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On the Cohomology of Four-Dimensional Almost Complex Lie Algebras

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Abstract. It is shown that the unimodularity condition for a four-dimensional Lie algebra $\mathfrak{g}$ with $H^2(\mathfrak{g}) \neq \{0\}$ is equivalent with a certain decomposition of the group $H^2(\mathfrak{g})$ taking place with respect to any almost complex structure J on $\mathfrak{g}$. One direction of this result was proved by Li and Tomassini in [8]. This note proves the other direction.
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1. Introduction

For any almost complex manifold (M, J), Li and Zhang introduced in [9] certain cohomology subgroups naturally defined by the almost complex structure. Particularly interesting are the subgroups H_J^+ and H_J^- of the second cohomology group $H^2(M, \mathbb{R})$. These subgroups contain the cohomology classes having a closed 2-form representative which is J-invariant, respectively, J-anti-invariant. It was shown in [4], Theorem 2.3, that on any compact 4-dimensional almost complex manifold (M^4, J), the subgroups H_J^+ and H_J^- induce a direct sum decomposition of the second cohomology group $H^2(M^4, \mathbb{R})$. This is specifically a 4-dimensional result, see [5], [2], [1] for counter-examples in dimension greater than 6. In dimension 4, however, this decomposition can be seen as an extension to non-integrable compact almost complex 4-manifolds of the Dolbeault degree two cohomology decomposition for compact complex surfaces.

For an almost complex Lie algebra $(\mathfrak{g}, J)$, one similarly defines the subgroups $H_J^+(\mathfrak{g})$, $H_J^-(\mathfrak{g})$ of the second Chevalley-Eilenberg cohomology group with real coefficients $H^2(\mathfrak{g})$. Among other interesting results in [8], Li and Tomassini prove a Lie algebra version of the cohomology decomposition observed in [4].

Theorem 1.1. *([8], Theorem 3.3) If the four-dimensional Lie algebra $\mathfrak{g}$ is unimodular then for any almost complex structure J on $\mathfrak{g}$ there is a direct sum*

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cohomology decomposition

$$H^2(\mathfrak{g}) = H_J^+(\mathfrak{g}) \oplus H_J^-(\mathfrak{g}) .$$

In this note we prove the converse of the above result for 4-dimensional Lie algebras with $H^2(\mathfrak{g}) \neq \{0\}$. The precise statement is in Theorem 2.2 in the next section. Combining with the above result of Li and Tomassini, we obtain the following:

Theorem 1.2. *A four-dimensional Lie algebra $\mathfrak{g}$ with $H^2(\mathfrak{g}) \neq \{0\}$ is unimodular if and only if for any almost complex structure J on $\mathfrak{g}$ there is a direct sum cohomology decomposition*

$$H^2(\mathfrak{g}) = H_J^+(\mathfrak{g}) \oplus H_J^-(\mathfrak{g}) .$$

Recall that a Lie algebra $\mathfrak{g}$ is called *unimodular* if $\mathrm{tr}(\mathrm{ad}_\xi) = 0$ for every $\xi \in \mathfrak{g}$. This is equivalent to the fact that a volume form on $\mathfrak{g}$ (i.e. a non-vanishing top-dimensional form) cannot be exact(see [6], p. 81 (6.3)), which in turn is expressed in dimension 4 by $H^4(\mathfrak{g}) = \mathbb{R}$. Consequently, the condition $H^4(\mathfrak{g}) = \{0\}$ is the cohomological characterization of four-dimensional *non-unimodular* Lie algebras. Therefore, Theorem 1.2 can be interpreted in cohomological terms as saying that the structure of the (non-trivial) second Chevalley-Eilenberg group $H^2(\mathfrak{g})$ with respect to almost complex structures is predicted by the top Chevalley-Eilenberg group $H^4(\mathfrak{g})$.

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2. Notations and Preliminaries

Given a Lie algebra $\mathfrak{g}$, the Lie bracket induces the Chevalley-Eilenberg differential d on the spaces of forms $\Lambda^k(\mathfrak{g}^*)$ on $\mathfrak{g}$. On $\Lambda^1(\mathfrak{g}^*)$, d is defined by

$$d\alpha(u,v) = -\alpha([u,v]), \quad \alpha \in \Lambda^1(\mathfrak{g}^*), \ u, v \in \mathfrak{g},$$

and then is extended to $\Lambda^*(\mathfrak{g}^*)$ by the Leibniz rule. The cohomology of the differential complex $(\Lambda^*(\mathfrak{g}^*), d)$ is the Chevalley-Eilenberg cohomology with real coefficients of $\mathfrak{g}$. The r-th cohomology group is $H^r(\mathfrak{g}) = \mathcal{Z}^r/\mathcal{B}^r$, where $\mathcal{Z}^r$, $\mathcal{B}^r$ denote the spaces of closed, respectively, exact r-forms on $\mathfrak{g}$.

Let now J be an almost complex structure on the Lie algebra $\mathfrak{g}$, that is, an endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ with $J^2 = -1$. The induced action of J on the bundle of 2-forms, $\alpha(\cdot,\cdot) \to \alpha(J\cdot, J\cdot)$, is an involution. This induces the decomposition of $\Lambda^2(\mathfrak{g}^\star)$ into J-invariant and J-anti-invariant forms, respectively, the ± 1-eigenspaces of the above action

$$\Lambda^2(\mathfrak{g}^\star) = \Lambda_J^+(\mathfrak{g}^\star) \oplus \Lambda_J^-(\mathfrak{g}^\star) .$$

We denote by $\mathcal{Z}_J^\pm = \mathcal{Z}^2 \cap \Lambda_J^\pm(\mathfrak{g}^\star)$ the spaces of closed J-invariant, respectively, closed J-anti-invariant 2-forms. The following are Lie algebra analogues of definitions introduced by Li and Zhang [9] for almost complex manifolds:

Definition 2.1. Let $(\mathfrak{g}, J)$ be an almost complex complex Lie algebra.

(i) The subgroups $H_J^{\pm}(\mathfrak{g})$ of $H^2(\mathfrak{g})$ are defined as the sets of cohomology classes that can be represented by J-invariant forms, respectively, by J-anti-invariant forms:

$$H_J^+(\mathfrak{g})=\{\mathfrak{a}\in H^2(\mathfrak{g})|\exists\alpha\in\mathcal{Z}_J^+,\ [\alpha]=\mathfrak{a}\},\ H_J^-(\mathfrak{g})=\{\mathfrak{a}\in H^2(\mathfrak{g})|\exists\alpha\in\mathcal{Z}_J^-,\ [\alpha]=\mathfrak{a}\}.$$

$$H_J^{\pm}(\mathfrak{g}) = \{\mathfrak{a} \in H^2(\mathfrak{g}) \mid \exists\ \alpha \in \mathcal{Z}_J^{\pm},\ [\alpha] = \mathfrak{a}\}.$$

(ii) J is said to be C^∞-*pure* if $H_J^+(\mathfrak{g}) \cap H_J^-(\mathfrak{g}) = \{0\}$.

(iii) J is said to be C^∞-*full* if $H_J^+(\mathfrak{g}) + H_J^-(\mathfrak{g}) = H^2(\mathfrak{g})$.

(iv) J is said to be C^∞-*pure-and-full* if $H_J^+(\mathfrak{g}) \oplus H_J^-(\mathfrak{g}) = H^2(\mathfrak{g})$.

With these definition, Theorem 1.1 of Li and Tomassini mentioned in the introduction states that on a four-dimensional unimodular Lie algebra any almost complex structure is C^∞-pure-and-full. We prove the following converse:

Theorem 2.2. *Let $\mathfrak{g}$ be a non-unimodular four-dimensional Lie algebra and assume that $H^2(\mathfrak{g}) \neq \{0\}$. Then $\mathfrak{g}$ admits an almost complex structure which is not C^∞-pure-and-full. More precisely, if $\mathfrak{g}$ admits symplectic forms, then there exists an almost complex structure on $\mathfrak{g}$ which is not C^∞-pure. If $\mathfrak{g}$ admits no symplectic forms, then there exists an almost complex structure on $\mathfrak{g}$ which is not C^∞-full.*

3. Proof of Theorem 2.2

Let $\mathfrak{g}$ be a four-dimensional Lie algebra oriented by a fixed a volume form $\mu \in \Lambda^4(\mathfrak{g}^\star)$. The wedge product and the fixed volume form induce a bilinear form of signature (3,3) on the 6-dimensional space of 2-forms:

$$\Phi_\mu : \Lambda^2(\mathfrak{g}^\star) \times \Lambda^2(\mathfrak{g}^\star) \to \mathbb{R}, \quad \alpha \wedge \beta = \Phi_\mu(\alpha, \beta)\ \mu,\ \ \forall\ \alpha, \beta \in \Lambda^2(\mathfrak{g}^\star). \qquad (1)$$

As observed by Donaldson in the introduction of [3], a special feature of dimension four is that various geometric structures can be characterized in terms of subspaces of the space of 2-forms and their behavior with respect to Φ_μ. Important for us will be the following well known lemma:

Lemma 3.1. *Let $\mathfrak{g}$ be a four-dimensional Lie algebra, oriented by a fixed volume form $\mu \in \Lambda^4(\mathfrak{g}^\star)$. The map $\pm J \to \Lambda_J^-$ is a two-to-one correspondence between (positively oriented) almost complex structures J on $\mathfrak{g}$ and 2-dimensional planes in $\Lambda^2(\mathfrak{g}^\star)$, positive definite with respect to Φ_μ.*

This lemma will enable us to construct almost complex structures on the Lie algebra by defining two-dimensional subspaces of $\Lambda^2(\mathfrak{g}^\star)$, positive definite with respect to Φ_μ. Note that the sign ambiguity $\pm J$ is not relevant, as J and $-J$ induce the same action on $\Lambda^2(\mathfrak{g}^\star)$, so the subgroups $H_J^{\pm}(\mathfrak{g})$ are the same for $\pm J$.

The proof of Theorem 2.2 will proceed along the two cases mentioned in the statement, that is depending whether the Lie algebra $\mathfrak{g}$ admits symplectic forms or not. In the first case, we start with the following lemma.

Lemma 3.2. *Let $\mathfrak{g}$ be a four-dimensional non-unimodular Lie algebra with $H^2(\mathfrak{g}) \neq \{0\}$. Assume that $\mathfrak{g}$ admits a symplectic form ω_0, let $\mu = \omega_0 \wedge \omega_0$ be the induced volume form and denote by Φ_μ the bilinear form defined by (1). The following hold:*

(a) The Lie algebra $\mathfrak{g}$ admits a symplectic form α so that $[\alpha] \neq 0$ in $H^2(\mathfrak{g})$ and $\Phi_\mu(\alpha, \alpha) > 0$.

(b) With α as in part (a), there exists a 1-form $\theta \in \Lambda^1(\mathfrak{g}^\star)$ so that

$$\Phi_\mu(\alpha, \alpha + d\theta) = 0 \; . \tag{2}$$

(c) With α and θ as in parts (a), (b), there exists a 2-form $\beta \in \Lambda^2(\mathfrak{g}^\star)$ so that

$$\Phi_\mu(\beta, \beta) = \Phi_\mu(\alpha, \alpha), \;\; \Phi_\mu(\beta, \alpha) = 0, \;\; \Phi_\mu(\beta, \alpha + d\theta) = 0 \; . \tag{3}$$

Proof. (a) If $[\omega_0] \neq 0$ in $H^2(\mathfrak{g})$, then there is nothing to prove for part (a). Note however that non-unimodular Lie algebras may admit *exact* symplectic forms. But if ω_0 is exact, let α_0 be a closed, non-exact 2-form on $\mathfrak{g}$ (such form exists by the assumption that $H^2(\mathfrak{g}) \neq \{0\}$). Then for $\epsilon > 0$ small enough, $\alpha = \omega_0 + \epsilon\alpha_0$ satisfies the requirements of part (a).

(b) While it was not needed in (a), for (b) we'll use the assumption that $\mathfrak{g}$ is not unimodular. Thus, there exists a 3-form ψ so that $d\psi \neq 0$. By rescaling and eventually changing the sign of ψ, we can assume that $d\psi = -\alpha \wedge \alpha$. Since α is non-degenerate, the wedge product by α is an isomorphism between $\Lambda^1(\mathfrak{g}^\star)$ and $\Lambda^3(\mathfrak{g}^\star)$. There exists a 1-form θ so that $\psi = \theta \wedge \alpha$. As α is closed, the relation $d\psi = -\alpha \wedge \alpha$ is equivalent to (2).

(c) As the orthogonal complement of $Span(\alpha, \alpha + d\theta)$ with respect to Φ_μ is four-dimensional and Φ_μ has signature (3,3), there exists β in the orthogonal complement of $Span(\alpha, \alpha + d\theta)$ and such that $\Phi_\mu(\beta, \beta) = \Phi_\mu(\alpha, \alpha) > 0$. ∎

From Lemma 3.2 we immediately obtain the symplectic part of Theorem 2.2:

Proposition 3.3. *Let $\mathfrak{g}$ be a four-dimensional symplectic non-unimodular Lie algebra with $H^2(\mathfrak{g}) \neq \{0\}$. Then $\mathfrak{g}$ admits a non-C^∞-pure almost complex structure.*

Proof. With the notations from the previous Lemma, consider the almost complex structure J defined by the plane $\Lambda^-_J = Span(\alpha, \beta)$. It is clear that Φ_μ is positive definite on this plane. By parts (b) and (c) of Lemma 3.2, $\alpha + d\theta$ is orthogonal to Λ^-_J, hence $\alpha + d\theta \in \mathcal{Z}^+_J$. On the other hand, by construction, $\alpha \in \mathcal{Z}^-_J$. Thus, $0 \neq [\alpha] \in H^+_J(\mathfrak{g}) \cap H^-_J(\mathfrak{g})$. ∎

For the case when the Lie algebra $\mathfrak{g}$ does not admit any symplectic forms, we start again with a lemma.

Lemma 3.4. *Let $\mathfrak{g}$ be a four-dimensional non-unimodular Lie algebra with $H^2(\mathfrak{g}) \neq \{0\}$. Assume that $\mathfrak{g}$ does not admit any symplectic forms. Let μ be* *a volume form on* $\mathfrak{g}$ *and let* Φ_μ *be the bilinear from defined by (1). The following hold:*

(i) The space of closed 2-forms $\mathcal{Z}^2$ *is isotropic with respect to* Φ_μ*, i.e.*

$$\Phi_\mu(\phi,\rho)=0,\ \ \forall\phi,\rho\in\mathcal{Z}^2\ .$$

In particular, $\mathcal{Z}^2$ *is at most 3-dimensional and any element* $\phi\in\mathcal{Z}^2$ *is a simple form, i.e.* $\phi=a\wedge b$*, where* $a,b\in\Lambda^1(\mathfrak{g}^\star)$*.*

(ii) Given $\phi\in\mathcal{Z}^2$ *with* $[\phi]\neq 0$ *in* $H^2(\mathfrak{g})$*, there exists* $\alpha\in\Lambda^2(\mathfrak{g}^\star)$ *such that*

$$\Phi_\mu(\alpha,\alpha)>0,\ \ \Phi_\mu(\alpha,\phi)>0,\ \ \Phi_\mu(\alpha,\rho)=0,\ \forall\rho\in\mathcal{B}^2\ . \tag{4}$$

Proof. (i) The assumption that $\mathfrak{g}$ does not admit any symplectic forms implies that

$$\Phi_\mu(\phi,\phi)=0,\ \ \forall\phi\in\mathcal{Z}^2\ .$$

By polarization, this implies that $\mathcal{Z}^2$ is isotropic with respect to Φ_μ. Since Φ_μ has signature $(3,3)$, it follows that $\mathcal{Z}^2$ can be at most 3-dimensional. The fact that all elements of $\mathcal{Z}^2$ are simple forms follows from an observation of Li-Tomassini [8], Lemma 2.1.

(ii) Suppose now we fix $\phi\in\mathcal{Z}^2$ with $[\phi]\neq 0$ in $H^2(\mathfrak{g})$. Taking into account (i), by eventually changing the basis of the Lie algebra, we can assume $\phi=e^1\wedge e^2$ and

$$\mathcal{Z}^2\subseteq Span\{e^1\wedge e^2,e^1\wedge e^3,e^1\wedge e^4\}\text{ or }\mathcal{Z}^2\subseteq Span\{e^1\wedge e^2,e^1\wedge e^3,e^2\wedge e^3\}\ .$$

It follows that the space of exact 2-forms satisfies

$$\mathcal{B}^2\subseteq Span\{e^1\wedge e^3,e^1\wedge e^4\}\text{ or }\mathcal{B}^2\subseteq Span\{e^1\wedge e^3,e^2\wedge e^3\}\ .$$

Let $\alpha=e^1\wedge e^2\pm e^3\wedge e^4$, where the sign is chosen to satisfy the condition $\Phi_\mu(\alpha,\alpha)>0$. It is clear that the other conditions are also satisfied. ∎

The next proposition covers the non-symplectic case of Theorem 2.2 and concludes its proof.

Proposition 3.5. *Let* $\mathfrak{g}$ *be a four-dimensional non-unimodular Lie algebra with* $H^2(\mathfrak{g})\neq\{0\}$ *and assume that* $\mathfrak{g}$ *does not admit any symplectic form. Then* $\mathfrak{g}$ *carries a non-*C^∞*-full almost complex structure.*

Proof. First note that the assumption that $\mathfrak{g}$ does not admit any symplectic form implies that $H^-_J(\mathfrak{g})=\{0\}$ for any almost complex structure on $\mathfrak{g}$. It only remains to produce an almost complex structure J with $H^+_J(\mathfrak{g})\neq H^2(\mathfrak{g})$.

Fix $\phi\in\mathcal{Z}^2$ with $[\phi]\neq 0$ in $H^2(\mathfrak{g})$ and let $\alpha\in\Lambda^2(\mathfrak{g}^\star)$ satisfying the conditions from (4) in Lemma 3.4. Choose one other two form β, so that

$$\Phi_\mu(\beta,\beta)=\Phi_\mu(\alpha,\alpha),\ \ \Phi_\mu(\alpha,\beta)=0,$$

and define the almost complex structure J, by $\Lambda^-_J=Span(\alpha,\beta)$.

Relation (4) in Lemma 3.4 implies that $[\phi]\notin H^+_J(\mathfrak{g})$, as no representative of the class $[\phi]$ can be orthogonal to the space of J-anti-invariant 2-forms Λ^-_J. ∎

As pointed out in the proof of Proposition 3.5, if $\mathfrak{g}$ does not admit any symplectic form, then any almost complex structure J on $\mathfrak{g}$ satisfies $H_J^-(\mathfrak{g}) = \{0\}$, hence any J is automatically C^∞-pure. As a completion to Propositions 3.3, 3.5, one might ask whether it is true that any 4-dimensional symplectic non-unimodular Lie algebra with $H^2(\mathfrak{g}) \neq \{0\}$ admits non-C^∞-full almost complex structures. We don't know the answer to this, although one could probably do a case-by-case check using the classification of symplectic 4-dimensional Lie algebras (e.g. see [11]). The following example shows that there are even Kähler non-C^∞-full J's in the non-unimodular case.

Proposition 3.6. *The Lie algebra $\mathfrak{g} = \mathfrak{r}_2\mathfrak{r}_2$ admits an almost complex structure J which is not C^∞-full, but which is Kähler (that is, J is integrable and compatible with a symplectic form).*

Proof. The Lie algebra $\mathfrak{g} = \mathfrak{r}_2\mathfrak{r}_2$ has structure equations

$$de^1 = 0 \quad de^2 = -e^1 \wedge e^2 \quad de^3 = 0 \quad de^4 = -e^3 \wedge e^4 \ .$$

The spaces of exact 2-forms, respectively, of closed 2-forms are given by

$$\mathcal{B}^2 = Span(e^1 \wedge e^2, e^3 \wedge e^4), \ \ \mathcal{Z}^2 = Span(e^1 \wedge e^2, e^1 \wedge e^3, e^3 \wedge e^4) \ ,$$

so the cohomology group $H^2(\mathfrak{g}) = Span([e^1 \wedge e^3])$. If J is given by $Je^1 = e^2$, $Je^3 = e^4$, it is immediately checked that J is integrable and, in fact, is Kähler with respect to the compatible (exact) symplectic form $e^1 \wedge e^2 + e^3 \wedge e^4$. On the other hand, it is clear that the class $[e^1 \wedge e^3]$ does not admit a J-invariant nor a J-anti-invariant closed form representative. Thus $H_J^+ = H_J^- = \{0\}$, so J is not C^∞-full. ∎

As final remarks, note that the difference between the results of [8] in the unimodular case and those in the non-unimodular case presented here was not unexpected. A key observation in [8] is that on unimodular Lie algebras endowed with a Riemannian metric the Hodge decomposition of forms holds as for compact Riemannian manifolds. This clearly is not true for non-unimodular Lie algebras.

In dimensions 6 and higher, the problems considered here are much less understood. There are known examples of almost complex unimodular Lie algebras (nilpotent, or completely solvable) which are not C^∞-pure and full (see again [1] and the references therein). Such examples exist even with integrable J, [2]. Finding necessary and sufficient conditions for C^∞-pure-and-full almost complex Lie algebras in dimensions 6 and higher is an interesting problem. One could also ask whether the statement of Theorem 2.2 remains true for non-unimodular Lie algebras of any even dimension.

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