Stable Homeomorphism Theorem, which implies that the homeomorphism group of the sphere $S(R)$ has two connected components and hence the orientation-preserving homeomorphism $h|S(R)$ is isotopic to the identity homeomorphism. This isotopy can be used to show that the homeomorphism h can be replaced by a homeomorphism of $\overline{B}(R)$, which coincides with $\varphi_{x,y}|\overline{U}_z$ on $\overline{U}_z$ and is identity on the sphere $S(R)$. Now the Alexander trick [2] (see also Lemma 5.6 in [12]) guarantees that h is isotopic to the identity homeomorphism of $\bar{B}(R)$ and hence $\varphi_{x,y}|U_z = h|U_z$ is open-isotopic to the identity embedding $i : U_z \to E = \mathbb{R}^n$. This means that the E-manifold M is E-orientable. ■

An $\mathbb{R}^n_+$-manifold M is defined to be *orientable* if its interior $M \setminus \partial M$ is an orientable $\mathbb{R}^n$-manifold. Theorem 1 implies:

Corollary 1. *A topological manifold M is orientable if and only if its interior $M \setminus \partial M$ is isotopically orientable.*

2. Isotopical orientability of spaces and manifolds with a compatible algebraic structure

Let us recall that a map $f : X \to Y$ between topological spaces is *open* if for every open set $U \subset X$ the image $f(U)$ is open in Y. It is clear that each injective open continuous map is a topological embedding.

Theorem 2. *A locally compact locally connected metrizable space S is isotopically orientable if S admits a continuous binary operation $\cdot : S \times S \to S$ and a point $e \in S$ such that*

1) *the right shift $\rho_e : S \to S$, $\rho_e : x \mapsto xe$, is injective and open;*
2) *for every $a \in S$ the left shift $\lambda_a : S \to S$, $\lambda_a : x \mapsto ax$, is injective and open;*
3) *for every $a \in S$ and neighborhood $O_e \subset S$ of e there is a neighborhood $U_a \subset S$ of a such that $\bigcap_{x\in U_a} xO_e$ is a neighborhood of the point ae in S.*

Proof. Since the space S is locally connected, the connected components of S are closed-and-open in S. Now we see that it suffices to prove that each connected component X of S is an E-orientable E-manifold for any open neighborhood E of the point e.

The metrizable space X, being connected and locally compact, can be written as the countable union $X = \bigcup_{n\in\omega} K_n$ of compact subsets $K_n \subset X$ such that $K_0 = \emptyset$ and each K_n lies in the interior K°_{n+1} of K_{n+1} in X.

By our hypothesis, the right shift $\rho_e : X \to S$ is an open embedding. Since the image $Xe = \rho_e(X)$ is homeomorphic to X, it suffices to check that Xe is an E-orientable E-manifold.

Fix a metric d generating the topology of the space S. For a point $x \in S$ and a real number $r > 0$ denote by $B_x(r) = \{y \in S : d(x,y) < r\}$ the open r-ball centered at x and let $\overline{B}_x(r)$ be the closure of $B_x(r)$ in S. The r-balls $B_e(r)$ and $\overline{B}_e(r)$ centered at the point e will be denoted by $B(r)$ and $\bar{B}(r)$, respectively.

Find $\varepsilon_0 > 0$ such that the ball $B(\varepsilon_0)$ has compact closure in the neighborhood E of e. Using the condition (3) of the theorem and the compactness of the sets K_n and $K_n \setminus K^\circ_{n-1}$, $n \in \mathbb{N}$, it is easy to construct a decreasing sequence of positive real numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$ the following conditions are satisfied:

(a) $\overline{B}_x(\varepsilon_n) \subset K^\circ_{n+1} \setminus K_{n-2}$ for every $x \in K_n \setminus K^\circ_{n-1}$;
(b) $x \cdot \overline{B}(\varepsilon_n) \subset (K^\circ_{n+1} \setminus K_{n-2}) \cdot e$ for every $x \in K_n \setminus K^\circ_{n-1}$;
(c) $x \cdot \overline{B}(\varepsilon_n) \subset \bigcap_{y \in B_x(\varepsilon_n)} y \cdot B(\varepsilon_0)$ for every $x \in K_n$;
(d) for all points $x, y \in K_n$ with $x \cdot \overline{B}(\varepsilon_n) \cap y \cdot \overline{B}(\varepsilon_n) \neq \emptyset$, we get $d(x,y) < \varepsilon_{n-1}$;
(e) for all points $x, y \in K_n$ with $d(x,y) < \varepsilon_n$ there is a continuous path $\pi : [0,1] \to X$ of diameter $< \varepsilon_{n-1}$ such that $\pi(0) = x$ and $\pi(1) = y$.

Now we shall construct an atlas $\{(U_{xe}, \varphi_{xe}) : xe \in Xe\}$ witnessing that Xe is an E-orientable E-manifold.

For every point $x \in X$ find $n_x \in \mathbb{N}$ with $x \in K_{n_x} \setminus K_{n_x-1}$ (we recall that $K_0 = \emptyset$). The property (b) guarantees that $x \cdot \overline{B}(\varepsilon_{n_x+5}) \subset x \cdot \overline{B}(\varepsilon_{n_x}) \subset K_{n_x+1} \cdot e \subset Xe$ and hence

$$U_{xe} := x \cdot B(\varepsilon_{n_x+5}) = \lambda_x(B(\varepsilon_{n_x+5})) \subset Xe$$

is an open neighborhood of the point xe in Xe. Define the chart

$$\varphi_{xe}\colon U_{xe} \to B(\varepsilon_{n_x+5}) \subset E \text{ by the formula } \varphi_{xe} = \lambda_x^{-1}|U_{xe}$$

and observe that the map $\varphi_{xe} : U_{xe} \to E$ is an open embedding being the inverse map to the open embedding $\lambda_x|B(\varepsilon_{n_x+5}) : B(\varepsilon_{n_x+5}) \to U_{xe} \subset Xe$.

We claim that the atlas $\{(U_{xe}, \varphi_{xe}) : xe \in Xe\}$ witnesses that Xe is an E-oriented E-manifold. Take any two points $xe, ye \in Xe$ such that $U_{xe} \cap U_{ye} \neq \emptyset$ and consider the transition function

$$\varphi_{xe,ye} : \varphi_{xe}(U_{xe} \cap U_{ye}) \to \varphi_{ye}(U_{xe} \cap U_{ye}) \subset E, \quad \varphi_{xe,ye} : z \mapsto \varphi_{ye} \circ \varphi_{xe}^{-1}(z) = \lambda_y^{-1} \circ \lambda_x(z)$$

defined on the open subset $\varphi_{xe}(U_{xe} \cap U_{ye})$ of E. Given any point $z \in \varphi_{xe}(U_{xe} \cap U_{ye})$, we need to find a neighborhood $U_z \subset \varphi_{xe}(U_{xe} \cap U_{ye})$ such that the map

$$\varphi_{xe,ye}|U_z : U_z \to E$$

is open-isotopic to the identity embedding $\mathrm{id} : U_z \to E$. Fix any neighborhood U_z of z with compact closure $\overline{U}_z$ in $\varphi_{xe}(U_{xe} \cap U_{ye})$.

Let $n_x, n_y \in \mathbb{N}$ be the unique numbers such that $x \in K_{n_x} \setminus K_{n_x-1}$ and $y \in K_{n_y} \setminus K_{n_y-1}$. We claim that $|n_x - n_y| \le 2$. The property (b) guarantees

$$U_{xe} = x \cdot B(\varepsilon_{n_x+5}) \subset x \cdot B(\varepsilon_{n_x}) \subset (K_{n_x+1} \setminus K_{n_x-2}) \cdot e \quad \text{and} \quad U_{ye} \subset (K_{n_y+1} \setminus K_{n_y-2}) \cdot e.$$

Since $U_{xe} \cap U_{ye} \neq \emptyset$, the injectivity of the shift ρ_e implies

$$(K_{n_x+1} \setminus K_{n_x-2}) \cap (K_{n_y+1} \setminus K_{n_y-2}) \neq \emptyset,$$

which yields $|n_x - n_y| \le 2$. Then for the number $n = \max\{n_x, n_y\}$ we get $x, y \in K_n \setminus K_{n-3}$. Also $U_{xe} = x \cdot B(\varepsilon_{n_x+5}) \subset x \cdot B(\varepsilon_{n+3})$.

Since $\emptyset \ne U_{xe} \cap U_{ye} = x \cdot B(\varepsilon_{n_x+5}) \cap y \cdot B(\varepsilon_{n_y+5}) \subset x \cdot B(\varepsilon_{n+3}) \cap y \cdot B(\varepsilon_{n+3})$, the item (d) ensures that $d(x,y) < \varepsilon_{n+2}$. By the item (e), the points x, y can be linked by a continuous path $\pi : [0,1] \to S$ of diameter $< \varepsilon_{n+1}$ such that $\pi(0) = x$, $\pi(1) = y$.

Now we are able to prove that the map $\varphi_{xe,ye}|U_z : U_z \to E$ is open-isotopic to the identity embedding $U_z \to E$. For this consider the continuous map $g : [0,1] \times \overline{B}(\varepsilon_0) \to [0,1] \times S$, $g : (t, z) \mapsto (t, \pi(t) \cdot z)$, defined on the compact set $[0,1] \times \overline{B}(\varepsilon_0)$. The injectivity of the left shifts on S implies that the map g is injective and hence is a topological embedding.

The choice of the path π guarantees that $\pi([0,1]) \subset B_x(\varepsilon_{n+1})$. Then condition (c) implies that

$$x\overline{U}_z = \varphi_{xe}^{-1}(\overline{U}_z) \subset U_{xe} \subset x \cdot B(\varepsilon_{n+1}) \subset \bigcap_{t \in [0,1]} \pi(t) \cdot B(\varepsilon_0) = \bigcap_{t \in [0,1]} \lambda_{\pi(t)}(B(\varepsilon_0)).$$

Consequently, the map $g^{-1}|[0,1] \times x\overline{U}_z : [0,1] \times x\overline{U}_z \to [0,1] \times \overline{B}(\varepsilon_0)$ is a topological embedding and the map

$$h : [0,1] \times \overline{U}_z \to [0,1] \times \overline{B}(\varepsilon_0), \quad h : (t, u) \mapsto (t, \lambda_{\pi(t)}^{-1} \circ \lambda_x(u))$$

is a well-defined topological embedding of $[0,1] \times \overline{U}_z$ into $[0,1] \times \overline{B}(\varepsilon_0) \subset [0,1] \times E$. Since for every $t \in [0,1]$ the map $\lambda_{\pi(t)}^{-1} \circ \lambda_x | U_z : U_z \to E$ is an open embedding, the family $\big(\lambda_{\pi(t)}^{-1} \circ \lambda_x | U_z\big)_{t \in [0,1]}$ is an open-isotopy linking the transition map $\varphi_{xe,ye} = \lambda_{\pi(1)}^{-1} \circ \lambda_x | U_z$ with the identity map $\lambda_{\pi(0)}^{-1} \circ \lambda_x | U_z$ of U_z. ∎

Applying Theorems 1 and 2 to topological manifolds, we get the following theorem.

Theorem 3. *A topological manifold M is orientable provided M admits a continuous binary operation $\cdot : M \times M \to M$ such that*

1) *for some $e \in M \setminus \partial M$ the right shift $\rho_e : M \to M$ is injective;*
2) *for every $x \in M \setminus \partial M$ the left shift $\lambda_x : M \to M$ is injective.*

Proof. To show that an $\mathbb{R}^n_+$-manifold M is orientable, we need to check that its interior $N = M \setminus \partial M$ is orientable.

By the Open Domain Principle, each continuous injective map $f : U \to \mathbb{R}^n$ defined on an open subset $U \subset \mathbb{R}^n$ is an open topological embedding. This implies that the shifts $\rho_e : N \to M$ and $\lambda_x : N \to M$, $x \in N$, are open topological embeddings. Moreover, $\rho_e(N) \subset N$ and $\lambda_x(N) \subset N$ for all $x \in N$, which means that the interior N of M is closed under the binary operation. Now we see that the binary operation restricted to N satisfies the conditions (1) and (2) of Theorem 2. It remains to check the condition (3). Fix any open neighborhood $O_e \subset N$ of the point e. Given any point $x \in N$, we should find a neighborhood $U_x \subset N$ such that $\bigcap_{u \in U_x} uO_e$ is a neighborhood of x in N. Since $xO_e = \lambda_x(O_e)$ is an open neighborhood of the point xe in the $\mathbb{R}^n$-manifold N, we can find a homeomorphism $\psi : O_{xe} \to \mathbb{R}^n$ defined on some open neighborhood $O_{xe} \subset \lambda_x(O_e)$ of xe such that

$\psi(xe) = \vec{0}$. Consider the open unit ball $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$ and its closure $\bar{B} = \{x \in \mathbb{R}^n : \|x\| \le 1\}$ in $\mathbb{R}^n$. Then $U_e = \lambda_x^{-1}(\psi^{-1}(B))$ is an open neighborhood of e whose closure $\bar{U}_e = \lambda_x^{-1}(\psi^{-1}(\bar{B})) \subset \lambda_x^{-1}(O_{xe}) \subset O_e$ is homeomorphic to $\bar{B}$ and $x\bar{U}_e = \lambda_x(\bar{U}_e) = \psi^{-1}(\bar{B}) \subset O_{xe}$. Consider the homeomorphism $h : \bar{U}_e \to \bar{B}$, $h : z \mapsto \psi(\lambda_x(z))$. Using the compactness of $\bar{U}_e$ and the continuity of the binary operation, we can find a neighborhood $U_x \subset N$ of x such that $U_x \cdot \bar{U}_e \subset O_{xe}$ and $\|\psi(u \cdot z) - \psi(x \cdot z)\| < \frac{1}{2}$ for all $u \in U_x$ and $z \in \bar{U}_e$. We claim that the intersection $\bigcap_{u \in U_x} uO_e$ contains the neighborhood $\psi^{-1}(\frac{1}{2}B)$ of xe. Given any points $u \in U_x$ and $b_0 \in \frac{1}{2}B = \{b \in \mathbb{R}^n : \|b\| < \frac{1}{2}\}$, we shall check that $\psi^{-1}(b_0) \in uO_e$. For this consider the continuous map $f : \bar{B} \to \bar{B}$ defined by the formula

$$f(b) = b_0 + b - \psi \circ \lambda_u(h^{-1}(b)) = b_0 + \psi(x \cdot h^{-1}(b)) - \psi(u \cdot h^{-1}(b))$$

for $b \in \bar{B}$. The Brouwer Fixed Point Theorem [10, 7.3.18] guarantees that the map f has a fixed point $b \in \bar{B}$. Then the point $z = h^{-1}(b) \in \bar{U}_e \subset O_e$ has the property $f(h(z)) = f(b) = b = b_0 + b - \psi \circ \lambda_u(z)$, which implies $b_0 = \psi \circ \lambda_u(z)$ and $\psi^{-1}(b_0) = \lambda_u(z) = uz \in uO_e$. Therefore, $\bigcap_{w \in U_x} uO_e \supset \psi^{-1}(\frac{1}{2}B)$ is a neighborhood of x in N. Now it is legal to apply Theorem 2 and conclude that the $\mathbb{R}^n$-manifold N is isotopically orientable. By Theorem 1, it is orientable and then the topological manifold $M = N \cup \partial M$ is orientable as well. ■

A binary operation $\cdot : X \times X \to X$ is called a *left-loop operation* if it has a right unit $e \in X$ (which means that $xe = x$ for all $x \in X$) and for every $a \in X$ the left shift $\lambda_a : X \to X$, $\lambda_a : x \mapsto ax$, is bijective. A *topological left-loop* is a topological space X endowed with a continuous left-loop operation $\cdot : X \times X \to X$ such that the map $X \times X \to X \times X$, $(x, y) \mapsto (x, xy)$, is a homeomorphism, see [14]. A topological left-loop with a two-sided unit is called a topological lop, see [3], [4], [5]. By Theorem 3.2 of [5], a topological space X is homeomorphic to a topological left-loop if and only if it is homeomorphic to a topological lop. Applying Theorem 3 to left-loop operations we get the following corollary implying that topological manifolds homeomorphic to topological left-loops are orientable (for topological loops this orientability result was proved in Theorem 1.44 [19]).

Corollary 2. *Each topological manifold admitting a continuous left-loop operation is orientable.*

A binary operation $\cdot : X \times X \to X$ on a set X is called *left cancellative* (resp. *right cancellative*) if for any points $x, y, z \in X$ the equality $zx = zy$ (resp. $xz = yz$) implies $x = y$. This is equivalent to saying that for every $z \in X$ the left shift $\lambda_z : X \to X$ (resp. the right shift $\rho_z : X \to X$) is injective. A binary operation is *cancellative* if it is both left and right cancellative. The structure of cancellative topological semigroups on manifolds was studied by Brown, Houston [7] and Hofmann, Weiss [15].

Applying Theorem 3 to cancellative operations, we get the following generalization of the mentioned result on orientability of Lie (quasi)groups (see Corollary 1.45 of [19]).

Corollary 3. *Each topological manifold admitting a cancellative continuous binary operation is orientable.*

We say that an element $e \in X$ is a *right unit* for a binary operation on a set X is $xe = x$ for all $x \in X$.

Corollary 4. *A topological manifold M is orientable if it admits a left cancellative continuous binary operation possessing a right unit $e \in M \setminus \partial M$.*

Remark 1. Corollary 3 implies that the Möbius band $\mathbb{M}$ admits no cancellative continuous binary operation. Yet, $\mathbb{M}$ has a natural structure of an abelian topological inverse monoid, see [14]. To construct such a structure on $\mathbb{M}$, consider the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ endowed with the operation of multiplication of complex numbers and the unit interval $[0,1]$ endowed with the operation of minimum. Then the product $\mathbb{T} \times [0,1]$ is a commutative compact topological inverse monoid, $(1,1)$ is the unit of $\mathbb{T} \times [0,1]$, and $\mathbb{T} \times \{0\}$ is the minimal ideal of $\mathbb{T} \times [0,1]$. On $\mathbb{T} \times [0,1]$ consider the congruence $\sim$ identifying the points $(z,0)$ and $(-z,0)$ for $z \in \mathbb{T}$. Then the quotient semigroup $\mathbb{M} = \mathbb{T} \times [0,1]$ is homeomorphic to the Möbius band and is a commutative compact topological inverse monoid.

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