

Nilpotent Orbits: Finiteness, Separability and Howe’s Conjecture

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Abstract. This paper is about nilpotent orbits of reductive groups over local non-Archimedean fields. We will try to identify for which groups there are only finitely many nilpotent orbits, for which groups the nilpotent orbits are separable and for which groups Howe’s conjecture holds. For split reductive groups we get a classification in terms of the root data and the characteristic of the underlying local field. For this classification the proof of the failure of Howe’s conjecture for split reductive groups for which the characteristic of the field is bad and the proof of Howe’s conjecture for the projective linear group are the key results. For general reductive groups we get some partial results, among which there is a proof of Howe’s conjecture for groups for which all nilpotent orbits are separable. *Mathematics Subject Classification 2010:* 20G25, 22E50, 17B45.
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1. Introduction

Let \mathbb{F} be a local non-Archimedean field of characteristic p and \mathcal{G} a connected reductive group defined over \mathbb{F} . Let $G = \mathcal{G}(\mathbb{F})$ and \mathfrak{g} the Lie algebra of G . In this article we investigate Howe’s conjecture, the geometry of the nilpotent orbits in \mathfrak{g} , and the relation between the nilpotent orbits and Howe’s conjecture.

For $\omega \subset \mathfrak{g}$ define $J(\omega)$ to be the set of G -invariant distributions with support contained in the closure of ${}^G\omega = \text{Ad}(G)\omega$. For L a \mathcal{O} -lattice in \mathfrak{g} define $J_L(\omega)$ to be the image of $J(\omega)$ in the distributions of \mathfrak{g}/L under the canonical map $\phi_L : \mathfrak{g} \rightarrow \mathfrak{g}/L$.

Conjecture 1 (Howe). *For all compact $\omega \subset \mathfrak{g}$ and all \mathcal{O} -lattices L of \mathfrak{g} :*

$$\dim J_L(\omega) < \infty.$$

This conjecture has been proved by Howe in [How74] for $G = GL_n(\mathbb{F})$.

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Later it has been proved by Harish-Chandra, see [HC99], for general G in the case that $\text{char } \mathbb{F} = 0$. The main results on Howe's conjecture in this article are:

- Howe's conjecture fails for a split reductive group for which the characteristic is bad.
- Howe's conjecture holds for groups which split over a tamely ramified field extension and for which all nilpotent orbits are separable.
- For a split reductive group G Howe's conjecture holds if and only if the characteristic is good for G and the adjoint map $Ad : G \rightarrow Ad(G)$ is separable.

Rather surprisingly, there are reductive group for which Howe's conjecture holds and which have infinitely many nilpotent orbits. Probably there are only finitely many nilpotent orbits with a non-empty intersection with every neighborhood of 0, see for example in $SO_3(\mathbb{F})$, with $\text{char } \mathbb{F} = 2$ (Corollary 38). To prove Howe's conjecture for certain groups, we will just adapt the proof in [HC99].

The proof in [HC99] of the local summability of the character of an admissible representation and the local upper bound $|D(g)|^{-\frac{1}{2}}$ depends on Howe's conjecture. Howe's conjecture is used to prove that the character of a representation is locally a linear combination of Fourier transforms of nilpotent orbital integrals. Although Howe's conjecture is not a necessary condition for the summability of such characters (see e.g. [Lem05]), it is quite hard to do without.

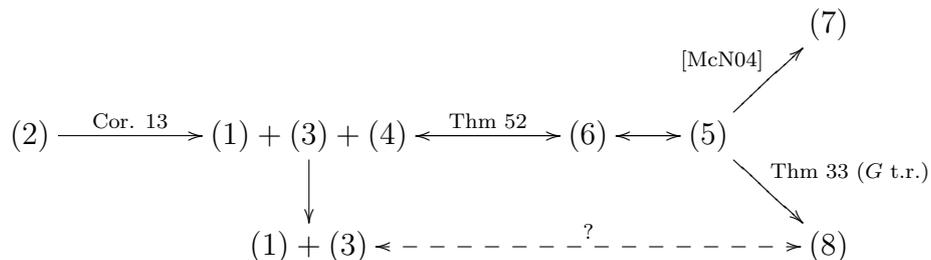
Regarding the nilpotent orbits, we will show that if a split reductive group has finitely many nilpotent orbits, then all nilpotent orbits are separable. The converse statement has been proven in [McN04].

To be more precise, in this article we investigate the relation between the following statements for G (we will clarify the first six statements in §2):

1. p is good;
2. p is very good;
3. p does not divide the virtual number of components of $Z(G)$;
4. p does not divide the virtual order of $\pi_1(G_{\text{der}})$;
5. all the nilpotent orbits are separable;
6. the regular nilpotent orbit is separable;
7. the number of nilpotent orbits is finite;
8. Howe's conjecture holds for G .

If $\text{char } \mathbb{F} = 0$ (including $\mathbb{F} = \mathbb{C}, \mathbb{R}$), then all these statements hold for G . In case \mathbb{F} has positive characteristic these statements depend on G and p .

For general G we will prove the following implications



where (5) implies (8) under the assumption that G splits over a tamely ramified field extension of \mathbb{F} . The question whether (5) implies (8) without the tameness condition on G is still open. The question if (7) and (5) are equivalent and the question if (1) + (3) is equivalent to (8) are still open.

If moreover G is \mathbb{F} -split, then we get the following implications

$$\begin{array}{ccccccc}
 & & & (2) & & & \\
 & & & \downarrow & & & \\
 (5) & \longleftrightarrow & (6) & \longleftrightarrow & (1) + (3) + (4) & \xleftarrow{\text{Thm 66}} & (7) \\
 & & & \downarrow & & & \\
 & & & (1) + (3) & \xleftarrow{\text{Thm 49}} & (8) &
 \end{array}$$

Besides the proofs of these implications we will also give counter examples for the non-implications. That (1) + (3) does not imply (4) can be seen by the example PGL_p . That (1) + (3) + (4) does not imply (2) can be seen by the example GL_p . That (1) + (4) does not imply (3) can be seen by the example SL_p . That (3) + (4) does not imply (1) can be seen in the simple groups of exceptional type. Thus for \mathbb{F} -split groups we have determined all the implications and non-implications between every possible combination of these 8 properties.

The first 4 statements are related to p and the root datum of G and the last 4 statements are related to the adjoint action of G on its Lie algebra \mathfrak{g} . The proofs of the implications from a collection of statements about the root datum to a statement about the adjoint action are mostly based on known proofs in the case that \mathbb{F} has characteristic 0. The proofs of the implications from a statement about the adjoint action to a collection of statements about the root datum are different. In this case we assume that one of the statements about the root datum does not hold and then show that the statement about the adjoint action does not hold. For example, we will show that $\neg(7)$ is a consequence of $\neg(1)$ or $\neg(3)$ or $\neg(4)$. The strategy is to make a surjective function from a part of the regular nilpotent elements of the Lie algebra to $\mathbb{F}/\mathbb{F}^{(p)}$ or $\mathbb{F}^\times/(\mathbb{F}^\times)^p$, which is G -invariant. For example, in $SL_2(\mathbb{F})$ with $\text{char } \mathbb{F} = 2$ we take the function

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mapsto x \pmod{(\mathbb{F}^\times)^2}.$$

The proof of $\neg(1)$ or $\neg(3)$ implies $\neg(8)$ is based on the existence of such functions.

In the following table we list the properties of some \mathbb{F} -split groups. In the column *nHwC* are the set of primes P such that Howe's conjecture does not hold for $G(\mathbb{F})$ if and only if $\text{char } \mathbb{F} \in P$. In the column *INO* are the set of primes P such that the group $G(\mathbb{F})$ has infinitely many nilpotent orbits if and only if

char $\mathbb{F} \in P$.

G	bad p	$\kappa_v(G)$	$\rho_v(G)$	$nHwC$	INO
GL_n	—	1	1	—	—
SL_n	—	n	1	$p n$	$p n$
PGL_n	—	1	n	—	$p n$
SO_{2n+1}	2	1	2	2	2
SO_{2n}	2	2	2	2	2
Sp_{2n}	2	2	1	2	2
F_4	2, 3	1	1	2, 3	2, 3
G_2	2, 3	1	1	2, 3	2, 3
E_8	2, 3, 5	1	1	2, 3, 5	2, 3, 5

The obvious direction for generalizing the theory about Howe's conjecture and on the (in)finiteness of nilpotent orbits of this article is to look at reductive groups that are not \mathbb{F} -split. The proofs of this article depend heavily on the case by case consideration of the irreducible root systems. It would be nice to find unified proofs.

2. Notations

Unless otherwise stated, \mathbb{F} is a local non-Archimedean field with uniformizer π and ring of integers \mathcal{O} . We define $p := \text{char } \mathbb{F}$. For $n \in \mathbb{N}$ we define $\mathbb{F}^{(n)} := \{x^n : x \in \mathbb{F}\}$ and $\mathcal{O}^{(n)} := \{x^n : x \in \mathcal{O}\}$.

A prime number p is bad for a root system R if

1. $p = 2$ and R has a component not of type A_n ;
2. $p = 3$ and R has a component of type E_n, F_4 or G_2 ;
3. $p = 5$ and R has a component of type E_8 .

A prime number p is good for R if it is not bad. See [SS70, §4.1] for equivalent definitions of good primes.

A prime number p is very good for R if it is good and R does not have a component of type A_n with p a divisor of $n + 1$.

A prime number p is (very) good for G if it is (very) good for the root system of G .

A G -orbit $Ad(G)x$ in \mathfrak{g} is called separable if one of the following equivalent conditions holds:

1. The differential of the map $g \mapsto Ad(g)x$ is surjective.
2. $\dim\{g \in G \mid Ad(g)x = x\} = \dim\{y \in \mathfrak{g} \mid [y, x] = 0\}$.
3. The Lie algebra of $\{g \in G \mid Ad(g)x = x\}$ is equal to $\{y \in \mathfrak{g} \mid [y, x] = 0\}$.

2.1. $\kappa_v(G)$ & $\rho_v(G)$. Let T be a maximal torus of G . Let $R(G, T)$ be the roots of G relative to T and $R^\vee(G, T)$ be the coroots. Let $X^*(T)$ be the characters of T and $X_*(T)$ be the cocharacters of T . The two embeddings $R(G, T) \hookrightarrow X^*(T)$ and $R^\vee(G, T) \hookrightarrow X_*(T)$ induce the group homomorphisms $\Phi : X_*(T) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G, T), \mathbb{Z})$ and $\Phi^\vee : X^*(T) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G, T)^\vee, \mathbb{Z})$. We

call $\rho_v(G) := |\text{coker } \Phi^\vee|$ the virtual order of $\pi_1(G_{\text{der}})$. We call $\kappa_v(G) := |\text{coker } \Phi|$ the virtual number of components of $Z(G)$. To explain the names of these numbers, we look at complex tori and complex reductive groups.

Lemma 2. *Let \mathcal{T} and \mathcal{S} be two complex tori and $\phi : \mathcal{T} \rightarrow \mathcal{S}$. Let $\phi^* : X^*(\mathcal{S}) \rightarrow X^*(\mathcal{T})$ be the map $\epsilon \mapsto \epsilon \circ \phi$. Then*

$$|\ker \phi / (\ker \phi)^o| = |(\text{coker } \phi^*)_{\text{tor}}|,$$

where $(\text{coker } \phi^*)_{\text{tor}}$ is the torsion part of the cokernel of ϕ^* .

Proof. Choose the bases $\delta_1, \dots, \delta_m$ for $X^*(\mathcal{S})$ and $\epsilon_1, \dots, \epsilon_n$ for $X^*(\mathcal{T})$ in such a way that

$$\begin{aligned} \phi^*(\delta_1) &= d_1 \epsilon_1 \\ &\vdots \\ \phi^*(\delta_k) &= d_k \epsilon_k \\ \phi^*(\delta_{k+1}, \dots, \delta_m) &\in \langle d_1 \epsilon_1, \dots, d_k \epsilon_k \rangle \end{aligned}$$

then $\prod_{i=1}^k d_i = |(\text{coker } \phi^*)_t|$. Thus

$$\begin{aligned} \ker \phi &:= \{t \in \mathcal{T} \mid \epsilon_i(t)^{d_i} = 1 \text{ for all } 1 \leq i \leq k\} \\ (\ker \phi)^o &:= \{t \in \mathcal{T} \mid \epsilon_i(t) = 1 \text{ for all } 1 \leq i \leq k\}. \end{aligned}$$

Therefore

$$|\ker \phi / (\ker \phi)^o| = \left| \prod_{i=1}^k \mathbb{Z}/d_i \mathbb{Z} \right| = \prod_{i=1}^k d_i = |(\text{coker } \phi^*)_{\text{tor}}|. \quad \blacksquare$$

Corollary 3. *For a complex reductive group, \mathcal{G} , $|\text{coker } \Phi| = |\pi_0(Z(\mathcal{G}))|$ and $|\text{coker } \Phi^\vee| = |\pi_1(\mathcal{G}_{\text{der}})|$.*

Proof. Look at the adjoint map: $Ad : \mathcal{G} \rightarrow \mathcal{G}^{\text{ad}}$. Let \mathcal{T} be a maximal torus of \mathcal{G} and $\mathcal{T}^{\text{ad}} = Ad(\mathcal{T})$. Then $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z}) = X_*(\mathcal{T}^{\text{ad}})$ and Φ is the map corresponding with $Ad : \mathcal{T} \rightarrow \mathcal{T}^{\text{ad}}$:

$$\Phi(\epsilon) := Ad \circ \epsilon.$$

We define $\Phi^{\text{tr}} : X^*(\mathcal{T}^{\text{ad}}) \rightarrow X^*(\mathcal{T})$ as follows:

$$\Phi^{\text{tr}}(\epsilon) := \epsilon \circ Ad.$$

The cokernel of Φ^{tr} has a torsion group of order $|\text{coker } \Phi|$. Thus

$$|\text{coker } \Phi| = |Z(\mathcal{G})/Z(\mathcal{G})^o| = |\pi_0(Z(\mathcal{G}))|,$$

since $\mathcal{T} \cap \ker Ad = Z(\mathcal{G})$. Let \mathcal{G}_{sc} be the simply connected cover of \mathcal{G}_{der} . Let $\pi : \mathcal{G}_{\text{sc}} \rightarrow \mathcal{G}$ be the following morphism: $\mathcal{G}_{\text{sc}} \twoheadrightarrow \mathcal{G}_{\text{der}} \hookrightarrow \mathcal{G}$.

Let \mathcal{T}_{sc} be the maximal torus of \mathcal{G}_{sc} such that $\pi(\mathcal{T}_{\text{sc}}) = \mathcal{T} \cap \mathcal{G}_{\text{der}}$. Then $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z}) = X_*(\mathcal{T}_{\text{sc}})$ and Φ^\vee is the map corresponding with $\pi : \mathcal{T}_{\text{sc}} \rightarrow \mathcal{T}$. Thus

$$|\text{coker } \Phi^\vee| = |\ker \pi| = |\pi_1(\mathcal{G}_{\text{der}})|. \quad \blacksquare$$

2.2. Chevalley basis. The first part of this subsection is based on [BT84, §3.2].

Let G be a \mathbb{F} -split reductive group and T a maximal torus. Let \mathfrak{g} be the Lie algebra of G . Let $R := R(G, T)$ be the roots of G and T . Let R^+ be a set of positive roots of R and Δ be the set of corresponding simple roots.

We have for $\beta \in R$, the elements H_β and E_β in \mathfrak{g} , such that for all $\alpha, \beta \in R$:

$$\begin{aligned} [H_\alpha, H_\beta] &= 0 \\ [H_\alpha, E_\beta] &= \langle \alpha^\vee, \beta \rangle E_\beta \\ [E_\beta, E_\alpha] &= \begin{cases} N_{\beta, \alpha} E_{\beta+\alpha} & \text{if } \beta + \alpha \in R \\ H_\beta & \text{if } \alpha = -\beta \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where each $N_{\beta, \alpha} \in \mathbb{Z}$. For each $\beta \in R$ there exists a unique map $u_\beta : \mathbb{F} \rightarrow G$, such that $d\beta(1) = E_\beta$ and for all $t \in T$ and $x \in \mathbb{F}$, $tu_\beta(x)t^{-1} = u_\beta(\beta(t)x)$. Then β^\vee , the coroot of β , is equal to

$$\beta^\vee(\lambda) = u_\beta(\lambda)u_{-\beta}(-\lambda^{-1})u_\beta(\lambda)u_\beta(-1)u_{-\beta}(1)u_\beta(1).$$

Moreover $d\beta^\vee(1) = H_\beta$.

The set $\{H_\alpha : \alpha \in \Delta\} \cup \{E_\beta : \beta \in R\}$ is called a Chevalley basis. (The term ‘‘basis’’ is misplaced here, since if G is not semi-simple it does not span \mathfrak{g} and if $G = PGL_n$ and $\text{char } \mathbb{F} \mid n$ it is not linearly independent, see Lemma 43. However if the characteristic is 0 it is a basis for \mathfrak{g}' , the Lie algebra of $G_{\text{der}} = (G, G)$. The E_β are always linearly independent.)

The adjoint representation $Ad : G \rightarrow \text{End}(\mathfrak{g})$ is determined by the following formulas

$$\begin{aligned} Ad(u_\beta(\lambda))E_\alpha &= \begin{cases} E_\beta & \text{if } \beta = \alpha \\ E_{-\beta} + \lambda H_\beta - \lambda^2 E_\beta & \text{if } \alpha = -\beta \\ \sum_{i \geq 0} M_{\beta, \alpha, i} \lambda^i E_{i\beta + \alpha} & \text{otherwise} \end{cases} \\ Ad(t)E_\beta &= \beta(t)E_\beta \\ Ad(u_\beta(\lambda))H &= H - d\beta(H)\lambda E_\beta \\ Ad(t)H &= H \end{aligned}$$

for all $H \in \mathfrak{t}$, the Lie algebra of T and constants $M_{\beta, \alpha, i} \in \mathbb{F}$.

The \mathbb{F} -points of the image of the algebraic map Ad will be denoted by $Ad(G)$ or G^{ad} . From now on we fix a Chevalley basis on \mathfrak{g} .

3. Regular nilpotent orbits

In the first part of this short introduction to nilpotent orbits, especially regular nilpotent orbits, we will follow [Car85, §5.1]. Although [Car85, §5.1] treats regular unipotent elements, we can easily adapt it to regular nilpotent elements.

For each $\alpha \in R$, define $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : ad(t)x = \alpha(t)x\}$. We define the height function $ht : R \rightarrow \mathbb{Z}$ as follows:

$$ht\left(\sum_{\alpha \in \Delta} c_\alpha \alpha\right) := \sum_{\alpha \in \Delta} c_\alpha.$$

For $z \in \mathbb{Z}$ we define the following subspaces of \mathfrak{g} :

$$\mathfrak{n}_z := \bigoplus_{\alpha \in R | ht(\alpha) = z} \mathfrak{g}_\alpha$$

$$\mathfrak{n}_{\geq z} := \bigoplus_{\alpha \in R | ht(\alpha) \geq z} \mathfrak{g}_\alpha.$$

A nilpotent element of $n \in \mathfrak{g}$ is called a regular nilpotent element if and only if

$$\dim Z_G(n) = \dim T.$$

Proposition 4. *Let \mathbb{F} be an algebraically closed field. Let G be a connected reductive group (over \mathbb{F}). Then there exist regular nilpotent elements in \mathfrak{g} and any two are conjugated. Let $n \in \mathfrak{g}$ be nilpotent. The following conditions on n are equivalent:*

- (a) n is regular;
- (b) there is a unique Borel subgroup B of G such that n is in the Lie algebra of B ;
- (c) n is conjugated to an element of the form $\sum_{\alpha \in R^+} \lambda_\alpha E_\alpha$ with $\lambda_\alpha \neq 0$ for all $\alpha \in \Delta$.

Proof. We use the proof of [Car85, Proposition 5.1.2 & 5.1.3]. That there are only finitely many nilpotent classes is proven in [HS85, Theorem 1]. Then by the same proof of [Car85, Proposition 5.1.2] there exist regular nilpotent elements in \mathfrak{g} and any two are conjugated. The U -orbit of n is closed, since every orbit of a unipotent group is closed [Ste74, Proposition 2.5]. Therefore the proof of [Car85, Proposition 5.1.3] is also valid for nilpotent elements of \mathfrak{g} . ■

Corollary 5. *Let n, n' be regular nilpotent elements of the Lie algebra of B . If $g \in G$ is such that $gng^{-1} = n'$, then $g \in B$.*

If $n = \sum_{\alpha \in \Delta} c_\alpha E_\alpha$ and $n' = \sum_{\alpha \in \Delta} d_\alpha E_\alpha$, then the following statements are equivalent:

- 1. n and n' are conjugated by an element of $G(\mathbb{F})$;
- 2. there is a $t \in T$ such that $d_\alpha = \alpha(t)c_\alpha$.

Proof. By Proposition 4 $B = gBg^{-1}$, since n' is in the Lie algebra of B and gBg^{-1} . Thus $g \in N_G(B) = B$.

Assume that $n = \sum_{\alpha \in \Delta} c_\alpha E_\alpha$ and $n' = \sum_{\alpha \in \Delta} d_\alpha E_\alpha$. If n and n' are conjugated, then there exist $t \in T$ and $u \in U$ such that $Ad(tu)n = n'$. Since U acts trivial on $\mathfrak{n}/\mathfrak{n}_{\geq 1}$ and $Ad(t)E_\alpha = \alpha(t)E_\alpha$, the second statement follows. If $d_\alpha = \alpha(t)c_\alpha$, then $Ad(t)n = n'$. ■

Corollary 5 shows that ${}^G n \cap B = {}^B n$ for all regular $n \in \mathfrak{b}$.

Define Φ as follows:

$$\begin{aligned}\Phi &: X_*(T) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G, T), \mathbb{Z}) \\ \Phi &: \gamma \mapsto (\alpha \mapsto \langle \gamma, \alpha \rangle)\end{aligned}$$

The first reason for defining Φ is the following Proposition. Recall the virtual number of components of $Z(G)$ is $\kappa_v(G) := |\text{coker } \Phi|$.

Proposition 6. *If G is \mathbb{F} -split and $p|\kappa_v(G)$, then there are infinitely many rational regular nilpotent orbits in $\mathfrak{g}(\mathbb{F})$.*

Proof. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a basis for $R(G, T)$. Define for $1 \leq i \leq n$ the function $\epsilon_i \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G, T), \mathbb{Z})$ by:

$$\epsilon_i(\alpha_j) := \delta_{ij}.$$

So $\epsilon_1, \dots, \epsilon_n$ is a basis for $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G, T), \mathbb{Z})$. Let L be the image of Φ . Take a compatible basis for $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G, T), \mathbb{Z})$ and L : b_1, \dots, b_n and $d_1 b_1, \dots, d_n b_n$ with $d_i | d_{i+1}$. Since the cokernel is finite it has $\prod_{i=1}^n d_i$ elements. Define the matrix $M \in GL_n(\mathbb{Z})$ by

$$b_i = \sum_{j=1}^k M_{ij} \epsilon_j.$$

Look at the following subset of \mathfrak{n}_1 :

$$\mathfrak{n}'_1 := \left\{ \sum_{\alpha \in \Delta} c_{\alpha} E_{\alpha} : c_{\alpha} \in \mathbb{F}^{\times} \right\}.$$

Define π to be the following parametrization of \mathfrak{n}'_1 :

$$\pi : (\mathbb{F}^{\times})^n \rightarrow \mathfrak{n}'_1, \quad \pi(c_1, \dots, c_n) := \sum_{i=1}^n c_i E_{\alpha_i}.$$

Since $\sum_{\alpha \in \Delta} c_{\alpha} E_{\alpha}$ is a regular nilpotent element, it is in the same conjugacy class of $G(\mathbb{F})$ as $\sum_{\alpha \in \Delta} d_{\alpha} E_{\alpha}$ if and only if there is a $t \in T$ such that $d_{\alpha} = \alpha(t)c_{\alpha}$ for all $\alpha \in \Delta$, by Corollary 5.

Let $A \in GL_n(\mathbb{Z})$ and define $\phi_A : (\mathbb{F}^{\times})^n \rightarrow (\mathbb{F}^{\times})^n$ by:

$$\phi_A(x_1, \dots, x_n) := \left(\prod_{i=1}^n x_i^{a_{1i}}, \dots, \prod_{i=1}^n x_i^{a_{ni}} \right).$$

Now $\pi \circ \phi_A$ is also a parametrization of \mathfrak{n}'_1 and

$$\phi_A \pi^{-1}(t \pi(\phi_A^{-1}(x_1, \dots, x_n)) t^{-1}) = \left(\left(\prod_{i=1}^n \alpha_i(t)^{a_{1i}} \right) x_1, \dots, \left(\prod_{i=1}^n \alpha_i(t)^{a_{ni}} \right) x_n \right).$$

Define this action of T on $(\mathbb{F}^{\times})^n$ to be the action with respect to A .

Take $A := (M^{-1})^t$. We claim that for every $\gamma \in X_*(T)$ with $\Phi(\gamma) = \sum_{i=1}^n z_i d_i b_i$ one has the following action on $(\mathbb{F}^\times)^n$ with respect to A of $\gamma(s)$:

$$(x_1, \dots, x_n) \mapsto (s^{z_1 d_1} x_1, \dots, s^{z_n d_n} x_n).$$

To prove this claim, consider the factor in front of x_j :

$$\prod_{i=1}^n \alpha_i(\gamma(s))^{a_{ji}} = s^{\sum_{i=1}^n a_{ji} \langle \gamma, \alpha_i \rangle}$$

Evaluate the power of s :

$$\sum_{i=1}^n a_{ji} \langle \gamma, \alpha_i \rangle = \sum_{i=1}^n a_{ji} \sum_{k=1}^n z_k d_k b_k(\alpha_i) = \sum_{i=1}^n a_{ji} \sum_{k=1}^n z_k d_k m_{ki} = \sum_{k=1}^n z_k d_k \sum_{i=1}^n a_{ji} m_{ki}.$$

Since $A = (M^{-1})^t$ one has that $\sum_{i=1}^n a_{ji} m_{ki} = \delta_{jk}$, therefore

$$\begin{aligned} \sum_{i=1}^n a_{ji} \langle \gamma, \alpha_i \rangle &= z_j d_j, \text{ hence} \\ \prod_{i=1}^n \alpha_i(\gamma(s))^{a_{ji}} &= s^{z_j d_j}. \end{aligned}$$

Since $p \nmid \#\text{coker } \Phi$, then $p \mid d_n$. Identify \mathfrak{n}'_1 with $(\mathbb{F}^\times)^n$ via the parametrization $\pi \circ \phi_{M^t}$. Look at the n -th coordinate: $x_n \mapsto s^{z_\gamma d_n} x_n$ for every pair $\gamma \in X_*(T)$, $s \in \mathbb{F}^\times$. The images of the cocharacters generate the torus, so the orbit of the n -th coordinate under T is contained $\{s^{d_n} x_n : s \in \mathbb{F}^\times\}$. Hence if (x_1, \dots, x_n) is in the same orbit as (y_1, \dots, y_n) , then there is a $s \in \mathbb{F}^\times$ such that $s^{d_n} x_n = y_n$. Because $p \mid d_n$, the group $\mathbb{F}^\times / (\mathbb{F}^\times)^{d_n}$ is infinite. We conclude that if p divides the order of the cokernel, then there are infinitely many regular nilpotent orbits. \blacksquare

4. The virtual number of components of $Z(G)$

As we saw in the previous section, when p divides the virtual number of components of $Z(G)$ there are infinitely many regular nilpotent orbits. In this section we show that even more properties that hold when the characteristic is zero, do not hold anymore when $p \mid \kappa_v(G)$. After giving these counter examples for theorems that hold in characteristic zero, at the end of this section we show that $p \mid \kappa_v(G)$ for a restrictive class of reductive groups. By the way, the condition $p \mid \kappa_v(G)$ is based on the group $SL_n(\mathbb{F})$ with $p \mid n$. As turns out in the end for $p \geq 5$ the main example is $SL_n(\mathbb{F})$. In this section all the properties are geometric in nature, so we do not have to worry about rationality.

4.1. Separability and $\kappa_v(G)$.

Lemma 7. *Let $X := \sum_{\alpha \in \Delta} E_\alpha$. The map $[X, \cdot] : \mathfrak{t} \rightarrow \mathfrak{n}_1$ is not surjective if and only if $p \mid \kappa_v(G)$.*

Proof. Consider \mathfrak{t} as $X_*(T) \otimes \mathbb{F}$. Let X_1, \dots, X_n be a basis for $X_*(T)$. Let $Y \in X_*(T)$, then $[dY(1), E_\alpha] = \langle \alpha, Y \rangle E_\alpha$. The matrix M corresponding to Φ with respect to the basis X_1, \dots, X_n and the dual basis of Δ in $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z})$ is the same as the matrix corresponding to $[X, \cdot]$ with respect to the basis X_1, \dots, X_n and $(E_\alpha : \alpha \in \Delta)$. Let $d_1, \dots, d_{|\Delta|}$ be the integers on the diagonal of the Smith normal form of M . Then $\kappa_v(G) = \#\text{coker } \Phi = \prod_{i=1}^{|\Delta|} d_i$. Also there are $E_1, \dots, E_{|\Delta|}$ such that $\mathfrak{n}_1 = \langle E_1, \dots, E_{|\Delta|} \rangle$ and $[X, \mathfrak{t}] = \langle d_1 E_1, \dots, d_n E_{|\Delta|} \rangle$. ■

Recall an G -orbit $Ad(G)x$ in \mathfrak{g} is separable if and only if

$$\dim\{g \in G \mid Ad(g)x = x\} = \dim\{y \in \mathfrak{g} \mid [y, x] = 0\}.$$

Theorem 8. *If $p \mid \kappa_v(G)$, then the regular orbit is not separable.*

Proof. Let T be a torus and $X \in \mathfrak{n}_1$ a regular element. By Proposition 4, then $\dim Z_T(X) = \dim T - |\Delta|$. Thus if the orbit of X is separable, then $[X, \cdot] : \mathfrak{t} \rightarrow \mathfrak{n}_1$ has a kernel of dimension $\dim T - |\Delta|$. Thus $[X, \cdot]$ must be surjective. Since $p \mid \kappa_v(G)$ the map $[X, \cdot]$ is not surjective. Hence the orbit of X is not separable. ■

4.2. Ad and $\kappa_v(G)$. Let G be a reductive \mathbb{F} -group. Let $Ad(G)$ be the image of the adjoint mapping $Ad : G \rightarrow GL(\mathfrak{g})$. We will go back and forth between G and $Ad(G)$. Therefore we have a look at the adjoint map $Ad : G \rightarrow Ad(G)$. The adjoint map is defined over \mathbb{F} . We will show that $d(Ad)$ maps non-zero-nilpotent elements to non-zero nilpotent elements. Ad is separable (ie, $d(Ad)$ is surjective) if and only if $p \nmid \kappa_v(G)$. To distinguish the objects associated with $Ad(G)$ from the ones associated with G , the ones associated with $Ad(G)$ get a superscript ad : $G^{ad}, \mathfrak{g}^{ad}, \mathfrak{n}^{ad}$, ect.

Lemma 9. $d(Ad) : \mathfrak{n} \rightarrow \mathfrak{n}^{ad}$ is an isomorphism.

Proof. Take a Chevalley basis on \mathfrak{g} . Let $\alpha \in R(G, T)$. Let G_α^{ad} be the image of $Ad \circ u_\alpha : \mathbb{F} \rightarrow G^{ad}$. The action of $u_\alpha(x)$ on certain elements of \mathfrak{g} is as follows:

$$\begin{aligned} Ad(u_\alpha(x))E_{-\alpha} &= E_{-\alpha} + x d\alpha^\vee(1) - x^2 E_\alpha, \\ Ad(u_\alpha(x))H &= H - d\alpha(H)x E_\alpha. \end{aligned}$$

Since $\langle \alpha, \alpha^\vee \rangle = 2$, either $d\alpha^\vee(1) \neq 0$ or there exists a $H \in \mathfrak{t}$ such that $d\alpha(H) \neq 0$. Therefore $Ad \circ u_\alpha$ is an isomorphism between \mathbb{F} and its image in G . Because $t u_\alpha(x) t^{-1} = u_\alpha(\alpha(t)x)$ for $t \in T$ and $x \in \mathbb{F}$, also

$$t Ad(u_\alpha(x)) t^{-1} = Ad(u_\alpha(\alpha(t)x)),$$

for all $t \in T^{ad}$ and $x \in \mathbb{F}$. Thus $d(Ad) : \mathfrak{g}_\alpha \rightarrow \mathfrak{g}_\alpha^{ad}$ is an isomorphism. Therefore $d(Ad) : \mathfrak{n} \rightarrow \mathfrak{n}^{ad}$ is injective. Since $\dim \mathfrak{n} = \dim \mathfrak{n}^{ad}$, the Lemma follows. ■

Proposition 10. *The map $Ad : G \rightarrow Ad(G)$ is separable if and only if the characteristic of \mathbb{F} does not divide the virtual number of components of $Z(G)$.*

Proof. Let Δ be a system of positive roots for $R(G, T)$.

Define $n := |\Delta|$ and let $\alpha_1, \dots, \alpha_n$ be the roots in Δ . Take $\gamma_1, \dots, \gamma_n \in X_*(T)$ such that the image of Φ is generated by $\gamma_1, \dots, \gamma_n$. The number of elements in the cokernel of Φ is equal to the determinant of the matrix $M_{ij} := \langle \gamma_j, \alpha_i \rangle$. Since $d(Ad)$ is surjective on $\mathfrak{n}_+^{ad} \oplus \mathfrak{n}_-^{ad}$, we only have to look whether $Ad : T \rightarrow T^{ad}$ is separable. Identify T^{ad} with a torus of dimension n in such a way that the map Ad is as follows:

$$t \mapsto \begin{pmatrix} \alpha_1(t) & & 0 \\ & \ddots & \\ 0 & & \alpha_n(t) \end{pmatrix}$$

The Lie algebra of a torus S is canonically isomorphic to $X_*(S) \otimes_{\mathbb{Z}} \mathbb{F}$ by [Spr98, 4.4.11(4)]. With this isomorphism the map $d(Ad)$ is the linear map such that for $\gamma \in X_*(T)$, $d(Ad)(\gamma) = Ad \circ \gamma$. Now the images of $Ad \circ \gamma_1, \dots, Ad \circ \gamma_n$ generate the image of $d(Ad)$. Thus the image of \mathfrak{t} is generated by the vectors $\sum_{i=1}^n \langle \gamma_j, \alpha_i \rangle \chi_i$ for $j = 1, \dots, n$. This is surjective if and only if the corresponding matrix has non-zero determinant. The corresponding matrix is equal to M . Thus $p \nmid \#\kappa_v(G) = \#\text{coker } \Phi$ if and only if M is invertible if and only if Ad is separable. ■

4.3. Very good primes and $\kappa_v(G)$.

Lemma 11. *If $Y \subset X_*(T)$ such that $\Phi(Y)$ has finite index in $\text{Hom}(\mathbb{Z}R(G, T), \mathbb{Z})$, then $\#\text{coker } \Phi$ divides $\#\text{coker } \Phi|_Y$.*

Proof. The lemma follows from general abstract nonsense:

$$\begin{array}{ccc} A & & \text{coker } g \\ & \searrow f & \nearrow cg \\ & B & \\ & \nearrow g & \searrow cf \\ C & & \text{coker } f \end{array} \quad \begin{array}{c} \uparrow h \\ \vdots \\ \uparrow \end{array}$$

Since $cg \circ g \circ \iota = 0$ there is a unique morphism $h : \text{coker } f \rightarrow \text{coker } g$ such that $cg = cf \circ h$. Since cg is surjective, also h is surjective. Thus $\#\text{coker } g \mid \#\text{coker } f$. ■

Proposition 12. *If $p \mid \kappa_v(G)$, then p divides the determinant of the Cartan matrix of $R(G, T)$.*

Proof. Let Y be the subgroup of $X_*(T)$ generated by the coroots of $R(G, T)$. The order of the cokernel $Y \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G, T), \mathbb{Z})$ is equal to the determinant of the Cartan matrix. ■

Corollary 13. *If $p \mid \kappa_v(G)$, then p is not a very good prime for G . If moreover G does not contain a normal subgroup of type A_l , then p is a bad prime for G and $p \in \{2, 3\}$.*

Proof. By [Hum78, 11.4, Exercise 2] the determinants of the Cartan matrices for the irreducible root systems are:

$$A_l : l + 1; B_l : 2; C_l : 2; D_l : 4; E_6 : 3; E_7 : 2; E_8, F_4 \text{ and } G_2 : 1.$$

Compare this with the notion of a prime that is not a very good prime. Then p divides:

$$A_l : l + 1; B_l : 2; C_l : 2; D_l : 2; E_6 : 2, 3; E_7 : 2, 3; E_8 : 2, 3, 5; F_4 : 2, 3; G_2 : 2, 3. \quad \blacksquare$$

5. Howe's conjecture in bad characteristic

In this section we show that Howe's conjecture does not hold for \mathbb{F} -split groups in bad characteristic. The calculations in the actual group are postponed to the end of this section and the Appendix. Under the assumption that there exists a bad pair, we will construct sets of linearly independent distributions in $J_L(\omega)$ of arbitrary finite size. The support of these distributions is contained in the set of nilpotent elements. Two consequences of our method are the existence of infinitely many regular nilpotent orbits and the inseparability of the regular nilpotent orbit.

5.1. Reduction to bad pairs. Let G be a \mathbb{F} -split reductive group. Let T be a maximal \mathbb{F} -split torus. Let R^+ be a system of positive roots. Let U^+ be the unipotent subgroup corresponding to R^+ and \mathfrak{n} its Lie algebra. Let $B = TU^+$ be the corresponding Borel subgroup. The set of regular nilpotent elements of \mathfrak{n} is denoted by \mathfrak{n}' .

Let H_1, \dots, H_r and E_γ for $\gamma \in R$ be a Chevalley basis for \mathfrak{g} . Let, for $\gamma \in R$, $u_\gamma : \mathbb{F} \rightarrow U_\gamma$ be the corresponding parametrization of U_γ . ($du_\gamma(1) = E_\gamma$) Now \mathfrak{n} has as basis $E_\alpha : \alpha \in R^+$. For $x \in \mathfrak{n}$ and $\alpha \in R^+$ we define $x_\alpha \in \mathbb{F}$ such that $x = \sum_{\alpha \in R^+} x_\alpha E_\alpha$. Define $X_\alpha(x) := x_\alpha$.

Definition 14. Let $\eta : \mathbb{F} \rightarrow \mathfrak{n}'$ and $\chi : \mathfrak{n}' \rightarrow \mathbb{F}$ be polynomial functions. The pair (η, χ) is called a bad pair if it satisfies the following four conditions:

1. $\chi\eta(\alpha) = \alpha$ for all $\alpha \in \mathbb{F}$.
2. If $n, n' \in \mathfrak{n}'$ are conjugated by an element of $G(\mathbb{F})$, then $\chi(n) \equiv \chi(n') \pmod{\mathbb{F}^{(p)}}$. There is a $z \in p\mathbb{Z}$ such that $c^z \chi(n) = \chi(cn)$ for all $c \in \mathbb{F}^\times$ and $n \in \mathfrak{n}'$.
3. For $\gamma \in R^+$ and $\alpha \in \mathcal{O}^\times$, $\eta(\alpha)_\gamma \in \mathcal{O}$ and if moreover $\gamma \in \Delta$, then $\eta(\alpha)_\gamma \in \mathcal{O}^\times$.
4. $\chi \in \mathcal{O}[X_\gamma, X_\beta^{-1} : \gamma \in R^+, \beta \in \Delta] \subset \mathbb{F}[\mathfrak{n}']$, where $\mathbb{F}[\mathfrak{n}']$ is the algebra of all \mathbb{F} -regular functions on \mathfrak{n}' .

For the remainder of this subsection we assume that (η, χ) is a bad pair. Because $\mathbb{F}/\mathbb{F}^{(p)}$ is infinite, the first and second conditions of a bad pair already imply that there are infinitely many regular nilpotent orbits in \mathfrak{g} . We will use χ to define G -invariant distributions and η to show that they are linearly independent.

For $n \in \mathbb{N}$ we define a compact subgroup K_n of G as follows (see [MP94])

for a general construction):

$$\begin{aligned} U_{\gamma,n} &:= u_\gamma(v^{-1}[n, \infty)), \\ T_{i,n} &:= \{t \in T \mid \forall[\alpha \in X^*(T)] v(\alpha(t) - 1) \geq n\}. \end{aligned}$$

Define K_n to be the subgroup of G generated by the groups $U_{\gamma,n}$ and $T_{i,n}$. Define $K := K_0$. Now K is the group of \mathcal{O} -points of a split reductive \mathcal{O} -group scheme \mathcal{K} with generic fiber G , see [MP94, §3.2].

We may identify \mathfrak{t} with $X_*(T) \otimes_{\mathbb{Z}} \mathbb{F}$ by

$$X_*(T) \ni \gamma \mapsto d\gamma(1) \in \mathfrak{t}.$$

Let $\delta_1, \dots, \delta_s$ a basis for $X_*(T)$ and H'_1, \dots, H'_s the corresponding basis in \mathfrak{t} . Let L be the \mathcal{O} -lattice spanned by H'_1, \dots, H'_s and all E_γ .

For $m \in \mathfrak{g}$ we define $m_i \in \mathbb{F}$ and $m_\gamma \in \mathbb{F}$ such that

$$m = \sum_{i=1}^s m_i H'_i + \sum_{\gamma \in R} m_\gamma E_\gamma.$$

Now L is K -invariant. Thus K acts on $L/\pi^n L$. The group K_n acts trivially on $L/\pi^n L$, by the choice of K_n and L . Now $G(\mathbb{F}_q) := \mathcal{K}(\mathbb{F}_q) \cong K/K_1$ and $\mathfrak{g}(\mathbb{F}_q) = L/\pi L$. Let $B(\mathbb{F}_q) := (K \cap B)/K_1$ and $\mathfrak{n}(\mathbb{F}_q)$ its Lie algebra.

Lemma 15. *There exists a $N > 0$ such that for all $n \in \mathbb{N}_{>0}$, $k \in K$ and $\alpha \in \mathcal{O}^\times$:*

$$k\eta(\alpha)k^{-1} \in \mathfrak{n} + \pi^{Nn}L \Rightarrow k \in (B \cap K)K_n.$$

Proof. The map $\pi_0 : G(\mathcal{O}) \rightarrow \mathcal{K}(\mathbb{F}_q)$ induces a map on the Lie algebra: $\pi_0 : \mathfrak{g}(\mathcal{O}) \rightarrow \mathfrak{g}(\mathbb{F}_q)$ with kernel πL . Since $\pi_0(\eta(\alpha))$ is also a regular nilpotent element by condition 3 of definition 14 and $\pi_0(k\eta(\alpha)k^{-1}) \in \mathfrak{n}(\mathbb{F}_q)$, we have $\pi_0(k) \in B(\mathbb{F}_q)$. Thus $k \in (B \cap K)K_1$. Take for the moment a general $N \in \mathbb{N}_{>0}$. Because \mathfrak{n} and $\pi^{Nn}L$ are $(B \cap K)$ -invariant and $K_1 = (B \cap K_1)(U^- \cap K_1)$, we may assume $k \in U^- \cap K$.

Take $x_\gamma \in \mathcal{O}$ arbitrary for $\gamma \in R^-$. Define $u = \prod_{\gamma \in R^-} u_\gamma(x_\gamma)$. Let, for $i = 1, \dots, s$ and $\beta \in R^-$, $p_i, p_\beta \in \mathbb{F}[X_\gamma : \gamma \in R^-, Y, Y^{-1}]$ be such that $(u\eta(\alpha)u^{-1})_i = p_i(x_\gamma, \alpha, \alpha^{-1})$ and $(u\eta(\alpha)u^{-1})_\beta = p_\beta(x_\gamma, \alpha, \alpha^{-1})$.

Let I be the ideal generated by p_β for $\beta \in R^-$. Then $u\eta(\alpha)u^{-1} \in \mathfrak{n}$ if and only if $p_\beta(x_\gamma, \alpha, \alpha^{-1}) = 0$ for all $\beta \in R^-$. Because of Corollary 5 for $x_\gamma, \alpha \in \overline{\mathbb{F}}$:

$$u\eta(\alpha)u^{-1} \in \mathfrak{n} \Leftrightarrow u = 1 \Leftrightarrow \forall \gamma \in R^- [x_\gamma = 0],$$

where $u = \prod_{\gamma \in R^-} u_\gamma(x_\gamma)$. By the Nullstellensatz we have $X_\gamma \in \sqrt{I}$ for all $\gamma \in R^-$. Thus there exists a $m \in \mathbb{N}$ such that $X_\gamma^m \in I$ for all $\gamma \in R^-$. Therefore there are polynomials $f_{\gamma,\beta} \in \mathbb{F}[X_\gamma, Y, Y^{-1}]$ such that

$$X_\gamma^m = \sum_{\beta \in R^+} f_{\gamma,\beta} p_\beta.$$

Let M be the smallest $n \in \mathbb{N}_{\geq 0}$ such that $f_{\gamma,\beta}(x_\gamma, \alpha, \alpha^{-1}) \subset \pi^{-n}\mathcal{O}$ for all $\beta, \gamma \in R^-$, $x_\gamma \in \mathcal{O}$ and $\alpha \in \mathcal{O}^\times$.

Take $N := m + M$. Assume $u\eta(\alpha)u^{-1} \in \mathfrak{n} + \pi^{Nn}L$, then $v(p_\beta(x_\gamma, \alpha, \alpha^{-1})) \geq Nn$. Because

$$x_\gamma^m = \sum_{\beta \in R^-} f_{\gamma,\beta}(x_\gamma, \alpha, \alpha^{-1})p_\beta(x_\gamma, \alpha),$$

we have

$$v(x_\gamma^m) \geq Nn - M = mn + (n-1)M \geq mn.$$

Thus $v(x_\gamma) \geq n$. Hence if $k\eta(\alpha)k^{-1} \in \mathfrak{n} + \pi^{nN}L$ and $k \in U^- \cap K$, then $k \in K_n$. ■

Let δ_B be the modular function of B , thus

$$\delta_B(b) \int_B f(xb)dx = \int_B f(x)dx.$$

Proposition 16 (Rao). *Assume that $V \subset \mathfrak{n}$ is open and B -invariant. Then for all $f \in C_c^\infty(\mathfrak{g})$,*

$$\int_V f(bXb^{-1})dX = \delta_B(b) \int_V f(X)dX.$$

Moreover the distribution

$$D_V(f) := \int_V \int_K f(kXk^{-1})dkdX$$

is G -invariant.

Proof. Since $\delta|_B(b) = |\det(Ad b|_{\mathfrak{n}})|^{-1}$,

$$\int_{\mathfrak{n}} f(bXb^{-1})dX = \delta_B(b) \int_{\mathfrak{n}} f(X)dX.$$

Because V is open and B -invariant, we can apply this formula to $\int_V f(X)dX$. This proves the first statement of the Proposition.

The second statement follows from the first by [How74, Proposition 4]. The method described here is essentially in [Rao72]. ■

Let $\mathfrak{n}(\mathcal{O})$ be the \mathcal{O} -module generated by E_α , with $\alpha \in R^+$.

Corollary 17. *Let $\omega \subset \mathfrak{g}$ be open and compact. If $V \subset \mathfrak{n}$ is open and B -invariant, then $D_V \in J(\omega)$.*

Proof. By Proposition 16, D_V is a G -invariant distribution. The support of D_V is contained in $\overline{\mathfrak{n}^K}$. Since ω is open, there is a $m \in \mathbb{N}$ such that $\pi^m \mathfrak{n}(\mathcal{O}) \subset \omega$. Since $\mathfrak{n}(\mathcal{O})^T = \mathfrak{n}$, then $\text{supp } D_V \subset \overline{\mathfrak{n}^K} \subset \overline{(\omega^T)^K} \subset \overline{\omega^G}$. ■

For $\alpha \in \mathbb{F}^\times$ and $s \in \mathbb{N}$, define $V_{\alpha,s} \subset \mathfrak{n}$ as follows:

$$V_{\alpha,s} := \{n \in \mathfrak{n} \mid \chi(n) \equiv \alpha \pmod{(\pi^s \mathcal{O} + \mathbb{F}^{(p)})}\}.$$

Let $\Delta = \{\alpha_1, \dots, \alpha_m\}$. Define for $a_1, \dots, a_m \in \mathbb{F}$ the following nilpotent element:

$$n(a_1, \dots, a_m) := \sum_{i=1}^m a_i E_{\alpha_i}.$$

Take $z \in p\mathbb{Z}$ such that $\chi(cn) = c^z \chi(n)$ for all $n \in \mathfrak{n}'$ and $c \in \mathbb{F}$.

Lemma 18. *Let $N \in \mathbb{N}$ be the constant arising from Lemma 15. Let $n \in \mathbb{N}_{>0}$, $\alpha \in \mathcal{O}^\times$ and $\beta \in \mathcal{O}^\times$. If*

$$\int_{V_{\pi^{-znN}\beta, n}} \int_{k \in K} 1_{\pi^{-Nn}\eta(\alpha)+L}(kXk^{-1}) dk dX > 0,$$

then $\alpha \equiv \beta \pmod{\pi^n \mathcal{O} + \mathcal{O}^{(p)}}$.

Proof. Let $X \in V_{\pi^{-znN}\beta, n}$, $k' \in K$ and $l' \in L$ such that

$$k'Xk'^{-1} + l' = \pi^{-Nn}\eta(\alpha).$$

Since L is K -invariant, there exist a $k \in K$ and $l \in L$ such that

$$k\pi^{-nN}\eta(\alpha)k^{-1} + l = X \in V_{\pi^{-znN}\beta, n} \subset \mathfrak{n}.$$

Thus $k \in (K \cap B)K_n$ by Lemma 15, because $k\eta(\alpha)k^{-1} \in \mathfrak{n} + \pi^{nN}L$. Take $b_k \in K \cap B$ and $k_n \in K_n$ such that $k = k_n b_k$. Take $a_1, \dots, a_m \in \mathcal{O}^\times$ and $n_2 \in \mathfrak{n}_2(\mathcal{O})$ such that $b_k \eta(\alpha) b_k^{-1} = n(a_1, \dots, a_m) + n_2$. By assumption 2 of the bad pair, there exists a $\gamma \in \mathbb{F}$ such that $\chi(n(a_1, \dots, a_m) + n_2) = \alpha + \gamma^p$. Since $k_n \in K_n$ and $n(a_1, \dots, a_m) + n_2 \in L$, there exists a $l' \in L$ such that

$$k_n(n(a_1, \dots, a_m) + n_2)k_n^{-1} = n(a_1, \dots, a_m) + n_2 + \pi^n l'.$$

Thus

$$\begin{aligned} \chi(k\eta(\alpha)k^{-1} + \pi^{nN}l) &= \chi(k_n b_k \eta(\alpha) b_k^{-1} k_n^{-1} + \pi^{nN}l) \\ &= \chi(n(a_1, \dots, a_m) + n_2 + \pi^n l' + \pi^{nN}l). \end{aligned}$$

Since the a_i are in \mathcal{O}^\times and $\chi \in \mathcal{O}[X_\gamma, X_\beta^{-1} : \gamma \in R^+, \beta \in \Delta]$,

$$\begin{aligned} \chi(n(a_1, \dots, a_m) + n_2 + \pi^n l' + \pi^{nN}l) &\equiv \chi(n(a_1, \dots, a_m) + n_2) \\ &= \alpha + \gamma^p \pmod{\pi^n \mathcal{O}}. \end{aligned}$$

Because $\chi(\pi^{-znN}x) = \pi^{-znN}\chi(x)$ for all $x \in \mathfrak{g}$,

$$\chi(k\pi^{-nN}\eta(\alpha)k^{-1} + l) \equiv (\alpha + \gamma^p)\pi^{-znN} \pmod{\pi^{n-znN}\mathcal{O}}.$$

Since $k\pi^{-nN}\eta(\alpha)k^{-1} + l \in V_{\pi^{-znN}\beta, n}$,

$$\chi(k\pi^{-nN}\eta(\alpha)k^{-1} + l) \equiv \pi^{-znN}\beta \pmod{(\pi^{n-znN}\mathcal{O} + \mathbb{F}^{(p)})}.$$

Thus

$$\pi^{-znN}\beta \equiv \chi(k\pi^{-nN}\eta(\alpha)k^{-1} + l) \equiv \pi^{-znN}\alpha \pmod{(\pi^{n-znN}\mathcal{O} + \mathbb{F}^{(p)})}.$$

Then $\alpha \equiv \beta \pmod{(\pi^n\mathcal{O} + \mathbb{F}^{(p)})}$. Because $\mathbb{F}^{(p)} \cap \mathcal{O} = \mathcal{O}^{(p)}$ and $\alpha, \beta \in \mathcal{O}$, the Lemma follows. \blacksquare

Theorem 19. *Let G be a \mathbb{F} -split reductive group. If there exists a bad pair (η, χ) for G , then $\dim J_L(\omega) = \infty$.*

Proof. Take $n \in \mathbb{N}_{>0}$. Let $\alpha_1, \dots, \alpha_k$ be representatives of the cosets of $\pi^n\mathcal{O} + \mathcal{O}^{(p)}$ in \mathcal{O} . Define for $1 \leq i \leq k$ the following distribution and function:

$$D_i(f) := D_{V_{\pi^{-znN}\alpha_i, n}}(f) = \int_{V_{\pi^{-znN}\alpha_i, n}} \int_{k \in K} f(kXk^{-1}) dk dX,$$

$$f_i := 1_{\pi^{-nN}\eta(\alpha_i) + L}.$$

The distribution D_i are in $J(\omega)$ by Corollary 17. Let $c_i := D_i(f_i)$, then $c_i > 0$. Then $D_i(f_j) = c_i\delta_{ij}$, by Lemma 18. Therefore the distributions D_1, \dots, D_k are linearly independent. Since we may view the f_i 's as elements of $C_c^\infty(\mathfrak{g}/L)$, the distributions D_i remain linearly independent when we view them as distributions of \mathfrak{g}/L . Thus $\dim J_L(\omega) \geq k$. As n goes to infinity so does k . \blacksquare

5.2. The bad pair construction. In this section we assume that $p = \text{char } \mathbb{F}$ is bad for G . The construction of a bad pair is done in three steps. First we construct a bad pair in the case G is simple of adjoint type. Then we show that if there is a bad pair for $Ad(G)$, then we can construct a bad pair for G . In the third step we combine the results of the first and second step to construct a bad pair.

Define $X := \sum_{\beta \in \Delta} E_\beta$. Let $\alpha_1, \dots, \alpha_k$ be the roots of height $p+1$. They exists because p is bad for G . Define

$$n(a_1, \dots, a_k) := X + \sum_{i=1}^k a_i E_{\alpha_i},$$

for $a_i \in \mathbb{F}$. Recall that $\mathfrak{n}_i = \bigoplus_{\alpha: ht(\alpha)=i} \mathfrak{g}_\alpha$.

Theorem 20 (Springer). *Let G be a simple group. Let $X := \sum_{\alpha \in \Delta} E_\alpha$. Assume $p = \text{char } \mathbb{F}$ is bad for G . Then*

1. $[X, \cdot] : \mathfrak{n}_i \rightarrow \mathfrak{n}_{i+1}$ is surjective for $1 \leq i \leq p-1$;
2. $\dim \mathfrak{n}_{p+1}/[X, \mathfrak{n}_p] = 1$;
3. $|\Delta| = \dim \mathfrak{n}_1 = \dim \mathfrak{n}_i + 1$ for $2 \leq i \leq p+1$.

The author would like to thank the referee for pointing him at Springer's paper [Spr66]. In this paper the operation $[X, \cdot] : \mathfrak{g} \rightarrow \mathfrak{g}$ is studied in more detail, see for example [Spr66, (2.6) Theorem.].

Proof. See [Spr66, (2.11) Theorem]. ■

Corollary 21. *If p is bad for G , then the regular nilpotent orbit is not separable.*

Proof. Since X is a regular nilpotent element

$$\dim Z_G(X) = \dim Z_B(X) = \dim B - \dim U = \dim T.$$

Because $\dim Z_T(X) = \dim T - |\Delta|$, we have $\dim Z_U(X) = |\Delta|$.

Thus if the orbit of X is separable, then $[X, \cdot] : \mathfrak{n} \mapsto \mathfrak{n}_{\geq 2}$ has a kernel of dimension $|\Delta|$. Therefore $[X, \cdot]$ must be surjective. As Theorem 20 shows, this is not the case when p is bad for a simple group G . By passing to the adjoint group, the Corollary follows. ■

Proposition 22. *Assume that G is a simple group. There exists a surjective linear function $f : \mathbb{F}^k \rightarrow \mathbb{F}$ such that if $n(a_1, \dots, a_k)$ is conjugated to $n(b_1, \dots, b_k)$, then $f(a_1, \dots, a_k) \equiv f(b_1, \dots, b_k) \pmod{\mathbb{F}^{(p)}}$.*

Proof. Let $f : \mathfrak{n}_{p+1} \rightarrow \mathbb{F}$ be a linear function corresponding with the isomorphism $\mathfrak{n}_{p+1}/[X, \mathfrak{n}_p] \cong \mathbb{F}$. For $u \in U$ write $u = \prod_{\gamma \in R^+} u_\gamma(x_\gamma)$. By Theorem 20, we have $\dim \mathfrak{n}_i = \dim \mathfrak{n}_1 - 1$ for $2 \leq i \leq p+1$ and $n \mapsto [X, n]$ is a bijection from \mathfrak{n}_i to \mathfrak{n}_{i+1} for $2 \leq i \leq p-1$.

We will prove with induction on the height of the roots that there exist $c_\gamma, d_\gamma \in \mathbb{F}$ and $x \in \mathbb{F}$, such that for $i \leq p-1$, if $uXu^{-1} \equiv X \pmod{\mathfrak{n}_{\geq i+2}}$, then

$$x_\gamma = c_\gamma x^{ht(\gamma)}, \tag{1}$$

for $\gamma \in R^+$ with $ht(\gamma) \leq i-1$ and

$$uXu^{-1} \equiv X - [X, \sum_{\gamma \in R_{i+1}^+} x_\gamma E_\gamma] + \sum_{\gamma \in R_{i+2}^+} d_\gamma x^{i+1} E_\gamma \pmod{\mathfrak{n}_{\geq i+3}}. \tag{2}$$

Before we give the induction argument, first we restate (2).

The nilpotent element $uXu^{-1} \pmod{\mathfrak{n}_{\geq i+3}}$ only depends on the value of x_γ for the γ with height at most $i+1$. In expression (2) the dependence of the roots of height $i+1$ is taken care of with the term $-[X, \sum_{\gamma \in R_{i+1}^+} x_\gamma E_\gamma]$. So for the proof of (2) we need to show that

$$RM_i := uXu^{-1} - X + [X, \sum_{\gamma \in R_{i+1}^+} x_\gamma E_\gamma]$$

is equal to $\sum_{\gamma \in R_{i+2}^+} d_\gamma x^{i+1} E_\gamma$, when $uXu^{-1} \equiv X \pmod{\mathfrak{n}_{\geq i+2}}$.

The function $[X, \cdot] : \mathfrak{n}_1 \rightarrow \mathfrak{n}_2$ gives that $x := x_\gamma = x_\delta$ for all $\gamma, \delta \in \Delta$. By the Steinberg conjugacy formula [Spr98, Proposition 8.2.3] we have $d_\gamma \in \mathbb{F}$ such that $RM_1 = \sum_{\gamma \in R_3^+} d_\gamma x^2 E_\gamma$.

Assume that (1) and (2) hold for $i-1$.

The function $[X, \cdot] : \mathfrak{n}_i \rightarrow \mathfrak{n}_{i+1}$ is bijective. So for all $n_{i+1} \in \mathfrak{n}_{i+1}$ there is exactly one $u \in U_i$ such that $u(X + n_{i+1})u^{-1} = X \pmod{\mathfrak{n}_{\geq i+2}}$, namely the one

corresponding with the inverse of $[X, \cdot]$. Let $In : \mathfrak{n}_{i+1} \rightarrow \mathfrak{n}_i$ be the inverse of $[X, \cdot]$. Then the $x_\gamma \in \mathbb{F}$ for $\gamma \in R_i^+$ are such that

$$In(RM_{i-1}) = \sum_{\gamma \in R_i^+} x_\gamma E_\gamma.$$

By the induction hypotheses $RM_{i-1} = \sum_{\gamma \in R_{i+1}^+} d_\gamma x^i E_\gamma$ for some constants $d_\gamma \in \mathbb{F}$. Thus for every $\gamma \in R_i^+$ there exists a $c_\gamma \in \mathbb{F}$ such that $x_\gamma = c_\gamma x^i$. By the Steinberg conjugacy formula we have $d_\gamma \in \mathbb{F}$ such that $RM_i = \sum_{\gamma \in R_{i+2}^+} d_\gamma x^{i+1} E_\gamma$.

Assume that $un(a_1, \dots, a_k)u^{-1} \equiv n(b_1, \dots, b_k) \pmod{\mathfrak{n}_{\geq p+2}}$. Then certainly

$$uXu^{-1} \equiv un(a_1, \dots, a_k)u^{-1} \equiv n(b_1, \dots, b_k) \equiv X \pmod{\mathfrak{n}_{\geq p+1}}.$$

Thus by (2)

$$un(a_1, \dots, a_k)u^{-1} \equiv [X, n_p] + n(a_1 + d_1 x^p, \dots, a_k + d_k x^p) \pmod{\mathfrak{n}_{\geq p+2}},$$

with $d_i := d_{\alpha_i}$ and $n_p \in \mathfrak{n}_p$. Thus

$$n(a_1 + d_1 x^p, \dots, a_k + d_k x^p) - X \equiv n(b_1, \dots, b_k) - X \pmod{[X, \mathfrak{n}_p]}.$$

Since $f : \mathfrak{n}_{p+1} \rightarrow \mathbb{F}$ is a linear map with kernel $[X, \mathfrak{n}_p]$,

$$f(b_1 - a_1, \dots, b_k - a_k) = f(d_1 x^p, \dots, d_k x^p) = x^p f(d_1, \dots, d_k).$$

Define $c := f(d_1, \dots, d_k)$. Then $f(b_1 - a_1, \dots, b_k - a_k) \in c\mathbb{F}^{(p)}$ if and only if $n(a_1, \dots, a_k)$ is U -conjugated modulo $\mathfrak{n}_{\geq p+2}$ to $n(b_1, \dots, b_k)$.

Since being U -conjugated modulo $\mathfrak{n}_{\geq p+2}$ is an equivalence relation, we have that $c \in \mathbb{F}^{(p)}$. Now $c \neq 0$, because by Proposition 4 over an algebraically closed field the orbit of X in \mathfrak{n} contains $X + \bigoplus_{i=2}^{ht(R)} \mathfrak{n}_i$. Thus $c \in (\mathbb{F}^\times)^p$. \blacksquare

Lemma 23. *If G is simple of adjoint type and p is bad for G , then there exists a bad pair for G .*

Proof. Choose an $\alpha \in \Delta$, define

$$U_{\hat{\alpha}} := \prod_{\gamma \in \{\gamma \in R^+ - \{\alpha\} \mid ht(\gamma) \leq p-1\}} U_\alpha.$$

Recall that $X = \sum_{\alpha \in \Delta} E_\alpha$. The map $\Phi : X_*(T) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G, T), \mathbb{Z})$ is an isomorphism, because G is of adjoint type. For $\alpha \in \Delta$ take cocharacters $\omega_\alpha \in X_*(T)$ such that for all $\beta \in \Delta$, $\langle \omega_\alpha, \beta \rangle = \delta_{\alpha, \beta}$.

Let $n = \sum_{\alpha \in \Delta} c_\alpha E_\alpha$. Take $t = \prod_{\alpha \in \Delta} \omega_\alpha(c_\alpha)$, then $tXt^{-1} = n$. Let $s \in T$. Assume that $sXs^{-1} = n$, then for all $\alpha \in \Delta$, $c_\alpha = \alpha(s)$. Let $d_\alpha \in \mathbb{F}$ be such that $s = \prod_{\alpha \in \Delta} X_\alpha(d_\alpha)$, since the X_α are a basis for $X_*(T)$ such d_α 's exist. Thus $d_\alpha = \alpha(t) = c_\alpha$. Hence $s = \prod_{\alpha \in \Delta} \omega_\alpha(c_\alpha) = t$.

Thus for every $n \in \mathfrak{n}'$ there is exactly one $t \in T$ such that $tnt^{-1} \in X + \mathfrak{n}_{\geq 2}$. Therefore by the proof of Proposition 22, for every $n \in \mathfrak{n}'$, there exists a unique $b \in TU_{\hat{\alpha}}$ such that $bnb^{-1} = X + n(a_1, \dots, a_k) + n_{p+2}$, with $n_{p+2} \in \mathfrak{n}_{\geq p+2}$. Write

$n = \sum_{\alpha \in R} x_{\alpha} E_{\alpha}$. The a_1, \dots, a_k depend on x_{α} for $\alpha \in R_i^+$ with $i \leq p+1$. Let f_i be the rational functions such that $a_i = f_i(x_{\alpha})$. The f_i are homogeneous of degree $-p$:

$$\begin{aligned} bnb^{-1} &\equiv X + n(a_1, \dots, a_k) && \text{mod } \mathfrak{n}_{\geq p+2}, \\ b\lambda nb^{-1} &\equiv \lambda(X + n(a_1, \dots, a_k)) && \text{mod } \mathfrak{n}_{\geq p+2}, \\ tb\lambda nb^{-1}t^{-1} &\equiv X + \frac{\lambda}{\lambda^{p+1}}n(a_1, \dots, a_k) && \text{mod } \mathfrak{n}_{\geq p+2}, \end{aligned}$$

where $t \in T$ is such that $\gamma(t) = \frac{1}{\lambda}$ for all $\gamma \in \Delta$.

Define $\chi(n) := f(f_1(x_{\alpha}), \dots, f_k(x_{\alpha}))$ for $n \in \mathfrak{n}$. Choose a $g : \mathbb{F} \rightarrow \mathbb{F}^k$ to be a right inverse of f , ie. $fg = id$, such that $n(g(\mathcal{O})) \subset \mathfrak{n}_{p+1}(\mathcal{O})$. Define $\eta : \mathbb{F} \rightarrow \mathfrak{n}'$ by $\eta(a) := X + n(g(a))$. Now (χ, η) is a bad pair for G . ■

Now we will deduce from Lemma 23 the existence of a bad pair for reductive groups G for which p is bad.

Lemma 24. *If (η, χ) is a bad pair for $Ad(G)$, then there exists a bad pair for G .*

Proof. Let $Ad : G \rightarrow Ad(G)$ be the natural morphism. The linear map $d(Ad) : \mathfrak{n} \rightarrow \mathfrak{n}^{ad}$ is a bijection by Lemma 9. Let $da : \mathfrak{n}^{ad} \rightarrow \mathfrak{n}$ be its inverse. If $n, n' \in \mathfrak{n}$ are conjugated by G , then their image is conjugated by $Ad(G)$. If E_{α} is a Chevalley basis for G , then $d(Ad)(E_{\alpha})$ is one for $Ad(G)$. Thus $(da \circ \eta, \chi \circ d(Ad))$ is a bad pair for G . ■

Theorem 25. *Assume that G is a \mathbb{F} -split reductive group and p is bad for G . Then there exists a bad pair for G .*

Proof. By Lemma 24 we may assume that G is semi-simple of adjoint type. Then G is a direct product of simple connected normal subgroups. Since the characteristic is bad, there exists a bad pair for at least one of these subgroups by Lemma 23. Therefore there exists a bad pair for G . ■

Theorem 26. *If G is a \mathbb{F} -split reductive group and char \mathbb{F} is bad, then there are infinitely many nilpotent orbits and Howe's conjecture on the Lie algebra does not hold.*

Proof. This follows from Theorem 25 and Theorem 19. ■

5.3. The example $SO_5(\mathbb{F})$, char $\mathbb{F} = 2$. In this subsection \mathbb{F} has characteristic 2. We follow [Spr98, §7.4.7(6)] for the definition of $SO_5(\mathbb{F})$. Let $V = \mathbb{F}^5$ and let Q be the quadratic form on V defined by

$$Q(e_0, e_1, e_2, e_3, e_4) := e_0^2 + e_1e_3 + e_2e_4.$$

Now we define $SO_5(\mathbb{F})$ to be the subgroup of $t \in GL(V)$ with $Q(tv) = Q(v)$ for all $v \in V$. Then

$$T := \{t(t_1, t_2) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 & 0 \\ 0 & 0 & t_2 & 0 & 0 \\ 0 & 0 & 0 & t_1^{-1} & 0 \\ 0 & 0 & 0 & 0 & t_2^{-1} \end{pmatrix} : t_i \in \mathbb{F}^\times\}$$

is a maximal torus of SO_5 that is \mathbb{F} -split. Define, for $i = 1, 2$, the character ϵ_i of T by

$$\epsilon_i(t(t_1, t_2)) := t_i.$$

Then $R(G, T) = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j \mid i \neq j\}$. Let $R^+ := \{\epsilon_1 - \epsilon_2, \epsilon_2, \epsilon_1, \epsilon_1 + \epsilon_2\}$ be a system of positive roots and $\Delta := \{\epsilon_1 - \epsilon_2, \epsilon_2\}$ the corresponding set of simple roots. We take the following basis for \mathfrak{n} :

$$\begin{aligned} E_{\epsilon_1 - \epsilon_2} &:= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} & E_{\epsilon_1} &:= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ E_{\epsilon_2} &:= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & E_{\epsilon_1 + \epsilon_2} &:= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Now $E_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in R^+$.

Thus $X := E_{\epsilon_1 - \epsilon_2} + E_{\epsilon_2}$. Also $\mathfrak{n}_2 = \mathfrak{g}_{\epsilon_1}$ and $\mathfrak{n}_3 = \mathfrak{g}_{\epsilon_1 + \epsilon_2}$. The linear map $[X, \cdot] : \mathfrak{n}_2 \rightarrow \mathfrak{n}_3$ is 0, since both $E_{\epsilon_1 - \epsilon_2}$ and E_{ϵ_2} commute with E_{ϵ_1} . Thus according to Proposition 22 and its proof, for $d, d' \in \mathbb{F}$, $X + dE_{\epsilon_1 + \epsilon_2}$ is U -conjugated with $X + d'E_{\epsilon_1 + \epsilon_2}$ if and only if $d \equiv d' \pmod{\mathbb{F}^{(2)}}$. Now we follow Lemma 23. We take $U_{\hat{\epsilon}_2} := U_{\epsilon_1 - \epsilon_2}$. Define for $a, b, c, d \in \mathbb{F}$

$$n(a, b, c, d) := aE_{\epsilon_1 - \epsilon_2} + bE_{\epsilon_2} + cE_{\epsilon_1} + dE_{\epsilon_1 + \epsilon_2}.$$

Assume that $a, b \neq 0$, then there is by the proof of Lemma 23 a unique $g \in TU_{\hat{\epsilon}_2}$ such that $gn(a, b, c, d)g^{-1} = X + d'E_{\epsilon_1 + \epsilon_2}$ for some $d' \in \mathbb{F}$. Lets compute d' : first get the a and b to 1 by conjugating with $t := t((ab)^{-1}, b^{-1})$, then

$$tn(a, b, c, d)t^{-1} = n(1, 1, c(ab)^{-1}, da^{-1}b^{-2}).$$

By conjugating the result with

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & c(ab)^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c(ab)^{-1} & 1 \end{pmatrix}$$

we get $n(1, 1, 0, da^{-1}b^{-2})$. Thus $d' = \frac{d}{ab^2}$. Assume that $a', b' \neq 0$. Thus $n(a, b, c, d)$ is conjugated with $n(a', b', c', d')$ if and only if $\frac{d}{ab^2} \equiv \frac{d'}{a'b'^2} \pmod{\mathbb{F}^{(2)}}$. So we define $\chi : \mathfrak{n}' \rightarrow \mathbb{F}$ by $\chi(n(a, b, c, d)) := \frac{d}{ab^2}$ and we define $\eta : \mathbb{F} \rightarrow \mathfrak{n}'$ by $\eta(d) := n(1, 1, 0, d) = E_{\epsilon_1 - \epsilon_2} + E_{\epsilon_2} + dE_{\epsilon_1 + \epsilon_2}$. Then (η, χ) is a bad pair for SO_5 .

6. Howe's conjecture and $\kappa_v(G)$

In this section we assume that p divides $\kappa_v(G)$, i.e. the characteristic of \mathbb{F} divides the order of the cokernel of the map:

$$\begin{aligned} \Phi : X_*(T) &\rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G, T), \mathbb{Z}) \\ \gamma &\mapsto (\alpha \mapsto \langle \gamma, \alpha \rangle) \end{aligned}$$

We will follow the same strategy as in section 5. By the proof of Proposition 6 there exists integers $z_i \in \mathbb{Z}$ such that $\kappa : \mathfrak{n}' \rightarrow \mathbb{F}^\times$ defined by $\kappa(n) := \prod_{i=1}^m n_{\alpha_i}^{z_i}$ is surjective and $\kappa : \mathfrak{n}' \rightarrow \mathbb{F}^\times / (\mathbb{F}^\times)^p$ is B -invariant. Take a $\nu : \mathbb{F}^\times \rightarrow \mathfrak{n}$ such that ν is algebraic and $\kappa\nu$ is the identity. By the proof of Proposition 6 we can choose ν in such a way that for all $\alpha \in \mathcal{O}^\times$: $\nu(\alpha)_\gamma \in \mathcal{O}^\times$ for all $\gamma \in \Delta$ and $\nu(\alpha)_\gamma = 0$ for all $\gamma \in R - \Delta$. The pair (ν, κ) plays in this case a similar role as the bad pair (η, χ) in the bad characteristic case.

Let $N > 0$ such that for all $n \in \mathbb{N}_{>0}$, $k \in K$ and $\alpha \in \mathcal{O}^\times$:

$$k\nu(\alpha)k^{-1} \in \mathfrak{n} + \pi^{Nn}L \Rightarrow k \in (B \cap K)K_n.$$

By Lemma 15 such an N exists.

Define the following B -invariant open set of \mathfrak{n} :

$$V_{\alpha, s} := \{n \in \mathfrak{n}' \mid \kappa(n) \equiv \alpha \pmod{(1 + \pi^s \mathcal{O})(\mathbb{F}^\times)^p}\}.$$

Define $n(a_1, \dots, a_m) := \sum_{i=1}^m a_i E_{\alpha_i}$. Define $z := \sum_{i=1}^m z_i$. Then $\kappa(\pi^n x) = \pi^{zn} \kappa(x)$, for all $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$.

Lemma 27. *Let $n \in \mathbb{N}_{>0}$, $\alpha \in \mathcal{O}^\times$ and $\beta \in \mathcal{O}^\times$. If*

$$\int_{V_{\pi^{-zn} \beta, n}} \int_{k \in K} 1_{\pi^{-Nn} \nu(\alpha) + L}(kXk^{-1}) dk dX > 0,$$

then $\alpha \equiv \beta \pmod{(1 + \pi^n \mathcal{O})(\mathcal{O}^\times)^p}$.

Proof. Since the integral is positive, there exist a $k \in K$ and $l \in L$ such that

$$k\pi^{-nN} \nu(\alpha)k^{-1} + l \in V_{\pi^{-zn} \beta, n} + \mathfrak{n}_{\geq 2} \subset \mathfrak{n}.$$

Thus $k \in (K \cap B)K_n$ by Lemma 15, because $k\nu(\alpha)k^{-1} \in \mathfrak{n} + \pi^{nN}L$. Take $b_k \in K \cap B$ and $k_n \in K_n$ such that $k = k_n b_k$. Take $a_1, \dots, a_m \in \mathcal{O}^\times$ and $n_2 \in \mathfrak{n}_{\geq 2}(\mathcal{O})$ such that $b_k \nu(\alpha) b_k^{-1} = n(a_1, \dots, a_m) + n_2$. By the construction of κ , there exists a $\gamma \in \mathbb{F}^\times$ such that $\kappa(n(a_1, \dots, a_m)) = \alpha \gamma^p$. Since $k_n \in K_n$ and $n(a_1, \dots, a_m) + n_2 \in L$, there exists a $l' \in L$ such that

$$k_n(n(a_1, \dots, a_m) + n_2)k_n^{-1} = n(a_1, \dots, a_m) + n_2 + \pi^n l'.$$

Thus

$$\begin{aligned} \kappa(k\nu(\alpha)k^{-1} + \pi^{nN}l) &= \kappa(k_n b_k \nu(\alpha) b_k^{-1} k_n^{-1} + \pi^{nN}l) \\ &= \kappa(n(a_1, \dots, a_m) + n_2 + \pi^{nN}l) = \prod_{i=1}^m (a_i + \pi^{nN}l)^{z_i}, \end{aligned}$$

for some $l_i \in \mathcal{O}$. Since the a_i are in \mathcal{O}^\times ,

$$\prod_{i=1}^m (a_i + \pi^{nN}l)^{z_i} \equiv \prod_{i=1}^m a_i^{z_i} = \alpha \gamma^p \pmod{(1 + \pi^n \mathcal{O})}.$$

Because $\kappa(\pi^{-znN}x) = \pi^{-znN}\kappa(x)$ for all $x \in \mathfrak{g}$,

$$\kappa(k\pi^{-nN}\nu(\alpha)k^{-1} + l) \equiv (\alpha \gamma^p)\pi^{-znN} \pmod{(1 + \pi^n \mathcal{O})}.$$

Since $k\pi^{-nN}\nu(\alpha)k^{-1} + l \in V_{\pi^{-znN}\beta, n} + \mathfrak{n}_{\geq 2}$,

$$\kappa(k\pi^{-nN}\nu(\alpha)k^{-1} + l) \equiv \pi^{-znN}\beta \pmod{(\mathbb{F}^\times)^p(1 + \pi^n \mathcal{O})}.$$

Thus

$$\pi^{-znN}\beta \equiv \kappa(k\pi^{-nN}\nu(\alpha)k^{-1} + l) \equiv \pi^{-znN}\alpha \pmod{((\mathbb{F}^\times)^p(1 + \pi^n \mathcal{O}))}.$$

Then $\alpha \equiv \beta \pmod{(\mathbb{F}^\times)^p(1 + \pi^n \mathcal{O})}$. Because $(\mathbb{F}^\times)^p \cap \mathcal{O}^\times = (\mathcal{O}^\times)^p$ and $\alpha, \beta \in \mathcal{O}^\times$, the Lemma follows. \blacksquare

Theorem 28. *Let G be a \mathbb{F} -split reductive group. Assume that $\text{char } \mathbb{F} | \kappa_v(G)$, then Howe's conjecture does not hold.*

Proof. The proof is similar to the one of Theorem 19.

Let $\alpha_1, \dots, \alpha_k$ be representatives of the cosets of $(1 + \pi^n \mathcal{O})\mathcal{O}^\times$ in \mathcal{O}^\times . Define for $1 \leq i \leq k$ the following distribution and function:

$$\begin{aligned} D_i(f) &:= D_{V_{\pi^{-znN}\alpha_i, n}}(f) \\ f_i &:= 1_{\pi^{-nN}\nu(\alpha_i) + L} \end{aligned}$$

Let $c_i := D_i(f_i) > 0$. By Lemma 27 $D_i(f_j) = c_i \delta_{ij}$. Therefore $\dim J_L(\omega) \geq k$. \blacksquare

7. Howe's conjecture in good characteristic

Howe's conjecture does not hold when the characteristic is bad or $p | \kappa_v(G)$. In this section we investigate Howe's conjecture in good characteristic. Throughout this section we assume that p is good for G . The proofs of Howe's conjecture are based on [HC99, Part II].

7.1. Associated cocharacters to nilpotent elements. In this subsection we recall the theory of associated cocharacters. Let $\tau \in X_*(G)$. For $z \in \mathbb{Z}$, we define the following subspaces of \mathfrak{g} :

$$\begin{aligned}\mathfrak{g}(z; \tau) &:= \{X \in \mathfrak{g} \mid \forall [a \in \mathbb{F}] \tau(a)X\tau(a)^{-1} = a^z X\} \\ \mathfrak{g}(\geq z; \tau) &:= \bigoplus_{i \geq z} \mathfrak{g}(i; \tau) \\ \mathfrak{p}(\tau) &:= \mathfrak{g}(\geq 0; \tau) \\ \mathfrak{n}(\tau) &:= \mathfrak{g}(\geq 1; \tau).\end{aligned}$$

We sometimes abbreviate $\mathfrak{g}(z; \tau)$ ($\mathfrak{g}(\geq z; \tau)$) by $\mathfrak{g}(z)$ ($\mathfrak{g}(\geq z)$ resp.), in which case the cocharacter τ should be clear from the context.

Let $X \in \mathfrak{g}$ be nilpotent. It is called distinguished in \mathfrak{g} if each torus contained in $Z_G(X)$ is contained in the center of G .

A cocharacter τ of G is called associated to X if $X \in \mathfrak{g}(2, \tau)$ and if there exists a Levi subgroup L in G such that X is distinguished nilpotent in \mathfrak{l} and such that $\text{im } \tau \subset (L, L)$.

Following [McN04] we define $N(X) := \{g \in G \mid \text{Ad}(g)X \in \overline{\mathbb{F}}X\}$, where $\overline{\mathbb{F}}$ is the algebraic closure of \mathbb{F} .

Lemma 29. [McN04, Lemma 25] *Let S be any maximal torus of $N(X)$. Then there is a unique cocharacter in $X_*(S)$ associated with X .*

Theorem 30. [McN04, Theorem 26] *Let $X \in \mathfrak{g}$ be nilpotent. Assume that the G -orbit of X is separable. Then there exists a cocharacter τ associated to X which is defined over \mathbb{F} .*

Let τ be a cocharacter associated to X , we define

$$\begin{aligned}\mathfrak{p}_X &:= \mathfrak{g}(\geq 0; \tau) \\ \mathfrak{n}_X &:= \mathfrak{g}(\geq 1; \tau)\end{aligned}$$

By [Jan04, Proposition 5.9(a)] the Lie algebras \mathfrak{p}_X is independent of the choice of τ . Hence \mathfrak{n}_X is also independent of the choice of τ .

7.2. First proof of Howe's conjecture. All the proofs of Howe's conjecture in this paper are slight modifications of the proof of Howe's conjecture given in [HC99, Part II]. The proof of Howe's conjecture [HC99, Theorem 12.1] uses the following sections:

§10.2, §11.1 (excluding Theorem 11.3), §11.3, §12, §13.

The sections of [HC99] up to and including §12 are dedicated to proof the following: if a lattice L satisfies the condition $C(L)$, then Howe's conjecture holds for L . Here $C(L)$ (see §11.3) is a technical condition depending on L and the nilpotent elements $X \in \mathfrak{g}$, with $|X| = 1$. In §13 it is shown that for every well-adapted lattice L' , $C(L')$ holds. Now as stated in [HC99, Remark 10.7]: every lattice L

contains a well-adapted lattice L' . Thus Howe's conjecture holds for L' and hence for L .

All those sections are independent of the results of [HC99, Part I]. In all those sections of Part II except §13 the characteristic of \mathbb{F} does not have any role. Only §13 does not generalize verbatim to the case that \mathbb{F} has positive characteristic.

Lemma 31. *Assume that G splits over a tamely ramified extension of \mathbb{F} .*

Let L be a well-adapted lattice. Let $X_0 \in \mathcal{N} \cap S$. If there exists a cocharacter λ such that

$$X_0 \in \mathfrak{n}(\lambda) \text{ and } \mathfrak{n}(\lambda) \subset [X_0, \mathfrak{g}],$$

then $C(L)$ holds for X_0 .

Proof. The proof is basically the same as the proof that $C(L)$ holds for X_0 in [HC99, §13]. We only need to give some modifications to adjust it to the positive characteristic case.

In [HC99, §13.1], the subspaces \mathfrak{g}_r are defined using the semisimple element in an \mathfrak{sl}_2 -triple containing X_0 . In our setting, we will use instead $\mathfrak{g}_r = \mathfrak{g}(r; \lambda)$ where λ is a cocharacter associated to X_0 . The argument of §13.1 in loc.cit. then proceeds mutatis mutandum using $\lambda' = \lambda^n$ rather than $H'_0 = H_0^n$. Now the statements of Lemma 13.2 and Corollary 13.3 of [HC99] hold.

The remaining results of §13.1 are valid in our setting; and – as in [HC99] – they provided a proof of [HC99, Theorem 13.1] – i.e. of Lemma 31 of the present text –, modulo a proof of [HC99, Lemma 13.5].

The proof of Lemma 13.5 is considered in [HC99, §13.2]; since it depends in part on use of the exponential mapping, we must adapt this proof.

The only properties of the exponential map used to prove Lemma 13.5 are (3) and (4) in the second paragraph of [HC99, §13.2]. However by going through §13.2 one sees that the following is enough: there exists an open neighborhood U of 0 in \mathfrak{g} and a map $e : U \rightarrow G$ such that if $\epsilon > 0$ is small enough:

- (3') $\mathfrak{g}(\epsilon) \subset U$, and $K_\epsilon = e(\mathfrak{g}(\epsilon)) \subset K_\gamma$;
- (4') there exists a real number $a_3 > 0$ such that for $Z \in \mathfrak{g}(\epsilon)$ and $Y \in \mathfrak{g}$

$$|Ad(e(Z))Y - Y - [Z, Y]| \leq a_3|Z|^2|Y|.$$

(In [HC99], $K_\epsilon(X^\gamma)$ is a subgroup. But that is not required by the proof. Indeed, simply replace the final three lines of the proof on [HC99] p. 69 with the following:

However, $K_\epsilon(X^\gamma)$ is contained in the compact group $K_\gamma(X^\gamma)$. Hence by choosing a subsequence we may assume that $k'_n \rightarrow k'$ where $k' \in K_\gamma(X^\gamma)$. Then $p_1(X^{k'\gamma} - X^\gamma) = v$.)

Therefore we can replace the exponential map by a (not-necessarily G -invariant) map between a small open part of \mathfrak{g} and G . Let $e : \mathfrak{g}_{x,0+} \rightarrow G_{x,0+}$ be a mock exponential map as constructed in [Adl98, §1.5]. That (3') is satisfied follows immediately from the construction. We will use [Adl98, Proposition 1.6.3] for (4'):

Suppose $r > 0$, $Z \in \mathfrak{g}_{x,r}$ and $Y \in \mathfrak{g}_{x,s}$. Then

$$Ad(e(Z))Y - Y - [Z, Y] \in \mathfrak{g}_{x,2r+s}.$$

To prove (4') form [Adl98, Proposition 1.6.3] we first need to compare the norm, $|\cdot|$, on \mathfrak{g} with the grading $\mathfrak{g}_{x,r}$.

Since $\mathfrak{g}_{x,0}$ is open and compact, there exist constants $c_1, c_2 \in \mathbb{N}$ such that

$$\begin{aligned} |X| < q^{-c_1} &\Rightarrow X \in \mathfrak{g}_{x,0} \\ X \in \mathfrak{g}_{x,0} &\Rightarrow |X| < c_2 \end{aligned}$$

Let π be a uniformizer of \mathbb{F} , then $\pi^n \mathfrak{g}_{x,0} = \mathfrak{g}_{x,n}$ for all $n \in \mathbb{Z}$.

Assume that $|X| < q^s$, then $|\pi^{c_1+[s]} X| < q^{-c_1+s-[s]} < q^{-c_1}$. Thus $\pi^{c_1+[s]} X \in \mathfrak{g}_{x,0}$. Therefore $X \in \mathfrak{g}_{x,-c_1-[s]} \subset \mathfrak{g}_{x,-c_1-1-s}$.

Assume that $X \in \mathfrak{g}_{x,s}$, then $X \in \mathfrak{g}_{x,[s]}$. Thus $\pi^{-[s]} X \in \mathfrak{g}_{x,0}$. Therefore $|\pi^{-[s]} X| < c_2$. So $|X| < c_2 q^{-[s]} < c_2 q^{1-s}$.

Let $C_1 := c_1 + 1$ and $C_2 := qc_2$, then

$$\begin{aligned} |Z| < r &\Rightarrow Z \in \mathfrak{g}_{x,-C_1-\log_q(r)}, \text{ and} \\ Z \in \mathfrak{g}_{x,r} &\Rightarrow |Z| < C_2 q^{-r}. \end{aligned}$$

Define $d_1 = |Z|$ and $d_2 = |Y|$. Assume that $|Z|$ is small enough such that $-C_1 - \log_q(d_1) > 0$. Since $Z \in \mathfrak{g}_{x,-C_1-\log_q(d_1)}$ and $Y \in \mathfrak{g}_{x,-C_1-\log_q(d_2)}$, then

$$Ad(e(Z))Y - Y - [Z, Y] \in \mathfrak{g}_{x,l},$$

where $l = 2(-C_1 - \log_q(d_1)) - C_1 - \log_q(d_2)$. Thus

$$|Ad(e(Z))Y - Y - [Z, Y]| < C_2 q^{-l} = C_2 q^{3C_1} d_1^2 d_2 = C_2 q^{3C_1} |Z|^2 |Y|.$$

Hence (4') holds for the map $e : \mathfrak{g}_{x,0+} \rightarrow G_{x,0+}$ whenever ϵ is small enough. \blacksquare

Lemma 32. *Suppose that $\text{char}(\mathbb{F})$ is good for G and \mathbb{F} is algebraically closed. Let X be nilpotent. Let λ be a cocharacter associated with X . Then*

$$[\mathfrak{g}(-1), X] = \mathfrak{g}(1)$$

and

$$[\mathfrak{n}_X, X] = \mathfrak{g}(\geq 3).$$

Proof. We follow the same line as the proof of [Jan04, Proposition 5.9(c)].

Let G be a group satisfying the standard hypotheses:

1. The derived group of G is simply connected.
2. The characteristic of \mathbb{F} is good for G .
3. There exists a G -invariant nondegenerate bilinear form on \mathfrak{g} .

By [Jan04, Proposition 5.8 and Lemma 5.7]

$$[\mathfrak{g}(-1), X] = \mathfrak{g}(1)$$

and

$$[\mathfrak{n}_X, X] = \mathfrak{g}(\geq 3).$$

Now we show that the Lemma holds for G if and only if it holds for G_{der} . The cocharacter τ associated to X in G is also the cocharacter τ associated to X in G_{der} . Also $\mathfrak{g}(-1), \mathfrak{g}(1), \mathfrak{g}(\geq 3) \subset \mathfrak{g}'$.

When G is simply connected and the characteristic is very good, then G satisfies the standard hypotheses. The Lemma holds for GL_n by [How74, Lemma 2], thus for SL_n as well. Therefore the Lemma holds for all simply connected groups in good characteristic. Hence also for products of those groups.

Let $G = R(G)G_1, \dots, G_m$ with G_i the simple normal connected subgroups of G and $R(G) = Z(G)^\circ$ the radical of G . Let G'_i be the simply connected group belonging to G_i . Let $\pi : R(G) \prod_{i=1}^m G'_i \rightarrow G$ be the natural surjective homomorphism. Now $d\pi$ is surjective on the nilpotent elements and maps the associated cocharacter of a nilpotent element to the associated cocharacter of its image. Since the Lemma holds for $R(G) \prod_{i=1}^m G'_i$ it also holds for G . ■

Theorem 33. *Let G be a reductive group which splits over a tamely ramified field extension over \mathbb{F} . Assume the characteristic of \mathbb{F} to be good for G . If the nilpotent orbits of G in \mathfrak{g} are separable, then Howe's conjecture holds.*

Proof. As mentioned at the start of this subsection, it is enough to show that $C(L)$ holds for all $X_0 \in \mathcal{N} \cap S$ and well-adapted lattices L .

Let $X_0 \in \mathcal{N} \cap S$. Let λ be a cocharacter associated with X_0 defined over \mathbb{F} as promised by [McN04, Theorem 26] and the separability of the orbit of X_0 . Define $\mathfrak{g}_z := \mathfrak{g}(z; \lambda)$ and

$$\mathfrak{n} = \sum_{r \geq 1} \mathfrak{g}_r, \mathfrak{m} = \mathfrak{g}_0, \bar{\mathfrak{n}} = \sum_{r \geq 1} \mathfrak{g}_{-r}.$$

Now we give a proof of $\mathfrak{n} \subset [X_0, \mathfrak{g}]$. In the proof of [McN04, Proposition 34]:

“Well, by [Jan04, Proposition 5.9(c)], we have $\overline{\text{Ad}(P)X} = \bigoplus_{i \geq 2} \mathfrak{g}(i, \phi)$. Since the orbit of X is separable, the differential of the orbit map is surjective.”

Thus $\mathfrak{g}(\geq 2) \subset [\mathfrak{g}, X_0]$. Therefore with Lemma 32 we have $\mathfrak{n} \subset [X_0, \mathfrak{g}]$.

Thus $C(L)$ holds for all nilpotent elements X_0 with $|X_0| = 1$ by Lemma 31. Thus Howe's conjecture holds for G by [HC99, §12]. ■

Corollary 34. *If G is a simple group which splits over a tamely ramified extension over \mathbb{F} and \mathbb{F} is very good for G , then Howe's conjecture holds.*

Proof. By [Ric67] the nilpotent orbits are separable. ■

7.3. The case $SO_3(\mathbb{F})$ ($\text{char } \mathbb{F} = 2$). In this subsection $\text{char } \mathbb{F} = 2$.

Although there are infinitely many nilpotent conjugacy classes in $SO_3(\mathbb{F})$ and the nilpotent orbits are not separable, Howe's conjecture holds for $SO_3(\mathbb{F})$. We again follow [HC99], but have to make a few more modifications.

The next lemma and its proof are [HC99, Lemma 12.2], with ${}^G\mathfrak{n}$ instead of \mathcal{N} .

Lemma 35. *Let $\omega \subset \mathfrak{g}$ be a compact set.*

Let S be a split torus and K the stabilizer of 0 in the apartment of S (in the extended building). Take Φ^+ a system of positive roots of (G, S) . Let \mathfrak{n} be the Lie algebra for U^+ , $\bar{\mathfrak{n}}$ be the Lie algebra for U^- and \mathfrak{m} the Lie algebra of $M := Z_G(S)$.

There is a lattice Λ such that

$$Ad(G)\omega = \Lambda + Ad(KS)(\mathfrak{n} \cap \Lambda)$$

Proof. By Bruhat-Tits one has

$$G = KSFK$$

for some finite subgroup F of M .

Since $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{n}$ one has compact subsets $\omega_1, \omega_2, \omega_3$ in $\bar{\mathfrak{n}}$, \mathfrak{m} and \mathfrak{n} respectively, such that

$$Ad(FK)\omega \subset \omega_1 \oplus \omega_2 \oplus \omega_3.$$

Hence $Ad(G)\omega \subset Ad(KS)(\omega_1 \oplus \omega_2 \oplus \omega_3)$.

Now $Ad(S)\omega_1$ is contained in a compact lattice of $\bar{\mathfrak{n}}$, since $v(\alpha(s)) \geq 0$ for all $\alpha \in \Phi^-$ and $Ad(S)\omega_2 = \omega_2$. Therefore there is a lattice L such that

$$Ad(G)\omega \subset Ad(K)(L + Ad(S)(\mathfrak{n} \cap L)). \quad \blacksquare$$

Since ${}^G\mathfrak{n} = \mathcal{N}$ in characteristic 0, Lemma 12.2 of Harish-Chandra works with \mathcal{N} . For the group $SO_3(\mathbb{F})$ this is not the case. Therefore we shall work with ${}^G\mathfrak{n}$ instead of \mathcal{N} . We start with the definition of $SO_3(\mathbb{F})$.

Define $Q(e_0, e_1, e_2) := e_0^2 + e_1e_2$,

$$SO_3(\mathbb{F}) := \{g \in GL_3 \mid Q(gv) = Q(v)\}.$$

Let γ be the following cocharacter of SO_3 .

$$\gamma(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}$$

Let T be the following subgroup of SO_3 :

$$T := \{\gamma(t) : t \in \mathbb{F}^\times\}$$

Now T is a maximal torus of SO_3 .

The Lie algebra of SO_3 is of the following form:

$$\mathfrak{g} := \left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{F} \right\}.$$

With respect to the cocharacter γ we have a decomposition of the Lie algebra: $\mathfrak{g} := \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ with

$$\begin{aligned} \mathfrak{n} := \mathfrak{g}(1) &= \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b \in \mathbb{F} \right\} \\ \mathfrak{t} := \mathfrak{g}(0) &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} : c \in \mathbb{F} \right\} \\ \bar{\mathfrak{n}} := \mathfrak{g}(-1) &= \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{F} \right\} \end{aligned}$$

Take on \mathfrak{g} the following norm:

$$\left| \begin{pmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \right| = \max(|a|, |b|, |c|)$$

For the extended version of Howe's conjecture, Harish-Chandra needs to consider all nilpotent orbits. But for the regular Howe's conjecture we can restrict ourselves to one nilpotent orbit, namely the orbit of

$$n := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $N := {}^G n \cup \{0\} = {}^G \mathfrak{n}$.

Define, for $a, b \in \mathbb{F}$, the following elements of \mathfrak{so}_3 and SO_3 :

$$n_{a,b} := \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad u_b := \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & b^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \omega := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Lemma 36. *Let $(a, b), (c, d) \in \mathbb{F}^2 - (0, 0)$, then $n_{a,b}$ is in the same conjugacy class as $n_{c,d}$ if and only if there exists a $y \in \mathbb{F}$ such that $ab + y^2 = cd$.*

In particular $N = \{n_{a,b} \mid \exists (y \in \mathbb{F}) y^2 = ab\}$.

Proof. The conjugation action of the generators of $SO_3(\mathbb{F})$ on the nilpotent elements is as follows:

$$\begin{aligned} u_c n_{a,b} u_c &= n_{a,c^2 a + b} \\ \gamma(x) n_{a,b} \gamma(x)^{-1} &= n_{x^{-1} a, x b} \\ \omega n_{a,b} \omega &= n_{b,a} \end{aligned}$$

The lemma follows after some calculations. ■

Corollary 37. *The set N is closed in \mathfrak{g} and $cN = N$ for all $c \in \mathbb{F}^\times$.*

Proof. The nilpotent elements are closed in \mathfrak{g} . The function $Q : n_{a,b} \mapsto ab$ is a continuous function from \mathcal{N} to \mathbb{F} . Since $\mathbb{F}^{(2)}$ is closed in \mathbb{F} , so is $Q^{-1}(\mathbb{F}^{(2)})$. The latter is equal to N by Lemma 36. Since closed sets of closed subspaces are closed, N is closed in \mathfrak{g} . The second statement is obvious. ■

Corollary 38. 0 is in the closure of ${}^G n_{a,b}$ if and only if $ab \in \mathbb{F}^{(2)}$.

Proof. Assume that ab is not a square in \mathbb{F} . Let π be a uniformizer of \mathbb{F} . Write $ab = \sum_{n=-k}^{\infty} c_n \pi^n$. Let m be an odd integer such that $c_m \neq 0$, since ab is not a square in \mathbb{F} such a m exists. Let $y \in \mathbb{F}^{(2)}$. Write $ab + y = \sum_{n=-l}^{\infty} d_n \pi^n$, then $d_m = c_m$. Thus $v(ab + y) \leq m$ for every $y \in \mathbb{F}^{(2)}$. Thus every element in the conjugacy class of $n_{a,b}$ is at least at distance q^{-m} from 0 . Hence 0 is not in the closure of the G -orbit of $n_{a,b}$.

If ab is a square in \mathbb{F} , then either $n_{a,b}$ is conjugated to $n = n_{0,1}$ or $n_{a,b} = 0$. In both cases the closure of their G -orbit contains 0 . ■

Lemma 39. Let $X \in N$. There is a cocharacter τ such that $X \in \mathfrak{g}(1)$ and $\mathfrak{g}(1) \subset [X, \mathfrak{g}]$.

Proof. Since these statements are G -invariant, we may and will assume that $X = n_{0,1}$. In this case take $\tau := \gamma$. Clearly $X \in \mathfrak{n} \subset \mathfrak{g}(1)$. Now

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

thus $\mathfrak{n} \subset [X, \mathfrak{g}]$. ■

Theorem 40. Howe's conjecture holds in $SO_3(\mathbb{F})$.

Proof. We follow Harish-Chandra [HC99] again. Recall that the proof of Howe's conjecture is spread out over the following sections in [HC99]:

§10.2, §11.1 (excluding Theorem 11.3), §11.3, §12, §13.

In order to prove this Theorem we change these sections by replacing \mathcal{N} by $N = {}^G \mathfrak{n}$. In §11 and §12 three properties of \mathcal{N} are used (between brackets the Lemma's in [HC99] where the property is used):

1. $\mathcal{N} \cap S$ is compact (Lemma 11.9 & 12.3).
2. For all compact $\omega \subset \mathfrak{g}$ there exists a lattice L_1 , such that ${}^G \omega \subset L_1 + \mathcal{N}$ (Lemma 12.2).
3. If $c \in \mathbb{F}$ and $Y \in \mathcal{N}$, then $cY \in \mathcal{N}$ (Lemma 12.3).

By Corollary 37 (1) and (3) also hold for N and (2) is Lemma 35. With these modification §12 shows that we only need to prove $C(L)$ for $X \in N \cap S$ in order to prove Howe's conjecture for L .

Let L be a well-adapted lattice and $X \in N \cap S$. By Lemma 39 there exists a cocharacter τ such that

$$X \in \mathfrak{n}(\tau) \text{ and } \mathfrak{n}(\tau) \subset [X, \mathfrak{g}].$$

Thus $C(L)$ holds for X by Lemma 31. ■

This example shows that the separability of the nilpotent orbits is not a necessary condition for Howe's conjecture to hold.

7.4. The case $PGL_n(\mathbb{F})$ with char $\mathbb{F}|n$. In this subsection we generalize the results in the previous subsection to the group $PGL_n(\mathbb{F})$. This is the group consisting of the \mathbb{F} -points of the algebraic quotient of GL_n by its centrum of diagonal matrices Z . We have the exceptional isomorphism $PGL_2 \cong SO_3$. Let $G := PGL_n$. We identify \mathfrak{g} with $\mathfrak{gl}_n/\mathfrak{z}$. Now $\mathfrak{gl}_n/\mathfrak{z} := \{X + \mathfrak{z} : X \in \mathfrak{gl}_n\}$. Define $p := \text{char } \mathbb{F}$. The nilpotent elements of \mathfrak{g} are exactly those $X + \mathfrak{z}$ such that $X^{p^n} \in \mathfrak{z}$. We define the following G -invariant function ϕ on \mathcal{N} : for $X \in \mathfrak{g}(\mathbb{F})$ let $a \in \mathbb{F}$ be such that $X^{p^n} = aI_n$, with I_n the identity matrix. Then $\phi(X + \mathfrak{z}) := a + \mathbb{F}^{(p^n)}$. If $X + \mathfrak{z} = X' + \mathfrak{z}$, then $X - X' \in \mathfrak{z}(\mathbb{F})$. Thus ϕ is well defined.

Lemma 41. *The following statements hold for ϕ :*

1. ϕ is G -invariant.
2. $\mathbb{F}^{(p^{n-1})} \subset \text{Im } \phi$.
3. Let $X + \mathfrak{z}$ be a nilpotent element of \mathfrak{g} . Then $\phi(X) \in \mathbb{F}^{(p^n)}$ if and only if there exists a nilpotent matrix $n \in \mathfrak{gl}_n$ such that $n \in X + \mathfrak{z}$.

Proof. 1. trivial.

2. Let M_x be a block-diagonal matrix consisting of $\frac{n}{p}$ blocks with on each $(p \times p)$ -block the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & x \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Then $M_x^p = xI_n$ thus $M_x^{p^n} = x^{p^{n-1}}I_n$.

3. For a nilpotent matrix $n \in \mathfrak{gl}_n$ we have that $\phi(n + \mathfrak{z}) = 0$, thus the only if part is clear. Assume that $X^{p^n} = a^{p^n}I_n$, then $(X - aI_n)^{p^n} = X^{p^n} - a^{p^n}I_n = 0$. Thus $X - aI_n$ is nilpotent. ■

Corollary 42. *The number of nilpotent orbits is infinite.*

Proof. The group $\mathbb{F}^{(p^{n-1})}/\mathbb{F}^{(p^n)}$ is as group isomorphic to $\mathbb{F}/\mathbb{F}^{(p)}$, $\mathbb{F}/\mathbb{F}^{(p)}$ is infinite and $\mathbb{F}^{(p^{n-1})} \subset \text{Im } \phi$. ■

Thus not all nilpotent orbits are separable. In fact the orbit of

$$x := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dots & & 0 \end{pmatrix},$$

the superdiagonal entries of x are 1, is not separable, since the commutator with

$$x' := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & n & 0 \end{pmatrix},$$

the subdiagonal entries of x' are from left to right equal to $1, 2, \dots, n$, is equal to $(\text{char } \mathbb{F} | n)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(n-1) \end{pmatrix} = Id_n \in \mathfrak{z}.$$

Lemma 43. *If $p|n$, then $\{H_\alpha : \alpha \in \Delta\}$ are linearly dependent.*

Proof. Let T be the torus of diagonal matrices. For $i = 1, \dots, n$, define

$$\epsilon_i \begin{pmatrix} x_1 & 0 & \dots \\ 0 & \ddots & 0 \\ \dots & 0 & x_n \end{pmatrix} = x_i.$$

Let $\Delta = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n\}$, then

$$\sum_{i=1}^{n-1} H_{\epsilon_i - \epsilon_{i+1}} = [x, x'] = 0. \quad \blacksquare$$

Define $N := \{x + \mathfrak{z} : x \in \mathfrak{gl}_n \mid x \text{ is nilpotent}\}$.

Corollary 44. *N is a closed subset of \mathfrak{g} and $cN = N$ for $c \in \mathbb{F}^\times$.*

Proof. The map ϕ is continuous and $0 \in \mathbb{F}/\mathbb{F}^{(p^n)}$ is closed. Thus $N = \phi^{-1}(0)$ is closed in \mathcal{N} . Because \mathcal{N} is closed in \mathfrak{g} , so is N .

If $x \in \mathfrak{g}$ is nilpotent, then, for all $c \in \mathbb{F}$, cx is also nilpotent. Thus $cN = N$. \blacksquare

Lemma 45. *For every nilpotent element $X \in \mathfrak{gl}_n$ there exists a cocharacter γ , such that $X \in \mathfrak{n}(\gamma)$ and $\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)]$.*

Proof. We follow [How74] and its notation. See page 311 of loc. cit.. Define for $x \in \mathbb{F}$ the element $\gamma(x) \in M$ to be the transformation which acts on C_i by multiplication by x^i . Then $\mathcal{U} = \mathfrak{n}(\gamma)$. By [How74, Lemma 2] $\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)]$. ■

Corollary 46. *For every nilpotent element in N there exists a cocharacter γ , such that $X \in \mathfrak{n}(\gamma)$ and $\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)]$.*

Proof. Let $X \in GL_n$ and let $\gamma \in X_*(G)$ be the cocharacter of Lemma 45. Let $\varphi : GL_n \rightarrow PGL_n$ be the natural homomorphism. Because $d\varphi$ is surjective and $d\varphi(ad(x)X) = ad(\varphi(x))d\varphi(X)$, we have $\mathfrak{n}(\varphi\gamma) = d\varphi(\mathfrak{n}(\gamma))$ and $\mathfrak{p}(\varphi\gamma) = d\varphi(\mathfrak{p}(\gamma))$. We conclude that $\varphi\gamma$ is the desired cocharacter for $X + \mathfrak{z}$. ■

Theorem 47. *Howe's conjecture holds in PGL_n .*

Proof. We follow Harish-Chandra [HC99] again and mention the adjustments. Just like in the SO_3 case we replace \mathcal{N} by $N = {}^G\mathfrak{n}$. The proof of Harish-Chandra uses three properties of \mathcal{N} (between brackets the Lemma's in [HC99] where the property is used):

1. $\mathcal{N} \cap S$ is compact (Lemma 11.9 & 12.3).
2. For all compact $\omega \subset \mathfrak{g}$ there exists a lattice L_1 , such that ${}^G\omega \subset L_1 + \mathcal{N}$ (Lemma 12.2).
3. If $c \in \mathbb{F}$ and $Y \in \mathcal{N}$, then $cY \in \mathcal{N}$ (Lemma 12.3).

By Corollary 44 (1) and (3) also hold for N and (2) is Lemma 35.

Let L be a well-adapted lattice and $X \in N \cap S$. By Corollary 46 and Lemma 31 $C(L)$ holds for X . Thus Howe's conjecture holds in PGL_n . ■

7.5. The Howe's conjecture classification (\mathbb{F} -split case). In this subsection we determine exactly for which \mathbb{F} -split reductive groups Howe's conjecture holds.

Lemma 48. *Let G be a \mathbb{F} -split group. If T^{ad} is a \mathbb{F} -split torus of G^{ad} , then $Ad^{-1}(T^{ad})$ is a \mathbb{F} -split torus of G .*

Proof. Without loss of generality we assume that T^{ad} is a maximal \mathbb{F} -split torus of G^{ad} . Let S be a maximal split torus of G and B a Borel subgroup containing S . Then $S^{ad} := Ad(S)$ is a maximal split torus of G^{ad} and B^{ad} a Borel subgroup containing S^{ad} . Take $g \in G^{ad}(\mathbb{F})$ such that $gS^{ad}g^{-1} = T^{ad}$. Take $w^{ad} \in W^{ad}$ such that $g \in U_{(w^{ad})^{-1}}w^{ad}B^{ad}$. By multiplying g with a suitable element of S^{ad} , we may assume that $g \in U_{(w^{ad})^{-1}}w^{ad}U^{ad}$. Take $w \in W$ such that $Ad(w) = w^{ad}$, then

$$Ad : U_{w^{-1}}wU \rightarrow U_{(w^{ad})^{-1}}w^{ad}U^{ad}$$

is a bijection. Therefore there exists a $h \in G(\mathbb{F})$ such that $Ad(h) = g$. Thus

$$Ad(hSh^{-1}) = gS^{ad}g^{-1} = T^{ad}.$$

Thus $Ad^{-1}(T^{ad}) = hSh^{-1}$ is a \mathbb{F} -split torus. ■

Theorem 49. *Let G be a reductive \mathbb{F} -split group, then the following statements are equivalent:*

1. *The characteristic p of \mathbb{F} is good and $p \nmid \kappa_v(G)$.*
2. *For all compact subsets ω and lattices L in \mathfrak{g} :*

$$\dim J_L(\omega) < \infty.$$

Proof. If the characteristic p of \mathbb{F} is bad, then G has bad pairs. So in that case Howe's conjecture does not hold.

If $p \mid \kappa_v(G)$, then Howe's conjecture does not hold by Theorem 28.

Assume that p is good and $p \nmid \kappa_v(G)$. We will use the proof of Howe's conjecture given in [HC99, Part II].

Let A be a maximal \mathbb{F} -split torus of G and B a Borel subgroup containing A . Let \mathfrak{n} be the Lie algebra of the unipotent radical of B . Let $N := {}^G\mathfrak{n}$. For the moment assume that N has the following properties:

1. For all compact $\omega \subset \mathfrak{g}$ there exists a lattice L_1 , such that ${}^G\omega \subset L_1 + N$ (Lemma 12.2).
2. For all $X \in N$ there exists a \mathbb{F} -rational cocharacter $\gamma \in X_*(G)$ such that $X \in \mathfrak{n}(\gamma)$ and $\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)]$ (§13.1).
3. N is closed (Lemma 11.9 & 12.3).
4. If $c \in \mathbb{F}$ and $Y \in N$, then $cY \in N$ (Lemma 12.3).

By (1),(3),(4) and [HC99, §11.1 & §12], if $C(L)$ holds for all $X_0 \in N \cap S$, then Howe's conjecture holds for L . For $X_0 \in N \cap S$ and well-adapted lattices L , $C(L)$ holds by 2. and Lemma 31. Since every lattice contains a well-adapted lattice, Howe's conjecture holds for all lattices L . Thus it is enough to show that N posses these four properties.

Statement (1) is Lemma 35.

Let $G = R(G)G_1 \cdots G_m$ with G_i connected normal simple groups and $R(G)$ the radical of G . Let $G^{ad} = \prod_{i=1}^m G_i^{ad}$ be the adjoint group of G . Since p is good for G it is also good for all the groups G_i^{ad} . Either p is very good for G_i^{ad} or $G_i^{ad} = PGL_n$ with $p \nmid n$. Thus by Corollary 46 and the proof of Corollary 34 for all $X_i \in {}^{G_i^{ad}}\mathfrak{n}_i^{ad}$ there exists a cocharacter γ_i^{ad} defined over \mathbb{F} such that $X_i \in \mathfrak{n}_i^{ad}(\gamma_i^{ad})$ and $\mathfrak{n}_i^{ad}(\gamma_i^{ad}) = [X_i, \mathfrak{p}_i^{ad}(\gamma_i^{ad})]$. Since G^{ad} is a direct product of G_i^{ad} , for every $X \in {}^{G^{ad}}\mathfrak{n}^{ad}$ there exists a character γ such that $X \in \mathfrak{n}(\gamma)$ and $\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)]$.

Statement (2) will be shown for $X \in \mathcal{N} \cap \mathfrak{d}(Ad)^{-1}({}^{G^{ad}}\mathfrak{n}^{ad}) \supset N$.

Let $X \in \mathcal{N} \cap \mathfrak{d}(Ad)^{-1}({}^{G^{ad}}\mathfrak{n}^{ad})$. Let $X^{ad} := \mathfrak{d}(Ad)(X)$. Let γ^{ad} be a cocharacter of G^{ad} such that $X^{ad} \in \mathfrak{n}^{ad}(\gamma^{ad})$ and $\mathfrak{n}^{ad}(\gamma^{ad}) = [X^{ad}, \mathfrak{p}^{ad}(\gamma^{ad})]$. Let T be a maximal split torus of G such that γ^{ad} is a cocharacter of $T^{ad} = Ad(T)$. Take γ a cocharacter of T and $m \in \mathbb{N}_{>0}$ such that $Ad \circ \gamma = m\gamma^{ad}$. Let $k \in \mathbb{N}_{>0}$ be such that $Ad(\gamma^{ad}(t))X^{ad} = t^k X^{ad}$ for all $t \in \mathbb{F}^\times$. Then

$$Ad(\gamma(t))X = t^{km}X + X_z,$$

for some $X_z \in \mathfrak{z}$. Since $Ad(\gamma(t))X$ and $t^{km}X$ are nilpotent, $X_z = 0$. Thus $X \in \mathfrak{n}(\gamma)$. Since $p \nmid \kappa_v(G)$, $\mathfrak{d}(Ad) : \mathfrak{g} \rightarrow \mathfrak{g}^{ad}$ is surjective by Proposition 10. Hence

$d(Ad) : \mathfrak{p}(\gamma) \rightarrow \mathfrak{p}^{ad}(\gamma^{ad})$ is surjective. Also $d(Ad) : \mathfrak{n}(\gamma) \rightarrow \mathfrak{n}^{ad}(\gamma^{ad})$ is a bijection. Thus

$$\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)],$$

because $\mathfrak{n}^{ad}(\gamma^{ad}) = [X^{ad}, \mathfrak{p}^{ad}(\gamma^{ad})]$. Therefore for all $X \in \mathcal{N} \cap d(Ad)^{-1}({}^{G^{ad}}\mathfrak{n}^{ad})$ statement (2) holds.

Moreover since $X \in \mathfrak{n}(\gamma)$, also $X \in N$. Therefore

$$N = \mathcal{N} \cap d(Ad)^{-1}({}^{G^{ad}}\mathfrak{n}^{ad}).$$

Since \mathcal{N} and ${}^{G^{ad}}\mathfrak{n}^{ad}$ are closed and $d(Ad)$ is continuous, also N is closed.

Because $c\mathfrak{n} \subset \mathfrak{n}$ for all $c \in \mathbb{F}$, (4) follows. \blacksquare

Corollary 50. *If G is \mathbb{F} -split and has finitely many nilpotent orbits, then Howe's conjecture holds for G .*

Proof. If the characteristic p of \mathbb{F} is bad for G or if $p|\kappa_v(G)$, then there are infinitely many nilpotent orbits. \blacksquare

8. The separable classification

In this section we give a characterization of the reductive groups whose nilpotent orbits are all separable. As a consequence we get a large class of reductive groups for which the number of nilpotent orbits is finite and Howe's conjecture holds. We take a look at the cokernels of the following functions:

$$\begin{aligned} \Phi &: X_*(T) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z}), \\ \Phi^\vee &: X^*(T) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta^\vee, \mathbb{Z}). \end{aligned}$$

Lemma 51. *$p|\rho_v(G)$ if and only if the $H_\alpha := d\alpha^\vee(1) \in \mathfrak{t}$, for $\alpha \in \Delta$, are linearly dependent.*

Proof. We have the following isomorphism of vector spaces: $\mathfrak{t} \cong X_*(T) \otimes_{\mathbb{Z}} \mathbb{F}$. Let ϵ_i be a basis for $X^*(T)$ and ϵ_i^\vee a dual basis in $X_*(T)$. Let $\alpha^\vee \in X_*(T)$. Now $\alpha^\vee = \sum_{i=1}^m \langle \epsilon_i, \alpha^\vee \rangle \epsilon_i^\vee$. Hence $d\alpha^\vee(1) = \sum_{i=1}^m \langle \epsilon_i, \alpha^\vee \rangle d\epsilon_i^\vee(1)$. Let $\alpha_1^\vee, \dots, \alpha_n^\vee$ be the simple roots in Δ^\vee . Define M to be the $n \times m$ matrix with the following entries

$$M_{ij} := \langle \epsilon_j, \alpha_i^\vee \rangle.$$

Then M is the matrix corresponding with the map Φ^\vee . The matrix M^{tr} is the matrix corresponding with the linear span of the H_{α_i} 's.

Let (d_1, \dots, d_n) be the entries on the diagonal of the Smith normal form of M . Then $\rho_v(G) = \#\text{coker } \Phi^\vee = \prod_{i=1}^n d_i$. The linear span of the H_{α_i} 's is n -dimensional if and only if $p \nmid \prod_{i=1}^n d_i$. \blacksquare

Theorem 52. *The nilpotent orbits are separable if and only if the p is good and $p \nmid \kappa_v(G)$ and $p \nmid \rho_v(G)$.*

Proof \Rightarrow . If p is bad or divides $\kappa_v(G)$, then the regular nilpotent orbit is inseparable by Corollary 21 and Theorem 8. Assume that p divides $\rho_v(G)$. Let $X := \sum_{\alpha \in \Delta} E_\alpha$. Then:

$$\left[\sum_{\alpha \in \Delta} E_\alpha, \sum_{\alpha \in -\Delta} c_\alpha E_\alpha \right] = \sum_{\alpha \in \Delta} c_\alpha H_\alpha.$$

Now p divides the cokernel exactly when the $H_\alpha = d\alpha^\vee(1)$ are linearly dependent. Thus there exists a $Y \in \mathfrak{n}_{-1} - \{0\}$, such that $[X, Y] = 0$. Since $Z_G(X) \subset B$ and $\mathfrak{n}_{-1} \cap \mathfrak{b} = 0$, the orbit of X is not separable. \blacksquare

Before we prove the implication in the other direction, we first state a few lemma's.

Lemma 53. *If p is good for G , then*

$$\mathfrak{g} = \mathfrak{g}_A \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$$

with $G = G_A \prod_{i=1}^m G_i$ where G_i are all the closed normal connected simple groups not of type A and G_A is generated by $R(G)$ and the closed normal connected simple groups of type A in G .

Proof. We have $Ad(G) \cong G_A^{ad} \times G_1^{ad} \times \cdots \times G_m^{ad}$. Let $\Pi : Ad(G) \rightarrow G_1^{ad} \times \cdots \times G_m^{ad}$ be the corresponding projection map. Since p is good for G , it is very good for $G_c = G_1 \cdots G_m$. Thus the linear map $d(Ad) : \mathfrak{g}_c \rightarrow \mathfrak{g}_c^{ad}$ is surjective. Since $\dim G_c = \dim Ad(G_c)$, it is a bijection. Thus $d(\Pi \circ Ad) : \mathfrak{g}_c \rightarrow \mathfrak{g}_c^{ad}$ is a bijection. Therefore $\mathfrak{g}_c = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$, since p is very good for G_c . Moreover $\ker d(\Pi \circ Ad) \cap \mathfrak{g}_c = 0$. Since $\mathfrak{g}_A \subset \ker d(\Pi \circ Ad)$, also $\mathfrak{g}_A \cap \mathfrak{g}_c = 0$. Because $\dim \mathfrak{g}_A + \dim \mathfrak{g}_c = \dim \mathfrak{g}$, the Lemma follows. \blacksquare

Lemma 54. *If p is good for G and $p \nmid \kappa_v(G)$, then $d(Ad) : \mathfrak{g}_A \rightarrow \mathfrak{g}_A^{ad}$ is surjective.*

Proof. By Lemma 53 and its proof we have $\mathfrak{g} = \mathfrak{g}_A \oplus \mathfrak{g}_c$ and $d(Ad) : \mathfrak{g}_c \rightarrow \mathfrak{g}_c^{ad}$ is surjective. Since $p \nmid \kappa_v(G)$ the map $d(Ad) : \mathfrak{g} \rightarrow \mathfrak{g}_A^{ad} \oplus \mathfrak{g}_c^{ad}$ is surjective. Let $\Pi_A : Ad(G) \rightarrow G_A^{ad}$, then $d(\Pi_A \circ Ad) : \mathfrak{g} \rightarrow \mathfrak{g}_A^{ad}$ is surjective. Since \mathfrak{g}_c is contained in its kernel and $\mathfrak{g} = \mathfrak{g}_A \oplus \mathfrak{g}_c$, $d(Ad) : \mathfrak{g}_A \rightarrow \mathfrak{g}_A^{ad}$ is surjective. \blacksquare

Corollary 55. *If p is good for G and $p \nmid \kappa_v(G)$, then p does not divide the cokernel of the following map:*

$$\Phi_A : X_*(T_A) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta_A, \mathbb{Z}).$$

Proof. The map $d(Ad) : \mathfrak{g}_A \rightarrow \mathfrak{g}_A^{ad}$ is surjective, thus $p \nmid \#\text{coker } \Phi_A$ by Proposition 10. \blacksquare

Lemma 56. *Let $n = n_1 + \cdots + n_m$ and γ_i be the cocharacters $\gamma_i \in X_*(T \cap G_i^{ad})$ associated with n_i in \mathfrak{g}_i^{ad} . Let $\gamma \in X_*(T)$ be the cocharacter associated with n in G . Then $d(Ad) \circ \gamma = \sum_{i=1}^m \gamma_i$*

Proof. Clearly $\sum_{i=1}^m \gamma_i$ is a cocharacter associated with n in \mathfrak{g}^{ad} . Also $d(Ad) \circ \gamma$ is a cocharacter associated with n in \mathfrak{g}^{ad} . (See [Jan04, §5.6]) Since there is at most one cocharacter of $Ad(T)$ associated with n by [McN04, Corollary 22], they are equal. ■

Lemma 57. *Let $G = GL_m$ and $G^{ad} = PGL_m$. Let $n \in \mathfrak{g}^{ad}$ be a nilpotent element with associated cocharacter γ . Then*

$$\begin{aligned} [n, \mathfrak{g}^{ad}(k)] &= \mathfrak{g}^{ad}(k+2) && \text{for } k \geq -1 \\ [n, \cdot] : \mathfrak{g}^{ad}(k) &\rightarrow \mathfrak{g}^{ad}(k+2) \text{ is injective} && \text{for } k = -1 \text{ and } k \leq -3 \end{aligned}$$

Proof. For GL_m and $n \in \mathfrak{g}_m$ nilpotent $[n, \mathfrak{g}(k)] = \mathfrak{g}(k+2)$ for $k \geq -1$ and $[n, \cdot] : \mathfrak{g}(k) \rightarrow \mathfrak{g}(k+2)$ is injective for $k \leq -1$. Since the map

$$d(Ad) : \bigoplus_{k \geq 1} \mathfrak{g}(-k) \oplus \mathfrak{g}(k) \rightarrow \bigoplus_{k \geq 1} \mathfrak{g}^{ad}(-k) \oplus \mathfrak{g}^{ad}(k)$$

is a bijection, $[n, \mathfrak{g}^{ad}(k)] = \mathfrak{g}^{ad}(k+2)$ for $k \geq -1$ and $[n, \cdot] : \mathfrak{g}^{ad}(k) \rightarrow \mathfrak{g}^{ad}(k+2)$ is injective for $k = 1$ and $k \geq -3$. ■

Lemma 58. *Let $G = GL_m$. Let $n = \sum_{\alpha \in \Gamma} c_\alpha n_\alpha$, with $\Gamma \subset \Delta$. If $[n, m] \in \mathfrak{z}$ and $m \in \mathfrak{g}(-2)$, then $m = \sum_{\alpha \in -\Gamma} d_\alpha n_\alpha$ for some $d_\alpha \in \mathbb{F}$.*

Proof. Let $\gamma \in X_*(T)$ be such that there exists a $l \in \mathbb{N}_{\geq 0}$ such that for all $\alpha \in \Delta$:

$$\langle \alpha, \gamma \rangle = \begin{cases} l & \text{if } \alpha \in \Gamma, \\ 0 & \text{if } \alpha \notin \Gamma. \end{cases}$$

We know that $[n, \cdot] : \mathfrak{g}(-2; \tau) \rightarrow \mathfrak{g}(0; \tau)$ is injective for the associated cocharacter $\tau \in X_*(T)$ of n . Define $\mathfrak{g}(-2) := \mathfrak{g}(-2; \tau)$ and $\mathfrak{g}_i(-2) := \mathfrak{g}(-2; \tau) \cap \mathfrak{g}(il; \gamma)$. Then

$$\mathfrak{g}(-2) = \bigoplus \mathfrak{g}_i(-2)$$

and

$$[n, \mathfrak{g}_i(-2)] \subset \mathfrak{g}(0; \tau) \cap \mathfrak{g}(l(i+1); \gamma).$$

Because $\mathfrak{z} \subset \mathfrak{g}(0; \gamma)$ and $[n, \cdot]|_{\mathfrak{g}(-2)}$ is injective, then $m \in \mathfrak{g}(-l; \gamma)$. ■

Proof Theorem 52 \Leftarrow . Let $n \in \mathfrak{n}$. Take $n_A \in \mathfrak{n}_A$ and $n_i \in \mathfrak{n}_i$, such that

$$n = n_A + n_1 + \cdots + n_m.$$

Then

$$\begin{aligned} Z_G(n) &= Z_{G_A}(n_A) \prod_{i=1}^m Z_{G_i}(n_i) \\ Z_{\mathfrak{g}}(n) &= Z_{\mathfrak{g}_A}(n_A) \oplus \bigoplus_{i=1}^m Z_{\mathfrak{g}_i}(n_i) \end{aligned}$$

Since the G_i are simple and p is very good for G_i , the G_i -orbit of n_i is separable:

$$\dim Z_{\mathfrak{g}_i}(n_i) = \dim Z_{G_i}(n_i).$$

Thus we are left with showing that $\dim Z_{G_A}(n_A) = \dim Z_{\mathfrak{g}_A}(n_A)$. Since p is good for G , it is also good for G_A . By Corollary 55 and Lemma 51, p does not divide the order of the cokernels of Φ_A and Φ_A^\vee .

Thus without loss of generality we assume that G only consists of groups of type A and a center. Thus $G^{ad} = \prod_{i=1}^k PGL_{n_i}$.

Since $p \nmid \kappa_v(G)$ the map $d(Ad) : \mathfrak{g} \rightarrow \mathfrak{g}^{ad}$ is surjective. Let $n \in \mathfrak{g}$ be nilpotent and γ be a cocharacter associated with n . Define $P := P(\gamma)$. Then $Ad \circ \gamma$ is a cocharacter associated with $Ad(n)$. For G^{ad} the following holds:

$$\begin{aligned} [n, \mathfrak{g}^{ad}(k)] &= \mathfrak{g}^{ad}(k+2) && \text{for } k \geq -1 \\ [n, \cdot] : \mathfrak{g}^{ad}(k) &\rightarrow \mathfrak{g}^{ad}(k+2) \text{ is injective} && \text{for } k = -1 \text{ and } k \leq -3 \end{aligned}$$

Since $d(Ad)$ is surjective and injective on the nilpotent elements, then

$$\begin{aligned} [n, \mathfrak{p}] &= \mathfrak{g}(\geq 2) \\ [n, \cdot] : \mathfrak{g}(k) &\rightarrow \mathfrak{g}(k+2) \text{ is injective for } k = -1 \text{ and } k \leq -3. \end{aligned}$$

Therefore

$$\dim Z_G(n) = \dim Z_P(n) = \dim Z_{\mathfrak{p}}(n),$$

since $\overline{Ad P(n)} = \mathfrak{g}(\geq 2)$ and $[n, \mathfrak{p}] = \mathfrak{g}(\geq 2)$.

If $Z_{\mathfrak{g}}(n) \cap \mathfrak{g}(k) = 0$ for $k \leq -1$, then $Z_{\mathfrak{g}}(n) = Z_{\mathfrak{p}}(n)$.

For $k = -1$ and $k \leq -3$ the function $[n, \cdot] : \mathfrak{g}(k) \rightarrow \mathfrak{g}(k+2)$ is injective.

Thus $Z_{\mathfrak{g}}(n) \cap \mathfrak{g}(k) = 0$, for $k = -1$ and $k \leq -3$.

Thus we are done if the kernel of $[n, \cdot] : \mathfrak{g}(-2) \rightarrow \mathfrak{g}(0)$ is 0.

In G^{ad} every nilpotent element is conjugated with an element of the form $\sum_{\alpha \in \Gamma} E_\alpha$, with $\Gamma \subset \Delta$. Let $n = \sum_{\alpha \in \Gamma} E_\alpha$ with $\Gamma \subset \Delta$ and $m \in \mathfrak{g}(-2)$. If $[n, m] = 0$, then $[d(Ad)(n), d(Ad)(m)] = 0$. Thus $m = \sum_{\alpha \in -\Gamma} c_\alpha E_\alpha$ for some $c_\alpha \in \mathbb{F}$ by Lemma 58. Now

$$0 = [n, m] = \sum_{\alpha \in \Delta} c_\alpha H_\alpha.$$

Because $p \nmid \rho_v(G)$, the H_α are linearly independent. Thus $c_\alpha = 0$ for all $\alpha \in -\Gamma$, hence $m = 0$.

Thus every nilpotent orbit is separable. ■

9. On the number of nilpotent orbits

In this section we discuss when the number of nilpotent orbits is finite.

Theorem 59. *[McN04, Theorem 40] If p is good and all the nilpotent orbits are separable, then there are only finitely many nilpotent orbits.*

Corollary 60. *If p is good and $p \nmid \kappa_v(G)$ and $p \nmid \rho_v(G)$, then there are only finitely many nilpotent orbits.*

Proof. The condition in the Corollary is equivalent to the one in Theorem 59 by Theorem 52. \blacksquare

In this section we will prove the converse of Corollary 60. If G is \mathbb{F} -split and p is bad or divides $\kappa_v(G)$, then there are infinitely many regular nilpotent orbits by Theorem 26 and Proposition 6. So it is enough to prove that if G is \mathbb{F} -split, p is good, $p \nmid \kappa_v(G)$ and $p \mid \rho_v(G)$, then G has infinitely many nilpotent orbits. First a Theorem that we can easily deduce from the theory of the previous section.

Theorem 61. *If G is semi-simple and the characteristic of \mathbb{F} is not very good, then there are infinitely many nilpotent orbits.*

Proof. If the characteristic of \mathbb{F} is bad, then we have already showed that there are infinitely many nilpotent orbits. So without loss of generality we assume G has at least one normal simple subgroup of type A_n , with $p \mid n + 1$. Now the proof is split in two cases: $p \mid \kappa_v(G)$ and $p \nmid \kappa_v(G)$.

If $p \mid \kappa_v(G)$, then \mathfrak{g} has infinitely many nilpotent orbits by Proposition 6.

If $p \nmid \kappa_v(G)$, then $d(\text{Ad}) : \mathfrak{g} \rightarrow \mathfrak{g}^{\text{ad}}$ is an isomorphism by Theorem 10. Since there are infinitely many nilpotent orbits in \mathfrak{g}^{ad} by Corollary 42, there are also infinitely many nilpotent orbits in \mathfrak{g} . \blacksquare

Proposition 62. *Let G be a reductive group with only normal simple subgroups of type A for which p is not very good. Assume that $p \mid \rho_v(G)$. Let H be a reductive group with $G \triangleleft H$. Let \mathcal{N} be the set of nilpotent elements of \mathfrak{g} . Then there are infinitely many nilpotent H -orbits in \mathcal{N} .*

Proof. The proof of this proposition is distributed over two lemmas.

Lemma 63. *If $\alpha^\vee \in \Delta^\vee$, then $\text{Ad} \circ \alpha^\vee \in \Delta_{\text{ad}}^\vee$.*

Proof. The reader could verify this by taking the Chevalley basis on \mathfrak{g} . \blacksquare

Let $\Delta^\vee = \{\alpha_{11}^\vee, \dots, \alpha_{nm_n}^\vee\}$, such that α_{ij}^\vee is connected in the Dynkin diagram with $\alpha_{i'j'}^\vee$ if and only if $i = i'$ and $j = j + 1$.

Lemma 64. *If $\sum_{\alpha \in \Delta} c_\alpha d\alpha^\vee(1) = 0$, then for every i there exists a c_i such that $c_{\alpha_{ij}} = j c_i$ for all j .*

Proof. Because $\sum_{\alpha \in \Delta} c_\alpha d\alpha^\vee(1) = 0$, also

$$\sum_{\alpha \in \Delta_{\text{ad}}} c_\alpha d(\text{Ad} \circ \alpha^\vee)(1) = 0.$$

($d(\text{Ad})(d\alpha^\vee(1)) = d(\text{Ad} \circ \alpha^\vee)(1)$)

Since $\mathfrak{g}^{ad} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, with \mathfrak{g}_i the Lie algebra of PGL_{m_i+1} , then for every i :

$$\sum_{j=1}^{m_i} c_{\alpha_{ij}} d(Ad \circ \alpha_{ij}^\vee)(1) = 0.$$

A small calculation in \mathfrak{g}_i shows that there exists a c_i such that $c_{\alpha_{ij}} = jc_i$. \blacksquare

Since $p|\rho_v(G)$, by Lemma 64 there exist $c_i \in \mathbb{F}$ such that

$$\sum_{i=1}^n \sum_{j=1}^{m_i} jc_i d\alpha_{ij}^\vee(1) = 0$$

and at least one of the $c_i \neq 0$. Without loss of generality assume that $1, \dots, k$ are the i with $c_i \neq 0$.

Let $i \leq k$. Let $M_i(x)$ be the block matrix consisting of $\frac{m_i}{p}$ blocks of $p \times p$ -matrices, with on each block the following matrix

$$\begin{pmatrix} 0 & \cdots & 0 & c_i x \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \ddots & 0 & 1 & 0 \end{pmatrix}.$$

Thus the entries of M_i are as follows:

$$(M_i)_{kl} := \begin{cases} 1 & \text{if } k = l + 1 \text{ and } p \nmid l \\ c_i x & \text{if } l = k + p - 1 \text{ and } p \mid l \\ 0 & \text{otherwise} \end{cases}$$

Then $M_i(x)^p = c_i x I_{n_i}$.

Let $N(x)$ be the element in \mathfrak{g} corresponding with $M_1(x) \oplus \cdots \oplus M_k(x)$.

Then

$$N(x)^p = \sum_{i=1}^n \sum_{j=1}^{m_i} x jc_i d\alpha_{ij}^\vee(1) = 0.$$

Thus $N(x)$ is nilpotent.

Let q be a power of p such that $\mathcal{N}^{ad} = \{X \in \mathfrak{g}^{ad} \mid X^q = 0\}$.

Let $\phi' : \mathcal{N}^{ad} \rightarrow \mathbb{F}/\mathbb{F}^q$ be the following function. Take $X_i \in \mathfrak{gl}_{m_i+1}$ such that $X = \bigoplus_{i=1}^n X_i + \mathfrak{z}_i$. Then for each i we have a z_i such that $X_i^q = z_i I_{m_i+1}$ in \mathfrak{gl}_{m_i+1} . Define $\phi'(X) := z_1$.

If X'_i are also representatives for X , then $z'_i = z_i + a_i^q$ for $a_i \in \mathbb{F}$. Thus ϕ' is well-defined.

Since $G^{ad} = \prod_{i=1}^n PGL_{m_i+1}$, ϕ is also G^{ad} -invariant. Define $\phi : \mathcal{N} \rightarrow \mathbb{F}/\mathbb{F}^q$ by $\phi := \phi' \circ d(Ad)$. Then ϕ is H -invariant, since H acts on \mathcal{N} by conjugation and $G_1 \triangleleft H$. For $x \in \mathbb{F}$,

$$\phi(N(x)) = x^{\frac{q}{p}}.$$

Since $\mathbb{F}^{\frac{q}{p}}/\mathbb{F}^q \cong \mathbb{F}/\mathbb{F}^{(p)}$ is infinite and ϕ is H -invariant, there are infinitely many nilpotent H -orbits. \blacksquare

Theorem 65. *If p is good and $p \nmid \kappa_v(G)$, but $p \mid \rho_v(G)$, then there are infinitely many nilpotent orbits.*

When G is semisimple this is [Spr66, Theorem 5.9].

Proof. Let $G = R(G)G_1 \cdots G_l$ with G_i the minimal simple normal connected subgroups of G and $R(G)$ the radical of G . Assume that G_1, \dots, G_n are the groups of type A for which p is not very good. Define $G_A := R(G)G_1 \cdots G_n$ and $G_C := G_{n+1} \cdots G_l$. Because $p \nmid \kappa_v(G)$, the map $d(Ad) : \mathfrak{g} \rightarrow \mathfrak{g}^{ad}$ is surjective. Because p is very good for G_C , $d(Ad) : \mathfrak{g}_C \rightarrow \mathfrak{g}_C^{ad}$ is surjective. Since G_C is semi-simple, the map is even an isomorphism.

$$\mathfrak{g}^{ad} \cong \bigoplus_{i=1}^l \mathfrak{g}_i^{ad} = \mathfrak{g}_A^{ad} \oplus \mathfrak{g}_C^{ad}$$

Define $Ad_C : \mathfrak{g} \rightarrow \mathfrak{g}_C^{ad}$ by the composition of the projection and $d(Ad) : \mathfrak{g} \rightarrow \mathfrak{g}^{ad}$. Then $\mathfrak{g}_A \subset \ker Ad_C$ and $\ker Ad_C \cap \mathfrak{g}_C = 0$. Hence $\mathfrak{g}_A \cap \mathfrak{g}_C = 0$, thus $\mathfrak{g} = \mathfrak{g}_A \oplus \mathfrak{g}_C$. By Lemma 51 the $d\alpha^\vee(1)$'s are linearly dependent. Because of the decomposition of \mathfrak{g} , the $d\alpha^\vee(1) : \alpha \in \Delta_A$ are linearly dependent or the $d\alpha^\vee(1) : \alpha \in \Delta_C$ are linearly dependent. Since p is very good for G_C the $d\alpha^\vee(1) : \alpha \in \Delta_C$ are linearly independent. So the $d\alpha^\vee(1) : \alpha \in \Delta_A$ are linearly dependent. Therefore we can apply Proposition 62 with $H = G$ and $G = G_A$. ■

Theorem 66. *If G is \mathbb{F} -split, then the following are equivalent:*

1. *The number of nilpotent orbits is finite.*
2. *All the nilpotent orbits are separable.*
3. *The regular nilpotent orbit is separable.*
4. *p is good and $p \nmid \kappa_v(G)\rho_v(G)$.*

Proof. (2) implies (1) by [McN04, Theorem 40]. (1) implies (4) by Theorem 65, Theorem 26 and Proposition 6. (4) implies (2) by Theorem 52. By the proof of Theorem 52 also not (4) implies not (3). ■

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