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Lattices of Oscillator Groups

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Abstract. This paper is concerned with discrete, uniform subgroups (lattices) of oscillator groups, which are certain semidirect products of the Heisenberg group and the additive group $\mathbb{R}$ of real numbers. The present paper classifies the lattices of the oscillator group up to automorphisms of the ambient Lie group. Moreover, by introducing discrete oscillator groups we can classify the abstract isomorphism type of lattices of some oscillator groups.
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1. Introduction

Oscillator groups are certain semidirect products of the Heisenberg group and the additive group $\mathbb{R}$. They are interesting because they are the only simply connected solvable Lie groups, besides the abelian ones, which have a biinvariant Lorentzian metric. In addition, the quotient of an oscillator group by a lattice gives an example of a compact homogeneous Lorentzian manifold. Thus it is of interest to know the lattices of oscillator groups and an important duty to give a classification of them. In [8] Medina and Revoy already tried to classify the lattices of the oscillator group. But unfortunately there are several mistakes and inaccuracies that affect the classification result. Lattices have been classified for the Heisenberg group in [1] (see also [13] and [4]). There are also classifications for lattices of nilpotent Lie groups of special structure or small dimension (see for instance [2], [5], [3], [9] and [10]) and for certain 2-step solvable Lie groups in [11] and [6]. Now, our goal is to rectify the uncertainties in [8] and to characterize the lattices of oscillator groups up to automorphisms of the ambient Lie group by the help of a certain parameter space.

More precisely, we consider groups $Osc_n(\lambda_1, \dots, \lambda_n)$, i. e. $\mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}$ with the group multiplication given by

$$(z, \xi, t)(v, \eta, s) = \left(z + v + \frac{1}{2}\omega(\xi, \mathrm{e}^{tN_\lambda}\, \eta), \xi + \mathrm{e}^{tN_\lambda}\, \eta, t + s \right),$$

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where $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i \neq 0$ for $i = 1, \ldots, n$,

$$N_\lambda := \begin{pmatrix} \begin{array}{cc|} 0 & -\lambda_1 \\ \lambda_1 & 0 \\ \hline \end{array} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \begin{array}{|cc} \hline 0 & -\lambda_n \\ \lambda_n & 0 \end{array} \end{pmatrix}$$

and ω is a symplectic form on $\mathbb{R}^{2n}$ given by $\omega(\xi, \eta) := \xi^T N_{-1,\ldots,-1}\eta$. For the 4−dimensional oscillator groups it is well known that these groups are isomorphic to each other, compare [8].
We set

$$\mathbb{F}_+ := \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \frac{1}{2}, y > 0, x^2 + y^2 \geq 1 \right\}. \tag{1}$$

Theorem 1.1. *For $n = 1$ the set of lattices of $Osc_1(1)$ up to automorphisms of the ambient oscillator group $Osc_1(1)$ is given by the set of all tupels $(r, \lambda, x, y, \xi_0)$, where r is a positive integer, $\lambda = \lambda_0 + k\pi$ is an angle with $\lambda_0 \in \{\pi/3, \pi/2, 2\pi/3, \pi\}$ and $k \in \mathbb{N}$, (x, y) is a vector in $\mathbb{R}^2$ satisfying*

$$(x, y) \begin{cases} = (\frac{1}{2}, \frac{\sqrt{3}}{2}), & \text{if } \lambda_0 \in \{\frac{1}{3}\pi, \frac{2}{3}\pi\} \\ = (0, 1), & \text{if } \lambda_0 = \frac{1}{2}\pi \\ \in \mathbb{F}_+ & \text{if } \lambda_0 = \pi \end{cases}$$

and ξ_0 is an vector in $\mathbb{R}^2$ to extract from the list in Section 6 (dependent on r, λ, x and y).

This parameter space describes certain properties of the lattices. For a lattice L the parameter $r \in \mathbb{N}\backslash\{0\}$ classifies the restriction of the lattice to the normal subgroup $H_1(\omega)$ of the oscillator group construed as a lattice of the Heisenberg group. It is the uniquely determined positive power of the generator γ of $L \cap \mathfrak{z}(H_1(\omega))$ such that γ^r generates the commutator subgroup $[L \cap H_1(\omega), L \cap H_1(\omega)]$, compare [13] and [4]. The angle λ is the smallest positive number such that $(z, \xi, \lambda) \in L$ for some $(z, \xi) \in H_1(\omega)$. Moreover, λ describes a rotational invariance of the elements in the associated lattice $L \cap H_1(\omega)/\mathfrak{z}(L \cap H_1(\omega))$ in $\mathbb{R}^2$, which has to be satisfied, since for $(z, \xi, t) \in L$ and $(v, \eta, 0) \in L \cap H_1(\omega)$ the conjugation

$$(z, \xi, \lambda)(v, \eta, 0)(z, \xi, t)^{-1} = (\tilde{z}, \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \eta, 0)$$

is also in $L \cap H_1(\omega)$. Thus $\lambda = \lambda_0 + k\pi$ for $\lambda_0 \in \{\pi/3, \pi/2, 2\pi/3, \pi\}$ and $k \in \mathbb{N}$. The parameter (x, y) describes more precisely the associated lattice $L \cap H_1(\omega)/\mathfrak{z}(L \cap H_1(\omega))$. At this, we remark that the generators of the associated lattice have to be in precise proportion for a rotational invariance by an angle $\lambda_0 \in \{\pi/3, \pi/2, 2\pi/3\}$. Hence (x, y) is uniquely determined. This is not true for $\lambda_0 = \pi$. Here, we obtain an action of $GL(2, \mathbb{Z})$ on the upper half plane $\mathbb{H}$ of $\mathbb{C}$, which includes the well known action of the modular group $SL(2, \mathbb{Z})$. As a system of representatives of this action on $\mathbb{H}$ we can choose $\mathbb{F}_+$, the half of the fundamental domain of the quotient $SL(2,\mathbb{Z})\backslash\mathbb{H}$. However, the lattice L is not necessarily uniquely determined up to the ambient Lie groups by the parameters r, λ and (x,y). There is always a finite set of lattices satisfying r, λ and (x,y), which are not isomorphic to each other with respect to automorphisms of the oscillator group. The number and kind of lattices depends on the concrete values of r, λ and (x,y) and is prescribed by the parameter ξ_0.

The tupel (r,λ,x,y,ξ_0) defines a lattice of the 4-dimensional oscillator group $Osc_1(1)$ in a canonical way, namely the lattice $L(r,\lambda,x,y,\xi_0)$ generated by the elements $(0,\frac{1}{\sqrt{y}}e_1,0)$, $(0,\frac{x}{\sqrt{y}}e_1+\sqrt{y}e_2,0)$, $(\frac{1}{r},0,0)$ and $(0,\xi_0,\lambda)$, where $\{e_1,e_2\}$ denotes the standard basis of $\mathbb{R}^2$. However, there is another possibility to describe this lattice. We can also think of the lattice

$$L(\xi_0) := \langle (0,e_1,0),(0,e_2,0),(1,0,0),(0,\xi_0,1)\rangle$$

but now in a group $Osc_1(\omega_r,\lambda B_{x,y})$ isomorphic to the oscillator group and given by the multiplication

$$(z,\xi,t)(v,\eta,s) = (z+v+\frac{1}{2}\omega_r(\xi,\mathrm{e}^{t\lambda B_{x,y}\eta}),\xi+\mathrm{e}^{t\lambda B_{x,y}\eta},t+s)$$

for $(z,\xi,t)(v,\eta,s)\in Osc_1(\omega_r,\lambda B_{x,y})$. Here $\omega_r = r\omega$ and

$$B_{x,y} := \begin{pmatrix} \frac{x}{y} & -\frac{x^2}{y}-y \\ \frac{1}{y} & -\frac{x}{y} \end{pmatrix}.$$

It is not hard to see that $L(r,\lambda,x,y,\xi_0)$ in $Osc_1(1)$ and $L(\xi_0)$ in $Osc_1(\omega_r,\lambda B_x,y)$ are isomorphic. For instance we can choose

$$\varphi : Osc_1(1)\to Osc_1(\omega_r,\lambda B_{x,y}),\quad (z,\xi,t)\mapsto (rz,T_{x,y}\xi,t/\lambda),$$

where

$$T_{x,y} = \begin{pmatrix} \sqrt{y} & -x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}.$$

In this paper we will use the second visualisation. The reason for preferring the second visualisation is simple. We want to use the known facts about the classification of the lattices of the Heisenberg group (compare [13] and [4]). Therefor, we have to bear in mind that the automorphisms, used to classify the lattices of the Heisenberg group, have lifts to automorphisms of the oscillator group. This is in general not true and was ignored in [8]. It is only possible to find for any automorphism φ of the Heisenberg group an isomorphism between groups of oscillator type $Osc_1(1)$ and $Osc_1(\omega,\lambda B_{x,y})$ for certain parameters λ and (x,y) whose restriction on the Heisenberg group equals φ.

In addition, the second visualisation allows us to classify the lattices of oscillator groups of arbitrary dimension by Theorem 3.3, 3.7 and 3.9. Here it is not possible anymore to calculate the parameters explicitely, but we can describe the parameters with certain group actions. Remark that Theorem 3.9 differs from Theorem 3 in [8], since the automorphisms whose restriction to the Heisenberg group equals the identity were not calculated correctly in [8].

The second main theorem in this paper determines a set of representatives of the abstract isomorphism classes of lattices of $Osc_1(1)$. For this purpose we introduce discrete oscillator groups (see section 5 for definition) and use the one-to-one correspondence between the lattices of the 4-dimensional oscillator group and this discrete oscillator groups.
More precisely, for $r \in \mathbb{N}\backslash\{0\}$, $l,k \in \mathbb{Z}$ and $S \in SL(2,\mathbb{Z})$ of finite order we define the discrete oscillator group $G_r(S,l,k)$ as the semidirect product $\Gamma_r \rtimes_{\tilde{S}} \mathbb{Z}$. Here Γ_r is prescribed by generators $\{\alpha,\beta,\gamma\}$ where γ generates the center and $\alpha\beta\alpha^{-1}\beta^{-1} = \gamma^r$ is satisfied and $\tilde{S}(1)$ is the automorphism of Γ_r given by the 3×3-matrix

$$\begin{pmatrix} S & 0 \\ l\ k & 1 \end{pmatrix}$$

with respect to (α,β,γ).
Let $\lceil x \rceil$ denote the smallest integer larger or equal to x and $\lfloor x \rfloor$ the largest integer smaller or equal to x. We set

$$S_1 := Id, \quad S_2 := -Id, \quad S_3 := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad S_4 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S_6 := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

For $r \in \mathbb{N}\backslash\{0\}$ we define the groups

$$L_r^1 := G_r(S_1,0,0), \quad L_r^2 := G_r(S_2,0,0), \quad L_r^3 := G_r(S_3,0,-\lfloor \frac{r}{2} \rfloor),$$

$$L_r^4 := G_r(S_4,0,0), \quad L_r^6 := G_r(S_6,0,0).$$

Moreover, for an even r we define the additional groups

$$L_r^{2,+} = G_r(S_2,0,-1), \quad L_r^{4,+} := G_r(S_4,1,0),$$

and for an r devisible by 3 we define

$$L_r^{3,+} := G_r(S_3,1,-\lceil \frac{r+1}{2} \rceil).$$

Theorem 1.2. *Let L be a lattice of the 4–dimensional oscillator group. We choose the tupel (r,λ,x,y,ξ_0) for L which is uniquely determined by Theorem 1.1, set σ as the order of the matrix $\exp(\lambda B_{x,y})$ and $r_0 := r$ for $\sigma \neq 1$ or $r_0 := gcd(r, r\xi_1, r\xi_2)$ for $\sigma = 1$. Then*

$$L \simeq \begin{cases} L_{r_0}^{\sigma,+}, & \text{if } r_0 \in 2\mathbb{N}, \sigma \in \{2,4\}, \xi \neq 0 \\ & \text{or } r_0 \in 3\mathbb{N}, \sigma = 3, \xi_0 \in \{(\frac{1}{r},0),(\frac{1}{r},\frac{1}{2r}),(\frac{1}{r}+\frac{1}{2r},0)\} \\ L_{r_0}^{\sigma}, & \text{else.} \end{cases}$$

Moreover, $L_{r_0}^{\sigma}$ and $L_{r_0}^{\sigma,+}$ for $r_0 \in \mathbb{N}\backslash\{0\}$ and $\sigma \in \{1,2,3,4,6\}$ form a system of representatives of the isomorphism classes of discrete oscillator groups and thus a system of repesentatives of the lattices of $Osc_1(1)$ up to isomorphism as abstract groups.

Here, the parameter r_0 is compared with r (see above) the positive power of γ such that γ^{r_0} generates the subgroup $[L,L] \cap \mathfrak{z}(L)$. If the subgroup $[L,L]$ is abelian, then r equals r_0. In the other case the subgroup $[L,L]$ is contained in $\langle\gamma\rangle$ and thus r_0 is a factor of r. In particular, the parameter r is in general not determined by the abstract isomorphism type of the lattice.

Although the parameters $(x,y) \in \mathbb{F}_+$ are relevant to characterise a lattice of the oscillator group with $\lambda_0 = \pi$ up to automorphism, these parameters are irrelevant for describing this lattice as an abstract group. Now, the smallest positive power σ of the generator δ of the lattice not contained in the Heisenberg group such that δ^σ acts trivial on $H_1(\omega)/\mathfrak{z}(H_1(\omega))$ is important for the abstract isomorphism type. This power can be calculated by the equation in Theorem 1.2. Furthermore, Theorem 1.2 shows that the abstract isomorphism type is not necessarily determined by r_0 and σ.

In section 2 we give a definition of oscillator groups $Osc_n(\omega,B)$ and calculate there automorphism group. In the third section we classify the lattices of oscillator groups of arbitrary dimension by a certain parameter space, where the parameters are uniquely determines up to certain group actions. Section 4 contains the proof of Theorem 1.1 and Section 5 the proof of Theorem 1.2.

2. Oscillator groups and their automorphisms

We introduce the oscillator groups as a certain semidirect product of a Heisenberg group and the real numbers. Then we will see that every oscillator group is isomorphic to some $Osc_n(\omega,B)$, which are isomorphic to the groups $G_k(\lambda)$ considered in [8]. Finally we compute the automorphisms of $Osc_n(\omega,B)$ in Corollary 2.7, which rectifies the assertion in [8, p. 92].

Definition 2.1. For a symplectic form ω on $\mathbb{R}^{2n}$, let $H_n(\omega)$ denote the group $\mathbb{R} \times \mathbb{R}^{2n}$ with the multiplication given by

$$(z,\xi)(v,\eta) = \big(z+v+\frac{1}{2}\omega(\xi,\eta), \xi+\eta\big).$$

Each of these groups are isomorphic. We call them Heisenberg groups.

Let H be a Heisenberg group, $\mathfrak{h}$ its Lie algebra and $\mathfrak{z}$ the center of $\mathfrak{h}$. Suppose that p is a one-parameter subgroup of the automorphism group of H with trivial action on the center of the Heisenberg group and satisfying that the bilinear form

$$A : \mathfrak{h}/\mathfrak{z} \times \mathfrak{h}/\mathfrak{z} \to \mathfrak{z} \cong \mathbb{R}, \quad A(h_1+\mathfrak{z}, h_2+\mathfrak{z}) := \big[((\mathrm{d}_0 P)(1))(h_1), h_2\big]$$

is definite, where $P : \mathbb{R} \to Aut(\mathfrak{h})$ is the differential of the automorphism $p(t) \in Aut(H)$ at the point $0 \in \mathfrak{h}$, i. e. $P(t) := \mathrm{d}_0(p(t))$. Then the semidirect product $H \rtimes_p \mathbb{R}$ of H and $\mathbb{R}$ with respect to p is called oscillator group.

Lemma 2.2. *Let p be a one-parameter group of the automorphisms of $H_n(\omega)$. Then p satisfies the conditions in Definition 2.1 if and only if there is a $\delta \in \mathbb{R}^{2n}$ and a $B \in \mathfrak{gl}(2n,\mathbb{R})$ satisfying $\omega(B\xi,\eta) = -\omega(\xi,B\eta)$ for all $\xi,\eta \in \mathbb{R}^{2n}$ and*

$\omega(B\cdot,\cdot)$ is definite, such that

$$p(t) = \exp\left(t\begin{pmatrix}0 & \delta^T\\ 0 & B\end{pmatrix}\right) : \mathbb{R}\times\mathbb{R}^{2n} \to \mathbb{R}\times\mathbb{R}^{2n}.$$

If B satisfies $\omega(B\xi,\eta) = -\omega(\xi, B\eta)$ for all $\xi,\eta \in \mathbb{R}^{2n}$ and $\omega(B\cdot,\cdot)$ is definite, then B has purely imaginary eigenvalues and can be diagonalized over $\mathbb{C}$.

Proof. The equivalence follows directly from the definition and known results for automorphisms of the Heisenberg group. That is, in particular, that we can write each automorphism of $H_n(\omega)$ as a matrix

$$\begin{pmatrix}a & \epsilon^T\\ 0 & S\end{pmatrix},$$

where $S \in GL(2n,\mathbb{R})$, $\epsilon \in \mathbb{R}^{2n}$, $a \in \mathbb{R}\backslash\{0\}$ satisfying $S^*\omega = a\omega$ (see for instance [13]).
Let $\omega(B\cdot,\cdot)$ be positive definite (otherwise we consider -$\omega(B\cdot,\cdot)$) and consider the complexification of B and the complex bilinear extension of ω. One can check that $\omega(Bz,\overline{z}) \in \mathbb{R}$ and $\omega(Bz,\overline{z}) > 0$ for all $z \in \mathbb{C}^{2n}$. Suppose z is an eigenvector of B with its corresponding eigenvalue $\lambda \neq 0$, then we see $\mathbb{R} \ni \omega(Bz,\overline{z}) = \lambda\omega(z,\overline{z})$. On the other hand $\omega(z,\overline{z}) \in \mathrm{i}\mathbb{R}$. Hence, $\lambda \in \mathrm{i}\mathbb{R}$ and finally the first part follows.
Now, we prove the second part by induction over n. For $n = 1$ the assertion holds. Assuming the assertion holds for n, we will prove it for $n+1$. Let Z_1 be an eigenvector of B over $\mathbb{C}$ with corresponding eigenvalue $\mathrm{i}\lambda_1$, $\lambda_1 \neq 0$. Then $\omega(Z_1,\overline{Z_1}) = \frac{1}{\mathrm{i}\lambda_1}\omega(BZ_1,\overline{Z_1}) \neq 0$. In particular $\omega|_{\mathrm{span}\{Z_1,\overline{Z_1}\}}$ is nondegenerate. Also $\overline{Z_1}$ is an eigenvector of B with corresponding eigenvalue $-\mathrm{i}\lambda_1$, so B maps the subspace $\mathrm{span}\{Z_1,\overline{Z_1}\}$ onto itself. Since B is antisymmetric with respect to ω, B maps $\mathrm{span}\{Z_1,\overline{Z_1}\}^\perp$ into itself. Making use of the induction hypothesis on $B|_{\mathrm{span}\{Z_1,\overline{Z_1}\}^\perp}$ yields the assertion. ∎

Definition 2.3. The Lie group $Osc_n(\omega, B)$ is the oscillator group $H_n(\omega)\rtimes_p\mathbb{R}$, where $p(t) = \left(\begin{smallmatrix}1 & 0\\ 0 & \mathrm{e}^{tB}\end{smallmatrix}\right)$, $\omega(B\cdot,\cdot) = -\omega(\cdot,B\cdot)$ and $\omega(B\cdot,\cdot)$ definite. We also define for $\lambda = (\lambda_1,\ldots,\lambda_n)$, $\lambda_i \in \mathbb{R}$ the matrix

$$N_\lambda := \begin{pmatrix}\begin{array}{cc}0 & -\lambda_1\\ \lambda_1 & 0\end{array} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \begin{array}{cc}0 & -\lambda_n\\ \lambda_n & 0\end{array}\end{pmatrix}$$

and the symplectic form $\omega_\lambda(\xi,\eta) := \xi^T N_{-\lambda}\eta$. We denote $Osc_n(\omega_{(1,\ldots,1)}, N_\lambda)$ by $Osc(\lambda_1,\ldots,\lambda_n)$, where $\lambda_i\lambda_{i+1} > 0$ for $i = 1,\ldots,n-1$.

The group multiplication in $Osc_n(\omega, B)$ is given by

$$(z,\xi,t)(v,\eta,s) = \left(z + v + \frac{1}{2}\omega(\xi, \mathrm{e}^{tB}\eta), \xi + \mathrm{e}^{tB}\eta, t + s\right),$$

and inversion by

$$(z,\xi,t)^{-1} = (-z, -\mathrm{e}^{-tB}\,\xi, -t),$$

where $\xi,\eta\in\mathbb{R}^{2n}$ and $z,v,s,t\in\mathbb{R}$.

Lemma 2.4. *Each oscillator group* $H_n(\omega)\rtimes_p\mathbb{R}$ *is isomorphic to some* $Osc_n(\omega,B)$.

Furthermore, the oscillator groups $Osc_n(\omega,B)$ *and* $Osc(\lambda_1,\ldots,\lambda_n)$, *where* $0<\lambda_1\le\cdots\le\lambda_n$ *denote the positive imaginary parts of the eigenvalues of* B *with multiplicity, are isomorphic.*

Proof. First of all, we remark that the Lie algebra of the oscillator group $H_n(\omega)\rtimes_p\mathbb{R}$ is the semidirect sum of $\mathfrak{h}_n(\omega)$ and $\mathbb{R}$ with respect to the derivation $s\mapsto s\left(\begin{smallmatrix}0 & \delta^T\\ 0 & B\end{smallmatrix}\right)$.
Hence, there is a basis $\{X_1,Y_1,\ldots,X_n,Y_n\}$ of $\mathbb{R}^{2n}$ satisfying $\omega(X_i,Y_j)=\delta_{i,j}$, ($\delta_{i,j}$ denotes the Kronecker symbol) and $\omega(X_i,X_j)=\omega(Y_i,Y_j)=0$ for i,j=1,…,n such that the Lie algebra of the oscillator group $H_n(\omega)\rtimes_p\mathbb{R}$ is

$$\mathbb{R}Z\oplus\operatorname{span}\{X_1,Y_1,\ldots,X_n,Y_n\}\oplus\mathbb{R}T$$

with the non-zero brackets of $\{Z,X_1,Y_1,\ldots,X_n,Y_n,T\}$ given by

$$[X_i,Y_i]=Z,\quad [T,X_i]=BX_i+\delta_{2i-1}Z,\quad [T,Y_i]=BY_i+\delta_{2i}Z$$

for $i=1,\ \ldots,\ n$. Here δ_i denotes the i-th component of δ. Now

$$\phi: T\mapsto T+\sum_{j=1}^{n}(\delta_{2j}X_j-\delta_{2j-1}Y_j),\quad X_i\mapsto X_i,\quad Y_i\mapsto Y_i,\quad Z\mapsto Z$$

is an isomorphism from the Lie algebra of $H_n(\omega)\rtimes_p\mathbb{R}$ to $\mathfrak{osc}_n(\omega,B)$ and we proved the first part of the lemma. Now suppose that $\omega(B\cdot,\cdot)$ is positive definite (similarly for $\omega(B\cdot,\cdot)$ negative definite). Let $\{Z_1,\overline{Z_1},\ldots,Z_n,\overline{Z_n}\}$ be a basis of eigenvectors as in the proof of Lemma 2.2, such that $BZ_j=\mathrm{i}\lambda_jZ_j$ with $\lambda_j>0$ and $\lambda_1\le\cdots\le\lambda_n$. We set $\mu_j:=\frac{\mathrm{i}}{2}\omega(Z_j,\overline{Z_j})=\frac{\mathrm{i}}{2\lambda_j}\omega(BZ_j,\overline{Z_j})>0$. Then

$$Z\mapsto Z,\quad \frac{1}{\sqrt{\mu_j}}\operatorname{Re}(Z_j)\mapsto X_j,\quad \frac{1}{\sqrt{\mu_j}}\operatorname{Im}(Z_j)\mapsto Y_j,\quad T\mapsto -T$$

is an isomorphism from $\mathfrak{osc}_n(\omega,B)$ to $\mathfrak{osc}(\lambda_1,\ldots,\lambda_n)$. ∎

Note that $Osc(\lambda_1,\ldots,\lambda_n)$ is isomorphic to the group $G_k(\lambda)$ introduced in [8].

Lemma 2.5. *(Medina/Revoy: [8, p. 91])*
Let $0<\lambda_1\le\cdots\le\lambda_n$ *and* $0<\tilde\lambda_1\le\cdots\le\tilde\lambda_n$. *Then* $Osc(\lambda_1,\ldots,\lambda_n)$ *and* $Osc(\tilde\lambda_1,\ldots,\tilde\lambda_n)$ *are isomorphic if and only if there is a* $k>0$, *such that* $k\tilde\lambda_i=\lambda_i$ *for* $i=1,\ldots,n$.

Thus, each oscillator group of dimension $2n+2=4$ is isomorphic to $Osc(1)$.

Theorem 2.6. *Let $Osc_n(\omega, B)$ and $Osc_n(\omega, \tilde{B})$ be isomorphic and let $k > 0$ denote the number such that $k\tilde{\lambda}_i = \lambda_i$ for all positive imaginary parts of the eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_n$ of B and $0 < \tilde{\lambda}_1 \leq \cdots \leq \tilde{\lambda}_n$ of $\tilde{B}$ with multiplicity. A map $\phi : Osc_n(\omega, B) \to Osc_n(\omega, \tilde{B})$ is an isomorphism if and only if there are numbers $m \in \mathbb{R}$, $\mu \in \{\frac{1}{k}, -\frac{1}{k}\}$ and $a \in \mathbb{R}\backslash\{0\}$, a vector $b \in \mathbb{R}^{2n}$ and a matrix $S \in GL(2n, \mathbb{R})$ with $S^*\omega = a\omega$ and $SB = \mu\tilde{B}S$, such that*

$$\phi(z, \xi, t) = \left(az + \frac{1}{2}\omega(S\xi, \mathrm{e}^{t\mu\tilde{B}} b + b) + mt + \frac{1}{2}\omega(\mathrm{e}^{t\mu\tilde{B}} b, b), S\xi + \mathrm{e}^{t\mu\tilde{B}} b - b, \mu t\right). \tag{3}$$

Proof. One can check that a map satisfying condition (3) is an isomorphism. So we only verify the other implication. Let ϕ be an isomorphism. First of all, we check how elements of $\{0\} \times \{0\} \times \mathbb{R}$ will be mapped. Therefore, suppose $\phi(0, 0, t) = (z(t), \xi(t), \mu(t))$. Then

$$\begin{aligned}\big(z(s+t),&\xi(s+t), \mu(s+t)\big) = \phi(0,0,s)\phi(0,0,t) \\ &= \big(z(s) + z(t) + \frac{1}{2}\omega(\xi(s), \mathrm{e}^{\mu(s)\tilde{B}} \xi(t)), \xi(s) + \mathrm{e}^{\mu(s)\tilde{B}} \xi(t), \mu(s) + \mu(t)\big).\end{aligned}$$

We notice that μ is a linear mapping, i. e. $\mu(t) = \mu t$ for some $\mu \in \mathbb{R}\backslash\{0\}$.
Differentiating $\xi(s+t) = \xi(s) + \mathrm{e}^{\mu s\tilde{B}} \xi(t)$ with respect to s and setting $s = 0$, we get

$$\xi'(t) = \mu\tilde{B}b + \mu\tilde{B}\xi(t),$$

where $\mu\tilde{B}b = \xi'(0)$, $b \in \mathbb{R}^{2n}$. The solution of this ODE with $\xi(0) = 0$ is

$$\xi(t) = \mathrm{e}^{\mu t\tilde{B}} b - b.$$

Finally, the comparison of the first components shows that

$$z(s+t) + \frac{1}{2}\omega(b, \mathrm{e}^{(s+t)\mu\tilde{B}} b) = z(s) + \frac{1}{2}\omega(b, \mathrm{e}^{s\mu\tilde{B}} b) + z(t) + \frac{1}{2}\omega(b, \mathrm{e}^{t\mu\tilde{B}} b)$$

and thus

$$z(t) = mt - \frac{1}{2}\omega(b, \mathrm{e}^{t\mu\tilde{B}} b)$$

for some $m \in \mathbb{R}$.
Since $H_n(\omega)$ is the commutator subgroup of $Osc_n(\omega, B)$ and $Osc_n(\omega, \tilde{B})$, the isomorphism ϕ maps $H_n(\omega)$ onto itself. Hence $\phi(z, \xi, 0) = (az + \delta^T\xi, S\xi, 0)$, where $S^*\omega = a\omega$, $a \in \mathbb{R}\backslash\{0\}$ and $\delta \in \mathbb{R}^{2n}$ (compare to [13, p. 294]).
Thus $\phi(z, \mathrm{e}^{tB} \xi, 0)\phi(0, 0, t) = \phi(0, 0, t)\phi(z, \xi, 0)$ gives

$$\begin{aligned}&(\delta^T \mathrm{e}^{tB} \xi + az + mt + \frac{1}{2}\omega(\mathrm{e}^{t\mu\tilde{B}} b, b) + \frac{1}{2}\omega(S \mathrm{e}^{tB} \xi, \mathrm{e}^{t\mu\tilde{B}} b - b), S \mathrm{e}^{tB} \xi + \mathrm{e}^{t\mu\tilde{B}} b - b, \mu t) \\ &= (mt + \frac{1}{2}\omega(\mathrm{e}^{t\mu\tilde{B}} b, b) + \delta^T\xi + az + \frac{1}{2}\omega(\mathrm{e}^{t\mu\tilde{B}} b - b, \mathrm{e}^{\mu t\tilde{B}} S\xi), \mathrm{e}^{t\mu\tilde{B}} b - b + \mathrm{e}^{\mu t\tilde{B}} S\xi, \mu t).\end{aligned}$$

From the second component it follows, that $SB = \mu\tilde{B}S$.
Since $\det(SB) = \mu^{2n}\det(\tilde{B}S) = \mu^{2n}k^{2n}\det(BS)$, we get $\mu \in \{+\frac{1}{k}, -\frac{1}{k}\}$.
From the first component, we get $\delta^T \mathrm{e}^{tB} \xi + \omega(S \mathrm{e}^{tB} \xi, \mathrm{e}^{t\mu\tilde{B}} b - b) = \delta^T\xi$ for all $\xi \in \mathbb{R}^{2n}$, $t \in \mathbb{R}$. Differentiating and setting $t = 0$ gives $\delta^T B\xi - \omega(SB\xi, b) = 0$ for all ξ. This completes the proof. ∎

The next Corollary follows immediately

Corollary 2.7. *A map* $\phi : Osc_n(\omega, B) \to Osc_n(\omega, B)$ *is an automorphism if and only if there are numbers* $m \in \mathbb{R}$, $\mu \in \{+1, -1\}$ *and* $a \in \mathbb{R}\backslash\{0\}$, *a vector* $b \in \mathbb{R}^{2n}$ *and a matrix* $S \in GL(2n, \mathbb{R})$ *with* $S^*\omega = a\omega$ *and* $SB = \mu BS$, *such that*

$$\phi(z,\xi,t) = \left(az + \frac{1}{2}\omega(S\xi, \mathrm{e}^{t\mu B} b + b) + mt + \frac{1}{2}\omega(\mathrm{e}^{t\mu B} b, b), S\xi + \mathrm{e}^{t\mu B} b - b, \mu t\right). \tag{4}$$

So the automorphisms φ satisfying $\varphi|_{H_n(\omega)} = Id$ are of the form $(z,\xi,t) \mapsto (z+mt,\xi,t)$ with $m \in \mathbb{R}$. This result differs from that in [8, p. 92].

3. Lattices of oscillator groups

In this section we will describe lattices of oscillator groups up to equivalence under automorphisms of the ambient Lie group. For this classification we use three theorems. Now, Theorem 3.3 shows that we can describe each equivalence class by a certain data $r \in \mathbb{Z}_n^\sharp, B \in \mathbb{B}_r$ and $\xi \in \Xi_{r,B}$, which is defined in Definition 3.2. But to get some one-to-one correspondence between the isomorphism classes of lattices of the oscillator group and this data, we will define certain groups $\mathbb{S}_r$ and $\mathbb{S}_{r,B}$ with corresponding actions on $\mathbb{B}_r$ and $\Xi_{r,B}$ (Definition 3.6 and 3.8). Then we will prove in Theorem 3.7 and 3.9 the uniqueness of B modulo $\mathbb{S}_r$ and of ξ modulo $\mathbb{S}_{r,B}$. But first of all, we begin with an example.

Example 3.1. Let Γ_ω denote the subgroup in $Osc_n(\omega, B)$ generated by $\big\{(1,0,0),(0,e_i,0) \mid i = 1,\ldots,2n\big\}$. For each $z_0 \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}^{2n}$ the subgroup $L := \big\langle \Gamma_\omega \cup \{(z_0,\xi_0,1)\}\big\rangle$ is a lattice in $Osc_n(\omega,B)$ with $L \cap H_n(\omega) = \Gamma_\omega$ if and only if $\big(\omega(\xi_0, \mathrm{e}^B e_i), \mathrm{e}^B e_i, 0\big) \in \Gamma_\omega$ for $i = 1,\ldots,2n$. To see this, one can check that

$$(z,\xi,t)(v,\eta,0)(z,\xi,t)^{-1} = (v + \omega(\xi, \mathrm{e}^{tB}\eta), \mathrm{e}^{tB}\eta, 0) \tag{5}$$

for each $(v,\eta,0) \in H_n(\omega) \cap L$ and $(z,\xi,t) \in L$.

If L=$\langle \Gamma_\omega \cup \{(0,\xi,1)\}\rangle$ defines a lattice in $Osc_n(\omega,B)$ with $L \cap H_n(\omega) = \Gamma_\omega$, then we just call it $L(\xi)$. Furthermore, we set $\Gamma_r := \Gamma_{\omega_r}$ for $r \in (\mathbb{N}\backslash\{0\})^n$. Note that

$$\Gamma_r = \left\{(z,\xi,0) \,\middle|\, \xi \in \mathbb{Z}^{2n}, z \in \sum_{k=1}^{n} \frac{r_k}{2}\xi_{2k-1}\xi_{2k} + \mathbb{Z}\right\}.$$

In particular for $n = 1$, we have $\Gamma_r = \big\{(z,\xi,0) \mid \xi \in \mathbb{Z}^2, z \in \frac{1}{2}\xi_1\xi_2 + \mathbb{Z}\big\}$ for r odd and $\Gamma_r = \mathbb{Z}^3 \times \{0\}$ for r even. Moreover let $\Pi : Osc_n(\omega,B) \to \mathbb{R}$ denote the projection on the last component and for a lattice L of $Osc_n(\omega,B)$ we denote by $\Pi(L)$ the set $\Pi(L) := \big\{t \in \mathbb{R} \mid \exists\, z,\xi : (z,\xi,t) \in L\big\}$. Note that $\Pi(L)$ is a non-trivial discrete subgroup of $\mathbb{R}$ for each lattice L (compare to [8, p. 90]).

Definition 3.2. We define

$$\mathbb{Z}_n^\sharp := \left\{ r \in (\mathbb{N}\backslash\{0\})^n \mid r_k \text{ devides } r_{k+1}, k = 1, \dots, n-1 \right\}.$$

Then for $r \in \mathbb{Z}_n^\sharp$ we set

$$\mathbb{B}_r := \Big\{ B \in GL(2n, \mathbb{R}) \mid \omega_r(B\cdot, \cdot) \text{ is symmetric and negative definite and } \mathrm{e}^B \in SL(2n, \mathbb{Z}) \Big\}.$$

Moreover for $r \in \mathbb{Z}_n^\sharp$ and $B \in \mathbb{B}_r$, we set

$$\Xi_{r,B} := \left\{ \xi \in \mathbb{R}^{2n} \mid \omega_r(\xi, \mathrm{e}^B e_i) \in \sum_{k=1}^{n} \frac{1}{2} r_k (\mathrm{e}^B e_i)_{2k-1} (\mathrm{e}^B e_i)_{2k} + \mathbb{Z}, i = 1, \dots, 2n \right\}.$$

Theorem 3.3. *Let L be a lattice in $Osc_n(\omega, B)$. Then there exists a uniquely determined $r \in \mathbb{Z}_n^\sharp$, a linear map $\tilde{B} \in \mathbb{B}_r$, an isomorphism $\Phi : Osc_n(\omega, B) \to Osc_n(\omega_r, \tilde{B})$ and a $\xi_0 \in \Xi_{r,\tilde{B}}$, such that*

$$\Phi(L) = L(\xi_0).$$

Proof. At first, note that $L \cap H_n(\omega)$ is a lattice in $H_n(\omega)$, see [12, p. 50]. Furthermore, we know from Theorem 1.10 in [13, p. 303] that there is a uniquely determined $r = (r_1, \dots, r_n) \in \mathbb{Z}_n^\sharp$ and an isomorphism $\varphi : H_n(\omega) \to H_n(\omega_r)$, $\varphi(z, \xi) = (\delta^T \xi + az, S\xi)$, where $S^* \omega_r = a\omega$, such that $\varphi\big(L \cap H_n(\omega_r)\big) = \Gamma_r$. We choose b, such that $\delta^T \xi = \omega_r(S\xi, b)$ for all $\xi \in \mathbb{R}^{2n}$. Then

$$\overline{\varphi}(z, \xi, t) = \left(az + \frac{1}{2}\omega_r(S\xi, \mathrm{e}^{tSBS^{-1}} b + b) + \frac{1}{2}\omega_r(\mathrm{e}^{tSBS^{-1}} b, b), S\xi + \mathrm{e}^{tSBS^{-1}} b - b, t \right)$$

is an isomorphism from $Osc_n(\omega, B)$ to $Osc_n(\omega_r, SBS^{-1})$, mapping $L \cap H_n(\omega)$ onto Γ_r.

If $\omega_r(SBS^{-1}\cdot, \cdot)$ is positive definite, then we use the map $\phi_2 : (z, \xi, t) \mapsto (z, \xi, -t)$, which is an isomorphism from $Osc_n(\omega_r, SBS^{-1})$ to $Osc_n(\omega_r, -SBS^{-1})$ with $\phi_2|_{H_n(\omega_r)} = Id$.

Let t_0 denote the smallest positive element in $\Pi(\overline{\varphi}(L))$. So there is a z_0 and a ξ_0, such that $(z_0, \xi_0, t_0) \in \overline{\varphi}(L)$. The map

$$\phi : (z, \xi, t) \mapsto \left(z - \frac{z_0}{t_0} t, \xi, \frac{t}{t_0} \right)$$

is an isomorphism from $Osc_n(\omega_r, \pm SBS^{-1})$ to $Osc_n(\omega_r, \pm t_0 SBS^{-1})$ such that $\phi|_{H_n(\omega_r)} = Id$ and (z_0, ξ_0, t_0) maps to $(0, \xi_0, 1)$. Moreover $\tilde{B} := \pm t_0 SBS^{-1}$ has vanishing trace. Furthermore

$$\mathrm{e}^{\tilde{B}} e_i \in \mathbb{Z}^{2n} \text{ and } \omega_r(\xi_0, \mathrm{e}^{\tilde{B}} e_i) \in \textstyle\sum_{k=1}^n \frac{1}{2} r_k (\mathrm{e}^{\tilde{B}} e_i)_{2k-1} (\mathrm{e}^{\tilde{B}} e_i)_{2k} + \mathbb{Z},$$

since $(\omega_r(\xi_0, \mathrm{e}^{\tilde{B}} e_i), \mathrm{e}^{\tilde{B}}, 0) = (0, \xi_0, 1)(0, e_i, 0)(0, \xi_0, 1)^{-1} \in \Gamma_r$. Thus the theorem is proved. ■

Proposition 3.4. *Let* $\varphi : (z,\xi) \mapsto (az + \delta^T\xi, S\xi)$ *be an automorphism of* $H_n(\omega_r)$. *Then* $\varphi(\Gamma_r) = \Gamma_r$ *if and only if* $a = \pm 1$, $S \in GL(2n,\mathbb{Z})$ *and* $\delta_i \in \frac{1}{2}\sum_{k=1}^n r_k(Se_i)_{2k-1}(Se_i)_{2k} + \mathbb{Z}$ *for* $i = 1,\dots,2n$.

We need the following lemma to prove the Proposition.

Lemma 3.5. *Suppose* $S \in GL(2n,\mathbb{Z})$ *satisfying* $S^*\omega_0 = \pm\omega_0$, *where*

$$\omega_0(\xi,\eta) = \xi^T \begin{pmatrix} 0 & \operatorname{diag}(r_1,\dots,r_n) \\ -\operatorname{diag}(r_1,\dots,r_n) & 0 \end{pmatrix} \eta.$$

If $\delta_i \in \frac{1}{2}\sum_{k=1}^n r_k(Se_i)_k(Se_i)_{k+n} + \mathbb{Z}$ *for* $i = 1,\dots,2n$, *then for all* $\xi \in \mathbb{Z}^{2n}$ *we get*

$$\delta^T\xi \in \frac{1}{2}\sum_{k=1}^n r_k(S\xi)_k(S\xi)_{k+n} - \frac{1}{2}\sum_{k=1}^n r_k\xi_k\xi_{k+n} + \mathbb{Z}.$$

Proof. For short, we denote $R := \operatorname{diag}(r_1,\dots,r_n)$. Suppose $S \in GL(2n,\mathbb{Z})$, $S^*\omega_0 = \pm\omega_0$ and $\delta_i \in \frac{1}{2}\sum_{k=1}^n r_k(Se_i)_k(Se_i)_{k+n} + \mathbb{Z}$ for $i = 1,\dots,2n$. We show that

$$\alpha : \mathbb{R}^{2n} \to \mathbb{R}, \quad \xi \mapsto \frac{1}{2}\sum_{k=1}^n r_k(S\xi)_k(S\xi)_{k+n} - \frac{1}{2}\sum_{k=1}^n r_k\xi_k\xi_{k+n}$$

satisfies $\alpha(\lambda\xi+\tilde\xi) \in \lambda\alpha(\xi+\tilde\xi)+\mathbb{Z}$ for all $\xi,\tilde\xi \in \mathbb{Z}^{2n}$ and $\lambda \in \mathbb{Z}$. Therefore suppose

$$S = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.$$

Since $S^*\omega_0 = \pm\omega_0$ we get $A_1^TRA_3$ and $A_2^TRA_4$ are symmetric and $A_2^TRA_3 = A_4^TRA_1 \pm R$. Furthermore, suppose $\xi = (\xi_{11},\xi_{12})$ and $\tilde\xi = (\xi_{21},\xi_{22})$ for $\xi_{ij} \in \mathbb{Z}^n$ $i,j = 1,2$.
Then $\alpha(\xi+\tilde\xi)$

$$\begin{aligned}
=&\sum_k \frac{r_k}{2}(S(\xi+\tilde\xi))_k(S(\xi+\tilde\xi))_{k+n} - \sum_k \frac{r_k}{2}(\xi+\tilde\xi)_k(\xi+\tilde\xi)_{k+n}\\
=&\sum_k \frac{r_k}{2}\big((S\xi)_k(S\xi)_{k+n} + (S\tilde\xi)_k(S\tilde\xi)_{k+n}\big) + \sum_k \frac{r_k}{2}\big((S\xi)_k(S\tilde\xi)_{k+n} + (S\tilde\xi)_k(S\xi)_{k+n}\big)\\
&-\sum_k \frac{r_k}{2}\big((\xi)_k(\tilde\xi)_{k+n} + (\tilde\xi)_k(\xi)_{k+n}\big) - \sum_k \frac{r_k}{2}\big((\xi)_k(\xi)_{k+n} + (\tilde\xi)_k(\tilde\xi)_{k+n}\big)\\
=&\alpha(\xi)+\alpha(\tilde\xi) + \sum_k \frac{r_k}{2}[((A_1A_2)\xi)_k((A_3A_4)\tilde\xi)_k + ((A_1A_2)\tilde\xi)_k((A_3A_4)\xi)_k]\\
&-\sum_k \frac{r_k}{2}\big((\xi)_k(\tilde\xi)_{k+n} + (\tilde\xi)_k(\xi)_{k+n}\big)\\
=&\alpha(\xi)+\alpha(\tilde\xi) + \frac{1}{2}(\xi_{11}^TA_1^T + \xi_{12}^TA_2^T)R(A_3\xi_{21} + A_4\xi_{22})\\
&+\frac{1}{2}(\xi_{11}^TA_3^T + \xi_{12}^TA_4^T)R(A_1\xi_{21} + A_2\xi_{22}) - \frac{1}{2}\xi_{11}^TR\xi_{22} - \frac{1}{2}\xi_{12}^TR\xi_{21}\\
=&\alpha(\xi)+\alpha(\tilde\xi) + \xi_{11}^TA_1^TRA_3\xi_{21} + \xi_{11}^TA_1^TRA_4\xi_{22} + \xi_{12}^TA_4^TRA_1\xi_{21}\\
&+\xi 12TA_2^TRA_4\xi_{22} - \frac{1}{2}\xi_{11}^TR\xi_{22} \mp \frac{1}{2}\xi_{11}^TR\xi_{22} - \frac{1}{2}\xi_{12}^TR\xi_{21} \mp \frac{1}{2}\xi_{12}^TR\xi_{21}
\end{aligned}$$

$\in \alpha(\xi) + \alpha(\tilde{\xi}) + \mathbb{Z}$.
For $\xi \in \mathbb{Z}^{2n}$ and $\lambda \in \mathbb{Z}$ we also get

$$\begin{aligned}\alpha(\lambda\xi) &= \frac{1}{2}\sum_{k=1}^{n} r_k(\lambda S\xi)_k(\lambda S\xi)_{k+n} - \frac{1}{2}\sum_{k=1}^{n} r_k\lambda\xi_k\lambda\xi_{k+n} \\ &= \lambda^2\big(\frac{1}{2}\sum_{k=1}^{n} r_k(S\xi)_k(S\xi)_{k+n} - \frac{1}{2}\sum_{k=1}^{n} r_k\xi_k\xi_{k+n}\big) \\ &\in \lambda\big(\frac{1}{2}\sum_{k=1}^{n} r_k(S\xi)_k(S\xi)_{k+n} - \frac{1}{2}\sum_{k=1}^{n} r_k\xi_k\xi_{k+n}\big) + \mathbb{Z} \\ &= \lambda\alpha(\xi) + \mathbb{Z}.\end{aligned}$$

Now, we see for $\xi \in \mathbb{Z}^{2n}$ that $\delta^T\xi$

$$\begin{aligned}&= \sum_{i=1}^{2n}\xi_i\delta_i \in \frac{1}{2}\sum_{i=1}^{2n}\big(\xi_i\sum_{k=1}^{n} r_k(Se_i)_k(Se_i)_{k+n}\big) + \mathbb{Z} = \frac{1}{2}\sum_{i=1}^{2n}\xi_i\alpha(e_i) + \mathbb{Z} = \alpha(\xi) + \mathbb{Z} \\ &= \sum_{k=1}^{n} r_k(S\xi)_k(S\xi)_{k+n} - \sum_{k=1}^{n} r_k\xi_k\xi_{k+n} + \mathbb{Z},\end{aligned}$$

and we proved the assertion. ■

Proof of Proposition 3.4. Assume that the automorphism φ maps Γ_r onto itself. Then $a = \pm 1$, since $(a, 0), (\frac{1}{a}, 0) \in \Gamma_r$. Furthermore $\varphi(0, e_i) = (\delta^T e_i, Se_i) \in \Gamma_r$. Hence $\delta_i = \delta^T e_i$ is an element of $\frac{1}{2}\sum_{k=1}^{n} r_k(Se_i)_{2k-1}(Se_i)_{2k} + \mathbb{Z}$ and $Se_i \in \mathbb{Z}^{2n}$ for $i = 1, \ldots, 2n$. Moreover $(-a\delta^T S^{-1}e_i, S^{-1}e_i) \in \Gamma_r$, since $\varphi(-a\delta^T S^{-1}e_i, S^{-1}e_i) = (0, e_i) \in \Gamma_r$. Thus $S^{-1}e_i \in \mathbb{Z}^{2n}$ and $S \in GL(2n, \mathbb{Z})$.
For the other direction we see that $\varphi : (z, \xi) \mapsto (az + \delta^T\xi, S\xi)$ maps $(0, e_i)$ to $(\delta_i, Se_i) \in \Gamma_r$, $(1, 0)$ to $(\pm 1, 0) \in \Gamma_r$. We define a transformation matrix T by $Te_i = e_{2i-1}$ for $i \leq n$ and $Te_i = e_{2(i-n)}$ for $i > n$, which satisfies $T^*\omega_0 = \omega_r$. Then

$$\begin{aligned}\delta^T T^{-1}e_i &\in \frac{1}{2}\sum_{k=1}^{n} r_k(ST^{-1}e_i)_{2k-1}(ST^{-1}e_i)_{2k} + \mathbb{Z} \\ &= \frac{1}{2}\sum_{k=1}^{n} r_k(ST^{-1}e_i)_k(ST^{-1}e_i)_{k+n} + \mathbb{Z}\end{aligned}$$

and Lemma 3.5 shows

$$\begin{aligned}-a\delta^T S^{-1}e_i &= -a\delta^T T^{-1}TS^{-1}e_i \\ &\in -a\frac{1}{2}\big(\sum_{k=1}^{n} r_k(Te_i)_k(Te_i)_{k+n} - \sum_{k=1}^{n} r_k(TS^{-1}e_i)_k(TS^{-1}e_i)_{k+n}\big) + \mathbb{Z} \\ &= a\frac{1}{2}\sum_{k=1}^{n} r_k(S^{-1}e_i)_{2k-1}(S^{-1}e_i)_{2k} + \mathbb{Z} \\ &= -a\frac{1}{2}\sum_{k=1}^{n} r_k(S^{-1}e_i)_{2k-1}(S^{-1}e_i)_{2k} + \mathbb{Z}.\end{aligned}$$

Thus $(-a\delta^T S^{-1}e_i, S^{-1}e_i) \in \Gamma_r$ and $\varphi(-a\delta^T S^{-1}e_i, S^{-1}e_i) = (0, e_i)$. This proves that φ maps Γ_r onto Γ_r. ∎

Definition 3.6. We define the group

$$\mathbb{S}_r := \{S \in GL(2n, \mathbb{Z}) \mid S^*\omega_r = \pm\omega_r\}.$$

Moreover the formula $S.B := \mu SBS^{-1}$, where $S^*\omega_r = \mu\omega_r$, defines a left action of $\mathbb{S}_r$ on $\mathbb{B}_r$. Let $\mathbb{S}_r\backslash\mathbb{B}_r$ denote the set of all left cosets.

Theorem 3.7. *Assume $B, \tilde{B} \in \mathbb{B}_r$. Let $L(\xi_0)$ be a lattice in $Osc_n(\omega_r, B)$. There is an isomorphism $\varphi : Osc_n(\omega_r, B) \to Osc_n(\omega_r, \tilde{B})$ and an $\tilde{\xi} \in \mathbb{R}^{2n}$ such that $\varphi(L(\xi_0)) = L(\tilde{\xi})$ if and only if $[B] = [\tilde{B}] \in \mathbb{S}_r\backslash\mathbb{B}_r$.*

Proof. Let φ and $\tilde{\xi}$ be as in the theorem. We know from Theorem 2.6 that there is an $a, \mu \in \mathbb{R}\backslash\{0\}$ and an invertible matrix S such that $S^*\omega_r = a\omega_r$ and $SB = \mu BS$. Since $\Pi(\varphi(0, \xi_0, 1)) = \pm 1$ we get $\mu \in \{1, -1\}$. Furthermore $\varphi|_{H_n(\omega_r)}$ is an automorphism of $H_n(\omega_r)$ which maps Γ_r onto Γ_r. Hence $a \in \{1, -1\}$, $S \in GL(2n, \mathbb{Z})$. Finally $\mu = a$, since $\omega_r(B\cdot, \cdot)$ is negative definite and

$$a\omega_r(B\xi, \xi) = S^*\omega_r(B\xi, \xi) = \omega_r(SB\xi, S\xi) = \mu\omega_r(BS\xi, S\xi).$$

So one direction of the lemma is verified.
Now, assume $[B] = [\tilde{B}] \in \mathbb{S}_r\backslash\mathbb{B}_r$. Then there is an integer Matrix S satisfying $S^*\omega_r = \pm\omega_r$ and $\tilde{B} = \mu SBS^{-1}$. We set $m = 0$ and $\delta_i = \frac{1}{2}\sum_{k=1}^n r_k(Se_i)_{2k-1}(Se_i)_{2k}$. Then, by Proposition 3.4, $\phi : (z, \xi) \mapsto (\pm z + \delta^T\xi, S\xi)$ is an automorphism of $H_n(\omega_r)$ which maps Γ_r onto Γ_r. Now we set b such that $\omega_r(Se_i, b) = \delta_i$. Then

$$\tilde{\varphi} : (z, \xi, t) \mapsto (az + \frac{1}{2}\omega_r(S\xi, \mathrm{e}^{\mu t\tilde{B}}\, b + b) + \frac{1}{2}\omega_r(\mathrm{e}^{\mu t\tilde{B}}\, b, b), S\xi + \mathrm{e}^{\mu t\tilde{B}}\, b - b, \mu t)$$

is an isomorphism from $Osc_n(\omega_r, B)$ to $Osc_n(\omega_r, \tilde{B})$, satisfying $\tilde{\varphi}(\Gamma_r) = \Gamma_r$ and $\Pi(\tilde{\varphi}(0, 0, 1)) = \pm 1$, since $\tilde{\varphi}|_{H_n(\omega_r)} = \phi$. Then $\tilde{\varphi}(0, \xi_0, 1) = (\tilde{z}, \tilde{\xi}, 1)^\mu$ for some $\tilde{z} \in \mathbb{R}$ and $\tilde{\xi} \in \mathbb{R}^{2n}$. We define an automorphism of $Osc_n(\omega_r, \tilde{B})$ by $\hat{\varphi} : (z, \xi, t) \mapsto (z - \tilde{z}t, \xi, t)$ and finally $\varphi := \hat{\varphi} \circ \tilde{\varphi}$ is an isomorphism mapping $L(\xi_0)$ onto $L(\tilde{\xi})$. ∎

Definition 3.8. For $r \in (\mathbb{N}\backslash\{0\})^n$ and $B \in \mathbb{B}_r$ we define the subgroup

$$\begin{aligned}\mathbb{S}_{r,B} :=\Big\{&(S, b) \in GL(2n, \mathbb{Z}) \ltimes \mathbb{R}^{2n} \mid \exists\mu \in \{\pm 1\} : S^*\omega_r = \mu\omega_r, SB = \mu BS,\\ &\omega_r(Se_i, b) \in \sum_{k=1}^n \frac{1}{2} r_k(Se_i)_{2k-1}(Se_i)_{2k} + \mathbb{Z}, i = 1, \ldots, 2n\Big\}.\end{aligned}$$

Note that $\xi + \eta \in \Xi_{r,B}$ for $\eta \in \mathbb{Z}^{2n}$ and $\xi \in \Xi_{r,B}$. Hence this defines a right action of $\mathbb{Z}^{2n}$ on $\Xi_{r,B}$. Let ${}^{\Xi_{r,B}}/_{\mathbb{Z}^{2n}}$ denote the set of right cosets of this action. We define by

$$(S, b).\xi := \begin{cases} S\xi + \mathrm{e}^B\, b - b + \mathbb{Z}^{2n} & \text{if } S^*\omega_r = \omega_r \\ -S\,\mathrm{e}^{-B}\,\xi + \mathrm{e}^B\, b - b + \mathbb{Z}^{2n} & \text{if } S^*\omega_r = -\omega_r \end{cases}$$

a left action of the group $\mathbb{S}_{r,B}$ on $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$. By Lemma 3.5 and the transformation matrix T which satisfies $Te_i = e_{2i-1}$ for $i \leq n$ and $Te_i = e_{2(i-n)}$ for $i > n$ and $T^*\omega_0 = \omega_r$, this action leaves the set $\Xi_{r,B}/\mathbb{Z}^{2n}$ invariant. We denote the set of cosets by $\mathbb{S}_{r,B}\backslash\Xi_{r,B}/\mathbb{Z}^{2n}$.

Theorem 3.9. *There is a bijection between the equivalence classes of lattices $L(\xi)$ of $Osc_n(\omega_r, B)$ under the automorphisms of the ambient Lie group and the elements of $\mathbb{S}_{r,B}\backslash\Xi_{r,B}/\mathbb{Z}^{2n}$, mapping $L(\xi)$ to $[\xi]$.*

Proof. Let φ be an automorphism of $Osc_n(\omega_r, B)$, given as in formula (4), mapping $L(\xi_1)$ onto $L(\xi_2)$. Then $\varphi|_{H_n(\omega_r)}$ is an automorphism of $H_n(\omega_r)$ mapping Γ_r onto Γ_r and by Proposition 3.4 we get $a = \pm 1$, $S \in GL(2n, \mathbb{Z})$, $S^*\omega_r = a\omega_r$ and $\omega_r(Se_i, b) \in \sum_{k=1}^n \frac{r_k}{2}(Se_i)_{2k-1}(Se_i)_{2k} + \mathbb{Z}$. Since $\omega_r(BS\xi, S\xi)$ is negative definite and $\omega_r(BS\xi, S\xi) = \omega_r(\mu SB\xi, S\xi) = \mu a\omega_r(B\xi, \xi)$, we also get $\mu = a$. Then for $\mu = 1$ we see that

$$\begin{aligned}\varphi(0, \xi_1, 1) &= (\omega_r(S\xi_1, \mathrm{e}^B b + b) + m + \frac{1}{2}\omega_r(\mathrm{e}^B b, b), S\xi_1 + \mathrm{e}^B b - b, 1) \\ &= (k, 0, 0)(0, \eta_1 e_1, 0) \dots (0, \eta_{2n} e_{2n}, 0)(0, \xi_2, 1)\end{aligned}$$

for some $\eta_i, k \in \mathbb{Z}$. Thus $S\xi + \mathrm{e}^B b \in \xi_2 + \mathbb{Z}^{2n}$. For $\mu = -1$ we have instead that

$$\begin{aligned}&\varphi(0, -\mathrm{e}^{-B}\xi_1, -1) \\ &\quad= (\omega_r(-S\,\mathrm{e}^{-B}\xi_1, \mathrm{e}^B b + b) - m + \frac{1}{2}\omega_r(\mathrm{e}^B b, b), -S\,\mathrm{e}^{-B}\xi_1 + \mathrm{e}^B b - b, 1) \\ &\quad= (k, 0, 0)(0, \eta_1 e_1, 0) \dots (0, \eta_{2n} e_{2n}, 0)(0, \xi_2, 1)\end{aligned}$$

for some $\eta_i, k \in \mathbb{Z}$. Hence $-S\,\mathrm{e}^{-B}\xi + \mathrm{e}^B b \in \xi_2 + \mathbb{Z}^{2n}$. This shows that $[\xi_1] = [\xi_2] \in \mathbb{S}_{r,B}\backslash\Xi_{r,B}/\mathbb{Z}^{2n}$ and that the map between the equivalence classes of lattices $L(\xi)$ of $Osc_n(\omega_r, B)$ and the elements of $\mathbb{S}_{r,B}\backslash\Xi_{r,B}/\mathbb{Z}^{2n}$ is well-defined. The surjectivity follows directly, since $(\mathrm{e}^B, \xi) \in \mathbb{S}_{r,B}$ and thus $L := \langle(1, 0, 0), (0, e_i, 0), (0, \xi, 1)\rangle$ defines a lattice in $Osc_n(\omega_r, B)$ with $L \cap H_n(\omega_r) = \Gamma_r$. At least we show the injectivity. Suppose $[\xi_1] = [\xi_2]$. Then there are $\eta \in \mathbb{Z}^{2n}$, $S \in GL(2n, \mathbb{Z})$ satisfying $S^*\omega_r = \mu\omega_r$ for some $\mu \in \{\pm 1\}$ and a vector b with $\omega_r(Se_i, b) \in \sum_{k=1}^n \frac{r_k}{2}(Se_i)_{2k-1}(Se_i)_{2k} + \mathbb{Z}$ such that $\mu S\,\mathrm{e}^{\frac{1}{2}(\mu-1)B}\xi_1 + \mathrm{e}^B b - b = \xi_2 + \eta$. We set $a = \mu$ and

$$m = \mu\frac{1}{2}\omega_r(\eta, \xi_2) - \mu\omega_r\big(\mu S\,\mathrm{e}^{\frac{1}{2}(\mu-1)B}\xi_1, \mathrm{e}^B b + b\big) - \mu\frac{1}{2}\omega_r(\mathrm{e}^B b, b) + \mu\sum_{k=1}^n \frac{r_k}{2}\eta_{2k-1}\eta_{2k}$$

and define an automorphism φ with these S, b, μ, a and m as described in formula (4). This automorphism maps Γ_r onto itself (compare Proposition 3.4) and $(0, \mu\,\mathrm{e}^{\frac{1}{2}(\mu-1)B}\xi_1, \mu)$ to

$$\big(\sum_{k=1}^n \frac{r_k}{2}\eta_{2k-1}\eta_{2k} + \frac{1}{2}\omega_r(\eta, \xi_2), \eta + \xi_2, 1\big) = (0, \eta_1 e_1, 0) \dots (0, \eta_{2n} e_{2n}, 0)(0, \xi_2, 1).$$

Thus $\varphi(L(\xi_1)) = L(\xi_2)$. ∎

4. Lattices of the 4-dimensional oscillator group

In this section we concentrate on the lattices of the 4-dimensional oscillator group. As we already know from Theorem 3.3, 3.7 and 3.9, we can classify every lattice of oscillator groups up to automorphisms of the ambient oscillator group by a data (r, B, ξ), where $B \in \mathbb{B}_r$ and $\xi \in \Xi_{r,B}/\mathbb{Z}^{2n}$ are unique modulo $\mathbb{S}_r$ and $\mathbb{S}_{r,B}$. Now, we define the matrices $B_{x,y}$ so that we can give a certain fundamental domain for $\mathbb{S}_r\backslash\mathbb{B}_r$ in Theorem 4.4. Finally we give in Theorem 4.5 for each $B = \lambda B_{x,y}$ in that fundamental domain a system of representatives of $\mathbb{S}_{r,B}\backslash{}^{\Xi_{r,B}}/\mathbb{Z}^{2n}$ and thus obtain Theorem 1.1 as a corollary. Let us be more precise.

Definition 4.1. For $y \neq 0$ and $x \in \mathbb{R}$ we denote

$$B_{x,y} := \begin{pmatrix} \frac{x}{y} & -\frac{x^2}{y} - y \\ \frac{1}{y} & -\frac{x}{y} \end{pmatrix}.$$

The set of all $B_{x,y}$ is equal to the set of all matrices which are conjugate to N_1, and to the set of all 2×2-matrices with determinant 1 and trace 0.

Definition 4.2. We define

$$\mathbb{F}_+ := \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \frac{1}{2}, y > 0, x^2 + y^2 \geq 1\right\}$$

and

$$\mathbb{F} = \mathbb{F}_+ \cup \left\{(x, y) \in \mathbb{R}^2 \mid -\frac{1}{2} < x < 0 < y, x^2 + y^2 > 1\right\}.$$

Note that the map $\iota : B_{x,y} \mapsto x+iy$ is a bijection from $\left\{B_{x,y} \mid y > 0\right\}$ to the upper half plane of $\mathbb{C}$, satisfying $\iota(AB_{x,y}A^{-1}) = A(x+iy)$ for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$, where $Az = \frac{az+b}{cz+d}$. With this in mind, we rewrite the theorem in [7, p. 109]:

Remark 4.3. For all $B_{x',y'}$, $y' > 0$ there is a uniquely defined $(x, y) \in \mathbb{F}$ and an $S \in SL(2, \mathbb{Z})$, such that $SB_{x',y'}S^{-1} = B_{x,y}$.

Theorem 4.4. *Let L be a lattice of $Osc_1(\omega, B)$. Then there exist*

- *a uniquely determined $r \in \mathbb{N}\backslash\{0\}$,*
- *a uniquely determined $\lambda = \lambda_0 + k\pi$ with $k \in \mathbb{N}$ and $\lambda_0 \in \left\{\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi, \pi\right\}$*
- *and a uniquely determined*

$$(x, y) \begin{cases} = (\frac{1}{2}, \frac{\sqrt{3}}{2}), & \lambda_0 \in \{\frac{1}{3}\pi, \frac{2}{3}\pi\} \\ = (0, 1), & \lambda_0 = \frac{1}{2}\pi \\ \in \mathbb{F}_+, & \lambda_0 = \pi \end{cases},$$

and an isomorphism $\varphi : Osc_1(\omega, B) \to Osc_1(\omega_r, \lambda B_{x,y})$ satisfying $\varphi(L) \cap H_1(\omega_r) = \Gamma_r$ and $\Pi(\varphi(L)) = \mathbb{Z}$.
Conversely, for any such data $(r, \lambda, (x, y))$ there exists a lattice L in $Osc_1(\omega_r, \lambda B_{x,y})$ satisfying $L \cap H_1(\omega_r) = \Gamma_r$ and $\Pi(L) = \mathbb{Z}$.

Proof. Because of Theorem 3.3 we know that $r \in \mathbb{Z}_1^\sharp$ is uniquely determined. Moreover we know from Theorem 3.7 the existence of an $B = \lambda B_{x,y} \in \mathbb{B}_r$, which is unique modulo $\mathbb{S}_r$. So the procedure is to find a fundamental domain of $\mathbb{S}_r \backslash \mathbb{B}_r$. Be $\lambda B_{x,y} \in \mathbb{B}_r$. Then $x \in \mathbb{R}$ and $\lambda, y > 0$. We know that B and N_λ are conjugate. Hence e^B and $\mathrm{e}^{N_\lambda} = \left(\begin{smallmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{smallmatrix}\right)$ are conjugate. So $\mathrm{tr}(\mathrm{e}^B) = 2\cos\lambda$ and $\cos\lambda \in \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$, since $\mathrm{e}^B \in SL(2,\mathbb{Z})$. Thus $\lambda = \lambda_0 + k\pi \neq 0$, where $\lambda_0 \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi\right\}$ and $k \in \mathbb{N}$. Using Remark 4.3, we see that $[\lambda B_{x,y}] = [\lambda B_{x',y'}]$ for some $(x', y') \in \mathbb{F}$. Moreover $[\lambda B_{x',y'}] = [\lambda B_{-x',y'}]$, since $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathbb{S}_r$ and $S.B_{x',y'} = B_{-x',y'}$.
Now assume $[\lambda' B_{x',y'}] = [\lambda'' B_{x'',y''}]$ for $\lambda', \lambda'' > 0$, $(x', y'), (x'', y'') \in \mathbb{F}_1$. We want to see that $\lambda' = \lambda''$ and $(x', y') = (x'', y'')$. The first equation follows immediatly from $\lambda'^2 = \det(\lambda B_{x',y'}) = \det(\lambda'' B_{x'',y''}) = \lambda''^2$. If there is an $S \in SL(2,\mathbb{Z})$ satisfying $\lambda' S B_{x',y'} S^{-1} = \lambda' B_{x'',y''}$, then $(x', y') = (x'', y'')$ follows by Remark 4.3. If instead S satisfies $\lambda' S B_{x',y'} S^{-1} = -\lambda' B_{x'',y''}$. Then there is an $\tilde{S} \in SL(2,\mathbb{Z})$ satisfying $\lambda' \tilde{S} B_{-x',y'} \tilde{S}^{-1} = \lambda' B_{x'',y''}$, since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).B_{x',y'} = B_{-x',y'}$. Then $x' \in \{0, \frac{1}{2}\}$ because of Remark 4.3 and thus for x'=0 immediatly $(x', y') = (x'', y'')$. For $x' = \frac{1}{2}$ we see that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right).B_{-\frac{1}{2},y'} = B_{\frac{1}{2},y'}$ and finally by Remark 4.3 that $(x', y') = (x'', y'')$.
At last, we want to see how λ and (x, y) fit together. Suppose $B = \lambda B_{x,y} \in \mathbb{B}_r$, where $(x, y) \in \mathbb{F}_+$ and $\lambda = \lambda_0 + k\pi$ for $k \in \mathbb{N}$ and $\lambda_0 \in \left\{\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi\right\}$. Then

$$\mathrm{e}^B = \begin{pmatrix} \cos\lambda + \frac{x}{y}\sin\lambda & -\frac{x^2}{y}\sin\lambda - y\sin\lambda \\ \frac{1}{y}\sin\lambda & -\frac{x}{y}\sin\lambda + \cos\lambda \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{6}$$

Thus $\frac{1}{y}\sin\lambda \in \mathbb{Z}$ and $y = |\sin\lambda|$. So $x = \frac{1}{2}$ if $\cos\lambda = \pm\frac{1}{2}$, and $x = 0$ if $\cos\lambda = 0$. We get

$$B = \begin{cases} \lambda B_{0,1}, & \cos\lambda = 0 \\ \lambda B_{\frac{1}{2},\frac{\sqrt{3}}{2}}, & \cos\lambda = \pm\frac{1}{2} \end{cases}.$$

Conversely, it is obvious that $\mathrm{e}^{\lambda B_{x,y}} \in SL(2,\mathbb{Z})$ for the data $(r, \lambda, (x, y))$ described in the theorem. Hence there is a lattice satisfying the claimed conditions (see Example 3.1, where $\xi_0 = (0, 0)$, $\xi_0 = (0, \frac{1}{2})$ or $\xi_0 = (\frac{1}{2}, 0)$ depending on r and e^B). ■

Now we can prove Theorem 1.1 by using Theorem 4.4 and the following one.

Theorem 4.5. *Suppose $r \in \mathbb{N}\backslash\{0\}$ and $B = \lambda B_{x,y}$ with (x, y) and λ as in Theorem 4.4. Be $L(\xi)$ given in $Osc_1(\omega_r, B)$. Then there is a uniquely defined ξ_0 to extract from the list in Section 6 and an automorphism φ of $Osc_1(\omega_r, B)$, such that $\varphi(L(\xi)) = L(\xi_0)$.*

By Theorem 3.9 we know that it is sufficient to show that the given ξ_0 in the list in Section 6 form a system of representatives for $\mathbb{S}_{r,B} \backslash {}^{\Xi_{r,B}}/_{\mathbb{Z}^{2n}}$. In the first paragraph of the proof of Theorem 4.5 we show that every ξ in $\Xi_{r,B}$ has an ξ_0 as described in the list in Section 6, which is equivalent to ξ in $\mathbb{S}_{r,B} \backslash {}^{\Xi_{r,B}}/_{\mathbb{Z}^{2n}}$. This proof will be divided into parts, dependent on the value of λ. Here we always use e^B, which we can compute with equation (6) in the proof of Theorem 4.4. First we notice, depending on r, which are the elements of $\Xi_{r,B}$ and afterward, we give a $(S,b)\in \mathbb{S}_{r,B}$ which maps $\xi\in\Xi_{r,B}$ to one of the described ξ_0 in the list. In the second paragraph we show, that for every $\xi,\tilde{\xi}\in \Xi_{r,B}/\mathbb{Z}^{2n}$ with $\xi\neq\tilde{\xi}$ in the list in section 6 there is no $(S,b)\in\mathbb{S}_{r,B}$, such that $(S,b).\xi=\tilde{\xi}$. But first of all, we want to describe the elements of $\mathbb{S}_{r,B}$ more precisely.

Lemma 4.6. *Suppose r and $B=\lambda B_{x,y}$ as in Theorem 4.5 Then $(S,b)\in\mathbb{S}_{r,B}$ if and only if*

$$S=\begin{pmatrix} s_1 & s_2\\ s_3 & s_4\end{pmatrix}\in\left\{\mathrm{e}^{tB},\mathrm{e}^{tB}\begin{pmatrix}1 & -2x\\ 0 & -1\end{pmatrix}\mid t\in\mathbb{R}\right\}\cap GL(2,\mathbb{Z})$$

and $b\in(\frac{\mathbb{Z}}{r},\frac{\mathbb{Z}}{r}):=\frac{1}{r}\mathbb{Z}\times\frac{1}{r}\mathbb{Z}$, if r is even, respectively

$$b=(b_1,b_2)\in\begin{cases}(\frac{\mathbb{Z}}{r},\frac{\mathbb{Z}}{r}), & s_1s_2 \text{ and } s_3s_4 \text{ are even}\\ (\frac{\mathbb{Z}}{r},\frac{1}{2r}+\frac{\mathbb{Z}}{r}), & s_1s_2 \text{ is even and } s_3s_4 \text{ is odd}\\ (\frac{1}{2r}+\frac{\mathbb{Z}}{r},\frac{\mathbb{Z}}{r}), & s_3s_4 \text{ is even and } s_1s_2 \text{ is odd}\end{cases},$$

if r is odd.

Proof. It's not hard to show that such an (S,b) as defined in the lemma is an element of $\mathbb{S}_{r,B}$. Assume now that $(S,b)\in\mathbb{S}_{r,B}$. For

$$T=\begin{pmatrix}\sqrt{y} & \frac{x}{\sqrt{y}}\\ 0 & \frac{1}{\sqrt{y}}\end{pmatrix},$$

it holds that $TN_1T^{-1}=B_{x,y}$. Since $SB=\mu BS$ we see that $STN_1T^{-1}=\mu TN_1T^{-1}S$. Then $T^{-1}ST$ and N_1 (anti-)commute. Hence $T^{-1}ST\in O(2,\mathbb{R})$. Thus $T^{-1}ST=\mathrm{e}^{t\lambda N_1}\left(\begin{smallmatrix}1&0\\0&\mu\end{smallmatrix}\right)$ and $S=\mathrm{e}^{tB}T\left(\begin{smallmatrix}1&0\\0&\mu\end{smallmatrix}\right)T^{-1}$, for some $t\in\mathbb{R}$. If $\mu=1$, then $S=\mathrm{e}^{tB}$, if $\mu=-1$, then

$$S=\mathrm{e}^{tB}\begin{pmatrix}1 & -2x\\ 0 & -1\end{pmatrix}.$$

This shows that S is an integer matrix in

$$\left\{\mathrm{e}^{tB},\mathrm{e}^{tB}\begin{pmatrix}1 & -2x\\ 0 & -1\end{pmatrix}\mid t\in\mathbb{R}\right\}.$$

Finally we want to see how b looks like. Let r be even. Then

$$\omega_r\big((s_1,s_3),(b_1,b_2)\big),\omega_r\big((s_2,s_4),(b_1,b_2)\big)\in\mathbb{Z}.$$

Thus $b\in(\frac{1}{r}\mathbb{Z},\frac{1}{r}\mathbb{Z})$. Now let r be odd, then $\omega_r(Se_i,b)\in\frac{1}{2}(Se_i)_1(Se_i)_2+\mathbb{Z}$. We see that there are three cases to consider, depending on the values of S. Here, we only check the case that s_1s_2 is even and s_3s_4 is odd, especially s_1 is even (the case that s_2 is even runs similarly). Since $\det(S)$ is odd, we get that s_2 is odd. Then $\omega_r\big((s_1,s_3),(b_1,b_2)\big)\in\mathbb{Z}$ and $\omega_r\big((s_2,s_4),(b_1,b_2)\big)\in\mathbb{Z}+\frac{1}{2}$. Hence

$$r\begin{pmatrix}b_1\\ b_2\end{pmatrix}=\det(S)\begin{pmatrix}-s_2 & s_1\\ -s_4 & s_3\end{pmatrix}\begin{pmatrix}k_1\\ k_2+\frac{1}{2}\end{pmatrix},$$

for some $k_1, k_2 \in \mathbb{Z}$. Thus $rb_1 \in \mathbb{Z}$ and $rb_2 \in \mathbb{Z} + \frac{1}{2}$.
Finally, for each case we obtain the assertion. ∎

Proof of theorem 4.5. For further argumentation let $\xi = (\xi_1, \xi_2)$ denote an arbitrary vector in $\Xi_{r,B}$.
First part. Suppose $\lambda = \lambda_0 + 2k\pi$, $k \in \mathbb{N}$ and $\lambda_0 \in \{\frac{\pi}{3}, \frac{5\pi}{3}\}$. Let r be even. Then $\Xi_{r,B} = (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$. In addition $\det(\mathrm{e}^B - Id) = 1$. So $b := (\mathrm{e}^B - Id)^{-1}\xi \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$ and $(Id, b).0 = \xi$. Now let r be odd. For $\lambda_0 = \frac{\pi}{3}$ one can check that $\Xi_{r,B} = (\frac{1}{r}\mathbb{Z} + \frac{1}{2r}, \frac{1}{r}\mathbb{Z})$. Again, since $\det(\mathrm{e}^B - Id) = 1$, we see that $b := (\mathrm{e}^B - Id)^{-1}(\xi_1 - \frac{1}{2r}, \xi_2) \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$. Thus $(Id, b).(\frac{1}{2r}, 0) = \xi$. For $\lambda_0 = \frac{5}{3}\pi$ an analogous argumentation holds, except having $\frac{1}{2r}$ in the other component.

Suppose $\lambda = \lambda_0 + 2k\pi$, $k \in \mathbb{N}$ and $\lambda_0 \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. Then $\Xi_{r,B} = (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$. In addition $\det(\mathrm{e}^B - Id) = 2$. If $r\xi_1 - r\xi_2$ is even, then $(Id, (\mathrm{e}^B - Id)^{-1}\xi) \in \mathbb{S}_{r,B}$ and $(Id, (\mathrm{e}^B - Id)^{-1}\xi).0 = \xi$. If $r\xi_1 - r\xi_2$ is odd, then we set $S := Id$ and $b := (\mathrm{e}^B - Id)^{-1}(\xi_1 - \frac{1}{r}, \xi_2) \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$. Thus $(\frac{1}{r}, 0)$ is equivalent to ξ. If additionally r is odd, we instead set $b := (e^B - Id)^{-1}(1 + \xi_1, \xi_2) \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$. So $(Id, b).0 = \xi$.

Suppose $\lambda = \lambda_0 + 2k\pi$, $k \in \mathbb{N}$ and $\lambda_0 \in \{\frac{2\pi}{3}, \frac{4\pi}{3}\}$. Let r be even. Then $\Xi_{r,B} = (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$. If $r\xi_1 + r\xi_2 \equiv 0(3)$, then $(Id, \mathrm{e}^B - Id)^{-1}\xi) \in \mathbb{S}_{r,B}$ and maps 0 to ξ. Hence $(0,0)$ and ξ are equivalent. If $r\xi_1 + r\xi_2 \equiv 1(3)$, then $b := (\mathrm{e}^B - Id)^{-1}(\xi_1 - \frac{1}{r}, \xi_2) \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$ and $(Id, b).(\frac{1}{r}, 0) = \xi$. If $r\xi_1 + r\xi_2 \equiv 2(3)$, we set $b := (\mathrm{e}^B - Id)^{-1}(\xi_1 - \frac{1}{r}, \xi_2 - \frac{1}{r}) \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$ and $S := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus $(\frac{1}{r}, 0)$ and ξ are equivalent. If additionally 3 is not a factor of r, we set $b := (\mathrm{e}^B - Id)^{-1}(x + \xi_1, \xi_2) \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$, where $rx + r\xi_1 + r\xi_2 \equiv 0(3)$, and $S := Id$. So we get that $(0,0)$ is equivalent to ξ.
Now let r be odd. We obtain $\lambda_0 = \frac{2}{3}\pi$. We get $\Xi_{r,B} = (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z} + \frac{1}{2r})$. If $r\xi_1 + r\xi_2 - 1/2 \equiv 0(3)$, then $b := (\mathrm{e}^B - Id)^{-1}(\xi_1, \xi_2 - \frac{1}{2r}) \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$ and $S := Id$ shows that $(0, \frac{1}{2r})$ and ξ are equivalent. If $r\xi_1 + r\xi_2 - 1/2 \equiv 1(3)$, then we set $b := (\mathrm{e}^B - Id)^{-1}(\xi_1 - \frac{1}{r}, \xi_2 - \frac{1}{2r}) \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$ and $S := Id$. Then $(S, b).(\frac{1}{r}, \frac{1}{2r}) = \xi$. For $r\xi_1 + r\xi_2 - 1/2 \equiv 2(3)$ we set $b := (\mathrm{e}^B - Id)^{-1}(\xi_1, \xi_2 + \frac{1}{2r}) \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$, $S := -Id$ and get that $(0, \frac{1}{2r})$ and ξ are equivalent.
If additionally 3 is not a factor of r, then we set $b := (\mathrm{e}^B - Id)^{-1}(x + \xi_1, \frac{1}{2r} + \xi_2) \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$, where $rx + r\xi_1 + r\xi_2 - 1/2 \equiv 0(3)$, and see that $(Id, b).(0, \frac{1}{2r}) = \xi$.
For $\lambda_0 = \frac{4}{3}\pi$ there is an analogue argumentation, where $\frac{1}{2r}$ is in the other component.

Suppose $\lambda = \pi + 2\pi k$, $k \in \mathbb{N}$. Then $\Xi_{r,B} = (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$. Let r be odd. So there are $x', y' \in \{0, 1\}$, such that $b := (\mathrm{e}^B - Id)^{-1}(\xi_1 + x', \xi_2 + y') \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$ and (Id, b) maps $(0,0)$ to $\xi + (x', y') + \mathbb{Z}^{2n} = \xi$. Now let r be even. We set $\eta := \frac{1}{2r}(-1 + (-1)^{r\xi_1}, -1 + (-1)^{r\xi_2})$. Then $\eta \in \{(0,0), (\frac{1}{r}, 0), (0, \frac{1}{r}), (\frac{1}{r}, \frac{1}{r})\}$ and $(Id, (\mathrm{e}^B - Id)^{-1}(\xi_0 + \eta)) \in \mathbb{S}_{r,B}$ maps η to ξ.
If $x^2 + y^2 = 1$, then we set $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $b = (0,0)$. (S,b) shows that $(\frac{1}{r}, 0)$ and $(0, \frac{1}{r})$ are equivalent.
If $x = \frac{1}{2}$, then we set $S := \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$ and $b := (-\frac{1}{r}, -\frac{1}{r})$ and see that (S, b) maps

$(0,\frac{1}{r})$ to $(\frac{1}{r},\frac{1}{r})$. If $(x,y)=(\frac{1}{2},\frac{\sqrt{3}}{2})$, then both automorphisms can be used. Thus every $\xi\in(\frac{1}{r}\mathbb{Z},\frac{1}{r}\mathbb{Z})$ is equivalent to $(0,0)$ or $(\frac{1}{r},\frac{1}{r})$.

We come to the last case: Suppose $\lambda=2\pi k,\ k\in\mathbb{N}\backslash\{0\}$. We get $\Xi_{r,B}=(\frac{1}{r}\mathbb{Z},\frac{1}{r}\mathbb{Z})$. In this case we can neglect b, since $\mathrm{e}^B b-b=0$. Instead, it's more important to consider all the finitely many integer matrices in

$$\left\{\mathrm{e}^{tB_{x,y}},\mathrm{e}^{tB_{x,y}}\begin{pmatrix}1&-2x\\0&-1\end{pmatrix}\mid t\in\mathbb{R}\right\},$$

for $B_{x,y}$. At first we give some S, which (anti-) commute with all $B_{x,y}$. Afterward, we restrict our observation to the different cases.
For each $\xi\in\Xi_{r,B}$ there are $0\le k,l<r$ such that ξ and $(\frac{k}{r},\frac{l}{r})$ are equivalent. Moreover $(-Id,b)$ maps $(\frac{k}{r},\frac{l}{r})$ to $(\frac{r-k}{r},\frac{r-l}{r})-(1,1)+\mathbb{Z}^2$. Thus there is for each $\xi\in(\frac{1}{r}\mathbb{Z},\frac{1}{r}\mathbb{Z})$ an equivalent

$$\xi_0\in M_1:=\left\{(\frac{k}{r},\frac{l}{r})\mid 0\le k,l\le\frac{r}{2}\right\}\cup\left\{(\frac{k}{r},\frac{l}{r})\mid 0<k<\frac{r}{2}<l<r\right\}$$

and the first row is verified.
If additionally $x=0$, then $((\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}),b)$ maps $(\frac{k}{r},\frac{l}{r})$ to a $(0,-1)+(\frac{k}{r},\frac{r-l}{r})+\mathbb{Z}^2$. Hence the second row is verified.
If $x=\frac{1}{2}$ instead, then $((\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}),b)$ maps $(\frac{k}{r},\frac{l}{r})$, where $k>l$, to $(\frac{k-l}{r},\frac{r-l}{r})+\mathbb{Z}^2$. So we can narrow the set of all ξ_0 down to

$$\left\{(\frac{k}{r},\frac{l}{r})\mid 0\le k\le l\le\frac{r}{2}\right\}\cup\left\{(\frac{k}{r},\frac{l}{r})\mid 0<k<\frac{r}{2}<l<r\right\}\cup\left\{(\frac{k}{r},0)\mid 0\le k\le\frac{r}{2}\right\}.$$

Furthermore $(-(\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}),b$ maps $(\frac{k}{r},\frac{l}{r})$, where $l\ge k$, to $(\frac{l-k}{r},\frac{l}{r})+\mathbb{Z}^2$. Hence the third row follows.
For further argumentation suppose $x^2+y^2=1$. We know that there is for each $\xi\in(\frac{1}{r}\mathbb{Z},\frac{1}{r}\mathbb{Z})$ an equivalent $\xi_0\in M_1$. We can still restrict this set, since $((\begin{smallmatrix}0&-1\\-1&0\end{smallmatrix}),b)$ maps $(\frac{k}{r},\frac{l}{r})$ to $(\frac{l}{r},\frac{k}{r})+\mathbb{Z}^2$. So there is for each $\xi\in(\frac{1}{r}\mathbb{Z},\frac{1}{r}\mathbb{Z})$ an equivalent

$$\xi_0\in\left\{(\frac{k}{r},\frac{l}{r})\mid 0\le k\le l\le\frac{r}{2}\right\}\cup\left\{(\frac{k}{r},\frac{l}{r})\mid 0<k<\frac{r}{2}<l<r\right\}.$$

Additionally the element $((\begin{smallmatrix}0&1\\1&0\end{smallmatrix}),b)$ maps $(\frac{k}{r},\frac{l}{r})$ to $(\frac{r-l}{r},\frac{r-k}{r})-(1,1)+\mathbb{Z}^2$.
Hence for $x^2+y^2=1$ and $\xi\in(\frac{1}{r}\mathbb{Z},\frac{1}{r}\mathbb{Z})$ there is an equivalent

$$\xi_0\in M_2:=\left\{(\frac{k}{r},\frac{l}{r})\mid 0\le k\le l\le\frac{r}{2}\right\}\cup\left\{(\frac{k}{r},\frac{l}{r})\mid k+l\le r,0<k<\frac{r}{2}<l<r\right\}.$$

Now we want to see, which elements in M_2 are equivalent, if $(x,y)=(0,1)$ or $(x,y)=(\frac{1}{2},\frac{\sqrt{3}}{2})$.
So suppose $(x,y)=(0,1)$. Then $((\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}),b)$ maps $(\frac{k}{r},\frac{l}{r})$ to $(\frac{k}{r},\frac{r-l}{r})-(0,1)+\mathbb{Z}^2$ and the fourth row follows.

At last suppose $(x,y) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. The element $(\left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right), b)$ maps $(\frac{k}{r}, \frac{l}{r})$ to $(\frac{l-k}{r}, \frac{l}{r}) + \mathbb{Z}^2$. Hence we get for each $\xi \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$ an equivalent

$$\xi_0 \in \left\{ (\frac{k}{r}, \frac{l}{r}) \mid k + l \leq r, 0 < k < \frac{r}{2} < l < r, k \leq \frac{l}{2} \right\}$$
$$\cup \left\{ (\frac{k}{r}, \frac{l}{r})^T \mid 0 \leq k \leq \frac{l}{2} \leq l \leq \frac{r}{2} \right\}.$$

Additionally $(\left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right), b)$ maps $(\frac{k}{r}, \frac{l}{r})$ to $(0,-1) + (\frac{k}{r}, \frac{r-l+k}{r}) + \mathbb{Z}^2$. Thus the last row follows.

Second part. We use a proof by contradiction.
Assume that for an $r \in \mathbb{N}\backslash\{0\}$, $(x,y) \in \mathbb{F}_1$ and a $\lambda = \lambda_0 + k\pi$, where $\lambda_0 = \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi\right\}$ and $k \in \mathbb{N}$, there are ξ and $\tilde{\xi}$, $\xi \neq \tilde{\xi}$ from the list, which are equivalent. Then we check each $(S,b) \in \mathbb{S}_{r,B}$ and will note that none of them maps ξ onto $\tilde{\xi}$, and we get our contradiction.

Let us begin with $\lambda = \pi + 2\pi k$, $k \in \mathbb{N}$ and r even.
Assume $(S,b) \in \mathbb{S}_{r,B}$, then $b \in (\frac{1}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$ and $(S,b).\xi = S\xi - 2b + \mathbb{Z}^2$, since $\mathrm{e}^B = -Id$. Hence each $(S,b) \in \mathbb{S}_{r,B}$ maps $(0,0)$ to a vector in $\frac{2}{r}\mathbb{Z}^2$. Thus there is a contradiction for $\tilde{\xi} \in \left\{(\frac{1}{r}, 0), (0, \frac{1}{r}), (\frac{1}{r}, \frac{1}{r})\right\}$.
In an analogue way, we find a contradiction if $\xi = (\frac{1}{r}, 0)$. But we subdivide the proof, depending on the value of x and y.
If $x^2 + y^2 > 1$ and $x \neq \frac{1}{2}$, then each $(S,b) \in \mathbb{S}_{r,B}$ maps $(\frac{1}{r}, 0)$ to a vector in $(\frac{2}{r}\mathbb{Z} + \frac{1}{r}, \frac{2}{r}\mathbb{Z})$. Thus there is a contradiction for all $\tilde{\xi} \in \left\{(0, \frac{1}{r}), (\frac{1}{r}, \frac{1}{r})\right\}$. If $x^2 + y^2 > 1$ and $x = \frac{1}{2}$, then each $(S,b) \in \mathbb{S}_{r,B}$ maps $(\frac{1}{r}, 0)$ to a vector in $(\frac{1}{r}\mathbb{Z}, \frac{2}{r}\mathbb{Z})$. Thus there is a contradiction for $\tilde{\xi} = (\frac{1}{r}, \frac{1}{r})$.
Now, suppose $x^2 + y^2 = 1$ and $x \notin \left\{0, \frac{1}{2}\right\}$. Then each $(S,b) \in \mathbb{S}_{r,B}$ maps $(\frac{1}{r}, 0)$ to a vector with one entry in $\frac{2}{r}\mathbb{Z}$ and one in $\frac{2}{r}\mathbb{Z} + \frac{1}{r}$. Thus there is a contradiction for $\tilde{\xi} = (\frac{1}{r}, \frac{1}{r})$.
At last suppose $\xi = (0, \frac{1}{r})$ for $x^2 + y^2 > 1$ and $x \neq \frac{1}{2}$. Each $(S,b) \in \mathbb{S}_{r,B}$ maps $(0, \frac{1}{r})$ to a vector in $(\frac{2}{r}\mathbb{Z}, \frac{1}{r}\mathbb{Z})$. Thus there is a contradiction for $(\frac{1}{r}, \frac{1}{r})$.
Altogether, we verified that the ξ_0 from the list, which we refer to a lattice $L(\xi)$ in $Osc_1(\omega_r, \lambda B_{x,y})$, where $\lambda = \pi + 2\pi k$, $k \in \mathbb{N}$, $(x,y) \in \mathbb{F}_1$, is uniquely determined.

Now, we consider the case that $\lambda = 2\pi k$, $k \in \mathbb{N}\backslash\{0\}$, and get the same contradiction. But, first of all, we recall that $\xi \in \Xi_{r,B}$ and $\tilde{\xi} \in \Xi_{r,B}$ are equivalent if and only if there are $t_1, t_2 \in \mathbb{Z}$ and an integer matrix

$$S \in \left\{\mathrm{e}^{tB}, \mathrm{e}^{tB} \left(\begin{smallmatrix} 1 & -2x \\ 0 & -1 \end{smallmatrix}\right) \mid t \in \mathbb{R}\right\}$$

with $\det S = \mu$, such that $\tilde{\xi} = \mu S\xi + t_1 e_1 + t_2 e_2$.
So it suffices to fix a t_1 and t_2 for each integer matrix $S \in \left\{\mathrm{e}^{tB}, \mathrm{e}^{tB} \left(\begin{smallmatrix} 1 & -2x \\ 0 & -1 \end{smallmatrix}\right) \mid t \in \mathbb{R}\right\}$ and each ξ, such that $\mu S\xi + t_1 e_1 + t_2 e_2 \in [0,1)^2$ and show that $\mu S\xi + t_1 e_1 + t_2 e_2$ is equal to ξ or not an element in the set from the list.
We consider the case that $(x,y) = (0,1)$. Note that $\pm Id$, $\pm \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\pm \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are the only integer matrices in $\left\{\mathrm{e}^{tB}, \mathrm{e}^{tB} \left(\begin{smallmatrix} 1 & -2x \\ 0 & -1 \end{smallmatrix}\right) \mid t \in \mathbb{R}\right\}$ for this case.

We will denote by M the set $M := \left\{\left(\frac{k'}{r}, \frac{l'}{r}\right) | 0 \le k \le l \le \frac{r}{2}\right\}$ and set $\xi = (\frac{k}{r}, \frac{l}{r}) \in M$.

- If $S = Id$, then $S\xi = \xi$.
- Let $S = -Id$. For the cases that $k = l = 0$, $k = l = \frac{r}{2}$, or $k = 0$ and $l = \frac{r}{2}$, we get $S\xi = \xi$, $S\xi + e_1 + e_2 = \xi$ or $S\xi + e_1 = \xi$ respectively. Now, assume that $0 < k, l < \frac{r}{2}$, then $S\xi + e_1 + e_2 = (\frac{r-k}{r}, \frac{r-l}{r}) \in [0,1)^2$, but $\frac{r-k}{r} > \frac{r}{2}$. At last, if $k = 0$ and $0 < l < \frac{r}{2}$, then $S\xi + e_1 = (0, \frac{r-l}{r}) \in [0,1)^2$, but $\frac{r-l}{r} > \frac{r}{2}$.
- Suppose $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. First of all, we see that $S\xi + e_1 = \xi$, respectively $S\xi = \xi$ for $k = l = \frac{r}{2}$ or $k = l = 0$. If $l \notin \{0, \frac{r}{2}\}$, then $S\xi + e_1 = (\frac{r-l}{r}, \frac{k}{r}) \in [0,1)^2$, but $\frac{r-l}{r} > \frac{1}{2}$. For $l = \frac{r}{2}$ and $k < \frac{r}{2}$, we see that $S\xi + e_1 = (\frac{1}{2}, \frac{k}{r}) \in [0,1)^2$, but $\frac{k}{r} < \frac{1}{2}$.
- Similar arguments apply to the case $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.
- Let $S = -\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. It follows that $-S\xi = (\frac{l}{r}, \frac{k}{r}) \in [0,1)^2$. Hence, we get $(\frac{l}{r}, \frac{k}{r}) \notin M$ for $k < l$, and $-S\xi = \xi$, for $k = l$.
- Suppose $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We see again that $-S\xi = \xi$ for $k = l = 0$. If $k = 0$ and $l > 0$, then $-S\xi + e_1 \in [0,1)^2$, but $\frac{r-l}{r} > 0$. For $k, l \neq 0$ we get $-S\xi + e_1 + e_2 = (\frac{r-l}{r}, \frac{r-k}{r}) \in [0,1)^2$. If, additionally, $k < \frac{r}{2}$, then $\frac{r-k}{r} > \frac{r}{2}$. If, however, $k = l = \frac{r}{2}$, then $(\frac{r-l}{r}, \frac{r-k}{r}) = \xi$.
- Suppose $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. If $k \notin \{0, \frac{r}{2}\}$, then $-S\xi + e_1 = (\frac{r-k}{r}, \frac{l}{r}) \in [0,1)^2$, but $\frac{r-k}{r} > \frac{r}{2}$. If $k = l = \frac{r}{2}$, then $-S\xi + e_1 = \xi$ and if $k = 0$, then $-S\xi = \xi$.
- The same reasoning applies to the case $S = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Finally, for $(x, y) = (0, 1)$ it follows that $(\frac{k}{r}, \frac{l}{r}) \in M$ and $(\frac{k'}{r}, \frac{l'}{r}) \in M$, where $k \neq k'$ or $l \neq l'$ aren't equivalent.
The rest of the case $\lambda = 2\pi k$ runs as before.

For $\lambda = \lambda_0 + k\pi$, where $\lambda_0 \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}\right\}$ and $k \in \mathbb{N}$, we can use the same prove and obtain our assertion.
Finally, Theorem 4.5 is verified. ■

5. Abstract isomorphism classes and discrete oscillator groups

In this section we prove Theorem 1.2. Similar to the definition of oscillator groups we define a discrete oscillator group as a semidirect product of a group Γ which has an embedding such that the image of Γ is a lattice of the Heisenberg group and the group of integers with respect to an homomorphism $\tilde{S} : \mathbb{Z} \to Aut(\Gamma)$ which satisfies $\tilde{S}(1)\big|_{\mathfrak{z}(\Gamma)} = id : \mathfrak{z}(\Gamma) \to \mathfrak{z}(\Gamma)$ and $\overline{\tilde{S}(1)} : \Gamma/\mathfrak{z}(\Gamma) \to \Gamma/\mathfrak{z}(\Gamma)$ has finite order. Now, from [2] we know that every lattice of the Heisenberg group is isomorphic to Γ_r as an abstract group for exactly one $r \in \mathbb{N}\backslash\{0\}$. Here Γ_r is prescribed by generators $\{\alpha, \beta, \gamma\}$ where γ generates the center and $\alpha\beta\alpha^{-1}\beta^{-1} = \gamma^r$ is satisfied.

Compared to [13] we can prescribe every automorphism of Γ_r by a 3×3-matrix of the form

$$\begin{pmatrix} S & 0 \\ l\,k & d \end{pmatrix}$$

with respect to (α, β, γ), where $l, k \in \mathbb{Z}$, $S \in GL(2, \mathbb{Z})$ and $\det(S) = d$. This motivates the following notation. For $r \in \mathbb{N}\backslash\{0\}$, $l, k \in \mathbb{Z}$ and $S \in GL(2, \mathbb{Z})$ with $\det(S) = 1$ and finite order let $G_r(S, l, k)$ be the discrete oscillator group $\Gamma_r \rtimes_{\tilde{S}} \mathbb{Z}$ where $\tilde{S}(1)$ is given by the 3×3-matrix

$$\begin{pmatrix} S & 0 \\ l\,k & 1 \end{pmatrix}.$$

Every $G_r(S, l, k)$ has an embedding in the oscillator group such that the image of $G_r(S, l, k)$ is a lattice. Moreover, every lattice of the 4-dimensional oscillator group is isomorphic as an abstract group to $G_r(S, l, k)$ for some tupel (r, S, l, k). In addition, every $L(\xi_0) = \langle (0, e_1, 0), (0, e_2, 0), (1, 0, 0), (0, \xi_0, 1) \rangle \subset Osc_1(\omega_r, B)$ where $B = \lambda B_{x,y}$ for the tupel $(r, \lambda, x, y, \xi_0)$ given by Theorem 1.1 is isomorphic as an abstract group to $G_r(S, l, k)$ for some (r, S, l, k) in the following list:

r	S	l	k
$\in \mathbb{N}\backslash\{0\}$	S_6	$-\lfloor \frac{r}{2} \rfloor$	0
$\in \mathbb{N}\backslash\{0\}$	S_6^{-1}	0	$-\lceil \frac{r}{2} \rceil$
$\in \mathbb{N}\backslash\{0\}$	S_4	0	0
$\in 2\mathbb{N}\backslash\{0\}$	S_4	1	0
$\in \mathbb{N}\backslash\{0\}$	S_4^{-1}	0	0
$\in 2\mathbb{N}\backslash\{0\}$	S_4^{-1}	-1	0
$\in \mathbb{N}\backslash\{0\}$	S_3	0	$-\lfloor \frac{r}{2} \rfloor$
$\in 3\mathbb{N}\backslash\{0\}$	S_3	1	$-\lceil \frac{r+1}{2} \rceil$
$\in \mathbb{N}\backslash\{0\}$	S_3^{-1}	$-\lceil \frac{r}{2} \rceil$	0
$\in 3\mathbb{N}\backslash\{0\}$	S_3^{-1}	$-\lceil \frac{r+2}{2} \rceil$	0
$\in \mathbb{N}\backslash\{0\}$	S_2	0	0
$\in 2\mathbb{N}\backslash\{0\}$	S_2	0	-1
$\in 2\mathbb{N}\backslash\{0\}$	S_2	1	0
$\in 2\mathbb{N}\backslash\{0\}$	S_2	1	-1
$\in \mathbb{N}\backslash\{0\}$	S_1	$-r \le l \le 0$	$0 \le k \le r$

Here $S_1, \ldots, S_4, S_6$ is given by equation (2). More precisely, it is isomorphic to $G_r(S, l, k)$ with

$$S = \mathrm{e}^B,\ l = \omega_r(\xi_0, \mathrm{e}^B e_1) - \frac{1}{2} r (\mathrm{e}^B)_{1,1} (\mathrm{e}^B)_{2,1} \text{ and } k = \omega_r(\xi_0, \mathrm{e}^B e_2) - \frac{1}{2} r (\mathrm{e}^B)_{1,2} (\mathrm{e}^B)_{2,2}. \tag{7}$$

This follows from the fact that $L(\xi_0)$ as an abstract group is a discrete oscillator group $\Gamma_r \rtimes_{\tilde{S}} \mathbb{Z}$, where $\Gamma_r = \langle (0, e_1, 0), (0, e_2, 0), (1, 0, 0) \rangle$ and the action of $\mathbb{Z}$ is given by

$$\begin{pmatrix} S & 0 \\ l\,k & d \end{pmatrix} (0, e_i, 0) = \tilde{S}(1)(0, e_i, 0) = (0, \xi_0, 1)(0, e_i, 0)(0, \xi_0, 1)^{-1} = (\omega_r(\xi_0, \mathrm{e}^B e_i), \mathrm{e}^B e_i, 0).$$

Writing $(\omega_r(\xi_0, e^B e_i), e^B e_i, 0)$ as a product of $(0, e_1, 0)$, $(0, e_2, 0)$ and $(1, 0, 0)$ shows the equations (7).

Recall the definition of the discrete oscillator groups $L_r^1, \ldots, L_r^4, L_r^6, L_r^{2,+}, \ldots, L_r^{4,+}$ from the introduction. We can now show that every $G_r(S, l, k)$ with some (r, S, l, k) from the list is isomorphic to some $L_r^1, \ldots L_r^4, L_r^6, L_r^{2,+}, \ldots, L_r^{4,+}$. Therefor, it is not hard to see that the following mappings are isomorphisms:

$$G_r(S_6, -\lfloor\frac{r}{2}\rfloor, 0) \to L_r^6, \quad \alpha \mapsto \alpha\gamma^{-\lfloor\frac{r}{2}\rfloor}, \quad \beta \mapsto \beta\gamma^{\lfloor\frac{r}{2}\rfloor}, \quad \gamma \mapsto \gamma, \quad \delta \mapsto \delta$$

$$G_r(S_6^{-1}, 0, -\lceil\frac{r}{2}\rceil) \to L_r^6, \quad \alpha \mapsto \beta\gamma^{\lfloor\frac{r}{2}\rfloor}, \quad \beta \mapsto \alpha\gamma^{-\lfloor\frac{r}{2}\rfloor}, \quad \gamma \mapsto \gamma, \quad \delta \mapsto \delta^{-1}$$

$$G_r(S_4^{-1}, 0, 0) \to L_r^4, \quad \alpha \mapsto \alpha^{-1}, \beta \mapsto \beta^{-1}, \gamma \mapsto \gamma, \delta \mapsto \delta^{-1},$$

$$G_r(S_4^{-1}, -1, 0) \to L_r^{4,+}, \quad \alpha \mapsto \alpha^{-1}, \beta \mapsto \beta^{-1}\gamma^{-1}, \gamma \mapsto \gamma, \delta \mapsto \delta^{-1},$$

$$G_r(S_3^{-1}, -\lceil\frac{r}{2}\rceil, 0) \to L_r^3 \quad \alpha \mapsto \beta, \beta \mapsto \alpha, \gamma \mapsto \gamma^{-1}, \delta \mapsto \delta,$$

$$G_r(S_3^{-1}, -\lceil\frac{r+2}{2}\rceil, 0) \to L_r^{3,+} \quad \alpha \mapsto \beta\gamma, \beta \mapsto \alpha, \gamma \mapsto \gamma^{-1}, \delta \mapsto \delta$$

Moreover, $G_r(S_2, 0, -1)$, $G_r(S_2, 1, 0)$ and $G_r(S_2, 1, -1)$ are isomorphic since these groups have embeddings in $Osc_1(\omega_r, \pi B_{\frac{1}{2}, \frac{\sqrt{3}}{2}})$ such that the image equals $L(\frac{1}{r}, 0)$, $L(0, \frac{1}{r})$ and $L(\frac{1}{r}, \frac{1}{r})$ which are isomorphic, as we already know from the proof of Theorem 4.5. For groups of the form $G_r(Id, l, k)$ we have the following lemma.

Lemma 5.1. *Given $G_r(Id, l, k)$ then $r_0 := gcd(r, r\xi_1, r\xi_2)$ is the only positive integer such that $G_r(Id, l, k)$ and $G_{r_0}(Id, 0, 0) = L_{r_0}^1$ are isomorphic.*

Proof. Note that r_0 is the positive number such that γ^{r_0} generates the commutator subgroup. We choose z_1 as the positive number such that γ^{z_1} generates the subgroup $\langle\gamma^l, \gamma^k\rangle$. Then we set $x_2 = -\frac{k}{r_0 z_1}$ and $y_2 = -\frac{l}{r_0 z_1}$. Now, we can choose x_1 and y_1 as an integer solution of

$$\frac{r}{r_0} = x_1 \frac{k}{r_0 z_1} - y_1 \frac{l}{r_0 z_1}$$

and x_3, y_3 and z_3 as an integer solution of

$$x_3 \frac{l}{r_0} - y_3 \frac{k}{r_0} + z_3 \frac{r}{r_0} = 1.$$

Now, we define an isomorphism φ from $G_r(Id, l, k)$ to $G_{r_0}(Id, 0, 0)$ by

$$\alpha \mapsto \alpha^{x_1}\beta^{x_2}\delta^{x_3}, \quad \beta \mapsto \alpha^{y_1}\beta^{y_2}\delta^{y_3}, \quad \gamma \mapsto \gamma, \quad \delta \mapsto \alpha^{z_1}\delta^{z_3}.$$

It is not hard to see that φ is homomorphic and that this map is bijective, since

$$\det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & 0 & z_3 \end{pmatrix} = 1.$$

Since γ is unique up to inverse and γ^{r_0} the generator of the commutator subgroup, r_0 is unique. ■

At this point we have seen that indeed every lattice is isomorphic as an abstract group to $L^{\sigma}_{r_0}$ respectively $L^{\sigma,+}_{r_0}$ as claimed in theorem 1.2. We will now show that this $L^{\sigma}_{r_0}$ respectively $L^{\sigma,+}_{r_0}$ is the only group of this form which is isomorphic to L. Therefor, we show at first that r_0 is uniquely determined: For $S \neq Id$, $r = r_0$ is the power of γ such that γ^r generates the subgroup $\mathfrak{z}(L) \cap [L,L]$. Since γ is the uniquely determined generator of the center of Γ_r up to inverse, r is also uniquely determined. For $S = Id$, we already know that r_0 is uniquely determined from Lemma 5.1. Now, it remains to show that $L^1_{r_0}, \ldots, L^6_{r_0}$ are pairwise not isomorphic. Therefor, it suffices to show that the abelianizations of the groups $L^1_{r_0}, \ldots, L^6_{r_0}$ are not isomorphic: We compute

$$L^{1\ ab}_{r_0} = \mathbb{Z}_{r_0} \times \mathbb{Z}^3 \qquad L^{6\ ab}_{r_0} = \mathbb{Z}_{r_0} \times \mathbb{Z}$$

$$L^{2\ ab}_{r_0} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{r_0} \times \mathbb{Z} \qquad L^{2,+ab}_{r_0} = \mathbb{Z}_2 \times \mathbb{Z}_{2r_0} \times \mathbb{Z}$$

$$L^{3\ ab}_{r_0} = \begin{cases} \mathbb{Z}_3 \times \mathbb{Z}_{r_0} \times \mathbb{Z}, & \text{for even } r_0 \\ \mathbb{Z}_{3r_0} \times \mathbb{Z}, & \text{for odd } r_0 \end{cases} \qquad L^{3,+ab}_{r_0} = \begin{cases} \mathbb{Z}_{3r_0} \times \mathbb{Z}, & \text{for even } r_0 \\ \mathbb{Z}_3 \times \mathbb{Z}_{r_0} \times \mathbb{Z}, & \text{for odd } r_0 \end{cases}$$

$$L^{4\ ab}_{r_0} = \mathbb{Z}_2 \times \mathbb{Z}_{r_0} \times \mathbb{Z} \qquad L^{4,+ab}_{r_0} = \mathbb{Z}_{2r_0} \times \mathbb{Z}$$

and it immediately follows that $L^{\sigma_1,(+)}_{r_0}$ and $L^{\sigma_2,(+)}_{r_0}$ are not isomorphic for $\sigma_1 \neq \sigma_2$. Moreover, $L^2_{r_0}$, $L^{2,+}_{r_0}$, $L^4_{r_0}$ and $L^{4,+}_{r_0}$ are not isomorphic for an even r_0 and $L^3_{r_0}$ and $L^{3,+}_{r_0}$ are not isomorphic for an r_0 devisible by 3. This completes the proof of theorem 1.2.

6. List of ξ_0

$\lambda > 0$	$(x,y) \in \mathbb{F}_+$	ξ_0 for an even r	ξ_0 for an odd r
$\frac{1}{3}\pi + 2\pi k$	$x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$	$\{(0,0)\}$	$\{(\frac{1}{2r}, 0)\}$
$\frac{5}{3}\pi + 2\pi k$	$x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$	$\{(0,0)\}$	$\{(0, \frac{1}{2r})\}$
$\frac{1}{2}\pi + 2\pi k$	$x = 0, y = 1$	$\{(0,0), (\frac{1}{r}, 0)\}$	$\{(0,0)\}$
$\frac{3}{2}\pi + 2\pi k$	$x = 0, y = 1$	$\{(0,0), (\frac{1}{r}, 0)\}$	$\{(0,0)\}$
$\frac{2}{3}\pi + 2\pi k$	$x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$	$r \equiv 0(3)$: $\{(0,0), (\frac{1}{r}, 0)\}$ else: $\{(0,0)\}$	$r \equiv 0(3)$: $\{(0, \frac{1}{2r}), (\frac{1}{r}, \frac{1}{2r})\}$ else: $\{(0, \frac{1}{2r})\}$
$\frac{4}{3}\pi + 2\pi k$	$x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$	$r \equiv 0(3)$: $\{(0,0), (\frac{1}{r}, 0)\}$ else: $\{(0,0)\}$	$r \equiv 0(3)$: $\{(\frac{1}{2r}, 0), (\frac{1}{2r} + \frac{1}{r}, 0)\}$ else: $\{(\frac{1}{2r}, 0)\}$
$\pi + 2\pi k$	$x^2 + y^2 > 1$, $x \neq \frac{1}{2}$	$\{(0,0), (\frac{1}{r}, 0), (0, \frac{1}{r}),$ $(\frac{1}{r}, \frac{1}{r})\}$	$\{(0,0)\}$
	$x^2 + y^2 > 1$, $x = \frac{1}{2}$	$\{(0,0), (\frac{1}{r}, 0), (\frac{1}{r}, \frac{1}{r})\}$	
	$x^2 + y^2 = 1$, $x \neq \frac{1}{2}$	$\{(0,0), (\frac{1}{r}, 0), (\frac{1}{r}, \frac{1}{r})\}$	
	$x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$	$\{(0,0), (\frac{1}{r}, \frac{1}{r})\}$	
$2\pi k$	$x^2 + y^2 > 1$ $x \notin \{0, \frac{1}{2}\}$	$\{(\frac{k}{r}, \frac{l}{r}) \mid 0 \leq k, l \leq \frac{r}{2}\} \cup \{(\frac{k}{r}, \frac{l}{r}) \mid 0 < k < \frac{r}{2} < l < r\}$	
	$x^2 + y^2 > 1$ $x = 0$	$\{(\frac{k}{r}, \frac{l}{r}) \mid 0 \leq k, l \leq \frac{r}{2}\}$	
	$x^2 + y^2 > 1$ $x = \frac{1}{2}$	$\{(\frac{k}{r}, \frac{l}{r}) \mid 0 \leq, k \leq \frac{l}{2} \leq l \leq \frac{r}{2}\} \cup \{(\frac{k}{r}, 0) \mid 0 \leq k \leq \frac{r}{2}\} \cup$ $\{(\frac{k}{r}, \frac{l}{r}) \mid < k < \frac{r}{2} < l < r,\ k \leq \frac{l}{2}\}$	
	$x^2 + y^2 = 1$ $x \notin \{0, \frac{1}{2}\}$	$\{(\frac{k}{r}, \frac{l}{r}) \mid 0 \leq k \leq l \leq \frac{r}{2}\} \cup$ $\{(\frac{k}{r}, \frac{l}{r}) \mid 0 < k < \frac{r}{2} < l < r, k + l \leq r\}$	
	$x = 0, y = 1$	$\{(\frac{k}{r}, \frac{l}{r}) \mid 0 \leq k \leq l \leq \frac{r}{2}\}$	
	$x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$	$\{(\frac{k}{r}, \frac{l}{r}) \mid 0 \leq 2k \leq l \leq \frac{k+r}{2}\}$	

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