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Cohomology of the Heisenberg-Virasoro Conformal Algebra

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Abstract. We compute the cohomology groups of the Heisenberg-Virasoro conformal algebra with coefficients in its modules $\mathbb{C}, \mathbb{C}_a$ and $M_{\Delta,\alpha,\beta}$. In particular, we completely determine the cohomology groups with trivial coefficients both for the basic and reduced complexes.
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1. Introduction

The notion of Lie conformal algebra, introduced by Kac in [4], encodes an axiomatic description of the operator product expansion of chiral fields in conformal field theory. Lie conformal algebras are closely related to infinite-dimensional Lie algebras such as affine Kac-Moody algebras and the Virasoro algebra, as well as to vertex algebras. They have been extensively studied in [3, 5, 7, 8, 10, 11]. In particular, a general cohomology theory of conformal algebras with coefficients in an arbitrary conformal module was developed in [1], where explicit computations of cohomologies for the Virasoro conformal algebra and current conformal algebra were given. The cohomologies (with degree ≤ 3) of the general Lie conformal algebras gc_N were studied in [6]. However, the cohomology problem for non-semisimple Lie conformal algebras has not yet studied. It is highly nontrivial. In [9], we computed the cohomologies of the Lie conformal algebra of $W(2,2)$-type with trivial coefficients. It turned out that the nontrivial cohomology group is at most of degree 6, which is different from that in the Virasoro conformal algebra case.

In this paper, we aim to study the cohomology of the Heisenberg-Virasoro conformal algebra, which is another non-semisimple conformal algebra of rank 2. We obtain a different and interesting result on the cohomology groups with trivial

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coefficients. In addition, our methods may be useful to study the cohomologies of other non-semisimple conformal algebras of rank ≥ 2. This motivates us to present this study.

The Heisenberg-Virasoro conformal algebra was introduced in [8] as a Lie conformal algebra associated with the twisted Heisenberg-Virasoro Lie algebra. By definition, the Heisenberg-Virasoro conformal algebra is a free Lie conformal algebra with a $\mathbb{C}[\partial]$-basis $\{L, M\}$ and satisfying

$$[L_\lambda L] = (\partial + 2\lambda)L, \quad [L_\lambda M] = (\partial + \lambda)M, \quad [M_\lambda L] = \lambda M, \quad [M_\lambda M] = 0. \tag{1}$$

In this paper, we denote by $\mathcal{HV}$ the Heisenberg-Virasoro conformal algebra. It is easy to see that $\mathcal{HV}$ contains the Virasoro conformal algebra Vir as a subalgebra, which is a free $\mathbb{C}[\partial]$-module generated by L such that

$$\mathrm{Vir} = \mathbb{C}[\partial]L, \quad [L_\lambda L] = (\partial + 2\lambda)L. \tag{2}$$

Moreover, $\mathcal{HV}$ has a nontrivial abelian conformal ideal with one free generator M as a $\mathbb{C}[\partial]$-module. Thus $\mathcal{HV}$ is neither simple nor semisimple.

The paper is organized as follows. In Section 2, we recall the notions of Lie conformal algebra, conformal module and cohomology of Lie conformal algebras, and present the main results of this paper (see Theorem 2.6). Section 3 is devoted to the proof of the main theorem.

Throughout this paper, all vector spaces and tensor products are over the complex field $\mathbb{C}$. We use notations $\mathbb{Z}$ for the set of integers and $\mathbb{Z}_+$ for the set of nonnegative integers.

2. Preliminaries and Main results

In this section, we recall the definition of a Lie conformal algebra and a (conformal) module over it and cohomology with coefficients in an arbitrary module. Then we list our main results of this paper.

Definition 2.1. ([4]) A Lie conformal algebra $\mathcal{R}$ is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-bilinear map $\mathcal{R} \otimes \mathcal{R} \to \mathbb{C}[\lambda] \otimes \mathcal{R}$, $a \otimes b \mapsto [a_\lambda b]$, and satisfying the following axioms $(a, b, c \in \mathcal{R})$,

$$[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b] \quad \text{(conformal sesquilinearity)}, \tag{3}$$

$$[a_\lambda b] = -[b_{-\lambda-\partial} a] \quad \text{(skew-symmetry)}, \tag{4}$$

$$[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]] \quad \text{(Jacobi identity)}. \tag{5}$$

Definition 2.2. ([2]) A module V over a Lie conformal algebra $\mathcal{A}$ is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-bilinear map $\mathcal{A} \otimes V \to V[[\lambda]]$, $a \otimes v \mapsto a_\lambda v$, satisfying the following relations for $a, b \in \mathcal{A}$, $v \in V$,

$$a_\lambda(b_\mu v) - b_\mu(a_\lambda v) = [a_\lambda b]_{\lambda+\mu} v,$$
$$(\partial a)_\lambda v = -\lambda a_\lambda v, \; a_\lambda(\partial v) = (\partial + \lambda) a_\lambda v.$$

If $a_\lambda v \in V[\lambda]$ for all $a \in \mathcal{A}$, $v \in V$, then V is called conformal. If V is finitely generated over $\mathbb{C}[\partial]$, then V is simply called finite.

Since we only consider conformal modules, we will simply shorten the term "conformal module" to "module". The vector space $\mathbb{C}$ is viewed as a trivial module with trivial actions of ∂ and $\mathcal{A}$. For a fixed nonzero complex constant a, there is a natural $\mathbb{C}[\partial]$-module $\mathbb{C}_a$, such that $\mathbb{C}_a = \mathbb{C}$ and $\partial v = av$ for $v \in \mathbb{C}_a$. Then $\mathbb{C}_a$ becomes an $\mathcal{A}$-module with $\mathcal{A}$ acting by zero.

For the Virasoro conformal algebra Vir (cf. (2)), it was shown in [2] that all the free nontrivial Vir-modules of rank 1 over $\mathbb{C}[\partial]$ are the following ones ($\Delta, \alpha \in \mathbb{C}$):

$$M_{\Delta,\alpha} = \mathbb{C}[\partial]v, \quad L_\lambda v = (\partial + \alpha + \Delta\lambda)v. \tag{6}$$

The module $M_{\Delta,\alpha}$ is irreducible if and only if $\Delta \neq 0$. The module $M_{0,\alpha}$ contains a unique nontrivial submodule $(\partial + \alpha)M_{0,\alpha}$ isomorphic to $M_{1,\alpha}$. Moreover, the modules $M_{\Delta,\alpha}$ with $\Delta \neq 0$ exhaust all finite irreducible nontrivial Vir-modules.

From the proof of [8, Theorem 4.5 (1)], we have

Proposition 2.3. *All free nontrivial $\mathcal{HV}$-modules of rank 1 over $\mathbb{C}[\partial]$ are the following ones:*

$$M_{\Delta,\alpha,\beta} = \mathbb{C}[\partial]v,\ L_\lambda v = (\partial + \alpha + \Delta\lambda)v,\ M_\lambda v = \beta v,\ \textit{for some } \Delta, \alpha, \beta \in \mathbb{C}.$$

Definition 2.4. ([1]) An n-cochain ($n \in \mathbb{Z}_+$) of a Lie conformal algebra $\mathcal{A}$ with coefficients in an $\mathcal{A}$-module V is a $\mathbb{C}$-linear map

$$\gamma : \mathcal{A}^{\otimes n} \to V[\lambda_1, \cdots, \lambda\lambda_n], \quad a_1 \otimes \cdots \otimes a_n \mapsto \gamma_{\lambda_1,\cdots,\lambda_n}(a_1, \cdots, a_n)$$

satisfying the following conditions:

(1) $\gamma_{\lambda_1,\cdots,\lambda_n}(a_1, \cdots, \partial a_i, \cdots, a_n) = -\lambda_i \gamma_{\lambda_1,\cdots,\lambda_n}(a_1, \cdots, a_n)$ (conformal antilinearity),

(2) γ is skew-symmetric with respect to simultaneous permutations of a_i's and λ_i's (skew-symmetry).

As usual, let $\mathcal{A}^{\otimes 0} = \mathbb{C}$, so that a 0-cochain is an element of V. Denote by $\tilde{C}^n(\mathcal{A}, V)$ the set of all n-cochains. The differential d of an n-cochain γ is defined as follows:

$$(d\gamma)_{\lambda_1,\cdots,\lambda_{n+1}}(a_1, \cdots, a_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} a_{i_{\lambda_i}} \gamma_{\lambda_1,\cdots,\hat{\lambda_i},\cdots,\lambda_{n+1}}(a_1, \cdots, \hat{a_i}, \cdots, a_{n+1})$$
$$+ \sum_{i,j=1; i<j}^{n+1} (-1)^{i+j} \gamma_{\lambda_i+\lambda_j,\lambda_1,\cdots,\hat{\lambda_i},\cdots,\hat{\lambda_j},\cdots,+\lambda_{n+1}}([a_{i_{\lambda_i}} a_j], a_1, \cdots, \hat{a_i}, \cdots, \hat{a_j}, \cdots, a_{n+1}), \tag{7}$$

where γ is linearly extended over the polynomials in λ_i. In particular, if $\gamma \in V$ is a 0-cochain, then $(d\gamma)_\lambda(a) = a_\lambda \gamma$.

It was shown in [1] that the operator d preserves the space of cochains and satisfies $d^2 = 0$. Thus the cochains of a Lie conformal algebra $\mathcal{A}$ with coefficients in an $\mathcal{A}$-module V form a complex, called the *basic complex* and denoted by

$$\tilde{C}^\bullet(\mathcal{A}, V) = \bigoplus_{n \in \mathbb{Z}_+} \tilde{C}^n(\mathcal{A}, V). \tag{8}$$

Moreover, define a (left) $\mathbb{C}[\partial]$-module structure on $\tilde{C}^\bullet(\mathcal{A}, V)$ by

$$(\partial\gamma)_{\lambda_1,\cdots,\lambda_n}(a_1,\cdots,a_n) = (\partial_V + \sum_{i=1}^{n}\lambda_i)\gamma_{\lambda_1,\cdots,\lambda_n}(a_1,\cdots,a_n),$$

where ∂_V denotes the action of ∂ on V. Then $d\partial = \partial d$ and thus $\partial\tilde{C}^\bullet(\mathcal{A}, V) \subset \tilde{C}^\bullet(\mathcal{A}, V)$ forms a subcomplex. The quotient complex

$$C^\bullet(\mathcal{A}, V) = \tilde{C}^\bullet(\mathcal{A}, V)/\partial\tilde{C}^\bullet(\mathcal{A}, V) = \bigoplus_{n\in\mathbb{Z}_+} C^n(\mathcal{A}, V)$$

is called the *reduced complex*.

Definition 2.5. The basic cohomology $\tilde{\mathrm{H}}^\bullet(\mathcal{A}, V)$ of a Lie conformal algebra $\mathcal{A}$ with coefficients in an $\mathcal{A}$-module V is the cohomology of the basic complex $\tilde{C}^\bullet(\mathcal{A}, V)$ and the (reduced) cohomology $\mathrm{H}^\bullet(\mathcal{A}, V)$ corresponds to the reduced complex $C^\bullet(\mathcal{A}, V)$.

The main result of this paper is given as follows.

Theorem 2.6. *For the Heisenberg-Virasoro conformal algebra $\mathcal{HV}$, the following statements hold.*

(1) *For the trivial module $\mathbb{C}$,*

$$\dim \tilde{\mathrm{H}}^q(\mathcal{HV}, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0, \\ 3 & \text{if } q = 3, \\ 2 & \text{if } q = 4, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

$$\dim \mathrm{H}^q(\mathcal{HV}, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0, \\ 3 & \text{if } q = 2, \\ 5 & \text{if } q = 3, \\ 2 & \text{if } q = 4, \\ 0 & \text{otherwise}. \end{cases} \tag{10}$$

(2) $\mathrm{H}^\bullet(\mathcal{HV}, \mathbb{C}_a) = 0$ *if* $a \neq 0$.

(3) $\mathrm{H}^\bullet(\mathcal{HV}, M_{\Delta,\alpha,\beta}) = 0$ *if* $\alpha \neq 0$.

Remark 2.7. Theorem 2.6(1) in particular shows that there is a unique nontrivial universal central extension of the Heisenberg-Virasoro conformal algebra $\mathcal{HV}$ by a three-dimensional center $\mathbb{C}C_1 \oplus \mathbb{C}C_2 \oplus \mathbb{C}C_3$, which agrees with that of the twisted Heisenberg-Virasoro Lie algebra. The three independent reduced 2-cocycle $\bar{\phi}_1$, $\bar{\phi}_2$ and $\bar{\phi}_3$ are determined by (29)–(31) respectively, and the corresponding universal central extension $\widetilde{\mathcal{HV}}$ of $\mathcal{HV}$ is given by

$$\begin{aligned}
&[L_\lambda L] = (\partial + 2\lambda)L + \frac{\lambda^3}{12}C_1, \\
&[L_\lambda M] = (\partial + \lambda)M + \lambda^2 C_2, \\
&[M_\lambda L] = \lambda M - \lambda^2 C_2, \\
&[M_\lambda M] = \lambda C_3,
\end{aligned}$$

where C_1, C_2, C_3 are nonzero central elements of $\widetilde{\mathcal{HV}}$ with $\partial C_i = 0$, $i = 1, 2, 3$.

Remark 2.8. Denote by Lie$(\mathcal{HV})_-$ the annihilation Lie algebra of $\mathcal{HV}$. It can be easily checked that Lie$(\mathcal{HV})_-$ is isomorphic to the subalgebra spanned by $\{L_n, M_n \big| -1 \leq n \in \mathbb{Z}\}$ of the twisted Heisenberg-Virasoro algebra. Since $\tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C}) \cong \mathrm{H}^q(\mathrm{Lie}(\mathcal{HV})_-,\mathbb{C})$, we have actually determined the cohomology group of Lie$(\mathcal{HV})_-$ with trivial coefficients (cf. [1]).

3. Proof of Theorem 2.6

In this section, we prove Theorem 2.6, which will be done by several lemmas.

Keep notations in the previous section. An element γ in $\tilde{C}^q(\mathcal{A},V)$ is called a *q-cocycle* if $d(\gamma)=0$; a *q-coboundary* if there exists a $(q-1)$-cochain $\phi \in \tilde{C}^{q-1}(\mathcal{A},V)$ such that $\gamma = d(\phi)$. Two cochains γ_1 and γ_2 are called *equivalent* if $\gamma_1 - \gamma_2$ is a coboundary. Denote by $\tilde{D}^q(\mathcal{A},V)$ and $\tilde{B}^q(\mathcal{A},V)$ the spaces of q-cocycles and q-boundaries, respectively. By Definition 2.5,

$$\tilde{\mathrm{H}}^q(\mathcal{A},V) = \tilde{D}^q(\mathcal{A},V)/\tilde{B}^q(\mathcal{A},V) = \{\text{equivalent classes of } q\text{-cocycles}\}.$$

Lemma 3.1. $\tilde{\mathrm{H}}^0(\mathcal{HV},\mathbb{C}) = \mathrm{H}^0(\mathcal{HV},\mathbb{C}) = \mathbb{C}$.

Proof. For any $\gamma \in \tilde{C}^0(\mathcal{HV},\mathbb{C}) = \mathbb{C}$, $(d\gamma)_\lambda(X) = X_\lambda \gamma = 0$ for $X \in \mathcal{HV}$. This means $\tilde{D}^0(\mathcal{HV},\mathbb{C}) = \mathbb{C}$ and $\tilde{B}^0(\mathcal{HV},\mathbb{C}) = 0$. Thus $\tilde{\mathrm{H}}^0(\mathcal{HV},\mathbb{C}) = \mathbb{C}$ and $\mathrm{H}^0(\mathcal{HV},\mathbb{C}) = \mathbb{C}$ since $\partial\mathbb{C} = 0$. ■

Let $\gamma \in \tilde{C}^q(\mathcal{HV},\mathbb{C})$ with $q>0$. By Definition 2.4, γ is determined by its value on $X_1 \otimes \cdots \otimes X_q$ with $X_i \in \{L, M\}$. Since γ is skew-symmetric, we can always assume that the first k variables are L and the last $q-k$ variables are M in $\gamma_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)$. Thus we can regard $\gamma_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)$ as a polynomial in $\lambda_1,\cdots,\lambda_q$, which is skew-symmetric in $\lambda_1,\cdots,\lambda_k$ and also skew-symmetric in $\lambda_{k+1},\cdots,\lambda_q$. Therefore, $\gamma_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)$ is divisible by

$$\prod_{1\leq i<j\leq k}(\lambda_i-\lambda_j) \times \prod_{k+1\leq i<j\leq q}(\lambda_i-\lambda_j),$$

whose polynomial degree is $k(k-1)/2 + (q-k)(q-k-1)/2$.

Following [1], we define an operator $\tau : \tilde{C}^q(\mathcal{HV},\mathbb{C}) \to \tilde{C}^{q-1}(\mathcal{HV},\mathbb{C})$ by

$$(\tau\gamma)_{\lambda_1,\cdots,\lambda_{q-1}}(X_1,\cdots,X_{q-1}) = (-1)^{q-1}\frac{\partial}{\partial\lambda}\gamma_{\lambda_1,\cdots,\lambda_{q-1},\lambda}(X_1,\cdots,X_{q-1},L)|_{\lambda=0}, \quad (11)$$

where $X_1 = \cdots = X_k = L$, $X_{k+1} = \cdots = X_{q-1} = M$. By (7), (11) and skew-symmetry of γ,

$$\begin{aligned}
&((d\tau+\tau d)\gamma)_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)\\
&\quad = (-1)^q\frac{\partial}{\partial\lambda}\sum_{i=1}^q(-1)^{i+q+1}\gamma_{\lambda_i+\lambda,\lambda_1,\cdots,\hat{\lambda_i},\cdots,\lambda_q}([X_i{}_{\lambda_i}L],X_1,\cdots,\hat{X_i},\cdots,X_q)|_{\lambda=0}\\
&\quad = \frac{\partial}{\partial\lambda}\sum_{i=1}^q\gamma_{\lambda_1,\cdots,\lambda_i+\lambda,\cdots,\lambda_q}(X_1,\cdots,[X_i{}_{\lambda_i}L],\cdots,X_q)|_{\lambda=0}. \qquad (12)
\end{aligned}$$

By (1) and conformal antilinearity of γ, $[X_i{}_{\lambda_i}L]$ can be replaced by either $(\lambda_i - \lambda)X_i$ when $X_i = L$ or by $\lambda_i X_i$ when $X_i = M$ in (12).

Thus, equality (12) becomes

$$
\begin{aligned}
&((d\tau + \tau d)\gamma)_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)\\
&= \frac{\partial}{\partial\lambda}\sum_{i=1}^{k}(\lambda_i - \lambda)\gamma_{\lambda_1,\cdots,\lambda_{i-1},\lambda_i+\lambda,\lambda_{i+1},\cdots,\lambda_q}(X_1,\cdots,X_{i-1},X_i,X_{i+1},\cdots,X_q)|_{\lambda=0}\\
&\quad+\frac{\partial}{\partial\lambda}\sum_{i=k+1}^{q}\lambda_i\gamma_{\lambda_1,\cdots,\lambda_{i-1},\lambda_i+\lambda,\lambda_{i+1},\cdots,\lambda_q}(X_1,\cdots,X_{i-1},X_i,X_{i+1},\cdots,X_q)|_{\lambda=0}\\
&= (\deg\gamma - k)\gamma_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q),
\end{aligned}
\tag{13}
$$

where $\deg\gamma$ is the total degree of γ in $\lambda_1,\cdots,\lambda_q$. Therefore, only those homogeneous cochains whose degree as a polynomial is equal to k contribute to the cohomology of $\tilde{C}^\bullet(\mathcal{HV},\mathbb{C})$. This is our key observation. Consider the quadratic inequality

$$\frac{k(k-1)}{2} + \frac{(q-k)(q-k-1)}{2} \le k,$$

whose discriminant is $\Delta_k = -4k^2 + 12k + 1$. Since $\Delta_k \ge 0$ has $k = 0,1,2$ and 3 as the only integral solutions, we have

$$
k = \begin{cases} 0,1 & if\ q = 1,\\ 1,2 & if\ q = 2,\\ 1,2,3 & if\ q = 3,\\ 2,3 & if\ q = 4. \end{cases}
\tag{14}
$$

In particular, $\tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C}) = 0$ for $q \ge 5$.

Lemma 3.2. *Theorem 2.6 (1) holds.*

Proof. According to Lemma 3.1 and the discussions in the previous paragraph, we only need to compute $\tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C})$ for $q = 1,2,3,4$.

For $q = 1$, we need to consider $k = 0,1$ by (14). Let γ be a 1-cocycle. From the discussions below (13), we know $\gamma_\lambda(M)$ should be a constant, whereas $\gamma_\lambda(L)$ should be a constant factor of λ. By $d\gamma = 0$, it is easy to check that both $\gamma_\lambda(M)$ and $\gamma_\lambda(L)$ are zero. Hence, $\tilde{\mathrm{H}}^1(\mathcal{HV},\mathbb{C}) = 0$.

For $q = 2$, we need to consider $k = 1,2$ by (14). If $\gamma \in \tilde{\mathrm{H}}^2(\mathcal{HV},\mathbb{C})$, then $\deg(\gamma_{\lambda_1,\lambda_2}(L,L)) = 2$ and $\deg(\gamma_{\lambda_1,\lambda_2}(L,M)) = 1$ as polynomials in λ_1,λ_2. By skew-symmetry of γ, $\gamma_{\lambda_1,\lambda_2}(L,L)$ should be a constant factor of $\lambda_1^2 - \lambda_2^2$, which is a coboundary of a 1-cochain of the form $\varphi_{\lambda_1}(L) = \lambda_1$. Assume that $\gamma_{\lambda_1,\lambda_2}(L,M) = a\lambda_1 + b\lambda_2$ for some $a,b \in \mathbb{C}$. A straightforward computation gives

$$0 = (d\gamma)_{\lambda_1,\lambda_2,\lambda_3}(L,L,M) = -a\lambda_1(\lambda_1+\lambda_2)(\lambda_1-\lambda_2) + a\lambda_2\lambda_3 - a\lambda_1\lambda_3, \tag{15}$$

and thus $a = 0$. Set $\varphi_{\lambda_1}(M) = b$. Then $(d\varphi)_{\lambda_1,\lambda_2}(L,M) = b\lambda_2 = \gamma_{\lambda_1,\lambda_2}(L,M)$, namely, $\gamma_{\lambda_1,\lambda_2}(L,M)$ is also a coboundary. Therefore $\tilde{\mathrm{H}}^2(\mathcal{HV},\mathbb{C}) = 0$.

For $q = 3$, we need to consider $k = 1,2,3$ by (14). Let $\gamma \in \tilde{D}^3(\mathcal{HV},\mathbb{C})$ be a 3-cocycle. In the case of $k = 1$, we can assume that $\gamma_{\lambda_1,\lambda_2,\lambda_3}(L,M,M) = c(\lambda_2 - \lambda_3)$

for some $c \in \mathbb{C}$. One can check that it satisfies the following equation

$$\begin{aligned} & (d\gamma)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L,L,M,M) \\ = \; & c(-(\lambda_1-\lambda_2)\phi_{\lambda_1+\lambda_2,\lambda_3,\lambda_4}(L,M,M)+\lambda_3\phi_{\lambda_2,\lambda_1+\lambda_3,\lambda_4}(L,M,M) \\ & -\lambda_4\phi_{\lambda_2,\lambda_1+\lambda_4,\lambda_3}(L,M,M)-\lambda_3\phi_{\lambda_1,\lambda_2+\lambda_3,\lambda_4}(L,M,M) \\ & +\lambda_4\phi_{\lambda_1,\lambda_2+\lambda_4,\lambda_3}(L,M,M)) \\ = \; & c(-(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)+\lambda_3(\lambda_1+\lambda_3-\lambda_4) \\ & -\lambda_4(\lambda_1+\lambda_4-\lambda_3)-\lambda_3(\lambda_2+\lambda_3-\lambda_4)+\lambda_4(\lambda_2+\lambda_4-\lambda_3)) \\ = \; & 0. \end{aligned} \tag{16}$$

And it is a not coboundary, because it can be the coboundary of a 2-cochain $\varphi_{\lambda_1,\lambda_2}(M,M)$ of degree 0, which must be zero by skew-symmetry of φ. In the case when $k=2$, we suppose that

$$\gamma_{\lambda_1,\lambda_2,\lambda_3}(L,L,M)=(\lambda_1-\lambda_2)(a(\lambda_1+\lambda_2)+b\lambda_3), \text{ for some } a,b\in\mathbb{C}. \tag{17}$$

It satisfies $(d\gamma)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L,L,L,M)=0$. Setting $\varphi_{\lambda_1,\lambda_2}(L,M)=a\lambda_1$, we have

$$(d\varphi)_{\lambda_1,\lambda_2,\lambda_3}(L,L,M)=-a(\lambda_1-\lambda_2)(\lambda_1+\lambda_2+\lambda_3),$$

and

$$(d\varphi)_{\lambda_1,\lambda_2,\lambda_3}(L,L,M)+\gamma_{\lambda_1,\lambda_2,\lambda_3}(L,L,M)=(b-a)(\lambda_1-\lambda_2)\lambda_3.$$

Thus $\gamma_{\lambda_1,\lambda_2,\lambda_3}(L,L,M)$ is a constant factor of $(\lambda_1-\lambda_2)\lambda_3$, which is not a coboundary by (15). By [1, Theorem 7.1], $\gamma_{\lambda_1,\lambda_2,\lambda_3}(L,L,L)=(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)$ (up to a constant factor) is also a 3-cocycle, which is not a coboundary. Therefore, $\dim \tilde{\mathrm{H}}^3(\mathcal{HV},\mathbb{C})=3$ and $\tilde{\mathrm{H}}^3(\mathcal{HV},\mathbb{C})=\mathbb{C}\phi_1\oplus\mathbb{C}\phi_2\oplus\mathbb{C}\phi_3$, where

$$\phi_1 := \phi_{1\,\lambda_1,\lambda_2,\lambda_3}(L,M,M)=\lambda_2-\lambda_3, \tag{18}$$
$$\phi_2 := \phi_{2\,\lambda_1,\lambda_2,\lambda_3}(L,L,M)=(\lambda_1-\lambda_2)\lambda_3, \tag{19}$$
$$\phi_3 := \phi_{3\,\lambda_1,\lambda_2,\lambda_3}(L,L,L)=(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3), \tag{20}$$

and where, we take ϕ_1 for example, the skew-symmetric function $\phi_1:\mathcal{HV}\otimes\mathcal{HV}\otimes\mathcal{HV}\to\mathbb{C}[\lambda_1,\lambda_2,\lambda_3]$ has values $\lambda_2-\lambda_3$ on $L\otimes M\otimes M$ and 0 on $L\otimes L\otimes M$ and $L\otimes L\otimes L$.

For $q=4$, we need to consider $k=2,3$. Let $\gamma\in\tilde{D}^4(\mathcal{HV},\mathbb{C})$. By skew-symmetry of γ with whose degree as a polynomial taken into account, assume that

$$\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L,L,M,M)=e_1(\lambda_1-\lambda_2)(\lambda_3-\lambda_4), \quad \textit{for some} \;\; e_1\in\mathbb{C}. \tag{21}$$

It is easy to check that $(d\gamma)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L,L,L,M,M)=0$. Thus

$$\psi_1 := \psi_{1\,\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L,L,M,M)=(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)$$

is a 4-cocycle. If ψ_1 is the coboundary of a 3-cochain $\phi_{\lambda_1,\lambda_2,\lambda_3}(L,M,M)$ of degree 1, then $\phi_{\lambda_1,\lambda_2,\lambda_3}(L,M,M)$ must be a constant factor of $\lambda_2-\lambda_3$, whose coboundary is zero by (16). Hence ψ_1 is not a coboundary. Similarly, we have

$$\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L,L,L,M)=e_2(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3), \quad \textit{for some}\; e_2\in\mathbb{C},$$

is a cocycle. We should point out that

$$\psi_2 := \psi_{2\,\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L,L,L,M) = (\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3) \tag{22}$$

is not a coboundary. Because it can be the coboundary of a 3-cochain of the form $\phi_{\lambda_1,\lambda_2,\lambda_3}(L,L,M)$, which should be of degree 2 and thus of the form $(\lambda_1-\lambda_2)(a(\lambda_1+\lambda_2)+b\lambda_3)$, whose coboundary is zero by (17). Hence $\tilde{\mathrm{H}}^4(\mathcal{HV},\mathbb{C}) = \mathbb{C}\psi_1 \oplus \mathbb{C}\psi_2$.

By [1, Proposition 2.1], the map $\gamma \mapsto \partial\gamma$ gives an isomorphism $\tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C}) \cong \mathrm{H}^q(\partial\tilde{C}^\bullet)$ for $q \geq 1$. Therefore,

$$\mathrm{H}^q(\partial\tilde{C}^\bullet) = \begin{cases} \mathbb{C}(\partial\phi_1)\oplus\mathbb{C}(\partial\phi_2)\oplus\mathbb{C}(\partial\phi_3) & if\ q=3, \\ \mathbb{C}(\partial\psi_1)\oplus\mathbb{C}(\partial\psi_2) & if\ q=4, \\ 0 & otherwise. \end{cases} \tag{23}$$

The computation of $\mathrm{H}^\bullet(\mathcal{HV},\mathbb{C})$ is based on the short exact sequence of complexes

$$0 \longrightarrow \partial\tilde{C}^\bullet \stackrel{\iota}{\longrightarrow} \tilde{C}^\bullet \stackrel{\pi}{\longrightarrow} C^\bullet \longrightarrow 0 \tag{24}$$

where ι and π are the embedding and the natural projection, respectively. The exact sequence (24) gives the following long exact sequence of cohomology groups (cf. [1]):

$$\begin{aligned} \cdots \longrightarrow \mathrm{H}^q(\partial\tilde{C}^\bullet) &\stackrel{\iota_q}{\longrightarrow} \tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C}) \stackrel{\pi_q}{\longrightarrow} \mathrm{H}^q(\mathcal{HV},\mathbb{C}) \stackrel{\omega_q}{\longrightarrow} \\ \longrightarrow \mathrm{H}^{q+1}(\partial\tilde{C}^\bullet) &\stackrel{\iota_{q+1}}{\longrightarrow} \tilde{\mathrm{H}}^{q+1}(\mathcal{HV},\mathbb{C}) \stackrel{\pi_{q+1}}{\longrightarrow} \mathrm{H}^{q+1}(\mathcal{HV},\mathbb{C}) \longrightarrow \cdots \end{aligned} \tag{25}$$

where ι_q, π_q are induced by ι, π respectively and w_q is the $q-$th connecting homomorphism. Given $\partial\gamma \in \mathrm{H}^q(\partial\tilde{C}^\bullet)$ with a nonzero element $\gamma \in \tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C})$ of degree k, we have $\iota_q(\partial\gamma) = \partial\gamma \in \tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C})$. Since $\deg(\partial\gamma) = \deg(\gamma)+1 = k+1$, $\partial\gamma = 0 \in \tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C})$. Thus the image $\mathrm{im}(\iota_q)$ of ι_q is zero for any $q \in \mathbb{Z}_+$. Because $\ker(\pi_q) = \mathrm{im}(\iota_q) = \{0\}$ and $\mathrm{im}(\omega_q) = \ker(\iota_{q+1}) = \mathrm{H}^{q+1}(\partial\tilde{C}^\bullet)$, we obtain the following short exact sequence

$$0 \longrightarrow \tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C}) \stackrel{\pi_q}{\longrightarrow} \mathrm{H}^q(\mathcal{HV},\mathbb{C}) \stackrel{\omega_q}{\longrightarrow} \mathrm{H}^{q+1}(\partial\tilde{C}^\bullet) \longrightarrow 0. \tag{26}$$

Therefore,

$$\dim \mathrm{H}^q(\mathcal{HV},\mathbb{C}) = \dim \tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C}) + \dim \mathrm{H}^{q+1}(\partial\tilde{C}^\bullet), \text{ for all } q \geq 0. \tag{27}$$

Consequently, we obtain (10). Moreover, we can give a basis of $\mathrm{H}^q(\mathcal{HV},\mathbb{C})$ for $q = 2,3,4$. By (26), a basis of $\mathrm{H}^q(\mathcal{HV},\mathbb{C})$ can be obtained by combining the images of a basis of $\tilde{\mathrm{H}}^q(\mathcal{HV},\mathbb{C})$ with the pre-images of a basis of $\tilde{\mathrm{H}}^{q+1}(\mathcal{HV},\mathbb{C})$. Let φ be a nonzero $(q+1)$-cocycle of degree k such that $\partial\varphi \in \mathrm{H}^{q+1}(\partial\tilde{C}^\bullet)$. By (13), we have

$$d(\tau(\partial\varphi) = (d\tau+\tau d)(\partial\varphi) = (\deg(\partial\varphi)-k)(\partial\varphi) = ((k+1)-k)(\partial\varphi) = \partial\varphi. \tag{28}$$

Thus the pre-image $\omega_q^{-1}(\partial\varphi)$ of $\partial\varphi$ under the connecting homomorphism ω_p is $\tau(\partial\varphi)$, i.e., $\omega_q^{-1}(\partial\varphi) = \tau(\partial\varphi)$. Finally, let us finish the proof by giving a basis of

$\mathrm{H}^q(\mathcal{HV},\mathbb{C})$ for $q=2,3,4$. For $q=2$, we have known that $\tilde{\mathrm{H}}^2(\mathcal{HV},\mathbb{C})=0$ and $\mathrm{H}^3(\partial\tilde{C}^\bullet)=\mathbb{C}(\partial\phi_1)\oplus\mathbb{C}(\partial\phi_2)\oplus\mathbb{C}(\partial\phi_3)$. By (11), (18)–(20),

$$\begin{aligned}
\bar{\phi}_1: &= (\tau(\partial\phi_1))_{\lambda_1,\lambda_2}(M,M)\\
&= (-1)^2\frac{\partial}{\partial\lambda}(\partial\phi_1)_{\lambda_1,\lambda_2,\lambda}(M,M,L)|_{\lambda=0}\\
&= \frac{\partial}{\partial\lambda}(\lambda_1+\lambda_2+\lambda)(\lambda_1-\lambda_2)|_{\lambda=0}\\
&= \lambda_1-\lambda_2, && (29)\\
\bar{\phi}_2: &= (\tau(\partial\phi_2))_{\lambda_1,\lambda_2}(L,M)\\
&= (-1)^2\frac{\partial}{\partial\lambda}(\partial\phi_2)_{\lambda_1,\lambda_2,\lambda}(L,M,L)|_{\lambda=0}\\
&= -\frac{\partial}{\partial\lambda}(\lambda_1+\lambda_2+\lambda)(\lambda_1-\lambda)\lambda_2|_{\lambda=0}\\
&= \lambda_2^2, && (30)\\
\bar{\phi}_3: &= (\tau(\partial\phi_3))_{\lambda_1,\lambda_2}(L,L)\\
&= (-1)^2\frac{\partial}{\partial\lambda}(\partial\phi_3)_{\lambda_1,\lambda_2,\lambda}(L,L,L)|_{\lambda=0}\\
&= \frac{\partial}{\partial\lambda}(\lambda_1+\lambda_2+\lambda)(\lambda_1-\lambda_2)(\lambda_2-\lambda)(\lambda_1-\lambda)|_{\lambda=0}\\
&= -\lambda_1^3+\lambda_2^3. && (31)
\end{aligned}$$

Thus $\mathrm{H}^2(\mathcal{HV},\mathbb{C})=\mathbb{C}\bar{\phi}_1\oplus\mathbb{C}\bar{\phi}_2\oplus\mathbb{C}\bar{\phi}_3$. For $q=3$, we have

$$\begin{aligned}
\bar{\psi}_1: &= (\tau(\partial\psi_1))_{\lambda_1,\lambda_2,\lambda_3}(L,M,M)\\
&= (-1)^3\frac{\partial}{\partial\lambda}(\partial\psi_1)_{\lambda_1,\lambda_2,\lambda_3,\lambda}(L,M,M,L)|_{\lambda=0}\\
&= \frac{\partial}{\partial\lambda}(\lambda_1+\lambda_2+\lambda_3+\lambda)(\lambda_1-\lambda)(\lambda_3-\lambda_2)|_{\lambda=0}\\
&= \lambda_2^2-\lambda_3^2,\\
\bar{\psi}_2: &= (\tau(\partial\psi_2))_{\lambda_1,\lambda_2,\lambda_3}(L,L,M)\\
&= (-1)^3\frac{\partial}{\partial\lambda}(\partial\psi_2)_{\lambda_1,\lambda_2,\lambda_3,\lambda}(L,L,M,L)|_{\lambda=0}\\
&= \frac{\partial}{\partial\lambda}(\lambda_1+\lambda_2+\lambda_3+\lambda)(\lambda_1-\lambda_2)(\lambda_1-\lambda)(\lambda_2-\lambda)|_{\lambda=0}\\
&= -\lambda_1^3-\lambda_1^2\lambda_3+\lambda_2^3+\lambda_2^2\lambda_3.
\end{aligned}$$

Hence $\mathrm{H}^3(\mathcal{HV},\mathbb{C})=\mathbb{C}\phi_1\oplus\mathbb{C}\phi_2\oplus\mathbb{C}\phi_3\oplus\mathbb{C}\bar{\psi}_1\oplus\mathbb{C}\bar{\psi}_2$ and $\mathrm{H}^4(\mathcal{HV},\mathbb{C})=\mathbb{C}\psi_1\oplus\mathbb{C}\psi_2$ by (27). ■

Lemma 3.3. *Theorem 2.6 (2) holds.*

Proof. For $a\neq 0$, define an operator $\tau_2:\tilde{C}^q(\mathcal{HV},\mathbb{C}_a)\to\tilde{C}^{q-1}(\mathcal{HV},\mathbb{C}_a)$ by

$$(\tau_2\gamma)_{\lambda_1,\cdots,\lambda_{q-1}}(X_1,\cdots,X_{q-1})=(-1)^{q-1}\gamma_{\lambda_1,\cdots,\lambda_{q-1},\lambda}(X_1,\cdots,X_{q-1},L)|_{\lambda=0}, \quad (32)$$

for $X_1,\cdots,X_{q-1}\in\{L,M\}$. Note that $\partial\tilde{C}^q(\mathcal{HV},\mathbb{C}_a)=(a+\sum_{i=1}^q\lambda_i)\tilde{C}^q(\mathcal{HV},\mathbb{C}_a)$.

Thus

$$\begin{aligned} &((d\tau_2+\tau_2 d)\gamma)_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)\\ =\ &\big(\textstyle\sum_{i=1}^q\lambda_i\big)\gamma_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)\\ \equiv\ &-a\gamma_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)\ (\mathrm{mod}\ \partial\tilde{C}^q(\mathcal{HV},\mathbb{C}_a)). \end{aligned} \tag{33}$$

Let $\gamma\in\tilde{C}^q(\mathcal{HV},\mathbb{C}_a)$ be a q-cochain such that $d\gamma\in\partial\tilde{C}^{q+1}(\mathcal{HV},\mathbb{C}_a)$, namely, there is a $(q+1)$-cochain ϕ such that $d\gamma=(a+\sum_{i=1}^{q+1}\lambda_i)\phi$. By (32), we have $\tau_2 d\gamma=(a+\sum_{i=1}^q\lambda_i)\tau_2\phi\in\partial\tilde{C}^q(\mathcal{HV},\mathbb{C}_a)$. By (33), $\gamma\equiv -d(a^{-1}\tau_2\gamma)$ is a reduced coboundary. ■

Lemma 3.4. *Theorem 2.6(3) holds.*

Proof. In this case, $\partial\tilde{C}^q(\mathcal{HV},M_{\Delta,\alpha,\beta})=(\partial+\sum_{i=1}^q\lambda_i)\tilde{C}^q(\mathcal{HV},M_{\Delta,\alpha,\beta})$. Define an operator $\tau_3: C^q(\mathcal{HV},M_{\Delta,\alpha,\beta})\to C^{q-1}(\mathcal{HV},M_{\Delta,\alpha,\beta})$ by

$$(\tau_3\gamma)_{\lambda_1,\cdots,\lambda_{q-1}}(X_1,\cdots,X_{q-1})=(-1)^{q-1}\gamma_{\lambda_1,\cdots,\lambda_{q-1},\lambda}(X_1,\cdots,X_{q-1},L)|_{\lambda=0},$$

for $X_1,\cdots,X_{q-1}\in\{L,M\}$. We have

$$\begin{aligned} &((d\tau_3+\tau_3 d)\gamma)_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)\\ &=L_\lambda\gamma_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)|_{\lambda=0}+\big(\sum_{i=1}^q\lambda_i\big)\gamma_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)\\ &=\big(\partial+\alpha+\sum_{i=1}^q\lambda_i\big)\gamma_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)\\ &\equiv\alpha\gamma_{\lambda_1,\cdots,\lambda_q}(X_1,\cdots,X_q)\ (\mathrm{mod}\ \partial\tilde{C}^q(\mathcal{HV},M_{\Delta,\alpha,\beta})). \end{aligned} \tag{34}$$

If γ is a reduced q-cocycle, then, by (34), $\gamma\equiv d(\alpha^{-1}\tau_3\gamma)$ is a reduced coboundary, since $\alpha\neq 0$. Thus $\mathrm{H}^q(\mathcal{HV},M_{\Delta,\alpha,\beta})=0$ for all $q\geq 0$. ■

This completes the proof of Theorem 2.6.

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