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Limits of Jordan Lie Subalgebras

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Abstract. Let $\mathfrak{g}$ be a simple Lie algebra of rank n over $\mathbb{C}$. We show that the n-dimensional abelian ideals of a Borel subalgebra of $\mathfrak{g}$ are limits of Jordan Lie subalgebras. Combining this with a classical result by Kostant, we show that the $\mathfrak{g}$-module spanned by all n-dimensional abelian Lie subalgebras of $\mathfrak{g}$ is actually spanned by the Jordan Lie subalgebras.

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1. Introduction

To define a system for generalized Airy functions, Gel'fand, Retahk, and Serganova [8] considered a Jordan group, which is the centralizer of a maximal Jordan cell in $GL(n)$. We call its Lie algebra a Jordan Lie subalgebra. The system is a confluent version of an Aomoto-Gel'fand system ([1], [7], etc.) associated to a Cartan subalgebra of $\mathfrak{gl}_n$. Kimura and Takano [10] explained the process of confluence by taking limits of regular elements; a Cartan subalgebra is the centralizer of a semisimple regular element, and a Jordan Lie subalgebra is that of a nilpotent regular element. A natural question thus arises: describe the set of limits of Cartan subalgebras. Recall that an element X in a simple Lie algebra $\mathfrak{g}$ is said to be regular if the centralizer $\mathfrak{z}_{\mathfrak{g}}(X)$ has the minimal possible dimension, the rank of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a simple Lie algebra of rank n over $\mathbb{C}$, and G its adjoint group. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The question is to consider the closure $\overline{\mathrm{Ad}(G)\mathfrak{h}}$ in the Grassmannian $\mathrm{Gr}(n,\mathfrak{g})$ composed of n-dimensional subspaces of $\mathfrak{g}$. The centralizer of a regular element certainly belongs to $\overline{\mathrm{Ad}(G)\mathfrak{h}}$. In particular, a Jordan Lie subalgebra J that is the centralizer of a regular nilpotent element belongs to $\overline{\mathrm{Ad}(G)\mathfrak{h}}$. As a generalization of a regular nilpotent element, Ginzburg [9] defined and studied a principal nilpotent pair (also see [6]). We remark that its centralizer also belongs to $\overline{\mathrm{Ad}(G)\mathfrak{h}}$. More generally a wonderful nilpotent pair was studied in [15] and [18]; the $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$-graded component of its centralizer belongs to $\overline{\mathrm{Ad}(G)\mathfrak{h}}$.

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The theory of abelian ideals of a Borel subalgebra draw a strong attention to researchers in the representation theory (e.g., [5],[16]) after Kostant's remarkable paper [13] related the theory to the combinatorics of affine Weyl groups and the theory of discrete series.

In this paper, we show that the n-dimensional abelian ideals of a Borel subalgebra of $\mathfrak{g}$ belong to $\overline{\mathrm{Ad}(G)J}$. Combining this with a classical result by Kostant [12], we show that the $\mathfrak{g}$-module spanned by all n-dimensional abelian Lie subalgebras of $\mathfrak{g}$ is actually spanned by the Jordan Lie subalgebras.

In Section 2, after reviewing the classical result by Kostant [12], we state the main results in this paper. Then we introduce two types of deformation in Section 3: unipotent deformation and semisimple deformation. These are two basic techniques we employ.

In the subsequent sections, we prove the results type by type. For the classical types, we move some technical details into Appendix, to make the proofs clearer.

In Section 4, we treat the case of type A. Using the Weyl group action, we reduce the proof to a problem of the solvability of a system of inequalities, which is proved in Appendix A. In Section 5, we consider the other classical types in a uniform manner. In Section 6, we treat the exceptional types. To compute, we fix a Chevalley basis of $\mathfrak{g}$ as in [14, Proposition 4].

2. Main results

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank n, and G its adjoint group.

2.1. **Kostant's classical result.** For $k = 0, 1, \ldots, \dim \mathfrak{g}$, $\bigwedge^k \mathfrak{g}$ is a $\mathfrak{g}$-module by the adjoint representation. Let C_k be the subspace of $\bigwedge^k \mathfrak{g}$ spanned by all $\wedge^k \mathfrak{a}$ where $\mathfrak{a}$ is a k-dimensional abelian Lie subalgebra of $\mathfrak{g}$. Then C_k is a $\mathfrak{g}$-submodule of $\bigwedge^k \mathfrak{g}$.

Fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} \supseteq \mathfrak{h}$. Let Δ be the root system with respect to $\mathfrak{h}$, and Δ^+ the positive root system corresponding to $\mathfrak{b}$. As a $\mathfrak{g}$-module, C_k is characterized by the following theorem:

Theorem 2.1 (Kostant [12]). *Let $\mathfrak{a}$ be a k-dimensional abelian ideal of $\mathfrak{b}$. Then $\wedge^k \mathfrak{a}$ is a highest weight vector of C_k. Conversely any highest weight vector of C_k is of this form.*

Let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{b}$. Note that there exists a subset $\Delta(\mathfrak{a}) \subseteq \Delta^+$ such that

$$\mathfrak{a} = \bigoplus_{\alpha \in \Delta(\mathfrak{a})} \mathfrak{g}_\alpha \quad \text{and} \quad (\Delta^+ + \Delta(\mathfrak{a})) \cap \Delta \subseteq \Delta(\mathfrak{a}). \tag{2.1}$$

2.2. **Jordan Lie subalgebras.** Let $\alpha_1, \ldots, \alpha_n$ be the simple roots in Δ^+; we follow Bourbaki's notation [2].

Let

$$\{X_\alpha, H_i \mid \alpha \in \Delta,\ i = 1, 2, \ldots, n\}$$

be a Chevalley basis of $\mathfrak{g}$. Let $\Lambda := \sum_{i=1}^{n} X_{\alpha_i}$, and $J := \mathfrak{z}_{\mathfrak{g}}(\Lambda)$. Then Λ is a regular nilpotent element (cf. [11, Theorem 5.3]), and J is called a Jordan Lie subalgebra of $\mathfrak{g}$.

We have the following proposition (see [3, Lemma 2.5] and [9, (1.6)]):

Proposition 2.2. *In the Grassmannian* $\mathrm{Gr}(n, \mathfrak{g})$,

$$J = \lim_{t\to 0} \exp t^{-1} \mathrm{ad}\Lambda(\mathfrak{h}) \in \overline{\mathrm{Ad}(G)\mathfrak{h}}.$$

For $\alpha \in \Delta^+$, let $\mathrm{ht}(\alpha)$ denote the height of α. Then the nilradical $\mathfrak{n}$ of $\mathfrak{b}$ is graded by ht:

$$\mathfrak{n} = \bigoplus_{j>0} \mathfrak{g}_j \qquad \mathfrak{g}_j := \bigoplus_{\mathrm{ht}(\alpha)=j} \mathfrak{g}_\alpha.$$

The Jordan Lie subalgebra $J = \mathfrak{z}_{\mathfrak{g}}(\Lambda)$ is also graded by ht:

$$J = \bigoplus_j J \cap \mathfrak{g}_j.$$

The set of heights appearing in J is exactly the same as that of exponents of $\mathfrak{g}$ counting multiplicities (cf. [11, Theorem 6.7]).

In the following classical examples, we take the subset of diagonal matrices and that of upper triangular matrices as $\mathfrak{h}$ and $\mathfrak{b}$, respectively, and let $\varepsilon_i \in \mathfrak{h}^*$ denote the linear form taking the (i,i)-component. We denote by $E_{i,j}$ the matrix whose entries are 0 except for the (i,j)-entry 1. Let γ_0 denote the maximal root. The Jordan Lie subalgebras J below can be computed as follows: First it is easy to check that Z and Λ^i belong to $\mathfrak{g}$ for the indicated powers i. It is also clear that they commute with Λ. Since we know the heights appearing in J ([11, Theorem 6.7] loc. cit.), we see that they form a basis of J.

Example 2.3. Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, and $X_{\alpha_i} := E_{i,i+1}$ $(1 \le i \le n)$. Then $\Lambda = \sum_{i=1}^{n} E_{i,i+1}$, and

$$J = \bigoplus_{i=1}^{n} \mathbb{C}\Lambda^i.$$

We have $\gamma_0 = \sum_{i=1}^{n} \alpha_i$ and $\mathrm{ht}(\gamma_0) = n$.

Example 2.4. Let $F := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$, and let

$$\begin{aligned} \mathfrak{g} := \mathfrak{so}(2n+1, \mathbb{C}) &= \left\{ X \in \mathfrak{sl}(2n+1) \,|\, {}^t XF + FX = O \right\} \\ &= \left\{ \begin{bmatrix} A & \boldsymbol{x} & B \\ -{}^t\boldsymbol{y} & 0 & -{}^t\boldsymbol{x} \\ C & \boldsymbol{y} & -A' \end{bmatrix} \,|\, B' = -B,\, C' = -C \right\}, \end{aligned}$$

where A, B, C are $n \times n$ matrices, $\boldsymbol{x}, \boldsymbol{y}$ are column vectors of dimension n, and

$$A' := (a_{n+1-j,n+1-i}) \text{ for an } n \times n \text{ matrix } A = (a_{i,j}). \tag{2.2}$$

The simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n.$$

Let $X_{\alpha_i} := E_{i,i+1} - E_{2n+1-i,2n+2-i}$ $(1 \leq i \leq n)$.

Then

$$\Lambda = \sum_{i=1}^{n} E_{i,i+1} - \sum_{i=n+1}^{2n} E_{i,i+1},$$

and

$$J = \bigoplus_{k=1}^{n} \mathbb{C}\Lambda^{2k-1}.$$

We have $\gamma_0 = \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\sum_{k=2}^{n} \alpha_k$ and $\mathrm{ht}(\gamma_0) = 2n-1$.

Example 2.5. Let $F := \begin{bmatrix} 0 & & & & & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & & \\ -1 & & & & & 0 \end{bmatrix}$, and let

$$\begin{aligned} \mathfrak{g} := \mathfrak{sp}(2n, \mathbb{C}) &= \left\{ X \in \mathfrak{sl}(2n) \,|\, {}^tXF + FX = O \right\} \\ &= \left\{ \begin{bmatrix} A & B \\ C & -A' \end{bmatrix} \,|\, B' = B,\, C' = C \right\}, \end{aligned}$$

where A, B, C are $n \times n$ matrices (cf. (2.2) for A' etc.).

The simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = 2\varepsilon_n.$$

Let $X_{\alpha_i} := E_{i,i+1} - E_{2n-i,2n+1-i}$ $(1 \leq i \leq n-1)$, and $X_{\alpha_n} := E_{n,n+1}$.

Then $\Lambda = \sum_{i=1}^{n} E_{i,i+1} - \sum_{i=n+1}^{2n-1} E_{i,i+1}$, and

$$J = \bigoplus_{k=1}^{n} \mathbb{C}\Lambda^{2k-1}.$$

We have $\gamma_0 = 2\varepsilon_1 = 2\sum_{i=1}^{n-1} \alpha_i + \alpha_n$ and $\mathrm{ht}(\gamma_0) = 2n-1$.

Example 2.6. Let $F := \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}$, and let

$$\begin{aligned} \mathfrak{g} := \mathfrak{so}(2n, \mathbb{C}) &= \left\{ X \in \mathfrak{sl}(2n) \,|\, {}^tXF + FX = O \right\} \\ &= \left\{ \begin{bmatrix} A & B \\ C & -A' \end{bmatrix} \,|\, B' = -B,\, C' = -C \right\}, \end{aligned}$$

where A, B, C are $n \times n$ matrices.

The simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n.$$

Let $X_{\alpha_i} := E_{i,i+1} - E_{2n-i,2n+1-i}$ $(1 \leq i \leq n-1)$ and $X_{\alpha_n} := E_{n-1,n+1} - E_{n,n+2}$. Then

$$\Lambda \;=\; \sum_{i=1}^{n-1} E_{i,i+1} - \sum_{i=n+1}^{2n-1} E_{i,i+1} + E_{n-1,n+1} - E_{n,n+2},$$

and

$$J = \mathbb{C}Z \oplus \bigoplus_{k=1}^{n-1} \mathbb{C}\Lambda^{2k-1},$$

where $Z = E_{1,n} - E_{n+1,2n} - E_{1,n+1} + E_{n,2n}$.

The height of Λ^{2k-1} equals $2k-1$, and that of Z is $n-1$. We have $\gamma_0 = \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\sum_{k=1}^{n-2} \alpha_k + \alpha_{n-1} + \alpha_n$ and $\mathrm{ht}(\gamma_0) = 2n-3$.

2.3. **Main Theorem.** The following is the main theorem of this paper. The proof is given by a type-by-type consideration. Proposition 2.2 leads to the latter half of the statements in Theorem 2.7.

Theorem 2.7. *Let* $\mathfrak{a}$ *be an* n*-dimensional abelian ideal of* $\mathfrak{b}$. *Then* $\mathfrak{a} \in \overline{\mathrm{Ad}(G)J}$ *in* $\mathrm{Gr}(n, \mathfrak{g})$. *Hence* $\mathfrak{a} \in \overline{\mathrm{Ad}(G)\mathfrak{h}}$.

Corollary 2.8. *The subspace* C_n *of* $\wedge^n \mathfrak{g}$ *is spanned by any of the following:*

(1) $\{\wedge^n \mathfrak{a} \,|\, \mathfrak{a} \in \mathrm{Ad}(G)J\}$,

(2) $\{\wedge^n \mathfrak{a} \,|\, \mathfrak{a} \in \overline{\mathrm{Ad}(G)J}\}$,

(3) $\{\wedge^n \mathfrak{a} \,|\, \mathfrak{a} \in \mathrm{Ad}(G)\mathfrak{h}\}$,

(4) $\{\wedge^n \mathfrak{a} \,|\, \mathfrak{a} \in \overline{\mathrm{Ad}(G)\mathfrak{h}}\}$.

Proof. The subspace spanned by any set from (1) to (4) is a G-submodule of C_n, and thus a $\mathfrak{g}$-submodule. Hence (2) and (4) are clear from Theorems 2.1 and 2.7. Let C'_n be the subspace spanned by $\{\wedge^n \mathfrak{a} \,|\, \mathfrak{a} \in \mathrm{Ad}(G)J\}$. Since this is closed and includes $\{\wedge^n \mathfrak{a} \,|\, \mathfrak{a} \in \mathrm{Ad}(G)J\}$, we see

$$C'_n \supseteq \{\wedge^n \mathfrak{a} \,|\, \mathfrak{a} \in \overline{\mathrm{Ad}(G)J}\}.$$

Hence we obtain (1) from (2). Similarly we obtain (3). ∎

We use the following lemma very often throughout this paper.

Lemma 2.9. *Let* $\mathbb{C}^\times \ni t \mapsto \mathfrak{a}_t \in \mathrm{Gr}(n, \mathfrak{g})$ *and* $\mathbb{C}^\times \ni t \mapsto A_t \in \mathfrak{g} \setminus \{0\}$ *be morphisms. Suppose that* $A_t \in \mathfrak{a}_t$ *for all* $t \in \mathbb{C}^\times$, *and* $A := \lim_{t \to 0} A_t$ *exists in* $\mathfrak{g} \setminus \{0\}$. *Then* $A \in \lim_{t \to 0} \mathfrak{a}_t$.

Proof. Consider morphisms

$$\begin{aligned} P &: \mathfrak{g} \times (\mathfrak{g})^n \ni (Y, [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]) \mapsto Y \wedge \boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_n \in \bigwedge^{n+1} \mathfrak{g}, \\ P' &: (\mathfrak{g})^n \ni [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n] \mapsto \boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_n \in \bigwedge^{n} \mathfrak{g}. \end{aligned}$$

Then $P^{-1}(0)$ (respectively, $P'^{-1}(0)$) is closed and $\mathbb{C}^\times \times GL(n)$-stable (respectively, $GL(n)$-stable). Hence $P^{-1}(0) \cap ((\mathfrak{g} \setminus \{0\}) \times (\mathfrak{g}^n \setminus P'^{-1}(0)))$ is closed in $(\mathfrak{g} \setminus \{0\}) \times (\mathfrak{g}^n \setminus P'^{-1}(0))$ and $\mathbb{C}^\times \times GL(n)$-stable.

Thus its image

$$\{(\mathbb{C}Y, \mathfrak{a}) \mid Y \in \mathfrak{a}\}$$

under the canonical morphism is closed in $\mathbb{P}(\mathfrak{g}) \times \mathrm{Gr}(n, \mathfrak{g})$. Hence $(\mathbb{C}A, \lim_{t\to 0} \mathfrak{a}_t) = \lim_{t\to 0}(\mathbb{C}A_t, \mathfrak{a}_t)$ belongs to $\{(\mathbb{C}Y, \mathfrak{a}) \mid Y \in \mathfrak{a}\}$, i.e., $A \in \lim_{t\to 0} \mathfrak{a}_t$. ∎

We close this section with the following small example of Proposition 5.3 and Theorem 2.7:

Example 2.10. Let $\mathfrak{g} = \mathfrak{sp}(6, \mathbb{C})$. As in Example 2.5, let

$$J = \left\{ A = a\Lambda + b\Lambda^3 + c\Lambda^5 = \begin{bmatrix} 0 & a & 0 & b & 0 & c \\ 0 & 0 & a & 0 & -b & 0 \\ 0 & 0 & 0 & a & 0 & b \\ 0 & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

There are two 3-dimensional abelian ideals of the upper triangular Borel subalgebra:

$$\mathfrak{a}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & b & a & c \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}, \quad \mathfrak{a}_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & a & c \\ 0 & 0 & 0 & 0 & b & a \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

We have

$$[E_{25}, A] = \begin{bmatrix} 0 & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$\begin{aligned} &\exp(t^{-1}\mathrm{ad}E_{25})(t\Lambda) = t\Lambda - (E_{15} + E_{26}), \\ &\exp(t^{-1}\mathrm{ad}E_{25})(\Lambda^3) = \Lambda^3, \\ &\exp(t^{-1}\mathrm{ad}E_{25})(\Lambda^5) = \Lambda^5. \end{aligned}$$

By Lemma 2.9

$$\lim_{t\to 0}\exp(t^{-1}\mathrm{ad}E_{25})(J)=\left\{\begin{bmatrix}0&0&0&b&-a&c\\0&0&0&0&-b&-a\\0&0&0&0&0&b\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}\right\}=:K.$$

Let

$$d_1(t):=\mathrm{diag}(1,t,1,1,t^{-1},1),\quad d_2(t):=\mathrm{diag}(t,1,1,1,1,t^{-1}).$$

Then by Lemma 2.9

$$\lim_{t\to 0}\mathrm{Ad}(d_i(t))(K)=\mathfrak{a}_i\qquad (i=1,2).$$

Hence $\mathfrak{a}_1,\mathfrak{a}_2$ are contained in $\overline{\mathrm{Ad}(G)K}\subseteq\overline{\mathrm{Ad}(G)J}$.

3. Basic deformations

In this section, $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{b}$, and we suppose that,

$$\text{if } \alpha\in\Delta^+ \text{ and } \mathfrak{g}_\alpha\subseteq\mathfrak{a}, \text{ then } \mathfrak{g}_{\alpha+\beta}\subseteq\mathfrak{a} \text{ for all } \beta\in\Delta^+. \tag{3.1}$$

By (2.1), abelian ideals of $\mathfrak{b}$ satisfy (3.1).

We prepare two deformations: a unipotent deformation and a semisimple deformation, which are used many times in this paper.

Lemma 3.1. *Let $\beta\in\Delta^+$.*

1. *If $\alpha\in\Delta^+$ and $\mathfrak{g}_\alpha\subseteq\mathfrak{a}$, then*

$$\mathfrak{g}_\alpha\subseteq\lim_{t\to 0}\exp(t^{-1}\mathrm{ad}X_\beta)(\mathfrak{a}).$$

2. *Let $\Gamma\in\mathfrak{a}$. If $(\mathrm{ad}X_\beta)^i(\mathbb{C}\Gamma)\not\subseteq\mathfrak{a}$ and $(\mathrm{ad}X_\beta)^j(\mathbb{C}\Gamma)\subseteq\mathfrak{a}$ for all $j>i$, then*

$$(\mathrm{ad}X_\beta)^i(\mathbb{C}\Gamma)\subseteq\lim_{t\to 0}\exp(t^{-1}\mathrm{ad}X_\beta)(\mathfrak{a}).$$

Proof. (1) By the assumption (3.1), $(\mathrm{ad}X_\beta)^i(X_\alpha)\in\mathfrak{a}$ for all i. Suppose that k is the maximal i with $(\mathrm{ad}X_\beta)^i(X_\alpha)\neq 0$. Then

$$\exp(-t^{-1}\mathrm{ad}X_\beta)(X_\alpha)=\sum_{i=0}^{k}\frac{1}{i!}(-t^{-1}\mathrm{ad}X_\beta)^i(X_\alpha)\in\mathfrak{a}$$

for all $t\neq 0$. Since

$$X_\alpha=\exp(t^{-1}\mathrm{ad}X_\beta)\exp(-t^{-1}\mathrm{ad}X_\beta)(X_\alpha)\in\exp(t^{-1}\mathrm{ad}X_\beta)(\mathfrak{a})$$

for all $t \neq 0$, we have by Lemma 2.9

$$X_\alpha \in \lim_{t \to 0} \exp(t^{-1} \mathrm{ad} X_\beta)(\mathfrak{a}).$$

(2) Suppose that k is the maximal j with $(\mathrm{ad} X_\beta)^j(\Gamma) \neq 0$. We inductively define Laurent polynomials $a_j(t) \in \mathbb{C}[t, t^{-1}]$ by

$$\begin{aligned} a_0(t) &= a_1(t) = \cdots = a_i(t) = 0 \\ a_j(t) &= \frac{1}{j!} t^{-j} - \sum_{q=i+1}^{j-1} \frac{1}{(j-q)!} t^{q-j} a_q(t) \qquad (i+1 \leq j \leq k). \end{aligned}$$

Then

$$\begin{aligned} & \exp(t^{-1} \mathrm{ad} X_\beta)(\sum_{q=i+1}^{k} a_q(t)(\mathrm{ad} X_\beta)^q(\Gamma)) \\ = & \sum_{p,q} \frac{1}{p!} t^{-p} a_q(t)(\mathrm{ad} X_\beta)^{p+q}(\Gamma) \\ = & \sum_{j=i+1}^{k} \sum_{q=i+1}^{j} \frac{1}{(j-q)!} t^{-(j-q)} a_q(t)(\mathrm{ad} X_\beta)^j(\Gamma) \\ = & \sum_{j=i+1}^{k} \frac{1}{j!} t^{-j} (\mathrm{ad} X_\beta)^j(\Gamma). \end{aligned}$$

We have

$$\begin{aligned} & \lim_{t \to 0} \exp(t^{-1} \mathrm{ad} X_\beta)(t^i(\Gamma - \sum_{q=i+1}^{k} a_q(t)(\mathrm{ad} X_\beta)^q(\Gamma))) \\ = & \lim_{t \to 0} (\frac{1}{i!} (\mathrm{ad} X_\beta)^i(\Gamma) + o(1)) = \frac{1}{i!} (\mathrm{ad} X_\beta)^i(\Gamma). \end{aligned}$$

Hence by Lemma 2.9

$$(\mathrm{ad} X_\beta)^i(\mathbb{C}\Gamma) \subseteq \lim_{t \to 0} \exp(t^{-1} \mathrm{ad} X_\beta)(\mathfrak{a}). \qquad \blacksquare$$

Let H be the maximal torus of G with Lie algebra $\mathfrak{h}$. Let $\chi_1, \ldots, \chi_n$ be the characters of H corresponding to the simple roots $\alpha_1, \ldots, \alpha_n$, respectively. Let $\lambda_1, \ldots, \lambda_n$ be the 1-parameter subgroups of H such that $\chi_j(\lambda_i(t)) = t^{\delta_{ij}}$. For $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $\alpha = \sum_{j=1}^n d_j \alpha_j \in \Delta^+$, set

$$(\boldsymbol{m}, \alpha) = \sum_{j=1}^{n} m_j d_j. \tag{3.2}$$

Hence

$$\chi_\alpha(\prod_{j=1}^{n} \lambda_j^{m_j}(t)) = t^{(\boldsymbol{m}, \alpha)},$$

where χ_α is the character of H corresponding to α.

Lemma 3.2. *Let* $\Gamma = \sum_{\alpha \in \Delta^+} a_\alpha X_\alpha \in \mathfrak{a}$. *Suppose that* $\mathfrak{g}_\alpha \subseteq \mathfrak{a}$ *if* $(\boldsymbol{m}, \alpha) < c$ *and* $a_\alpha \neq 0$. *Then*

$$\sum_{(\boldsymbol{m},\alpha)=c} a_\alpha X_\alpha \in \lim_{t\to 0} \mathrm{Ad}(\prod_{j=1}^{n} \lambda_j^{m_j}(t))(\mathfrak{a}).$$

Proof. We have

$$\mathrm{Ad}(\prod_{j=1}^{n} \lambda_j^{m_j}(t))(\Gamma) = \sum_{\alpha\in\Delta^+} a_\alpha t^{(\boldsymbol{m},\alpha)} X_\alpha.$$

Hence

$$\mathrm{Ad}(\prod_{j=1}^{n} \lambda_j^{m_j}(t))(t^{-c}(\Gamma - \sum_{(\boldsymbol{m},\alpha)<c} a_\alpha X_\alpha)) = \sum_{(\boldsymbol{m},\alpha)=c} a_\alpha X_\alpha + o(1).$$

By Lemma 2.9, we have

$$\sum_{(\boldsymbol{m},\alpha)=c} a_\alpha X_\alpha \in \lim_{t\to 0} \mathrm{Ad}(\prod_{j=1}^{n} \lambda_j^{m_j}(t))(\mathfrak{a}).$$

■

4. The case $\mathfrak{sl}(n+1, \mathbb{C})$

In this section, let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, and $\mathfrak{b}$ the Lie subalgebra of upper triangular matrices.

Let $\mathfrak{a}$ be an n-dimensional abelian ideal of $\mathfrak{b}$. Then by (2.1) there exists a Young diagram $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l)$ with $|\mu| := \sum_{i=1}^{l} \mu_i = n$ $(\mu \vdash n)$ such that

$$\mathfrak{a} = \mathfrak{a}_\mu := \bigoplus_{k=1}^{l} \bigoplus_{j=1}^{\mu_k} \mathbb{C} E_{j,\, n-k+2}.$$

Example 4.1. Let $\mu = (\mu_1 \geq \mu_2 \geq \mu_3) = (4, 4, 1)$ and $n = 9$. Then the weight spaces of $\mathfrak{a}_\mu$ are the following places:

1				●	●	●
2					●	●
3					●	●
4					●	●
	5	6	7	8	9	10

in the upper right block of size $\mu_1 \times (n + 1 - \mu_1) = 4 \times 6$ of a square matrix of degree $n + 1 = 10$.

Besides $\mathfrak{a}_\mu$, we define another abelian Lie subalgebra $\mathfrak{a}'_\mu$ in the upper right block of size $\mu_1 \times (n + 1 - \mu_1)$ by

$$\mathfrak{a}'_\mu := \bigoplus_{k=1}^{l} \bigoplus_{j=1}^{\mu_k} \mathbb{C} E_{\mu_1+1-j,\, \mu_1+\sum_{i>k}\mu_i+1}.$$

Example 4.2. Let $\mu = (\mu_1 \geq \mu_2 \geq \mu_3) = (4,4,1)$ and $n = 9$. Then the weight spaces of $\mathfrak{a}'_\mu$ are the following places:

1		●				●
2		●				●
3		●				●
4	●	●				●
	5	6	7	8	9	10

.

Remark 4.3. $\mathfrak{a}'_\mu$ and $\mathfrak{a}_\mu$ are conjugate to each other by

$$\begin{bmatrix} P_\sigma & O \\ O & P_\tau \end{bmatrix},$$

where P_σ and P_τ are respectively the permutation matrices corresponding to $\sigma \in S_{[1,\mu_1]}$ and any $\tau \in S_{[\mu_1+1,n+1]}$ such that

$$\begin{cases} \sigma(i) = \mu_1 + 1 - i & (i = 1,2,\ldots,\mu_1) \\ \tau(\mu_1 + \sum_{i>k}\mu_i + 1) = n - k + 2 & (k = 1,2,\ldots,l). \end{cases}$$

Hence the statements $\mathfrak{a}_\mu \in \overline{\mathrm{Ad}(G)J}$ and $\mathfrak{a}'_\mu \in \overline{\mathrm{Ad}(G)J}$ are equivalent.

Lemma 4.4. *For each* $h = 1,2,\ldots,n$, *there exists a unique* $i(h)$ *such that* $E_{i(h),i(h)+h} \in \mathfrak{a}'_\mu$. *Explicitly,*

$$i(h) = \mu_1 + 1 - (h - \sum_{i>k}\mu_i), \tag{4.1}$$

or equivalently

$$i(h) + h = \mu_1 + \sum_{i>k}\mu_i + 1 \tag{4.2}$$

with k *satisfying* $\sum_{i>k}\mu_i < h \leq \sum_{i\geq k}\mu_i$. *Note that* $i(h) \leq \mu_1 < i(h) + h$. *We have*

$$\mathfrak{a}'_\mu = \bigoplus_{h=1}^{n} \mathbb{C}E_{i(h),i(h)+h}.$$

Proof. A weight of $\mathfrak{a}'_\mu$ corresponds to a place $(\mu_1 + 1 - j, \mu_1 + \sum_{i>k}\mu_i + 1)$. Then its difference of components equals

$$(\mu_1 + \sum_{i>k}\mu_i + 1) - (\mu_1 + 1 - j) = \sum_{i>k}\mu_i + j.$$

As j runs over $[1,\mu_k]$, they are all different, and they cover $\{1,2,\ldots,n\}$.

When $h = \sum_{i>k}\mu_i + j$, we have

$$i(h) = \mu_1 + 1 - j = \mu_1 + 1 - (h - \sum_{i>k}\mu_i). \qquad \blacksquare$$

For a vector $(z_1, z_2, \ldots, z_n) \in \mathbb{Q}^n$ and $h, j = 1, 2, \ldots, n$ with $j + h \leq n + 1$, put

$$z_j(h) := \sum_{i=j}^{j+h-1} z_i.$$

For $\mu \vdash n$, we consider the following system of inequalities:

$$\begin{cases} z_{i(h)}(h) < z_j(h) & (1 \leq h \leq n,\ j \leq n+1-h,\ j \neq i(h)), \\ z_i > 0 & (1 \leq i \leq n,\ i \neq \mu_1), \\ z_{\mu_1} = 0. \end{cases} \qquad (IE_\mu)$$

We give a proof of the following proposition in Appendix A:

Proposition 4.5. *For any $\mu \vdash n$, there exists a solution of the system (IE_μ) in $\mathbb{Z}^n$.*

Let $z = (z_1, \ldots, z_n) \in \mathbb{Z}^n$ be a solution of the system (IE_μ). Since $(n+1)z = ((n+1)z_1, \ldots, (n+1)z_n)$ also satisfies (IE_μ), we may assume $z_j \in (n+1)\mathbb{Z}$ for all j.

Define $w = (w_1, \ldots, w_{n+1}) \in \mathbb{Z}^{n+1}$ by

$$w_j := \frac{\sum_{k=j}^{n}(n+1-k)z_k - \sum_{k=1}^{j-1} k z_k}{n+1} \qquad (j = 1, \ldots, n+1). \tag{4.3}$$

Then $\sum_{j=1}^{n+1} w_j = 0$ and

$$w_j - w_{j+h} = \sum_{k=j}^{j+h-1} z_k = z_j(h). \tag{4.4}$$

Proposition 4.6. *Let $w \in \mathbb{Z}^{n+1}$ be the one defined in* (4.3)*, and let $t^w := \operatorname{diag}(t^{w_1}, t^{w_2}, \ldots, t^{w_{n+1}}) \in SL(n+1, \mathbb{C})$. Then*

$$\lim_{t \to 0} \operatorname{Ad}(t^w) J = \mathfrak{a}'_\mu.$$

Proof. Recall that $\Lambda := \sum_{i=1}^{n} E_{i,i+1}$, $\Lambda^h = \sum_{i=1}^{n+1-h} E_{i,i+h}$, and $J = \bigoplus_{h=1}^{n} \mathbb{C}\Lambda^h$. We have

$$\lim_{t \to 0} \operatorname{Ad}(t^w) t^{-z_{i(h)}(h)} \Lambda^h = \lim_{t \to 0} \sum_{j=1}^{n+1-h} t^{z_j(h) - z_{i(h)}(h)} E_{j,j+h} = E_{i(h),i(h)+h}.$$

Hence by Lemma 2.9 $\lim_{t\to 0} \operatorname{Ad}(t^w) J = \mathfrak{a}'_\mu$. ■

Proof of Theorem 2.7. For an n-dimensional abelian ideal $\mathfrak{a}$ of the Lie algebra of upper triangular matrices in $\mathfrak{sl}(n+1, \mathbb{C})$, there exists $\mu \vdash n$ such that $\mathfrak{a} = \mathfrak{a}_\mu$. By Remark 4.3 and Proposition 4.6, $\mathfrak{a}_\mu \in \overline{\operatorname{Ad}(G) J}$. ■

5. Main theorem for Types B, C, D

Let $\mathfrak{g}$ be a simple Lie algebra of Type B, C, or D. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots in Δ^+; we follow Bourbaki's notation [2]:

$$\begin{aligned}
(B_n) \qquad & \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (i < n), \quad \alpha_n = \varepsilon_n, \\
& \Delta^+ = \{\varepsilon_i, \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \,|\, i < j\}; \\
(C_n) \qquad & \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (i < n), \quad \alpha_n = 2\varepsilon_n, \\
& \Delta^+ = \{2\varepsilon_i, \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \,|\, i < j\}; \\
(D_n) \qquad & \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (i < n), \quad \alpha_n = \varepsilon_{n-1} + \varepsilon_n, \\
& \Delta^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \,|\, i < j\}.
\end{aligned}$$

In Examples 2.4, 2.5, and 2.6, we have seen the root vectors X_{α_i} for simple roots α_i, the regular nilpotent elements $\Lambda = \sum_{i=1}^{n} X_{\alpha_i}$, the Jordan Lie subalgebras $J = \mathfrak{z}_{\mathfrak{g}}(\Lambda)$, the maximal roots γ_0, and their heights.

We recall the diagram presentation of sets of positive roots of Type B, C, D, the left-right reversal of that in [4]. We assign a positive root to the 'place of the corresponding root vector' in the matrix presentation of Examples 2.4, 2.5, 2.6. More precisely, let T be the shifted filling of shape

$$(2n-1, 2n-3, \ldots, 1), (2n-1, 2n-3, \ldots, 1), (2n-2, 2n-4, \ldots, 2)$$

for Types B_n, C_n, D_n, respectively, with assigning the following t_{ij} to the place (i, j):

$$(B_n) \qquad t_{ij} = \begin{cases} \varepsilon_i + \varepsilon_j = \alpha_i + \cdots + 2(\alpha_j + \cdots + \alpha_n) \quad (i < j \leq n) \\ \varepsilon_i = \alpha_i + \cdots + \alpha_n \quad (j = n+1) \\ \varepsilon_i - \varepsilon_{2n-j+1} \\ \qquad = \alpha_i + \cdots + \alpha_{2n-j} \quad (n+1 < j \leq 2n-i), \end{cases}$$

$$(C_n) \qquad t_{ij} = \begin{cases} \varepsilon_i + \varepsilon_j \\ \qquad = \alpha_i + \cdots + 2(\alpha_j + \cdots + \alpha_{n-1}) + \alpha_n \quad (i \leq j \leq n) \\ \varepsilon_i - \varepsilon_{2n-j+1} \\ \qquad = \alpha_i + \cdots + \alpha_{2n-j} \quad (n+1 \leq j \leq 2n-i), \end{cases}$$

$$(D_n) \qquad t_{ij} = \begin{cases} \varepsilon_i + \varepsilon_j = \alpha_i + \cdots \\ \qquad +2(\alpha_j + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n \quad (i < j \leq n) \\ \varepsilon_i + \varepsilon_n = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n \quad (j = n) \\ \varepsilon_i - \varepsilon_{2n-j+1} = \alpha_i + \cdots + \alpha_{2n-j} \quad (n+1 \leq j \leq 2n-i), \end{cases}$$

where $1 \leq i \leq n$ for Types B_n, C_n and $1 \leq i \leq n-1$ for Type D_n.

For example, the diagram presentation of Type B_3 is as follows:

6	5	4	3	2	j/i
α_1	$\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2 + \alpha_3$	$\alpha_1 + \alpha_2 + 2\alpha_3$	$\alpha_1 + 2\alpha_2 + 2\alpha_3$	1
	α_2	$\alpha_2 + \alpha_3$	$\alpha_2 + 2\alpha_3$		2
		α_3			3

,

or

6	5	4	3	2	j/i
$\varepsilon_1 - \varepsilon_2$	$\varepsilon_1 - \varepsilon_3$	ε_1	$\varepsilon_1 + \varepsilon_3$	$\varepsilon_1 + \varepsilon_2$	1
	$\varepsilon_2 - \varepsilon_3$	ε_2	$\varepsilon_2 + \varepsilon_3$		2
		ε_3			3

.

The upper right corner of an element $t_{ij} \in T$ is defined as the set $\{t_{hk} \in T \,|\, h \leq i, k \leq j\}$. A set of boxes A of T is said to be a subdiagram of T if A contains the upper right corner of its elements.

When the type is of D_n, for $\Phi \subseteq T$, we denote by $\Phi^\bullet$ the set obtained from Φ by exchanging columns $n, n+1$.

Let $\mathfrak{b}$ be the Borel subalgebra corresponding to Δ^+.

Lemma 5.1 (Theorem 3.1 in [4]). *Let Φ be a subset of Δ^+. Then $\Phi = \Delta(\mathfrak{a})$ for an ideal $\mathfrak{a}$ of $\mathfrak{b}$ if and only if Φ is a subdiagram of T in case of B_n or C_n, and either Φ or $\Phi^\bullet$ is a subdiagram of T in case of D_n.*

Let $\mathfrak{a}$ be an n-dimensional abelian ideal of $\mathfrak{b}$. Recall that $\mathfrak{a}$ satisfies (2.1).

Lemma 5.2. *The set $\Delta(\mathfrak{a})$ consists of roots of form $\varepsilon_i + \varepsilon_j$ except*

1. $\Delta(\mathfrak{a}) = \{\varepsilon_1, \varepsilon_1 + \varepsilon_j \,|\, j \geq 2\}$ *in Type B_n,*
2. $\Delta(\mathfrak{a}) = \{\varepsilon_1 - \varepsilon_n, \varepsilon_1 + \varepsilon_j \,|\, j \geq 2\}$ *in Type D_n,*
3. $\Delta(\mathfrak{a}) = \{\varepsilon_2 + \varepsilon_3, \varepsilon_1 - \varepsilon_n, \varepsilon_1 + \varepsilon_j \,|\, n > j \geq 2\}$ *in Type D_n.*

Proof. This is easily checked by Lemma 5.1. ■

We prove the following proposition in Appendix.

Proposition 5.3. *There exist a nilpotent element $S \in \mathfrak{g}$ and an abelian Lie subalgebra $K \in \overline{\mathrm{Ad}(G)J}$ with a basis $\{\Lambda^{(i)} \,|\, \mathrm{ht}(\gamma_0) - (n-1) \leq i \leq \mathrm{ht}(\gamma_0)\}$ in case of Type B_n or C_n, and with a basis $\{\Lambda^{(i)} \,|\, \mathrm{ht}(\gamma_0) - (n-2) \leq i \leq \mathrm{ht}(\gamma_0)\} \cup \{Z\}$ in case of Type D_n, such that $K = \lim_{t\to 0} \exp(t^{-1}\mathrm{ad}S)(J)$, where $\Lambda^{(i)}$ is of the following form:*

$$\Lambda^{(i)} = \sum_{\mathrm{ht}(\alpha)=i} c_\alpha X_\alpha \qquad (c_\alpha \neq 0 \textit{ for any } \alpha).$$

Furthermore, in the case of Type D_n, we can take $\Lambda^{(n-1)}$ so that Z and $c_{\varepsilon_1-\varepsilon_n} X_{\varepsilon_1-\varepsilon_n} + c_{\varepsilon_1+\varepsilon_n} X_{\varepsilon_1+\varepsilon_n}$ are linearly independent.

By Proposition 5.3, to prove Theorem 2.7, it is sufficient to prove

$$\mathfrak{a} \in \overline{\mathrm{Ad}(G)K}. \tag{5.1}$$

We first consider the exceptional cases appearing in Lemma 5.2.

Proposition 5.4. *Let $\Delta(\mathfrak{a})$ be one of the following:*

1. $\{\varepsilon_1, \varepsilon_1 + \varepsilon_j \,|\, j \geq 2\}$ *in Type B_n,*
2. $\{\varepsilon_1 - \varepsilon_n, \varepsilon_1 + \varepsilon_j \,|\, j \geq 2\}$ *in Type D_n,*
3. $\{\varepsilon_2 + \varepsilon_3, \varepsilon_1 - \varepsilon_n, \varepsilon_1 + \varepsilon_j \,|\, n > j \geq 2\}$ *in Type D_n $(n \geq 5)$.*
4. *Any in Type D_4.*

Then we have (5.1), and hence Theorem 2.7 holds.

Proof. (1) The heights of K are greater than or equal to n (cf. Proposition 5.3). For a root α of height $\geq n$, the coefficient of α_1 is 1 or 0, and 1 exactly when $\alpha = \varepsilon_1, \varepsilon_1 + \varepsilon_j$ $(j \geq 2)$. In other words, for a root α of height $\geq n$,

$$\alpha(\lambda_1^{-1}(t)) = \begin{cases} t^{-1} & (\alpha = \varepsilon_1, \varepsilon_1 + \varepsilon_j \quad (j \geq 2)) \\ 1 & (\text{otherwise}). \end{cases}$$

Hence by Lemma 3.2, we see

$$\lim_{t \to 0} \mathrm{Ad}(\lambda_1^{-1}(t))(K) = \mathfrak{a}.$$

(2) Let

$$X_{\varepsilon_i - \varepsilon_j} := E_{i,j} - E_{2n+1-j,2n+1-i}, \quad X_{\varepsilon_i + \varepsilon_j} := E_{i,2n+1-j} - E_{j,2n+1-i}$$

for $i < j$. Similarly to the proof of (1), by Lemma 3.2,

$$\begin{aligned} & \lim_{t \to 0} \mathrm{Ad}(\lambda_1^{-1}(t))(K) \\ = \ & \langle Z, c_{\varepsilon_1 - \varepsilon_n} X_{\varepsilon_1 - \varepsilon_n} + c_{\varepsilon_1 + \varepsilon_n} X_{\varepsilon_1 + \varepsilon_n}, X_{\varepsilon_1 + \varepsilon_{n-1}}, \ldots, X_{\varepsilon_1 + \varepsilon_2} \rangle \\ = \ & \mathfrak{a}. \end{aligned}$$

Here $\langle A_1, \ldots, A_k \rangle$ means the $\mathbb{C}$-vector space spanned by $A_1, \ldots, A_k$, and the last equation holds by the latter half of Proposition 5.3.

(3) Let $\beta := \alpha_4 + \cdots + \alpha_{n-2} + \alpha_n$ if $n \geq 6$, and $\beta := \alpha_5$ if $n = 5$. Then $\mathrm{ht}(\beta) = n - 4$, and $\varepsilon_1 + \varepsilon_4 = \beta + (\alpha_1 + \cdots + \alpha_{n-2} + \alpha_{n-1})$. Since $\gamma := \alpha_1 + \cdots + \alpha_{n-2} + \alpha_{n-1}$ is the unique root of height $n-1$ such that $\beta + \gamma$ is a root, $\mathrm{ad}(X_\beta)(\Lambda^{(n-1)})$ and $\mathrm{ad}(X_\beta)(Z)$ are nonzero multiples of $X_{\varepsilon_1 + \varepsilon_4}$. Since no root of height n or $n+1$ remains as a root after added by β, we have $\mathrm{ad}(X_\beta)(\Lambda^{(n)}) = 0$ and $\mathrm{ad}(X_\beta)(\Lambda^{(n+1)}) = 0$. By $\mathrm{ht}(\beta) = n - 4$, $\mathrm{ad}(X_\beta)(\Lambda^{(n+j)}) = 0$ for $j \geq 2$. By Lemma 3.1

$$\begin{aligned} \lim_{t \to 0} \exp(t^{-1} \mathrm{ad} X_\beta)(K) &= \langle X_{\varepsilon_1 + \varepsilon_4}, \Lambda^{(n-1)}, \Lambda^{(n)}, \ldots, \Lambda^{(2n-3)} \rangle \\ &= \langle X_{\varepsilon_2 + \varepsilon_3}, \Lambda^{(n-1)}, \Lambda^{(n)}, \ldots, \Lambda^{(2n-3)} \rangle =: \mathfrak{a}_1. \end{aligned}$$

Here the last equation holds since $\Lambda^{(2n-5)} = c_{\varepsilon_1 + \varepsilon_4} X_{\varepsilon_1 + \varepsilon_4} + c_{\varepsilon_2 + \varepsilon_3} X_{\varepsilon_2 + \varepsilon_3}$. Again similarly to the proof of (1), by Lemma 3.2,

$$\begin{aligned} & \lim_{t \to 0} \mathrm{Ad}(\lambda_1^{-1}(t))(\mathfrak{a}_1) \\ = \ & \langle X_{\varepsilon_2 + \varepsilon_3}, c_{\varepsilon_1 - \varepsilon_n} X_{\varepsilon_1 - \varepsilon_n} + c_{\varepsilon_1 + \varepsilon_n} X_{\varepsilon_1 + \varepsilon_n}, X_{\varepsilon_1 + \varepsilon_{n-1}}, \ldots, X_{\varepsilon_1 + \varepsilon_2} \rangle \\ =: \ & \mathfrak{a}_2. \end{aligned}$$

Finally,

$$\lim_{t \to 0} \mathrm{Ad}(\lambda_n(t))(\mathfrak{a}_2) = \langle X_{\varepsilon_2 + \varepsilon_3}, X_{\varepsilon_1 - \varepsilon_n}, X_{\varepsilon_1 + \varepsilon_{n-1}}, \ldots, X_{\varepsilon_1 + \varepsilon_2} \rangle = \mathfrak{a}.$$

(4) Let $\mathfrak{g}$ be of type D_4. The filling T is the following:

7	6	5	4	3	2	j/i
$\varepsilon_1-\varepsilon_2$	$\varepsilon_1-\varepsilon_3$	$\varepsilon_1-\varepsilon_4$	$\varepsilon_1+\varepsilon_4$	$\varepsilon_1+\varepsilon_3$	$\varepsilon_1+\varepsilon_2$	1
	$\varepsilon_2-\varepsilon_3$	$\varepsilon_2-\varepsilon_4$	$\varepsilon_2+\varepsilon_4$	$\varepsilon_2+\varepsilon_3$		2
		$\varepsilon_3-\varepsilon_4$	$\varepsilon_3+\varepsilon_4$			3

.

By Lemma 5.1, there exist the following three cases:

(i) $\Delta(\mathfrak{a})=\{\varepsilon_1-\varepsilon_4,\varepsilon_1+\varepsilon_j\,|\,j=2,3,4\}$,

(ii) $\Delta(\mathfrak{a}')=\{\varepsilon_2+\varepsilon_3,\varepsilon_1+\varepsilon_j\,|\,j=2,3,4\}$,

(iii) $\Delta(\mathfrak{a}'')=\{\varepsilon_2+\varepsilon_3,\varepsilon_1-\varepsilon_4,\varepsilon_1+\varepsilon_j\,|\,j=2,3\}$.

The case (i) is included in (2). Similarly to the case (i), we have

$$\lim_{t\to 0}\mathrm{Ad}(\lambda_4^{-1}(t))(K)=\mathfrak{a}',\qquad \lim_{t\to 0}\mathrm{Ad}(\lambda_3^{-1}(t))(K)=\mathfrak{a}''.$$ ■

In the rest of this section, we fix an abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$ such that $\Delta(\mathfrak{a})$ is none of the ones in Proposition 5.4. For $\beta=\varepsilon_i+\varepsilon_j\in\Delta(\mathfrak{a})$ $(i\le j)$, put

$$i=i(\beta)\qquad\text{and}\qquad j=j(\beta). \tag{5.2}$$

Hence in particular

$$i(\beta)\le j(\beta).$$

Set

$$Y:=Y(\mathfrak{a}):=\{(i(\alpha),j(\alpha))\,|\,\alpha\in\Delta(\mathfrak{a})\}.$$

We sometimes identify $Y=Y(\mathfrak{a})$ with $\Delta(\mathfrak{a})$. For $\alpha,\beta\in\Delta(\mathfrak{a})$, note that $\alpha\le\beta$ if and only if β belongs to the upper right corner of α, or

$$\alpha\le\beta\Leftrightarrow i(\alpha)\ge i(\beta),\quad j(\alpha)\ge j(\beta). \tag{5.3}$$

Here recall that $\alpha\le\beta$ means $\beta-\alpha\in\mathbb{N}\Delta^+$. Hence Y is a subdiagram of T, of size n.

To prove (5.1) (or Theorem 2.7), we define a sequence of abelian Lie subalgebras $K=\mathfrak{a}_1,\mathfrak{a}_2,\dots,\mathfrak{a}_{n+1}=\mathfrak{a}$ of $\mathfrak{b}$ such that

$$\mathfrak{a}_{l+1}\in\overline{\mathrm{Ad}(G)\mathfrak{a}_l}\qquad(l=1,2,\dots,n).$$

Note that for $\alpha\in\Delta(\mathfrak{a})$

$$\begin{array}{rcll}\mathrm{ht}(\alpha)&=&2n+2-i(\alpha)-j(\alpha)&(B_n),\\ \mathrm{ht}(\alpha)&=&2n+1-i(\alpha)-j(\alpha)&(C_n),\\ \mathrm{ht}(\alpha)&=&2n-i(\alpha)-j(\alpha)&(D_n),\end{array} \tag{5.4}$$

Let M be the set of $(i,j)\in Y$ with minimal i among the elements in Y with the same height (or equivalently with the same $i+j$):

$$M:=\{(i,j)\in Y\,|\,(i',j')\in Y,\ i+j=i'+j'\Rightarrow i\le i'\}.$$

Put $L:=Y\setminus M$.

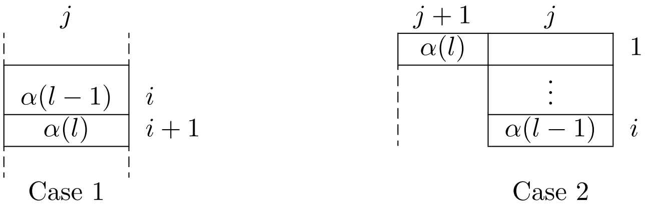


Figure 1: $\alpha(l-1)$ and $\alpha(l)$

We introduce a total order $\prec$ into $\Delta(\mathfrak{a})$ by

$$\alpha \succ \beta \qquad \Leftrightarrow \qquad \begin{cases} j(\alpha) < j(\beta) \\ \text{or} \\ j(\alpha) = j(\beta),\ i(\alpha) < i(\beta). \end{cases} \tag{5.5}$$

Then $\alpha \geq \beta$ implies $\alpha \succeq \beta$, and the maximal root is the biggest.

Enumerate the roots in $\Delta(\mathfrak{a})$ according to $\prec$ from the biggest to the smallest, starting with 1. Let $\alpha(k)$ be the k-th root in $\Delta(\mathfrak{a})$. (Hence $\alpha(1)$ is the maximal root γ_0.) Note that there exist two cases for $\alpha(l-1)$ and $\alpha(l)$ (see Figure 1).

For $l = 1, 2, \ldots, n+1$, put

$$Y(l) := \{\alpha(1), \alpha(2), \ldots, \alpha(l-1)\},\ L(l) := Y(l) \cap L,\ M(l) := Y(l) \cap M.$$

Then each $Y(l)$ is a subdiagram. We divide M into two sets M_1 and M_2:

$$M_1 := \{\alpha \in M \,|\, i(\alpha) = 1\}, \quad M_2 := M \setminus M_1.$$

Definition 5.5. A root $\alpha \in \Delta(\mathfrak{a})$ is called a *source* if there exist no $\beta \in \Delta(\mathfrak{a})$ and $\gamma \in \Delta^+$ such that $\alpha = \beta + \gamma$, i.e., a source is a minimal element of $\Delta(\mathfrak{a})$ with respect to $\leq$.

By (5.3) the following is obvious:

Proposition 5.6. *The set of sources equals*

$$\{\varepsilon_i + \varepsilon_j \in \Delta(\mathfrak{a}) \,|\, \varepsilon_{i+1} + \varepsilon_j, \varepsilon_i + \varepsilon_{j+1} \notin \Delta(\mathfrak{a})\}.$$

Example 5.7. Let $\mathfrak{g}$ be of type B_7, and let

$$\Delta(\mathfrak{a}) := \{\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \varepsilon_2 + \varepsilon_3, \varepsilon_1 + \varepsilon_4, \varepsilon_2 + \varepsilon_4, \varepsilon_3 + \varepsilon_4, \varepsilon_1 + \varepsilon_5\}.$$

The sources are $\varepsilon_3 + \varepsilon_4, \varepsilon_1 + \varepsilon_5$.

By the definition of $\prec$,

$$\alpha(1) = \varepsilon_1 + \varepsilon_2, \alpha(2) = \varepsilon_1 + \varepsilon_3, \ldots, \alpha(6) = \varepsilon_3 + \varepsilon_4, \alpha(7) = \varepsilon_1 + \varepsilon_5;$$

5	4	3	2	j/i
7	4	2	1	1
	5	3		2
	6			3

.

Here we put k in the box at (i,j) when $\alpha(k)=\varepsilon_i+\varepsilon_j$ $(i<j)$. We have

$$\begin{array}{rcl} M &=& \{\alpha(1),\alpha(2),\alpha(4),\alpha(6),\alpha(7)\} \\ L &=& \{\alpha(3),\alpha(5)\}, \end{array}$$

and

$$M_1=\{\alpha(1),\alpha(2),\alpha(4),\alpha(7)\}, \quad M_2=\{\alpha(6)\}.$$

For $\alpha\in\Delta(\mathfrak{a})$, let $s(\alpha)\in\Delta(\mathfrak{a})$ denote the biggest (with respect to $\prec$) source β with $\beta\leq\alpha$. In other words, $s(\alpha)$ is the source β with the smallest $j(\beta)$ satisfying $j(\beta)\geq j(\alpha)$. Hence, by (5.3),

$$i(s(\alpha))\geq i(\alpha), \quad j(s(\alpha))\geq j(\alpha), \tag{5.6}$$

and, since Y is a subdiagram, the following lemma is clear.

Lemma 5.8. *1. For $\alpha,\beta\in\Delta(\mathfrak{a})$, $j(\alpha)=j(\beta)$ implies $s(\alpha)=s(\beta)$.*

2. For $\alpha,\beta\in\Delta(\mathfrak{a})$, $j(\alpha)\leq j(\beta)$ implies $i(s(\alpha))\geq i(s(\beta))$.

3. For $\alpha\in M_2$, $j(\alpha)=j(s(\alpha))$.

Put $T_{\leq j}:=\{\alpha\in T\,|\,j(\alpha)\leq j\}$, and let j_0 be the biggest j with $T_{\leq j}\subseteq Y$.

For $1\leq l\leq n+1$, we define $1\leq t_l\leq\infty$, which plays an important role in the inductive proof of Theorem 5.12.

$$t_l:=\begin{cases}\infty & (l=1 \text{ or } \alpha(l-1)\in T_{\leq j_0}\setminus M_2)\\ \min\{i(s(\beta))\,|\,\beta\succeq\alpha(l-1)\} & (\text{otherwise}).\end{cases} \tag{5.7}$$

Example 5.9. In Example 5.7, $t_1=t_2=\cdots=t_6=\infty, t_7=3, t_8=1$.

Lemma 5.10. *We have*

$$t_1\geq t_2\geq\cdots,$$

and, if $t_l\neq\infty$, then $t_l=i(s(\alpha(l-1)))$. Moreover $t_{l+1}\neq t_l$ implies $t_l=\infty$ and $\alpha(l)\in M_2$, or $\alpha(l)\in M_1$.

Proof. Recall that there exist two cases for $\alpha(l-1)$ and $\alpha(l)$ (see Figure 1). First $t_{l+1}=\infty$ implies $t_l=\infty$, since $\alpha(l)\in Y_{\leq j_0}\setminus M_2$ implies $\alpha(l-1)\in Y_{\leq j_0}\setminus M_2$.

By Lemma 5.8, $i(s(\alpha(l)))\leq i(s(\alpha(l-1))$ for any l. Hence $t_{l+1}\leq t_l$, and $t_l=i(s(\alpha(l-1)))$ if $t_l\neq\infty$.

Finally suppose that $t_{l+1}\neq t_l$ and $\alpha(l)\notin M_1$. Then $\alpha(l-1)$ and $\alpha(l)$ are in Case 1 (Figure 1). By Lemma 5.8, $s(\alpha(l-1))=s(\alpha(l))$. Hence $t_l=\infty, t_{l+1}=i(s(\alpha(l))), \alpha(l-1)\notin M_2$, and $\alpha(l)\in M_2$. ■

Lemma 5.11. *$t_l\geq i(\alpha(l))$.*

Proof. We may suppose that $t_l \neq \infty$. By Lemma 5.10, $t_l = i(s(\alpha(l-1)))$. If $\alpha(l-1)$ and $\alpha(l)$ are in Case 2 (Figure 1), then $i(\alpha(l)) = 1$, and the assertion is clear. If $\alpha(l-1)$ and $\alpha(l)$ are in Case 1 (Figure 1), then $s(\alpha(l)) = s(\alpha(l-1))$ by Lemma 5.8, and hence by (5.3)

$$t_l = i(s(\alpha(l-1))) = i(s(\alpha(l))) \geq i(\alpha(l)). \qquad \blacksquare$$

Theorem 5.12. *Let $\mathfrak{g}$ be a simple Lie algebra of type B, C, or D. Then we have Theorem 2.7, i.e.,* $\mathfrak{a} \in \overline{\mathrm{Ad}(G)K}$.

Proof. We already proved the assertion in four cases (Proposition 5.4). We suppose that $\mathfrak{a}$ is none of those cases.

Set $\mathrm{ht}(Y) := \{\mathrm{ht}(\alpha) \,|\, \alpha \in Y\}$. Then, by the definition of M, for each $h \in \mathrm{ht}(Y)$, there exists a unique $\alpha \in M$ with $\mathrm{ht}(\alpha) = h$.

For k with $\mathrm{ht}(\gamma_0) - n + k < \min \mathrm{ht}(Y)$, Put

$$\Theta_k := \begin{cases} Z & \text{if } k = 1 \text{ in type } D_n, \\ \Lambda^{(\mathrm{ht}(\gamma_0)-n+k)} & \text{otherwise.} \end{cases}$$

Here recall Example 2.6 for Z and Proposition 5.3 for $\Lambda^{(k)}$. Then

$$K = \bigoplus_{\mathrm{ht}(\gamma_0)-n+k<\min(\mathrm{ht}(Y))} \mathbb{C}\Theta_k \oplus \bigoplus_{\alpha \in M} \mathbb{C}\Lambda^{(\mathrm{ht}(\alpha))}.$$

For k and $\Gamma = \sum_{\alpha\in\Delta^+} a_\alpha X_\alpha$, put

$$P_{\leq k}(\Gamma) := \sum_{i(\alpha)\leq k} a_\alpha X_\alpha. \tag{5.8}$$

Put

$$\begin{aligned} \overline{Y(l)} &:= Y(l) \cup \{\alpha \in M_2 \setminus Y(l) \,|\, s(\alpha) = s(\beta)\ (\exists \beta \in M_2(l))\}, \\ R(l) &:= \{k > \sharp L(l) \,|\, \mathrm{ht}(\gamma_0) - n + k < \min \mathrm{ht}(Y)\}. \end{aligned}$$

Note that $\overline{Y(l)}$ is a subdiagram. Indeed, $\overline{Y(l)}$ equals $Y(l')$ for some $l' \geq l$; if there exists $\beta \in M_2(l)$ with $s(\beta) \notin Y(l)$, then $s(\beta) = \alpha(l'-1)$.

Set

$$\mathfrak{a}_l := \Big(\bigoplus_{\alpha\in\overline{Y(l)}} \mathbb{C}X_\alpha\Big) \oplus \Big(\bigoplus_{\alpha\in M\setminus\overline{Y(l)}} \mathbb{C}P_{\leq t_l}(\Lambda^{(\mathrm{ht}(\alpha))})\Big) \oplus \Big(\bigoplus_{k\in R(l)} \mathbb{C}P_{\leq t_l}(\Theta_k)\Big).$$

Then $\mathfrak{a}_1 = K$, and $\mathfrak{a}_{n+1} = \mathfrak{a}$. Note that $\mathfrak{a}_l$ satisfies the assumption (3.1), since $\overline{Y(l)}$ is a subdiagram.

We show

$$\mathfrak{a}_{l+1} \in \overline{\mathrm{Ad}(G)\mathfrak{a}_l} \qquad (l = 1, 2, \ldots, n). \tag{5.9}$$

Then, inductively, we have Theorem 2.7.

The proof of (5.9) is divided into three cases according to $\alpha(l) \in M_1, L$, or M_2.

(Case 1: $\alpha(l) \in M_1$.) In this case, $\alpha(l-1)$ and $\alpha(l)$ are in Case 2 (Figure 1). We have $\overline{Y(l+1)} = \overline{Y(l)} \cup \{\alpha(l)\}$. Since each root appearing in $P_{\leq t_l}(\Lambda^{(\mathrm{ht}(\alpha(l)))})$ except $\alpha(l)$ belongs to $Y(l)$, the root vector $X_{\alpha(l)}$ belongs to $\mathfrak{a}_l$. If $t_{l+1} = t_l$, then $\mathfrak{a}_{l+1} = \mathfrak{a}_l$.

Next suppose that $t_{l+1} < t_l$. We prove that

$$\mathfrak{a}_{l+1} = \lim_{t \to 0} \mathrm{Ad}(\lambda_{t_{l+1}}^{-1}(t))\mathfrak{a}_l. \tag{5.10}$$

First we show $t_{l+1} \leq i(\alpha(l-1))$. If $t_{l+1} < t_l < \infty$, then $\alpha(l-1)$ is a source, and $t_l = i(\alpha(l-1))$ by Lemma 5.10. If $t_l = \infty$ and $t_{l+1} < \infty$, then

$$(i(\alpha(l-1)) + 1, j(\alpha(l-1)) + 1)$$

does not belong to Y. Since $j(\alpha(l)) = j(\alpha(l-1)) + 1$, this implies $t_{l+1} = i(s(\alpha(l))) \leq i(\alpha(l-1))$. Hence we have proved $t_{l+1} \leq i(\alpha(l-1))$.

If the coefficient of $\alpha_{t_{l+1}}$ in a root α is 2, then α is of form $\varepsilon_i + \varepsilon_j$ with $i \leq j \leq t_{l+1}$. Since $i \leq j \leq t_{l+1} \leq i(\alpha(l-1)) \leq j(\alpha(l-1))$, we have $\alpha(l-1) \leq \alpha$, and hence $X_\alpha \in \mathfrak{a}_l$. Hence the linear combinations with roots whose coefficients of $\alpha_{t_{l+1}}$ are 1 survive under $\lim_{t\to 0} \mathrm{Ad}(\lambda_{t_{l+1}}^{-1}(t))$ by Lemma 3.2;

$$P_{\leq t_{l+1}}(\Lambda^{(h)}), P_{\leq t_{l+1}}(\Theta_k) \in \lim_{t \to 0} \mathrm{Ad}(\lambda_{t_{l+1}}^{-1}(t))\mathfrak{a}_l.$$

Hence we have proved (5.10).

(Case 2: $\alpha(l) \in L$.) In this case, $\alpha(l-1)$ and $\alpha(l)$ are in Case 1 (Figure 1), and $t_{l+1} = t_l$ by Lemma 5.10. We have $\overline{Y(l+1)} = \overline{Y(l)} \cup \{\alpha(l)\}$.

Suppose that $l = 3$ and $\mathfrak{g}$ is of type D_n. Then $K = \mathfrak{a}_1 = \mathfrak{a}_2 = \mathfrak{a}_3$ and $\alpha(3) = \varepsilon_2 + \varepsilon_3$. Let $\beta := \alpha_4 + \cdots + \alpha_{n-2} + \alpha_n$ if $n \geq 6$, and $\beta := \alpha_5$ if $n = 5$. As in the proof of Proposition 5.4 (3), $\mathbb{C}[X_\beta, Z] = \mathbb{C}X_{\varepsilon_1+\varepsilon_4}$. Hence $\langle [X_\beta, Z], \Lambda^{(2n-5)} \rangle = \langle X_{\varepsilon_1+\varepsilon_4}, X_{\varepsilon_2+\varepsilon_3} \rangle$ and

$$\lim_{t \to 0} \exp(t^{-1}\mathrm{ad}X_\beta)(\mathfrak{a}_3) = \mathfrak{a}_4.$$

Suppose that $l \neq 3$ or $\mathfrak{g}$ is not of type D_n. Let $h := \mathrm{ht}(\Theta_{\sharp L(l)+1})$. Recall that $\Theta_{\sharp L(l)+1} = \Lambda^{(h)}$ and $h = \mathrm{ht}(\gamma_0) - n + \sharp L(l) + 1$.

Put $i := i(\alpha(l))$ and $j := j(\alpha(l))$. Note that $i \geq 2$, since $\alpha(l) \in L$. We express $\alpha(l)$ as a sum of simple roots in the following order:

$$\begin{aligned}
(B_n) \qquad \alpha(l) &= \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k \leq n} \alpha_k \\
&= \alpha_i + \cdots + \alpha_j + \cdots + \alpha_n + \alpha_n + \cdots + \alpha_j, \\
(C_n) \qquad \alpha(l) &= \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k < n} \alpha_k + \alpha_n \\
&= \alpha_i + \cdots + \alpha_j + \cdots + \alpha_n + \cdots + \alpha_j, \\
(D_n) \qquad \alpha(l) &= \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k \leq n-2} \alpha_k + \alpha_{n-1} + \alpha_n \\
&= \alpha_i + \cdots + \alpha_j + \cdots + \alpha_{n-2} + \alpha_n + \alpha_{n-1} + \cdots + \alpha_j,
\end{aligned}$$

In the above, let γ be the sum of the first h simple roots, and β the rest. Note that β and γ can be defined since we have $h < \min \mathrm{ht}(Y) \leq \mathrm{ht}(\alpha(l))$, and that they are in fact roots since $h = n + \sharp L(l)$ for B_n, C_n ($n - 2 + \sharp L(l)$ for D_n respectively).

We prove that

$$\mathfrak{a}_{l+1} = \lim_{t \to 0} \exp t^{-1} \mathrm{ad} X_\beta(\mathfrak{a}_l). \tag{5.11}$$

First we show

$$[X_\beta, \mathbb{C} P_{\le t_l}(\Theta_{\sharp L(l)+1})] = \mathbb{C} X_{\alpha(l)}. \tag{5.12}$$

By Lemma 5.11, $t_l \ge i(\alpha(l)) = i$. Hence γ certainly appears in $P_{\le t_l}(\Theta_{\sharp L(l)+1})$. If $\gamma' \ne \gamma$ is a root of height h, and if $\gamma' + \beta$ is also a root, then γ' should be of the form $\alpha_{j-1} + \alpha_{j-2} + \cdots + \alpha_{j-h}$. In Case B or C, no such root appears in $P_{\le t_l}(\Theta_{\sharp L(l)+1})$, since $h \ge n$. In Case D, we have $h \ge n - 2$. If $h \ge n - 1$, then γ' cannot exist. If $h = n - 2$, then $\sharp L(l) = 0$, and hence $\alpha(l)$ is the first root in L; $\alpha(l) = \varepsilon_2 + \varepsilon_3$. If γ' exists, then $1 \le j - h = 3 - (n - 2)$. Namely $n \le 4$. Hence the only possible case is the one when $n = 4, \gamma = \alpha_2 + \alpha_4, \beta = \alpha_3$, and $\gamma' = \alpha_2 + \alpha_1$, which we excluded in the beginning. Hence we have proved (5.12). Since $X_\alpha \in \mathfrak{a}_l$ for all $\alpha > \alpha(l)$, by Lemma 3.1

$$X_{\alpha(l)} \in \lim_{t \to 0} \exp t^{-1} \mathrm{ad} X_\beta(\mathfrak{a}_l).$$

As in the previous paragraph, for $k \ge 1$,

$$[X_\beta, P_{\le t_l}(\Lambda^{(h+k)})] \in \mathbb{C} X_\alpha,$$

where $\alpha = \alpha(l) + \alpha_{i-1} + \alpha_{i-2} + \cdots + \alpha_{i-k}$. Hence, even if α is a root, we have

$$[X_\beta, P_{\le t_l}(\Lambda^{(h+k)})] \in \mathfrak{a}_l \qquad (k \ge 1) \tag{5.13}$$

since $\alpha > \alpha(l)$. Hence by Lemma 3.1

$$P_{\le t_{l+1}}(\Lambda^{(h+k)}) = P_{\le t_l}(\Lambda^{(h+k)}) \in \lim_{t \to 0} \exp t^{-1} \mathrm{ad} X_\beta(\mathfrak{a}_l). \tag{5.14}$$

Since $\overline{Y(l)}$ is a subdiagram, $[X_\beta, X_\alpha] \in \mathfrak{a}_l$ for $\alpha \in \overline{Y(l)}$. Hence by Lemma 3.1

$$X_\alpha \in \lim_{t \to 0} \exp t^{-1} \mathrm{ad} X_\beta(\mathfrak{a}_l) \qquad \text{for } \alpha \in \overline{Y(l)}. \tag{5.15}$$

Hence we have proved (5.11).

(Case 3: $\alpha(l) = \varepsilon_{i_l} + \varepsilon_{j_l} \in M_2$. ($i_l < j_l$ for B, D; $i_l \le j_l$ for C.))

In this case, $\alpha(l-1)$ and $\alpha(l)$ are in Case 1 (Figure 1), and $s(\alpha(l-1)) = s(\alpha(l))$ (Lemma 5.8). We have $\overline{Y(l+1)} = \overline{Y(l)} \cup \{\alpha \in M_2 \,|\, s(\alpha) = s(\alpha(l))\}$. By Lemma 5.10, there exist two cases:

(a) $t_{l+1} = t_l < \infty$,

(b) $t_{l+1} < \infty, t_l = \infty, \alpha(l-1) \notin M_2$.

Since $\alpha(l) = \varepsilon_{i_l} + \varepsilon_{j_l} \in M_2$ implies that there exist no $\alpha = \varepsilon_{i_l} + \varepsilon_j \in Y$ for $j > j_l$, we see $j(s(\alpha(l))) = j_l$.

First consider Case (a); suppose that $t_{l+1} = t_l < \infty$. If there exist $\beta \in M_2$ such that $\alpha(l) \prec \beta$ and $s(\beta) = s(\alpha(l))$, then $\mathfrak{a}_{l+1} = \mathfrak{a}_l$.

Suppose that there exist no such $\beta \in M_2$. Then $s(\alpha(l)) = \varepsilon_{t_{l+1}} + \varepsilon_{j_l}$ by Lemma 5.10. Put

$$\boldsymbol{m} = -a\boldsymbol{e}_{t_{l+1}} - b\boldsymbol{e}_{j_l} \in \mathbb{Z}^n,$$

where $a > b > 0$. Then $(\boldsymbol{m}, \varepsilon_i + \varepsilon_j)$ (cf. (3.2)) are as follows:

$\cdots > j_l$	$j_l \geq \cdots$	$t_{l+1} \geq \cdots$	j/i
			$\vdots$
$-a-b$	$-a-2b$	$-2a-2b$	t_{l+1}
			$\vdots$
$-b$	$-2b$	$-a-2b$	j_l
0	$-b$	$-a-b$	$\vdots$

.

We prove that

$$\mathfrak{a}_{l+1} = \lim_{t\to 0} \operatorname{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathfrak{a}_l). \tag{5.16}$$

Let $\alpha \in M_2$ satisfy $s(\alpha) = s(\alpha(l))$. Then $j(\alpha) = j(\alpha(l)) = j_l$, and $i_l = i(\alpha(l)) \leq i(\alpha) \leq i(s(\alpha(l))) = t_{l+1}$, since $\alpha \preceq \alpha(l)$. Hence α is the unique root in $P_{\leq t_l}(\Lambda^{(\mathrm{ht}(\alpha))})$ with $\boldsymbol{m}$-weight $-a-2b$ ($-2a-2b$ if $t_{l+1} = j_l$ in Type C) outside of $Y(l)$. Hence by Lemma 3.2

$$X_\alpha \in \lim_{t\to 0} \operatorname{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathbb{C}P_{\leq t_l}(\Lambda^{(\mathrm{ht}(\alpha))})).$$

Let $\alpha \in M \setminus Y(l)$ and $s(\alpha) \neq s(\alpha(l))$. Then $\alpha \prec \alpha(l)$, and $j(\alpha) > j(\alpha(l))$ by Lemma 5.8. Since $\alpha(l) \in M_2$, we have $i(\alpha) < i(\alpha(l))$. We show

$$\mathrm{ht}(\alpha) > \mathrm{ht}(\alpha(l)). \tag{5.17}$$

Suppose otherwise. Then $j(\alpha) \geq i(\alpha(l)) + j(\alpha(l)) - i(\alpha)$. We see that $\gamma := \varepsilon_{i(\alpha)} + \varepsilon_{i(\alpha(l))+j(\alpha(l))-i(\alpha)}$ is a root and $\gamma \geq \alpha$. Thus $\gamma \in Y$ and $\mathrm{ht}(\gamma) = \mathrm{ht}(\alpha(l))$, which contradicts the fact that $\alpha(l) \in M$.

By (5.17), all roots in $P_{\leq t_l}(\Lambda^{(\mathrm{ht}(\alpha))})$ with $\boldsymbol{m}$-weight $-a-2b$ ($-2a-2b$ if $t_{l+1} = j_l$ in Type C) are in $Y(l)$. Hence by Lemma 3.2 the linear combination with $\boldsymbol{m}$-weight $-a-b$ survives;

$$P_{\leq t_{l+1}}(\Lambda^{(\mathrm{ht}(\alpha))}) \in \lim_{t\to 0} \operatorname{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathbb{C}P_{\leq t_l}(\Lambda^{(\mathrm{ht}(\alpha))}) + \bigoplus_{\beta \in Y(l)} \mathbb{C}X_\beta).$$

Similarly, by $t_{l+1} = t_l$ and the above table,

$$P_{\leq t_{l+1}}(\Theta_k) = \lim_{t\to 0} \operatorname{Ad}(\lambda^{\boldsymbol{m}}(t))(P_{\leq t_l}(\Theta_k)) \in \lim_{t\to 0} \operatorname{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathfrak{a}_l).$$

Clearly $\lim_{t\to 0} \operatorname{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathbb{C}X_\alpha) = \mathbb{C}X_\alpha$. Hence in Case (a)

$$\mathfrak{a}_{l+1} = \lim_{t\to 0} \operatorname{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathfrak{a}_l).$$

Next consider Case (b); suppose that $t_{l+1} < \infty, t_l = \infty, \alpha(l-1) \notin M_2$. In this case, $s(\alpha(l)) = \varepsilon_{t_{l+1}} + \varepsilon_{j_l}$, and $t_{l+1} = j_l - 1$ for Types B, D and $t_{l+1} = j_l$ for Type C, respectively.

Again we put

$$\boldsymbol{m} = -a\boldsymbol{e}_{t_{l+1}} - b\boldsymbol{e}_{j_l} \in \mathbb{Z}^n,$$

where $a > b > 0$, and we show

$$\mathfrak{a}_{l+1} = \lim_{t\to 0} \mathrm{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathfrak{a}_l). \tag{5.18}$$

For $\alpha \in M_2$ satisfying $s(\alpha) = s(\alpha(l))$,

$$X_\alpha \in \lim_{t\to 0} \mathrm{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathbb{C}P_{\le t_l}(\Lambda^{(\mathrm{ht}(\alpha))}))$$

as in Case (a).

For $\alpha \in M \setminus Y(l)$ and $s(\alpha) \ne s(\alpha(l))$,

$$P_{\le t_{l+1}}(\Lambda^{(\mathrm{ht}(\alpha))}) \in \lim_{t\to 0} \mathrm{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathbb{C}P_{\le t_l}(\Lambda^{(\mathrm{ht}(\alpha))}) + \bigoplus_{\beta\in Y(l)} \mathbb{C}X_\beta)$$

as in Case (a) as well.

In this case, $\mathrm{ht}(s(\alpha(l)) = \min \mathrm{ht}Y$. Hence the possible $\boldsymbol{m}$-weights of roots appearing in Θ_k are $-a-b, -b, 0$ (Types B, D) and $-a-b, 0$ (Type C), respectively. The roots with $\boldsymbol{m}$-weight $-a-b$ are exactly those roots appearing $P_{\le t_{l+1}}(\Theta_k)$. Hence

$$P_{\le t_{l+1}}(\Theta_k) = \lim_{t\to 0} \mathrm{Ad}(\lambda^{\boldsymbol{m}}(t))(P_{\le t_l}(\Theta_k)) \in \lim_{t\to 0} \mathrm{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathfrak{a}_l).$$

Since $\lim_{t\to 0} \mathrm{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathbb{C}X_\alpha) = \mathbb{C}X_\alpha$ is clear, we see

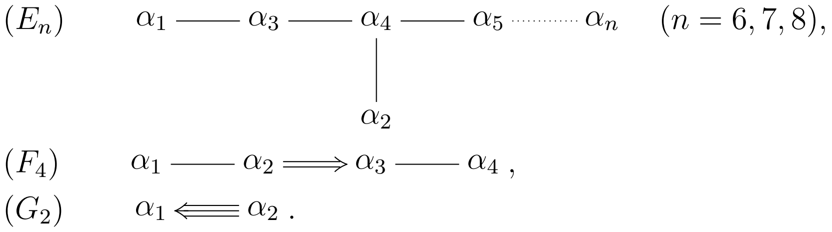
$$\mathfrak{a}_{l+1} = \lim_{t\to 0} \mathrm{Ad}(\lambda^{\boldsymbol{m}}(t))(\mathfrak{a}_l)$$

in Case (b) as well.

We have thus finished the proof of the theorem. ■

6. Exceptional types

Recall Bourbaki's notation [2]:



In the exceptional types, we fix a Chevalley basis

$$\{X_\alpha, H_i \,|\, \alpha \in \Delta,\ i = 1, 2, \ldots, n\}$$

of $\mathfrak{g}$ as in [14, Proposition 4]. In particular, we have

$$[X_{\alpha_i}, X_\beta] = (p+1)X_{\beta+\alpha_i} \tag{6.1}$$

for non-simple $\beta \in \Delta^+$ with $\beta + \alpha_i \in \Delta^+$, where p is the nonnegative integer satisfying $\beta - p\alpha_i \in \Delta^+$ and $\beta - (p+1)\alpha_i \notin \Delta^+$. Note that $p = 0$ in Type E.

Lemma 6.1. *The following is the list of n-dimensional abelian $\mathfrak{b}$-ideals:*

(E_6) $\mathfrak{a}_i = \mathfrak{a}' \bigoplus \mathfrak{a}'_i \qquad (i = 1, 2, 3)$, *where*
$\mathfrak{a}' = \langle X_{\underset{2}{12321}}, X_{\underset{1}{12321}}, X_{\underset{1}{12221}}, X_{\underset{1}{11221}}, X_{\underset{1}{12211}} \rangle$, $\mathfrak{a}'_1 = \langle X_{\underset{1}{01221}} \rangle$, $\mathfrak{a}'_2 = \langle X_{\underset{1}{11211}} \rangle$, $\mathfrak{a}'_3 = \langle X_{\underset{1}{12210}} \rangle$.

(E_7) $\mathfrak{a}_i = \langle X_{\underset{2}{234321}}, X_{\underset{2}{134321}}, X_{\underset{2}{124321}}, X_{\underset{2}{123321}}, X_{\underset{2}{123221}} \rangle \bigoplus \mathfrak{a}'_i \quad (i = 1, 2, 3)$, *where* $\mathfrak{a}'_1 = \langle X_{\underset{2}{123211}}, X_{\underset{2}{123210}} \rangle$, $\mathfrak{a}'_2 = \langle X_{\underset{1}{123321}}, X_{\underset{2}{123211}} \rangle$, $\mathfrak{a}'_3 = \langle X_{\underset{1}{123321}}, X_{\underset{1}{123221}} \rangle$.

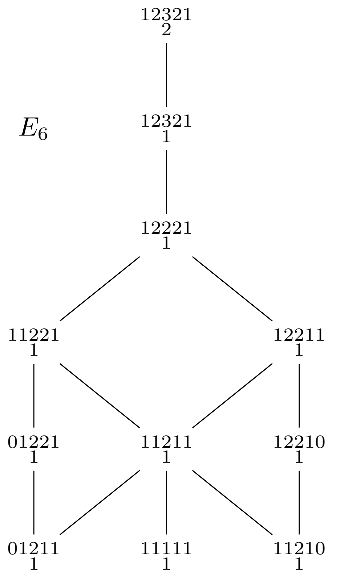
(E_8) $\mathfrak{a}_1 = \mathfrak{a}' \bigoplus \langle X_{\underset{3}{1354321}} \rangle, \quad \mathfrak{a}_2 = \mathfrak{a}' \bigoplus \langle X_{\underset{2}{2454321}} \rangle$, *where* $\mathfrak{a}' :=$
$\langle X_{\underset{3}{2465432}}, X_{\underset{3}{2465431}}, X_{\underset{3}{2465421}}, X_{\underset{3}{2465321}}, X_{\underset{3}{2464321}}, X_{\underset{3}{2454321}}, X_{\underset{3}{2354321}} \rangle$.

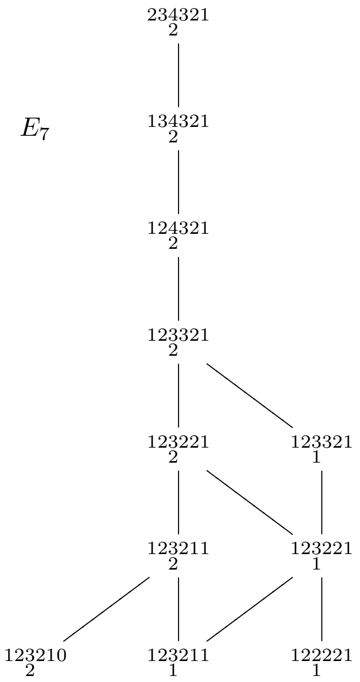
(F_4) $\langle X_{2342}, X_{1342}, X_{1242}, X_{1232} \rangle$.

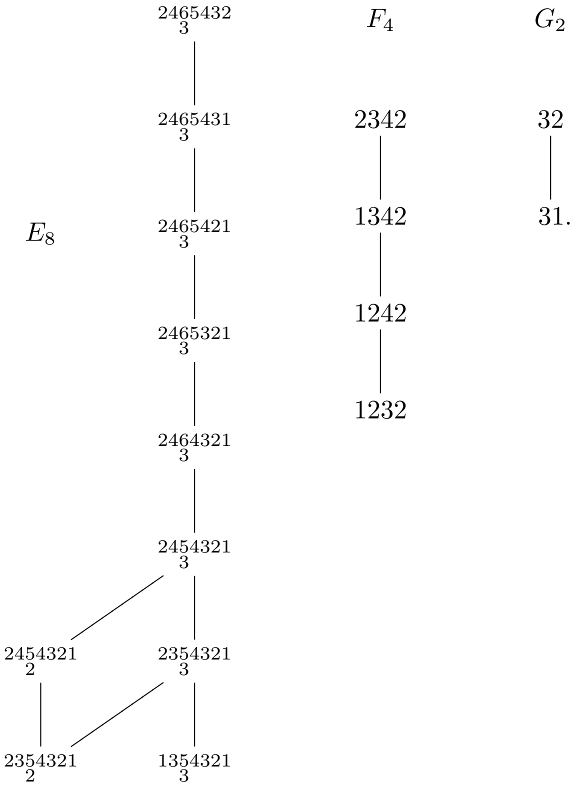
(G_2) $\langle X_{32}, X_{31} \rangle$,

where X_{32} means $X_{3\alpha_1+2\alpha_2}$ in Type G_2, etc.

Proof. This is easily checked by the following Hasse diagrams of positive roots of height greater than $\mathrm{ht}(\gamma_0) - \mathrm{rank}(\mathfrak{g})$:







∎

Lemma 6.2. *Let* $\Lambda = \sum_{i=1}^{n} X_{\alpha_i}$, *and* $J := \mathfrak{z}_{\mathfrak{g}}(\Lambda)$. *Then*

(E_6) $J = \langle \Lambda = f_1, f_4, f_5, f_7, f_8, f_{11} \rangle$, *where*

- $f_4 := X_{\substack{01111\\0}} - X_{\substack{00111\\1}} - X_{\substack{11110\\0}} + X_{\substack{11100\\1}}$,
- $f_5 := X_{\substack{01111\\1}} - X_{\substack{01210\\1}} + X_{\substack{11110\\1}} - 2X_{\substack{11111\\0}}$,
- $f_7 := X_{\substack{01221\\1}} - X_{\substack{11211\\1}} + X_{\substack{12210\\1}}$,
- $f_8 := X_{\substack{11221\\1}} - X_{\substack{12211\\1}}$,
- $f_{11} := X_{\substack{12321\\2}}$.

(E_7) $J = \langle \Lambda = f_1, f_5, f_7, f_9, f_{11}, f_{13}, f_{17} \rangle$, *where*

- $f_5 := X_{\substack{012100\\1}} - X_{\substack{111100\\1}} - X_{\substack{011110\\1}} + 2X_{\substack{111110\\0}} - 2X_{\substack{011111\\0}} + 3X_{\substack{001111\\1}}$,
- $f_7 := X_{\substack{122100\\1}} - X_{\substack{112110\\1}} + X_{\substack{012210\\1}} - X_{\substack{012111\\1}} + 2X_{\substack{111111\\1}}$,
- $f_9 := X_{\substack{122111\\1}} - X_{\substack{112211\\1}} + X_{\substack{012221\\1}}$,
- $f_{11} := X_{\substack{123210\\2}} - X_{\substack{123211\\1}} + X_{\substack{122221\\1}}$,
- $f_{13} := X_{\substack{123221\\2}} - X_{\substack{123321\\1}}$,
- $f_{17} := X_{\substack{234321\\2}}$.

(E_8) $J = \langle \Lambda = f_1, f_7, f_{11}, f_{13}, f_{17}, f_{19}, f_{23}, f_{29} \rangle$, *where*

- $f_7 := X_{\substack{1221000\\1}} - X_{\substack{1121100\\1}} + X_{\substack{0122100\\1}} - X_{\substack{0121110\\1}} + 2X_{\substack{1111110\\1}} - 2X_{\substack{1111111\\0}} + X_{\substack{0111111\\1}}$,

- $f_{11} := X_{\substack{1232100\\2}} - X_{\substack{1232110\\1}} + X_{\substack{1222210\\1}} + X_{\substack{1222111\\1}} - 2X_{\substack{1122211\\1}} + 2X_{\substack{0122221\\1}}\,,$
- $f_{13} := X_{\substack{1222221\\1}} - X_{\substack{1232211\\1}} + X_{\substack{1233210\\1}} - X_{\substack{1232210\\2}} + 2X_{\substack{1232111\\2}}\,,$
- $f_{17} := X_{\substack{2343210\\2}} - X_{\substack{1343211\\2}} + X_{\substack{1243221\\2}} - X_{\substack{1233321\\2}}\,,$
- $f_{19} := X_{\substack{2343221\\2}} - X_{\substack{1343321\\2}} + X_{\substack{1244321\\2}}\,,$
- $f_{23} := X_{\substack{2454321\\2}} - X_{\substack{2354321\\3}}\,,$
- $f_{29} := X_{\substack{2465432\\3}}\,.$

(F_4) $J = \langle \Lambda,\ 2X_{0122} - X_{1121} + X_{1220},\ X_{1222} - X_{1231},\ X_{2342} \rangle\,.$

(G_2) $J = \langle \Lambda, X_{32} \rangle\,.$

Here f_m *indicates that this is an element of height* m*.*

Proof. It is easy to check that the above elements commute with Λ by (6.1). The linear independence leads to the assertion. ■

Proposition 6.3. *Let* $\mathfrak{g}$ *be a simple Lie algebra of exceptional type. Then Theorem 2.7 holds.*

Proof. We show the proof only for Type E_6. For the other exceptional types, the proofs go similarly, and hence we give only examples of steps from J to the abelian ideals.

Let $\mathfrak{g}$ be the simple Lie algebra of Type E_6. By Lemma 6.1, we have three 6-dimensional abelian $\mathfrak{b}$-ideals: $\mathfrak{a}_i$ $(i = 1, 2, 3)$.

Since $\mathrm{ad}(X_{\substack{12221\\1}})(\Lambda) = -\sum_{i=1}^{6} [X_{\alpha_i}, X_{\substack{12221\\1}}] = -X_{\substack{12321\\1}}$, we have

$$\lim_{t\to 0} \exp t^{-1}\mathrm{ad}(X_{\substack{12221\\1}})(J) = \langle f_4, f_5, f_7, f_8, X_{\substack{12321\\1}}, X_{\substack{12321\\2}} \rangle =: J_1.$$

Then from $\mathrm{ad}(X_{\substack{01111\\1}})(f_4) = X_{\substack{12221\\1}}$ we obtain

$$\lim_{t\to 0} \exp t^{-1}\mathrm{ad}(X_{\substack{01111\\1}})(J_1) = \langle f_5, f_7, f_8, X_{\substack{12221\\1}}, X_{\substack{12321\\1}}, X_{\substack{12321\\2}} \rangle =: J_2.$$

Next we have

$$\lim_{t\to 0} \exp t^{-1}\mathrm{ad}(X_{\substack{00111\\0}})(J_2) = \langle f_7 \rangle \oplus \mathfrak{a}' =: J_3.$$

We have

$$\begin{aligned} \lim_{t\to 0} \mathrm{Ad}(\lambda_1(t))(J_3) &= \mathfrak{a}_1, \\ \lim_{t\to 0} \mathrm{Ad}(\lambda_1^{-1}(t)\lambda_6^{-1}(t))(J_3) &= \mathfrak{a}_2, \\ \lim_{t\to 0} \mathrm{Ad}(\lambda_6(t))(J_3) &= \mathfrak{a}_3. \end{aligned}$$

Hence we have proved $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \in \overline{\mathrm{Ad}(G)J}$.

To give examples of steps from J to the abelian ideals, we introduce the notations

$$\mathfrak{a} \xrightarrow{X} \mathfrak{a}', \quad \mathfrak{a} \xrightarrow{\lambda_i} \mathfrak{a}'$$

for

$$\mathfrak{a}' = \lim_{t \to 0} \exp t^{-1} \mathrm{ad}(X)(\mathfrak{a}), \quad \mathfrak{a}' = \lim_{t \to 0} \mathrm{Ad}(\lambda_i(t))(\mathfrak{a}),$$

respectively. Then the steps we have seen in Type E_6 are

$$\begin{tikzcd}
 & & & & \mathfrak{a}_1 & \\
J \arrow[r, "\overset{X_{12221}}{1}"] & J_1 \arrow[r, "\overset{X_{01111}}{1}"] & J_2 \arrow[r, "\overset{X_{00111}}{0}"] & J_3 \arrow[ur, "\lambda_1"] \arrow[rr, "\lambda_1^{-1}\lambda_6^{-1}"] \arrow[dr, "\lambda_6"] & & \mathfrak{a}_2 \\
 & & & & \mathfrak{a}_3 & .
\end{tikzcd}$$

In Types G_2 and F_4, we may take the steps

$$J \xrightarrow{X_{21}} \langle X_{32}, X_{31} \rangle, \quad J \xrightarrow{X_{1242}} J_1 \xrightarrow{X_{0121}} J_2 \xrightarrow{X_{0001}} \langle X_{2342}, X_{1342}, X_{1242}, X_{1232} \rangle,$$

respectively. In Type E_7, we may take the steps

$$J \overset{\overset{X_{124321}}{2}}{\longrightarrow} J_1 \overset{\overset{X_{123210}}{1}}{\longrightarrow} J_2 \overset{\overset{X_{012210}}{1}}{\longrightarrow} J_3 := \langle f_9, f_{11}, f_{13}, \alpha \,|\, \mathrm{ht}(\alpha) \geq 14 \rangle$$

$$\begin{tikzcd}
J_3 \arrow[r, "\overset{X_{001100}}{1}"] \arrow[dr, "\substack{X_{001110} \\ 1}"'] & J_4' \arrow[r, "\lambda_2^{-1}"] & \mathfrak{a}_1 \\
 & J_4 \arrow[r, "\overset{X_{000001}}{0}"] \arrow[dr, "\substack{X_{001000} \\ 0}"'] & \mathfrak{a}_2 \\
 & & \mathfrak{a}_3.
\end{tikzcd}$$

In Type E_8, we may take the steps

$$J \overset{\overset{X_{2465421}}{3}}{\longrightarrow} J_1 \overset{\overset{X_{2343321}}{2}}{\longrightarrow} J_2 \overset{\overset{X_{1233221}}{1}}{\longrightarrow} J_3 \overset{\overset{X_{1232110}}{2}}{\longrightarrow} J_4 \overset{\overset{X_{1111110}}{1}}{\longrightarrow} J_5$$

$$\begin{tikzcd}
J_5 \arrow[r, "\overset{X_{0011000}}{1}"] \arrow[dr, "\substack{X_{0111100} \\ 0}"'] & J_6 \arrow[r, "\lambda_2^{-1}"] & \mathfrak{a}_1 \; . \\
 & \mathfrak{a}_2 &
\end{tikzcd}$$

■

A. The proof of Proposition 4.5

First we treat two fundamental cases.

Lemma A.1. *If* $\mu = (n)$*, then* $(z_1, \ldots, z_n) = (1, 1, \ldots, 1, 0)$ *is a solution of* (IE_μ).

Proof. In this case, $i(h) = n + 1 - h$ and $z_{i(h)}(h) = z_{n+1-h} + z_{n+2-h} + \cdots + z_n$. Hence $(1, 1, \ldots, 1, 0)$ is a solution. ■

Lemma A.2. *Suppose that n is even and $\mu = (n/2, n/2)$. Then*

z_1	z_2	$\cdots$	$z_{(n-2)/2}$	$z_{n/2}$	$z_{(n+2)/2}$	$z_{(n+4)/2}$	$\cdots$	z_{n-1}	z_n
2	2	$\cdots$	2	0	3	2	$\cdots$	2	1

is a solution of (IE_μ).

Proof. In this case,

$$i(h) = \begin{cases} \frac{n}{2} + 1 - h & (h \le \frac{n}{2}) \\ n + 1 - h & (h > \frac{n}{2}) \end{cases},$$

and

$$z_{i(h)}(h) = \begin{cases} \sum_{i=\frac{n}{2}+1-h}^{\frac{n}{2}} z_i & (h \le \frac{n}{2}) \\ \sum_{i=n+1-h}^{n} z_i & (h > \frac{n}{2}) \end{cases}.$$

Hence the values of z_i in the statement is a solution. ■

Lemma A.3. *For any $\mu \vdash n$, there exists a solution of the system (IE_μ).*

Proof. We prove the assertion by induction on n. For $n = 1$, there is nothing to prove.

Let $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_l) \vdash n$.

(Case 1: The case $\mu_2 < \mu_1$)

Define $\mu' = (\mu'_1 \ge \mu'_2 \ge \cdots \ge \mu'_l) \vdash n - 1$ by

$$\mu'_i := \begin{cases} \mu_i & (i > 1) \\ \mu_1 - 1 & (i = 1). \end{cases}$$

Let $z' = (z'_1, \ldots, z'_{n-1})$ be a solution of $(IE_{\mu'})$. Then define $z = (z_1, \ldots, z_n)$ by

$$z_i := \begin{cases} \sum_{j=1}^{n-1} z'_j & (i = 1) \\ z'_{i-1} & (i > 1). \end{cases}$$

Then it is easy to see that $z = (z_1, \ldots, z_n)$ is a solution of the system (IE_μ).

(Case 2: The case $\mu_2 = \mu_1 =: m$ and $n + 1 \ge 3m$.)

Let $\tilde{\mu} := (\mu_2 \ge \mu_3 \ge \cdots \ge \mu_l)$. Then $\tilde{\mu} \vdash n - m$. In this case, $\tilde{i}(h) = i(h)$ for $h \le n - m$ by (4.1). For $h > n - m$, we have $k = 1$ in Lemma 4.4, and we have $i(h) = i(h - m)$ and $i(h) + h - 1 = n$ by (4.1) and (4.2).

Let $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_{n-m})$ be a solution of $(IE_{\tilde{\mu}})$. Define $z = (z_1, \ldots, z_n)$ by

$$z_i := \begin{cases} \tilde{z}_i & (i \le n - m) \\ \sum_{j=1}^{n-m} \tilde{z}_j & (i = n + 1 - m) \\ \tilde{z}_{i-m} & (i > n + 1 - m). \end{cases}$$

Note that $i - m > m = \mu_1$ when $i > n + 1 - m$, since $n + 1 \ge 3m$. Hence, in particular, $\tilde{z}_{i-m} > 0$ for $i > n + 1 - m$. Then it is straightforward to check that $z = (z_1, \ldots, z_n)$ is a solution of the system (IE_μ).

(Case 3: The case $\mu_2 = \mu_1 =: m$ and $n+1 < 3m$.)

Note that $n \geq 2m$, or equivalently $n+1-m > m$. Let $\tilde{\mu} := (\mu_2 \geq \mu_3 \geq \cdots \geq \mu_l)$. Then $\tilde{\mu} \vdash n-m$. Let $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_{n-m})$ be a solution of $(IE_{\tilde{\mu}})$. Take $\delta \in \mathbb{Q}$ so that

$$0 < \delta < \min_{j \neq i(h)} (\tilde{z}_j(h) - \tilde{z}_{i(h)}(h)).$$

Define $z = (z_1, \ldots, z_n)$ by

$$z_i := \begin{cases} \tilde{z}_i & (i \leq n-m) \\ \sum_{j=1}^{n-m} \tilde{z}_j & (i = n+1-m) \\ \delta & (i = 2m) \\ \tilde{z}_{i-m} & (i > n+1-m,\, i \neq 2m). \end{cases}$$

Then it is also straightforward to check that $z = (z_1, \ldots, z_n)$ is a solution of the system (IE_μ). ■

B. The proof of Proposition 5.3 (Type B)

In this section, let $\mathfrak{g} := \mathfrak{so}(2n+1, \mathbb{C})$ (cf. Example 2.4).

$$\mathfrak{so}(2n+1,\mathbb{C}) = \left\{ \begin{bmatrix} A & \boldsymbol{x} & B \\ -{}^t\boldsymbol{y} & 0 & -{}^t\boldsymbol{x} \\ C & \boldsymbol{y} & -A' \end{bmatrix} \mid B' = -B,\, C' = -C \right\}.$$

Recall that

$$\Lambda = \sum_{i=1}^{n} E_{i,i+1} - \sum_{i=n+1}^{2n} E_{i,i+1} = \sum_{i=1}^{n} \widetilde{E}_{i,i+1}, \quad J = \bigoplus_{k=1}^{n} \mathbb{C}\Lambda^{2k-1},$$

where $\widetilde{E}_{i,j} = E_{i,j} - E_{2n+2-j,2n+2-i} \in \mathfrak{g}$. Then

$$\Lambda^{2k-1} = \sum_{i \leq n+1-(2k-1)} \widetilde{E}_{i,i+2k-1} + \sum_{i=n-2k+3}^{n+1-k} (-1)^{n-i} \widetilde{E}_{i,i+2k-1}.$$

First suppose that n is odd. Let $n = 2m+1$. Put

$$S := \sum_{i=1}^{m+1} a_i \widetilde{E}_{i,n+i}. \tag{B.1}$$

Note that the height of S is n. By a simple computation, for $k \leq m$,

$$\begin{aligned} [S, \Lambda^{2k-1}] &= \sum_{i=1}^{m-2k+2} (-a_i - a_{i+2k-1}) \widetilde{E}_{i,i+n+2k-1} \\ &\quad + \sum_{i=m-2k+3}^{m-k+1} (a_{n-i-2k+3} - a_i) \widetilde{E}_{i,i+n+2k-1}. \end{aligned}$$

Proof of Proposition 5.3, odd n case. Take, for example, $a_i = i$ in (B.1). Then

$$\begin{aligned}[S, \Lambda^{2k-1}] &= \sum_{i=1}^{m-2k+2} -(2i+2k-1)\widetilde{E}_{i,i+n+2k-1} \\ &\quad + \sum_{i=m-2k+3}^{m-k+1} (n+3-2k-2i)\widetilde{E}_{i,i+n+2k-1}.\end{aligned}$$

By the consideration of height, we have $[S, [S, \Lambda^{2k-1}]] = 0$ for all k and $[S, \Lambda^{2k-1}] = 0$ for $2k-1 \geq n$. Hence

$$\lim_{t\to 0} \exp t^{-1}\mathrm{ad}S(\mathbb{C}\Lambda^{2k-1}) = \begin{cases} \mathbb{C}[S, \Lambda^{2k-1}] & (k \leq m) \\ \mathbb{C}\Lambda^{2k-1} & (k > m), \end{cases}$$

and $K := \langle \Lambda^{(l)} \,|\, n \leq l \leq 2n-1 \rangle$ with

$$\Lambda^{(l)} := \begin{cases} [S, \Lambda^{l-n}] & (l\text{: even}) \\ \Lambda^{l} & (l\text{: odd, } l \geq n) \end{cases}$$

satisfies $K = \lim_{t\to 0} \exp t^{-1}\mathrm{ad}S(J)$ and the condition in Proposition 5.3. ■

Next suppose that n is even. Let $n = 2m$. Put

$$S := \sum_{i=1}^{m+1} a_i \widetilde{E}_{i,i+n-1}. \tag{B.2}$$

Note that the height of S is $n-1$. By a simple computation, for $k \leq m$,

$$\begin{aligned}[S, \Lambda^{2k-1}] &= (a_1 + a_{2k})\widetilde{E}_{1,n+2k-1} - \sum_{i=2}^{m-2k+2} (a_i + a_{i+2k-1})\widetilde{E}_{i,i+n+2k-2} \\ &\quad - \sum_{i=m-2k+3}^{m-k+1} (a_i - a_{n-i-2k+4})\widetilde{E}_{i,i+n+2k-2}.\end{aligned}$$

Proof of Proposition 5.3, even n case. Take, for example, $a_i = i$ in (B.2). Then

$$\begin{aligned}[S, \Lambda^{2k-1}] &= (2k+1)\widetilde{E}_{1,n+2k-1} - \sum_{i=2}^{m-2k+2} (2i+2k-1)\widetilde{E}_{i,i+n+2k-2} \\ &\quad + \sum_{i=m-2k+3}^{m-k+1} (n+4-2k-2i)\widetilde{E}_{i,i+n+2k-2}.\end{aligned}$$

By the consideration of height, we have $[S, [S, \Lambda^{2k-1}]] = 0$ for all $k \geq 2$, $[S, [S, \Lambda]] \in \mathbb{C}\Lambda^{2n-1}$, and $[S, \Lambda^{2k-1}] = 0$ for $2k-1 \geq n+1$. Hence by Lemma 3.1

$$\begin{aligned} &[S, \Lambda^{2k-1}] \in \lim_{t\to 0} \exp t^{-1}\mathrm{ad}S(J) \qquad (2k-1 < n+1), \\ &\Lambda^{2k-1} \in \lim_{t\to 0} \exp t^{-1}\mathrm{ad}S(J) \qquad (2k-1 \geq n+1), \end{aligned}$$

and $K := \langle \Lambda^{(l)} \,|\, n \leq l \leq 2n-1 \rangle$ with

$$\Lambda^{(l)} := \begin{cases} [S, \Lambda^{l-(n-1)}] & (l\text{: even}) \\ \Lambda^{l} & (l\text{: odd, } l > n) \end{cases}$$

satisfies $K = \lim_{t\to 0} \exp t^{-1}\mathrm{ad}S(J)$ and the condition in Proposition 5.3. ■

C. The proof of Proposition 5.3 (Type C)

In this section, let $\mathfrak{g} := \mathfrak{sp}(2n, \mathbb{C})$ (cf. Example 2.5).

$$\mathfrak{sp}(2n,\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ C & -A' \end{bmatrix} \mid B' = B,\, C' = C \right\}.$$

Recall that $\Lambda = \sum_{i=1}^{n} E_{i,i+1} - \sum_{i=n+1}^{2n-1} E_{i,i+1}$, and

$$J = \bigoplus_{k=1}^{n} \mathbb{C}\Lambda^{2k-1},$$

where

$$\begin{aligned} \Lambda^{2k-1} \quad = \quad & \sum_{i \le n-(2k-1)} E_{i,i+2k-1} + \sum_{i=\max\{n-(2k-2),1\}}^{\min\{n,2n-(2k-1)\}} (-1)^{n-i} E_{i,i+2k-1} \\ & - \sum_{i=n+1}^{2n-(2k-1)} E_{i,i+2k-1}. \end{aligned}$$

Define an abelian Lie subalgebra K as follows:

$$K := (\bigoplus_{k=\frac{n+1}{2}}^{n} \mathbb{C}\Lambda^{2k-1}) \oplus (\bigoplus_{k=1}^{\frac{n-1}{2}} \mathbb{C} \begin{bmatrix} O & \Lambda_A^{2k-1} \\ O & O \end{bmatrix}) \quad \text{if } n \text{ is odd,}$$

$$K := (\bigoplus_{k=\frac{n}{2}+1}^{n} \mathbb{C}\Lambda^{2k-1}) \oplus (\bigoplus_{k=1}^{\frac{n}{2}} \mathbb{C} \begin{bmatrix} O & \Lambda_A^{2k-2} \\ O & O \end{bmatrix}) \quad \text{if } n \text{ is even,}$$

where $\Lambda_A := \sum_{i=1}^{n-1} E_{i,i+1}$. Put

$$S := \begin{bmatrix} O & -\frac{1}{2}I \\ O & O \end{bmatrix} \in \mathfrak{g} \quad \text{if } n \text{ is odd, and}$$

$$S := \begin{bmatrix} \frac{1}{2}E_{1,n} & -\frac{1}{2}\sum_{i=1}^{n-1} E_{i+1,i} \\ O & -\frac{1}{2}E_{1,n} \end{bmatrix} \in \mathfrak{g} \quad \text{if } n \text{ is even.}$$

Note that the height of S is n for n odd and $n-1$ for n even.

Proof of Proposition 5.3. By a straightforward computation, we have

$$\lim_{t\to 0} \exp(t^{-1}\mathrm{ad}S)(\mathbb{C}\Lambda^{2k-1}) = \begin{cases} \mathbb{C}\Lambda^{2k-1} & (2k-1 \ge n) \\ \mathbb{C}\begin{bmatrix} O & \Lambda_A^{2k-1} \\ O & O \end{bmatrix} & (2k-1 < n) \end{cases}$$

for n odd, and

$$\lim_{t\to 0} \exp(t^{-1}\mathrm{ad}S)(\mathbb{C}\Lambda^{2k-1}) = \begin{cases} \mathbb{C}\Lambda^{2k-1} & (2k-1 > n) \\ \mathbb{C}\begin{bmatrix} O & \Lambda_A^{2k-2} \\ O & O \end{bmatrix} & (2k-1 < n) \end{cases}$$

for n even. Hence

$$K = \lim_{t\to 0} \exp(t^{-1}\mathrm{ad}S)(J).$$

It is clear that K satisfies the condition in Proposition 5.3. ■

D. The proof of Proposition 5.3 (Type D)

In this section, let $\mathfrak{g} := \mathfrak{so}(2n, \mathbb{C})$ (cf. Example 2.6).

$$\mathfrak{so}(2n, \mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ C & -A' \end{bmatrix} \mid B' = -B,\, C' = -C \right\}.$$

Put

$$\widetilde{E}_{i,j} = E_{i,j} - E_{2n+1-j,2n+1-i}.$$

Recall that

$$\Lambda = \sum_{i=1}^{n-1} \widetilde{E}_{i,i+1} + \widetilde{E}_{n-1,n+1}, \quad J = \mathbb{C}Z \bigoplus \oplus_{k=1}^{n-1} \mathbb{C}\Lambda^{2k-1},$$

where $Z = \widetilde{E}_{1,n} - \widetilde{E}_{1,n+1}$. We have

$$\Lambda^{2k-1} = \sum_{i=1}^{n-(2k-1)} \widetilde{E}_{i,i+2k-1} + \widetilde{E}_{n-(2k-1),n+1} + 2\sum_{i=1}^{k-1} (-1)^i \widetilde{E}_{n-(2k-1)+i,n+1+i}.$$

The height of Λ^{2k-1} equals $2k-1$, and that of Z $n-1$. Note that, when n is even,

$$\Lambda^{n-1} = \widetilde{E}_{1,n} + \widetilde{E}_{1,n+1} + 2\sum_{i=1}^{\frac{n}{2}-1} (-1)^i \widetilde{E}_{1+i,n+1+i}.$$

Let $1 \le i < n$, $1 < j < 2n$, $i < j$, $i < 2n+1-j$. Then the height of $\widetilde{E}_{i,j}$ equals $j-i$ for $j \le n$ and $j-i-1$ for $j > n$. Hence, $\mathbb{C}\widetilde{E}_{1,n}$ and $\mathbb{C}\widetilde{E}_{i,i+n}$ $(i < \frac{n+1}{2})$ are all the root spaces of height $n-1$, and, for $h \ge n$, $\mathbb{C}\widetilde{E}_{i,i+h+1}$ $(i < n - \frac{h}{2})$ are all the root spaces of height h.

First suppose that n is odd. Put

$$S := \sum_{i=1}^{2} a_i \widetilde{E}_{i,i+n-2} + \sum_{i=2}^{\frac{n+1}{2}} b_i \widetilde{E}_{i,i+n-1}. \tag{D.1}$$

By a simple computation,

$$\begin{aligned} [S, \Lambda] &= (a_1 - a_2)\widetilde{E}_{1,n} + (a_1 - b_2)\widetilde{E}_{1,n+1} \\ &\quad -(a_2 + b_2 + b_3)\widetilde{E}_{2,n+2} - \sum_{i=3}^{\frac{n-1}{2}} (b_i + b_{i+1})\widetilde{E}_{i,n+i}, \end{aligned}$$

and for $k \ge 2$

$$\begin{aligned} [S, \Lambda^{2k-1}] &= (2a_1 - b_{2k})\widetilde{E}_{1,n+2k-1} - (a_2 + b_2 + b_{2k+1})\widetilde{E}_{2,n+2k} \\ &\quad - \sum_{i=3}^{\frac{n+3-4k}{2}} (b_i + b_{i+2k-1})\widetilde{E}_{i,i+n+2k-2} \\ &\quad - \sum_{i=\frac{n+5-4k}{2}}^{\frac{n-2k+1}{2}} (b_i - b_{n+3-i-2k})\widetilde{E}_{i,i+n+2k-2}. \end{aligned}$$

Proof of Proposition 5.3, odd n case. Take, for example, $a_1 = 2, a_2 = 1, b_2 = 1$, and $b_i = -i \ (i \geq 3)$ in (D.1). Then

$$[S, \Lambda] = \widetilde{E}_{1,n} + \widetilde{E}_{1,n+1} + \widetilde{E}_{2,n+2} + \sum_{i=3}^{\frac{n-1}{2}} (2i+1)\widetilde{E}_{i,n+i},$$

and for $k \geq 2$

$$\begin{aligned} [S, \Lambda^{2k-1}] &= (4+2k)\widetilde{E}_{1,n+2k-1} + (2k-1)\widetilde{E}_{2,n+2k} \\ &\quad + \sum_{i=3}^{\frac{n+3-4k}{2}} (2i+2k-1)\widetilde{E}_{i,i+n+2k-2} \\ &\quad - \sum_{i=\frac{n+5-4k}{2}}^{\frac{n+1-2k}{2}} (n+3-2k-2i)\widetilde{E}_{i,i+n+2k-2}. \end{aligned}$$

By the consideration of height, we have

$$[S, \Lambda^{2k-1}] = 0 \text{ for } 2k-1 \geq n \quad \text{and} \quad [S, Z] \in \mathbb{C}\Lambda^{2n-3}.$$

Hence

$$\lim_{t \to 0} \exp t^{-1} \mathrm{ad} S(J) = \langle Z, \Lambda^{(l)} \mid n-1 \leq l \leq 2n-3 \rangle =: K,$$

where

$$\Lambda^{(l)} = \begin{cases} [S, \Lambda^{l-(n-2)}] & (l\text{: even}) \\ \Lambda^l & (l\text{: odd}, l \geq n). \end{cases}$$

■

Next suppose that n is even. Let

$$S := a\widetilde{E}_{1,n} + \sum_{i=1}^{\frac{n}{2}} b_i \widetilde{E}_{i,i+n}. \tag{D.2}$$

By a simple computation,

$$\begin{aligned} [S, \Lambda^{2k-1}] &= -(a + b_1 + b_{2k})\widetilde{E}_{1,n+2k} - \sum_{i=2}^{\frac{n+2-4k}{2}} (b_i + b_{i+2k-1})\widetilde{E}_{i,i+n+2k-1} \\ &\quad - \sum_{i=\frac{n+4-4k}{2}}^{\frac{n-2k}{2}} (b_i - b_{n+2-2k-i})\widetilde{E}_{i,i+n+2k-1}. \end{aligned}$$

Proof of Proposition 5.3, even n case.

Take, for example, $a = 0$ and $b_i = -i$ in (D.2). Then

$$\begin{aligned} [S, \Lambda^{2k-1}] &= (2k+1)\widetilde{E}_{1,n+2k} + \sum_{i=2}^{\frac{n+2-4k}{2}} (2i+2k-1)\widetilde{E}_{i,i+n+2k-1} \\ &\quad - \sum_{i=\frac{n+4-4k}{2}}^{\frac{n-2k}{2}} (n+2-2k-2i)\widetilde{E}_{i,i+n+2k-1}. \end{aligned}$$

By the consideration of height, we have

$$[S, \Lambda^{2k-1}] = 0 \text{ for } 2k \geq n \quad \text{and} \quad [S, Z] = 0.$$

Hence

$$\lim_{t \to 0} \exp t^{-1} \mathrm{ad} S(J) = \langle Z, \Lambda^{(l)} \,|\, n-1 \leq l \leq 2n-3 \rangle =: K,$$

where

$$\Lambda^{(l)} = \begin{cases} [S, \Lambda^{l-(n-1)}] & (l\text{: even}) \\ \Lambda^{l} & (l\text{: odd}, \ l \geq n-1). \end{cases}$$

■

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