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Dirac Operators and Cohomology for Lie Superalgebra of Type I

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Abstract. Vogan raised the idea of Dirac cohomology to study representations of semisimple Lie groups and Lie algebras. He conjectured that the infinitesimal characters of Harish-Chandra modules are determined by their Dirac cohomology. Huang and Pandžić proved this conjecture and initiated the research on Dirac cohomology for Lie superalgebras based on Kostant's results. The aim of the present paper is to study Dirac cohomology of unitary representations for the basic classical Lie superalgebra of Type I and its relation to nilpotent Lie superalgebra cohomology.
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Introduction

After Dirac discovered a matrix-valued first-order differential operator and had a remarkable success in the understanding of elementary particles, there have been various analogues of differential operators called Dirac operators. One striking example is the Dirac operator used to construct discrete series representations by Parthasarathy [18] and Atiyah-Schmid [1]. Relative to Parthasarathy's geometric setting, Vogan introduced an algebraic version of the Dirac operator; and conjectured that the infinitesimal character of $(\mathfrak{g}, K)$-modules is determined by their Dirac cohomology [22]. The conjecture was proved in [5]. During the past twelve years, there have been many results of this nature. The Dirac cohomology turned out to be involved deeply with a few classical subjects of representation theory, like the discrete series and branching laws (see [7, 10]). The relation between Dirac cohomology and nilpotent Lie algebra cohomology (Kostant's $\mathfrak{u}$-cohomology [14]) is also interesting. It was shown in [8] that the Dirac cohomology of unitary modules is up to a twist isomorphic to $\mathfrak{u}$-cohomology for Hermitian types. Similar isomorphisms were obtained by Huang and the author in [9] for all the simple highest weight modules in the setting of cubic Dirac operator (see [15, 17]).

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On the other hand, Kac's foundational papers [11, 12, 13] about Lie superalgebras and their representations had led to an enormous amount of work involving a growing list of researchers. A distinguished feature of the representation theory of Lie superalgebras is that Lie superalgebras have typical and atypical irreducible finite-dimensional representations. The nilpotent Lie superalgebra cohomology groups play important roles in the determination of formal characters of atypical representations (see [19, 20, 2, 21]). Dirac cohomology for Lie superalgebras was introduced by Huang and Pandžić in [6]. They defined Dirac cohomology for Lie superalgebras of Riemannian type (see [16]) and proved an analogue of Vogan's conjecture in the case of basic classical Lie superalgebras.

The aim of the present paper is to study Dirac cohomology of unitary representations for basic classical Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of type I and its relation to nilpotent Lie superalgebra cohomology. More precisely, $\mathfrak{g}$ has a natural consistent $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, namely it is one of the Lie superalgebras $A(m|n)$ and $C(n)$. Since the results obtained in [6] also hold for $\mathfrak{gl}(m|n)$, for convenience, we will make no explicit distinction between $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$ and focus on the cases when $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{osp}(2|2n)$. It is shown that the Weil represention $M(\mathfrak{g}_1)$ of the Weyl algebra $W(\mathfrak{g}_1)$ can be identified with the corresponding polynomial algebra up to a twist as $\mathfrak{g}_0$ representations. With this, we have a Hodge decomposition for any irreducible unitary representation V of $\mathfrak{g}$. The Dirac cohomology of V is up to a twist equal to its $\mathfrak{g}_+$-cohomology and $\mathfrak{g}_-$-homology. Note that for $\mathfrak{gl}(m|n)$, Cheng and Zhang [3] found an explicit formula for the $\mathfrak{g}_+$-cohomology of unitarizable tensor representations. Therefore their calculation also gives the Dirac cohomology of the associated representations.

An outline of this paper is as follows. In Sections 2 and 3, we recall the basic notions and properties of Lie superalgebras and corresponding Dirac cohomology. In Section 4, a correspondence between Weil representation and related polynomial algebra for Lie superalgebras $\mathfrak{g}$ of type I is proved. A Hodge decomposition for $\mathfrak{g}_+$-cohomology and $\mathfrak{g}_-$-homology of unitary representations of $\mathfrak{g}$ is given in Section 5.

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1. Lie superalgebras of Riemannian type

In this section we outline the fundamental results on Lie superalgebras used in this paper, referring to [16] and [7] for full details. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a complex Lie superalgebra with a bracket $[\cdot,\cdot]$. A bilinear form B on $\mathfrak{g}$ is called supersymmetric if it is symmetric on $\mathfrak{g}_0$ and skew-symmetric on $\mathfrak{g}_1$, and $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are orthogonal. The form B is called invariant if $B([X,Y],Z) = B(X,[Y,Z])$ for all $X, Y, Z \in \mathfrak{g}$. We say that $\mathfrak{g}$ is of Riemannian type if there exists a nondegenerate supersymmetric invariant bilinear form B on $\mathfrak{g}$.

We call a subspace of $\mathfrak{g}_1$ a Lagrangian subspace if it is maximal isotropic. Fix a pair of complementary Lagrangian subspaces with bases ∂_i, x_i for $i = 1, \cdots, n$, such that

$$B(\partial_i, x_j) = \frac{1}{2}\delta_{ij}.$$

This notation is chosen so that the Weyl algebra $W(\mathfrak{g}_1)$ is generated by ∂_i and x_i, with commutation relations:

$$[x_i, x_j]_W = 0; \quad [\partial_i, \partial_j]_W = 0; \quad [\partial_i, x_j]_W = \delta_{ij}.$$

The subscript W is used to distinguish the commutators in $W(\mathfrak{g}_1)$ from the (totally different) bracket in $\mathfrak{g}$. With $\partial_i = \partial/\partial x_i$, we see that $W(\mathfrak{g}_1)$ can be identified with the algebra of differential operators with polynomial coefficients in variables x_i.

For the basis $\partial_1, \cdots, \partial_n, x_1, \cdots, x_n$ of $\mathfrak{g}_1$, the dual basis with respect to B is $2x_1, \cdots, 2x_n, -2\partial_1, \cdots, -2\partial_n$. The Casimir element of $\mathfrak{g}$ can then be defined as

$$\Omega_{\mathfrak{g}} = \sum_k W_k^2 + 2\sum_i (x_i\partial_i - \partial_i x_i),$$

where W_k is an orthonormal basis of $\mathfrak{g}_0$ with respect to B. It is easy to check that $\Omega_{\mathfrak{g}}$ is contained in the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

The adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ defines a map

$$\nu : \mathfrak{g}_0 \to \mathfrak{sp}(\mathfrak{g}_1).$$

We can embed $\mathfrak{sp}(\mathfrak{g}_1)$ into the Weyl algebra $W(\mathfrak{g}_1)$ as a Lie subalgebra consisting of quadratic elements. Start with the symmetrization map $\sigma : S(\mathfrak{g}_1) \to W(\mathfrak{g}_1)$, where $S(\mathfrak{g}_1)$ is the symmetric algebra of $\mathfrak{g}_1$. It is a linear isomorphism obtained by first embedding $S(\mathfrak{g}_1)$ into the subset of symmetric tensors in the tensor algebra $T(\mathfrak{g}_1)$, and then projecting to $W(\mathfrak{g}_1)$. Next we show that $\sigma(S^2(\mathfrak{g}_1))$ is isomorphic to $\mathfrak{sp}(\mathfrak{g}_1)$. In fact, one can verify that

$$\sigma(x_ix_j) = x_ix_j, \quad \sigma(\partial_i\partial_j) = \partial_i\partial_j, \quad \sigma(\partial_i x_j) = \partial_i x_j - \frac{1}{2}\delta_{ij} = x_j\partial_i + \frac{1}{2}\delta_{ij} \tag{1}$$

in the basis (∂_i, x_j). Considering the action of $\sigma(S^2(\mathfrak{g}_1))$ on $\mathfrak{g}_1 \subset W(\mathfrak{g}_1)$ by commutators $[\cdot,\cdot]_W$ in $W(\mathfrak{g}_1)$, the associated matrices in the basis (∂_i, x_j) are

$$\begin{aligned} \sigma(x_ix_j) &\longleftrightarrow -E_{n+i\ j} - E_{n+j\ i}; \\ \sigma(\partial_i\partial_j) &\longleftrightarrow \quad E_{i\ n+j} + E_{j\ n+i}; \\ \sigma(\partial_i x_j) &\longleftrightarrow -E_{ij} + E_{n+j\ n+i}, \end{aligned}$$

where E_{kl} is the matrix with 1 in the k-th row and l-th column and 0 elsewhere. Therefore $\sigma(S^2(\mathfrak{g}_1)) \simeq \mathfrak{sp}(\mathfrak{g}_1)$. In view of the map ν mentioned above, we obtain a Lie algebra morphism

$$\alpha : \mathfrak{g}_0 \to W(\mathfrak{g}_1).$$

Since $\alpha(\mathfrak{g}_0) \in \sigma(S^2(\mathfrak{g}_1))$, one can assume that

$$\alpha(X) = \sum_{i,j} a_{ij}\sigma(x_ix_j) + \sum_{i,j} b_{ij}\sigma(\partial_i\partial_j) + \sum_{i,j} c_{ij}\sigma(\partial_i x_j), \quad X \in \mathfrak{g}_0.$$

To determine the coefficients, we apply $[\cdot, \partial_k]_W$ to both sides and get

$$[\alpha(X), \partial_k]_W = [X, \partial_k] = -\sum_i a_{ik}x_i - \sum_j a_{kj}x_j - \sum_i c_{ik}\partial_i.$$

Then we apply $B(\cdot, \partial_l)$ and obtain $B([X, \partial_k], \partial_l) = 1/2(a_{lk} + a_{kl})$, that is,

$$a_{lk} + a_{kl} = 2B(X, [\partial_k, \partial_l]).$$

Similarly, we get

$$b_{lk} + b_{kl} = 2B(X, [x_k, x_l])$$

and

$$c_{kl} = -2B(X, [x_k, \partial_l]).$$

With (1) in hand, the explicit formula for α is

$$\begin{aligned}\alpha(X) = &\sum_{i,j} (B(X, [\partial_i, \partial_j])x_i x_j + B(X, [x_i, x_j])\partial_i \partial_j) \\ &- \sum_{i,j} 2B(X, [x_i, \partial_j])x_j \partial_i - \sum_i B(X, [\partial_i, x_i]), \quad X \in \mathfrak{g}_0.\end{aligned} \tag{2}$$

Now we can define a diagonal embedding

$$\mathfrak{g}_0 \to U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$$

given by

$$X \to X \otimes 1 + 1 \otimes \alpha(X) = X_\Delta.$$

We denote by $\mathfrak{g}_{0\Delta}$ the image of $\mathfrak{g}_0$. Denote by $U(\mathfrak{g}_{0\Delta})$ the image of $U(\mathfrak{g}_0)$ and by $Z(\mathfrak{g}_{0\Delta})$ the image of the center $Z(\mathfrak{g}_0)$ of $U(\mathfrak{g}_0)$. Let $\Omega_{\mathfrak{g}_0}$ be the Casimir element for $\mathfrak{g}_0$. We denote by $\Omega_{\mathfrak{g}_{0\Delta}}$ the image of $\Omega_{\mathfrak{g}_0}$. Then

$$\Omega_{\mathfrak{g}_{0\Delta}} = \sum_k (W_k^2 \otimes 1 + 2W_k \otimes \alpha(W_k) + 1 \otimes \alpha(W_k)^2).$$

Kostant [16] proved that $C := \sum_k \alpha(W_k)^2$ is a constant which is equal to 1/8 of the trace of Ω_0 on $\mathfrak{g}_1$.

2. Dirac cohomology for $(\mathfrak{g}, \mathfrak{g}_0)$

In this section, we present the definition and fundamental results on Dirac cohomology for Lie superalgebras ([6, 7]).

The Dirac operator D is defined to be the following element of $U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$:

$$D = 2\sum_{i=1}^{n} (\partial_i \otimes x_i - x_i \otimes \partial_i).$$

Then D is $\mathfrak{g}_0$-invariant and is independent of the choice of basis of $\mathfrak{g}_1$. The property of this Dirac operator is analogous to the case of reductive Lie algebras.

Proposition 2.1 ([7], Proposition 10.2.2). *Let $D \in U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$ be the Dirac operator. Then*

$$D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{g}_{0\Delta}} - C,$$

where C is the constant mentioned above.

Recall that the Weyl algebra $W(\mathfrak{g}_1)$ can be identified with the algebra of differential operators with polynomial coefficients in the x_i's, where $i = 1, \cdots, n$. Then we have a natural representation of $W(\mathfrak{g}_1)$ on the polynomial algebra $\mathbb{C}[x_1, \cdots, x_n]$. This is the Weil (or metaplectic) representation which we denote by $M(\mathfrak{g}_1)$. Note that $M(\mathfrak{g}_1)$ is $\mathbb{Z}_2$-graded. Let $M^+(\mathfrak{g}_1)$ and $M^-(\mathfrak{g}_1)$ be the submodules of $M(\mathfrak{g}_1)$ generated by homogeneous polynomials of even and odd degrees respectively.

Definition 2.2. Let V be a representation of $\mathfrak{g}$. Consider the action of $D \in U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$ on $V \otimes M(\mathfrak{g}_1)$:

$$D : V \otimes M(\mathfrak{g}_1)^{\pm} \to V \otimes M(\mathfrak{g}_1)^{\mp}$$

The Dirac cohomology of V is the $\mathfrak{g}_0$-module

$$H_D(V) := \mathrm{Ker}D / \mathrm{Ker}D \cap \mathrm{Im}D$$

In particular, the $\mathbb{Z}_2$-grading of $M(\mathfrak{g}_1)$ implies a $\mathbb{Z}_2$-grading of $H_D(V)$, that is, $H_D(V) = H_D^+(V) \oplus H_D^-(V)$ with even part $H_D^+(V)$ and odd part $H_D^-(V)$.

Proposition 2.3. *Let $0 \to U \xrightarrow{i} V \xrightarrow{p} W \to 0$ be a short exact sequence of $\mathfrak{g}$-module and suppose that D^2 is a semisimple operator for U, V and W. Then there exists a six-term exact sequence:*

$$\begin{array}{ccccc} H_D^+(U) & \longrightarrow & H_D^+(V) & \longrightarrow & H_D^+(W) \\ \uparrow & & & & \downarrow \\ H_D^-(W) & \longleftarrow & H_D^-(V) & \longleftarrow & H_D^-(U), \end{array}$$

where the horizontal arrow are induced by i and p; the vertical are the connecting homomorphisms.

Proof. Tensoring the short exact sequence $0 \to U \xrightarrow{i} V \xrightarrow{p} W \to 0$ by $M(\mathfrak{g}_1)$, the corresponding homomorphisms will still be denoted by i and p. We only need to give the definition of connecting homomorphisms since the proof of exactness is quite similar to the case of ordinary long exact sequence for Lie cohomology. Let $w \in W \otimes M(\mathfrak{g}_1)$ represent a Dirac cohomology class. Then $Dw = 0$. The surjectivity of p gives rise to a $v \in V \otimes M(\mathfrak{g}_1)$ such that $pv = w$. Since D^2 is semisimple and $D^2 w = 0$, we can assume that $D^2 v = 0$. Observe that $pDv = Dpv = Dw = 0$. One obtains $Dv = iu$ for some $u \in U \otimes M(\mathfrak{g}_1)$. In view of $D^2 v = 0$, we see that $Du = 0$, that is, u defines a cohomology class, which is the image of the class of w under the connecting homomorphism. Moreover, the parity changed when we applied D, and this gives the two vertical homomorphisms. ∎

Recall that we say a Lie superalgebra $\mathfrak{g}$ is classical if $\mathfrak{g}_0$ is reductive, in which case the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is completely reducible. We call $\mathfrak{g}$ a basic classical Lie superalgebra if it is also of Riemannian type. The analogues of major results in [5] for Dirac operators of Lie superalgebras were obtained in [6] and summarized in [7].

Theorem 2.4 ([7], Corollary 10.3.4 and Theorem 10.4.7). *Let $\mathfrak{g}$ be a basic classical Lie superalgebra with a Cartan subalgebra $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$. Let W be the Weyl group of $(\mathfrak{g}_0, \mathfrak{h}_0)$. For any $z \in Z(\mathfrak{g})$, there exists an algebra homomorphism $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{g}_0) \cong Z(\mathfrak{g}_{0\Delta})$ and a $\mathfrak{g}_0$-invariant $a \in U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$, such that*

$$z \otimes 1 - \zeta(z) = Da + aD.$$

Moreover, ζ fits into the following commutative diagram:

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\ \zeta\ } & Z(\mathfrak{g}_0) \\ \gamma\big\downarrow & & \gamma_0\big\downarrow \\ S(\mathfrak{h}_0)^W & \xrightarrow{\ \mathrm{id}\ } & S(\mathfrak{h}_0)^W, \end{array}$$

where the vertical maps γ and γ_0 are Harish-Chandra monomorphism and isomorphism respectively.

For $\lambda \in \mathfrak{h}_0^*$, denote by $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ the character $\chi_\lambda(z) = \lambda(\gamma(z))$ for $z \in Z(\mathfrak{g})$. Similarly, denote by $\chi^0_\lambda : Z(\mathfrak{g}_0) \to \mathbb{C}$ the character $\chi^0_\lambda(z_0) = \lambda(\gamma_0(z_0))$ for $z_0 \in Z(\mathfrak{g}_0)$.

Corollary 2.5 ([7], Corollary 10.4.8). *Let V be a representation of a basic classical Lie superalgebra $\mathfrak{g}$, with infinitesimal character χ_λ. Suppose that a $\mathfrak{g}_0$-module with $Z(\mathfrak{g}_0)$-infinitesimal character χ^0_μ for $\mu \in \mathfrak{h}_0^*$ is contained in the Dirac cohomology $H_D(V)$. Then $\chi_\lambda(z) = \chi^0_\mu(\zeta(z))$.*

3. Cohomology of nilpotent Lie superalgebras

Let $\mathfrak{g}$ be a basic classical Lie superalgebra. We say that $\mathfrak{g}$ is of type I if $\mathfrak{g}_1$ is a direct sum of two irreducible representations of $\mathfrak{g}_0$. Otherwise $\mathfrak{g}_1$ is irreducible and said to be of type II. By Proposition 2.3.9, 2.4.4 and 2.5.5 in [12], a basic classical Lie superalgebra of type I is either $A(m,n)$ or $C(n)$. For convenience, we assume that $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{osp}(2,2n)$ from now on.

Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. Let $\Delta \subset \mathfrak{h}_0^*$ be the root system of $(\mathfrak{g}, \mathfrak{h}_0)$, with root space $\mathfrak{g}_\alpha$ corresponding to each $\alpha \in \Delta$. Fix a positive root system Δ^+. The set Δ^+ decomposes as $\Delta^+_0 \cup \Delta^+_1$, where Δ^+_0 and Δ^+_1 are the sets of the even and odd positive roots respectively. Denote the corresponding Borel subalgebra by $\mathfrak{b}$, with corresponding nilradical $\mathfrak{n}$. Set

$$\mathfrak{g}_+ := \sum_{\alpha \in \Delta^+_1} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}_- := \sum_{\alpha \in \Delta^+_1} \mathfrak{g}_{-\alpha}.$$

Both of them are $\mathfrak{g}_0$-invariant super commutative subalgebras of $\mathfrak{g}$: that is, $[\mathfrak{g}_+, \mathfrak{g}_+] = [\mathfrak{g}_-, \mathfrak{g}_-] = 0$. The odd space $\mathfrak{g}_1 = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Put

$$\rho_0 = \frac{1}{2} \sum_{\beta \in \Delta^+_0} \beta, \quad \rho_1 = \frac{1}{2} \sum_{\beta \in \Delta^+_1} \beta \quad \text{and} \quad \rho = \rho_0 - \rho_1.$$

Let $B(,)$ be a nondegenerate invariant supersymmetric bilinear form on $\mathfrak{g}$ (see e.g., [12], Proposition 1.1.2 and Proposition 2.5.5) such that

$$B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \quad \text{for} \quad \alpha + \beta \neq 0. \tag{3}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be all the odd positive roots in Δ_1^+. Then we can choose $\partial_i \in \mathfrak{g}_{\alpha_i}, x_j \in \mathfrak{g}_{-\alpha_j}$ such that $B(\partial_i, x_j) = \frac{1}{2}\delta_{ij}$. Let $\mathbb{C}[x_1, \cdots, x_k]$ be the polynomial algebra generated by $x_1, \cdots, x_k$. Then it is a graded $\mathfrak{g}_0$-module under the natural adjoint action.

Recall the formula (2) of Lie algebra morphism $\alpha : \mathfrak{g}_0 \to W(\mathfrak{g}_1)$. Put

$$\alpha_1(X) = -\sum_{i,j} 2B(X, [x_i, \partial_j])x_j\partial_i$$

and

$$\alpha_2(X) = -\sum_i B(X, [\partial_i, x_i]).$$

for $X \in \mathfrak{g}_0$. Since $[\partial_i, \partial_j] = [x_i, x_j] = 0$ in this case, we have

$$\alpha(X) = \alpha_1(X) + \alpha_2(X).$$

Lemma 3.1. *Let f be a polynomial in $\mathbb{C}[x_1, \cdots, x_k]$. Then for any $X \in \mathfrak{g}_0$,*

$$\alpha_1(X)f = [X, f],$$

where the left side is given by the action of differential operators in $W(\mathfrak{g}_1)$ and the right side is given by the usual adjoint action.

Proof. By linearity, it suffices to consider the case when f is a monomial, that is,

$$f = \prod_l x_l^{q_l}, \qquad q_l \in \mathbb{Z}^{\geq 0} \text{ for } l = 1, \cdots, k.$$

Since $\mathfrak{g}_-$ is $\mathfrak{g}_0$-invariant, we can obtain

$$[X, x_i] = -\sum_j 2B([X, x_i], \partial_j)x_j = -\sum_j 2B(X, [x_i, \partial_j])x_j.$$

Therefore, $\alpha_1(X) = \sum_i [X, x_i]\partial_i$. It follows that

$$\begin{aligned}
\alpha_1(X)f &= \sum_i [X, x_i]\partial_i \prod_l x_l^{q_l} \\
&= \sum_i [X, x_i]\frac{q_i}{x_i} \prod_l x_l^{q_l} \\
&= \sum_i x_1^{q_1} \cdots [X, x_i^{q_i}] \cdots x_k^{q_k} \\
&= [X, \prod_l x_l^{q_l}] = [X, f].
\end{aligned}$$

■

Lemma 3.2. *Let $\mathbb{C}_{-\rho_1}$ be the one-dimensional $\mathfrak{g}_0$-module with weight $-\rho_1$. Given $v \in \mathbb{C}_{-\rho_1}$, then*

$$\alpha_2(X)v = X \cdot v.$$

Proof. It suffices to show this for $X \in \mathfrak{h}_0$. Indeed, if $X \in \mathfrak{h}_0$, then

$$\begin{aligned}\alpha_2(X)v &= -\sum_i B(X, [\partial_i, x_i])v = -\sum_i B([X, \partial_i], x_i)v \\ &= -\sum_i \alpha_i(X) B(\partial_i, x_i)v = -\frac{1}{2}\sum_i \alpha_i(X)v \\ &= -\rho_1(X)v = X \cdot v.\end{aligned}$$

■

It follows immediately from Lemma 3.1 and Lemma 3.2 that the action of $\alpha(\mathfrak{g}_0)$ on $M(\mathfrak{g}_1)$ and the adjoint action $\operatorname{ad}\mathfrak{g}_0$ on $\mathbb{C}[x_1, \cdots, x_k]$ differ by a twist of the one-dimensional character $\mathbb{C}_{-\rho_1}$. We have

Proposition 3.3. *There exists a $\mathfrak{g}_0$-module isomorphism*

$$M(\mathfrak{g}_1) \simeq \mathbb{C}[x_1, \cdots, x_k] \otimes \mathbb{C}_{-\rho_1}.$$

We see that the symmetric algebras $S(\mathfrak{g}_\pm)$ of $\mathfrak{g}_\pm$ are graded with $S(\mathfrak{g}_\pm) = \sum_{i=0}^{\infty} S^i(\mathfrak{g}_\pm)$, where $S^i(\mathfrak{g}_\pm)$ are homogeneous of degree i in $\mathfrak{g}_\pm$. We can identify $\mathfrak{g}_+^*$ with $\mathfrak{g}_-$ by the pairing $2B(\cdot,\cdot) : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{C}$. The identification is $\mathfrak{g}_0$-invariant since B is invariant. Upon identifying $S(\mathfrak{g}_+^*)$ with $S(\mathfrak{g}_-)$, and then the polynomial algebra $\mathbb{C}[x_1, \cdots, x_k]$, the cohomology group $H^i(\mathfrak{g}_+, V)$ is given by the complex $C = (\{V \otimes S^i(\mathfrak{g}_-)\}, d)$, where $d = \sum_i \partial_i \otimes x_i$ is $\mathfrak{g}_0$-invariant. On the other hand, the homology group $H_i(\mathfrak{g}_-, V)$ is given by the complex $(\{V \otimes S^i(\mathfrak{g}_-)\}, \delta)$, with $\mathfrak{g}_0$-invariant differential operator $\delta = \sum_i x_i \otimes \partial_i$. Then the following lemma is an immediate consequence of Proposition 3.3.

Proposition 3.4. *If we consider d and δ as operators on $V \otimes M(\mathfrak{g}_1)$, then as $\mathfrak{g}_0$-modules, the cohomology of d is identified with $H^*(\mathfrak{g}_+, V) \otimes \mathbb{C}_{-\rho_1}$, while the homology of δ is identified with $H_*(\mathfrak{g}_-, V) \otimes \mathbb{C}_{-\rho_1}$.*

4. Hodge decomposition for $\mathfrak{g}_+$-cohomology

Now we consider Hermitian forms on $V \otimes M(\mathfrak{g}_1)$ for a unitarizable module V. Let ω be an adjoint operation of $\mathfrak{g}$. Recall that we say a $\mathbb{Z}_2$-graded $\mathfrak{g}$-module V is unitary if it admits a positive definite contravariant Hermitian form $\langle \cdot, \cdot \rangle_V$, where contravariance means that $\langle av, v' \rangle_V = \langle v, \omega(a)v' \rangle_V$ for all $a \in \mathfrak{g}$ and $v, v' \in V$. Here we refer to [4] for the two possible choices of ω which correspond to star and grade star representations of $\mathfrak{g}$ respectively.

In each case, once ω is fixed, it is not difficult to change B, ∂_i and x_j up to scalars such that

$$\omega(\partial_i) = x_i, \omega(x_i) = \partial_i \tag{4}$$

while maintaining $B(\partial_i, x_j) = \frac{1}{2}\delta_{ij}$. On the other hand, there is a unique positive definite contravariant Hermitian form $\langle \cdot, \cdot \rangle_M$ on $M(\mathfrak{g}_1)$, with $\langle 1, 1 \rangle_M = 1$, where contravariance means that $\langle af, f' \rangle_M = \langle f, \omega(a) f' \rangle_M$ for all $a \in \mathfrak{g}$ and $f, f' \in M$. Here, we should emphasize that the form is given explicitly by

$$\begin{aligned} \langle \prod_k x_k^{p_k}, \prod_k x_k^{q_k} \rangle_M &= \prod_k p_k! \qquad \text{if } p_k = q_k \text{ for all } k \\ &= 0 \qquad\qquad \text{otherwise,} \end{aligned}$$

where $p_k, q_k \in \mathbb{Z}^{\geq 0}$.

Let V be a $(\mathfrak{g}, \mathfrak{g}_0)$-module, that is, viewed as a $\mathfrak{g}_0$-module, V is a direct sum of finite-dimensional simple modules with finite multiplicities. We consider the tensor product Hermitian form on $V \otimes M(\mathfrak{g}_1)$; this form will be denoted by $\langle \cdot, \cdot \rangle$.

Lemma 4.1. *Let V be a unitary $(\mathfrak{g}, \mathfrak{g}_0)$-module. With respect to the form $\langle \cdot, \cdot \rangle$ on $V \otimes M(\mathfrak{g}_1)$, the operators d and δ are adjoints of each other. Hence the Dirac operator $D = 2(d - \delta)$ is anti self-adjoint.*

Proof. In view of (4), we have $\omega(\partial_i) = x_i$ and $\omega(x_i) = \partial_i$. Then the lemma follows from the fact that $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_M$ are contravariant. ■

Recall that $D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{g}_{0\Delta}} - C$. Suppose that $\Omega_{\mathfrak{g}}$ acts on V by a constant. Since $\Omega_{\mathfrak{g}_{0\Delta}}$ acts by a scalar on each irreducible $\mathfrak{g}_0$-submodule in $V \otimes M(\mathfrak{g}_1)$, the same is true for D^2.

Lemma 4.2. *If the $(\mathfrak{g}, \mathfrak{g}_0)$-module V has infinitesimal character, then we have* $V \otimes M(\mathfrak{g}_1) = \operatorname{Ker} D^2 \oplus \operatorname{Im} D^2$.

Proof. Both V and $M(\mathfrak{g}_1)$ are direct sums of finite-dimensional irreducible $\mathfrak{g}_0$-modules, so is the tensor product $V \otimes M(\mathfrak{g}_1)$. Then $V \otimes M(\mathfrak{g}_1)$ is a direct sum of eigenspaces for $D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{g}_{0\Delta}} - C$. The zero eigenspace is $\operatorname{Ker} D^2$ and the sum of all the nonzero eigenspaces is $\operatorname{Im} D^2$. ■

Lemma 4.3. *For Dirac operator D, we have*

(i) $\operatorname{Ker} D^2 = \operatorname{Ker} D = \operatorname{Ker} d \cap \operatorname{Ker} \delta$;

(ii) *With respect to the form $\langle \cdot, \cdot \rangle$, $\operatorname{Im} d$ is orthogonal to $\operatorname{Ker} \delta$ and $\operatorname{Im} \delta$, $\operatorname{Im} \delta$ is orthogonal to $\operatorname{Ker} d$.*

Proof. (i) For any $a \in V \otimes M(\mathfrak{g}_1)$, it follows from $\langle Da, Da \rangle = \langle -D^2 a, a \rangle$ that $Da = 0$ if and only if $D^2 a = 0$. On the other hand, $\operatorname{Ker} d \cap \operatorname{Ker} \delta \subseteq \operatorname{Ker} D$ since $D = 2(d - \delta)$. Conversely, if $Da = 0$, then $da = \delta a$ and $\delta da = \delta^2 a = 0$. So $\langle da, da \rangle = \langle a, \delta da \rangle = 0$. Hence $da = 0$. Similarly we get $\delta a = 0$. (ii) It is an easy consequence of the fact that d and δ are adjoints of each other. ■

Theorem 4.4. *Let V be an irreducible unitary $(\mathfrak{g}, \mathfrak{g}_0)$-module. Then*

(i) $V \otimes M(\mathfrak{g}_1) = \operatorname{Ker} D \oplus \operatorname{Im} d \oplus \operatorname{Im} \delta$;

(ii) $\operatorname{Ker} d = \operatorname{Ker} D \oplus \operatorname{Im} d$;

(iii) $\operatorname{Ker} \delta = \operatorname{Ker} D \oplus \operatorname{Im} \delta$.

In particular, there exists $\mathfrak{g}_0$-module isomorphisms:

$$H_D(V) \simeq H^*(\mathfrak{g}_+, V) \otimes \mathbb{C}_{-\rho_1} \simeq H_*(\mathfrak{g}_-, V) \otimes \mathbb{C}_{-\rho_1}$$

Proof. (i) In view of Lemma 4.3, we see that $\operatorname{Ker} D$, $\operatorname{Im} d$ and $\operatorname{Im} \delta$ are disjoint subspaces of $V \otimes M(\mathfrak{g}_1)$. Since $D = 2(d - \delta)$, one has $\operatorname{Im} D^2 \subseteq \operatorname{Im} D \subseteq \operatorname{Im} d \oplus \operatorname{Im} \delta$. It follows from Lemma 4.2 and Lemma 4.3 that

$$V \otimes M(\mathfrak{g}_1) = \operatorname{Ker} D^2 \oplus \operatorname{Im} D^2 \subseteq \operatorname{Ker} D \oplus \operatorname{Im} d \oplus \operatorname{Im} \delta.$$

Then (i) follows and $\operatorname{Im} D^2 = \operatorname{Im} D = \operatorname{Im} d \oplus \operatorname{Im} \delta$. The formula (ii) is an obvious consequence of (i) and Lemma 4.3, so is (iii). Thus

$$H_D(V) = \operatorname{Ker} D \simeq \operatorname{Ker} d / \operatorname{Im} d \simeq \operatorname{Ker} \delta / \operatorname{Im} \delta.$$

The theorem is now evident from Proposition 3.4. ■

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