

Local Integrability of Characters on $\mathrm{GL}(2)$, Orbital Integrals, Germs

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Abstract. The character $\mathrm{tr} \pi$ of an irreducible admissible representation π of the group $G(F)$ of F -points of a reductive connected linear algebraic group G over a local non-Archimedean field F has been shown by Harish-Chandra to be locally constant on the regular set and *locally integrable*, that is, representable by a function χ with such properties, when the characteristic of F is 0. His method was extended to $G = \mathrm{GL}(n)$ and its inner forms for all characteristics. Earlier this result had been proven for $G = \mathrm{GL}(2)$ and F of any characteristic, characteristic two being the difficult case, in Jacquet-Langlands, by a direct and relatively elementary approach. We give here another proof by explicit computation, in this case of $\mathrm{GL}(2)$ and F of any characteristic, especially two, which we believe extends to other low rank groups. Our computation gives an explicit evaluation of the orbital integral of the characteristic function χ_K of the maximal compact subgroup K . We use this to compute the coefficients in the germ expansion of the orbital integrals on G , and observe that the germ expansion of the orbital integral of χ_K extends to all of K .

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1. Introduction

Let F be a non-Archimedean local field. Let G be a reductive connected group over F . Let $G(F)$ be the group of F -points of G . It is a locally compact group. Let π be an admissible irreducible representation of $G(F)$ in a complex vector space V . By Schur's lemma it has a central character ω , thus $\pi(zg) = \omega(z)\pi(g)$ for all z in $Z(F)$ and g in $G(F)$, where Z is the center of G . Denote by H the convolution algebra of complex valued locally constant measures on $G(F)$ which transform under the center by ω^{-1} and are compactly supported on $G(F)/Z(F)$. Fixing a Haar measure dg on $G(F)/Z(F)$ we can view H as a space of fdg , with functions f having these properties. The convolution operator $\pi(fdg) = \int f(g)\pi(g)dg$ (g in $G(F)/Z(F)$) is of finite rank, thus $\mathrm{tr} \pi(f)$ is finite. Our aim is

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to show that there exists a locally integrable function χ_π on $G(F)$, which transform under $Z(F)$ by ω , with

$$\mathrm{tr} \pi(fdg) = \int_{G(F)/Z(F)} f(g)\chi_\pi(g)dg.$$

Consider only the case of $G = \mathrm{GL}(2)$. It is well known and easy to prove that, when π is induced (in a normalized way) from a character η of the diagonal subgroup $A(F)$, extended by 1 on the upper unipotent subgroup $N(F)$, to the upper triangular subgroup, its character is given by

$$\chi_\pi(g^{-1} \mathrm{diag}(a, b)g) = (\eta(\mathrm{diag}(a, b)) + \eta(\mathrm{diag}(b, a)))/\Delta^{1/2},$$

where $\Delta^{1/2} = |a - b|$ on the regular split set, and χ_π is zero on the regular elliptic set. The character of a one dimensional representation of $G(F)$ is itself, and the Steinberg representation is the quotient of an induced representation by a one dimensional subrepresentation. All other irreducible representations are cuspidal, and we work only with them.

An element γ in $G(F)$ is called *regular* if its eigenvalues are distinct. The singular elements are the scalars, their products with unipotent elements, and the inseparable elements – which exist only in characteristic two – namely the γ which generate an inseparable quadratic extension over F . Their measure is 0. The centralizer in $G(F)$ of a regular γ is a torus, isomorphic to the multiplicative group of an algebra E of degree two over F , isomorphic to $F \oplus F$ or to a separable quadratic field extension of F . In the first case the torus is conjugate in $G(F)$ to the diagonal subgroup $A(F)$, otherwise it is a torus $T(F) \simeq E^\times$, where E is a separable quadratic field extension of F . Fix a set S of representatives of the conjugacy classes of tori in $G(F)$ (resp. S' for the elliptic tori, so that $S = S' \cup \{A\}$). If $\gamma \in G(F)$, its eigenvalues a, b are roots of $x^2 - \mathrm{tr}(\gamma)x + \det(\gamma)$ and

$$D(\gamma) = \frac{(a - b)^2}{ab} = \frac{(\mathrm{tr} \gamma)^2}{\det \gamma} - 4$$

lies in F , so we put $\Delta(\gamma) = |D(\gamma)|$, where $|\cdot| = |\cdot|_F$ is the normalized absolute value on F . Thus $|\pi| = q^{-1}$ where π is a generator of the maximal ideal in the ring $\mathcal{O} = \mathcal{O}_F$ of integers in F , and q is the cardinality of the residue field \mathcal{O}/π . We put $\mathrm{Int}(g)x = \mathrm{Int} g(x) = gxg^{-1}$. Note that

$$D(\gamma) = \det(\mathrm{Ad}(\gamma) - I)|_{\mathrm{Lie} G / \mathrm{Lie} T},$$

where $\mathrm{Lie} T$ is the Lie algebra of the normalizer T of γ in G . The Weyl integration formula asserts

$$\int_{G(F)/Z(F)} f(g)dg = \sum_{T \in S} \frac{1}{2} \int_{T(F)/Z(F)} \Delta(t) \left\{ \int_{G(F)/T(F)} f(\mathrm{Int}(g)t) \frac{dg}{dt} \right\} dt$$

for an integrable function f on $G(F)/Z(F)$. In characteristic two, the set S is infinite. If $G^{\mathrm{reg}}(F)$ is the set of regular elements in $G(F)$, and if we set $T^{\mathrm{reg}}(F) = T(F) \cap G^{\mathrm{reg}}(F)$, then $G^{\mathrm{reg}}(F) = \bigcup_{T \in S} T^{\mathrm{reg}}(F)^{G(F)}$ where

$$T^{\mathrm{reg}}(F)^{G(F)} = (\mathrm{Int} G(F))(T^{\mathrm{reg}}(F)).$$

The sets $T^{\text{reg}}(F)^{G(F)}$, $G^{\text{reg}}(F)$, $\cup_{T \in \mathcal{S}'} T^{\text{reg}}(F)^{G(F)}$ are open in $G(F)$. When E/F is a ramified separable quadratic extension, its discriminant is an ideal of the form $\pi_F^{2t} \mathcal{O}_F$; $t \geq 1$. Put $c(E) = q_F^{-t}$. For g in $G^{\text{reg}}(F)$ we write $c(g)$ for $c(E)$ where E is the quadratic extension $F(g)$.

Theorem 1.1. *The function $\Delta^{-1/2}c$ is locally constant on $G^{\text{reg}}(F)$. It is bounded away from zero on any compact subset of $G(F)$. It is locally integrable on $G(F)/Z(F)$ and $G(F)$. The character of a cuspidal (irreducible) representation exists as a locally integrable function whose absolute value is bounded by a multiple of $\Delta^{-1/2}c$. It is continuous on $G^{\text{reg}}(F)$.*

If the character χ_π of π exists and χ is a character (multiplicative function) of F^\times into \mathbb{C}^\times , then the character $\chi_{\pi\chi}$ of $\pi\chi$ ($g \mapsto \pi(g)\chi(g)$) also exists and $\chi_{\pi\chi}(g) = \chi(\det g)\chi_\pi(g)$. So it suffices to prove the theorem for a unitarizable π . As in [4], we start with

Proposition 1.2. *Let ξ be a vector of length one in the space of the cuspidal π . Then*

$$\text{tr } \pi(fdg) = d(\pi) \int_{G(F)/Z(F)} \left[\int_{G(F)/Z(F)} f(h)(\pi(\text{Int}(g)h)\xi, \xi) dh \right] dg$$

for all f in H , where $d(\pi)$ is the formal degree of π .

The formal degree $d(\pi) > 0$ of π is defined by the Schur orthogonality relations: if $\{\xi_i\}$ is an orthonormal basis of π then

$$\int_{G(F)/Z(F)} (\pi(g)\xi, \xi_j)(\xi_i, \pi(g)\xi) dg = d(\pi)^{-1}(\xi_i, \xi_j)$$

where (\cdot, \cdot) denotes the inner product on the unitarizable π . See [5] for a proof.

Proof. Since f is locally constant, the convolution operator $\pi(f)$ is of finite rank, hence we can choose a basis for the space of π such that all but finitely many coefficients $\pi(f)_{ij} = (\pi(f)\xi_i, \xi_j)$ are zero. Since π is unitarizable, $\pi(g)^* = \pi(g^{-1})$, hence

$$(\pi(g^{-1})\pi(f)\pi(g)\xi, \xi) = (\pi(f)\pi(g)\xi, \pi(g)\xi),$$

which equals

$$\sum_i (\pi(f)\pi(g)\xi, \xi_i)(\xi_i, \pi(g)\xi) = \sum_i \sum_j (\pi(g)\xi, \xi_j)\pi(f)_{ji}(\xi_i, \pi(g)\xi),$$

where all sums are finite. Hence

$$\int_{G(F)/Z(F)} (\pi(g)^{-1}\pi(f)\pi(g)\xi, \xi) dg = \sum_{i,j} \pi(f)_{ji} \int_{G(F)/Z(F)} (\pi(g)\xi, \xi_j)(\xi_i, \pi(g)\xi) dg.$$

The integrals on the right exist since the representation is square-integrable. By the Schur orthogonality relations we get

$$d(\pi)^{-1} \sum_{ij} \pi(f)_{ji}(\xi_i, \xi_j) = d(\pi)^{-1} \sum_i \pi(f)_{ii} = d(\pi)^{-1} \text{tr } \pi(fdg).$$

Since

$$(\pi(g)^{-1}\pi(f)\pi(g)\xi, \xi) = \int_{G(F)/Z(F)} f(h)(\pi(g)^{-1}\pi(h)\pi(g)\xi, \xi)dh,$$

the proposition follows. ■

The integral of Proposition 1.2 is an iterated, not a double integral.

Any g in $G(F)$ can be written as $k_1 \begin{pmatrix} \pi^p & 0 \\ 0 & \pi^s \end{pmatrix} k_2$ with $k_1, k_2 \in K = G(\mathcal{O})$, and $p \leq s$. The elements π^p and π^s are the elementary divisors of g . Write C_r for the set of g with $s - p \leq r$, and C'_r for its compact image in $G(F)/Z(F)$; C_r is the inverse image of C'_r . Any compact set in $G(F)/Z(F)$ is contained in C'_r for large enough r . Thus the integral of Proposition 1.2 is the limit as $r \rightarrow \infty$ of

$$\int_{C'_r} \left[\int_{G(F)/Z(F)} f(h)(\pi(\text{Int}(g)h)\xi, \xi)dh \right] dg.$$

As C'_r is compact this integral is absolutely convergent and equals

$$\int_{G(F)/Z(F)} f(h) \left[\int_{C'_r} (\pi(\text{Int}(g)h)\xi, \xi) dg \right] dh.$$

To show that the character of a cuspidal representation exists as a locally integrable function bounded by $\Delta^{-1/2}c$ we show that the sequence of functions

$$\varphi_r(h) = \int_{C'_r} (\pi(\text{Int}(g)h)\xi, \xi) dg$$

is dominated locally by a multiple of $\Delta^{-1/2}c$ and converges almost everywhere on $G(F)$. Then we write $\chi_\pi(h) = d(\pi) \lim_{r \rightarrow \infty} \varphi_r(h)$ when the limit exists.

2. Proof of Theorem

To show the continuity of χ_π on the regular set one uses

Lemma 2.1. *Let C_A be a compact subset of $G_{\text{ell}}^{\text{reg}}(F)$ (set of $g \in G(F)$ with eigenvalues not in F). Let C_B be a compact set in $G(F)$. Then the image in $G(F)/Z(F)$ of*

$$\{g \in G(F) : \text{Int}(g)C_A \cap Z(F)C_B \neq \emptyset\}$$

is compact.

For a proof see [4, §7, p. 129, l. 1 to l. -3].

Since π is cuspidal, by definition its matrix coefficient $(\pi(g)\xi, \xi)$ is compactly supported in $G(F) \bmod Z(F)$, for any vector ξ in the space of π . This support is then contained in $Z(F)C_B$ for some compact set C_B in $G(F)$. Let C_A be a compact subset of $G_{\text{ell}}^{\text{reg}}(F)$. By Lemma 2.1, the image in $G(F)/Z(F)$ of $\{g \in G(F); (\pi(\text{Int}(g)h)\xi, \xi) \neq 0 \text{ for some } h \in C_A\}$ lies in a compact C'_D . Hence the integral

$$\int_{G(F)/Z(F)} (\pi(\text{Int}(g)h)\xi, \xi) dg = \int_{C'_D} (\pi(g^{-1}hg)\xi, \xi) dg$$

converges for all h in C_A . If r is large enough then $C'_r \supseteq C'_D$ and

$$\varphi_r(h) = \int_{G(F)/Z(F)} (\pi(\text{Int}(g)h)\xi, \xi) dg$$

for $h \in C_A$. Hence the sequence $\{\varphi_r\}$ converges uniformly on any compact set C_A in $G_{\text{ell}}^{\text{reg}}(F)$ and its limit $d(\pi)^{-1}\chi_\pi(h)$ is continuous on $G_{\text{ell}}^{\text{reg}}(F)$. We obtained

Proposition 2.2. *For $h \in G_{\text{ell}}^{\text{reg}}(F)$, the integral $\int_{G(F)/Z(F)} (\pi(\text{Int}(g)h)\xi, \xi) dg$ exists and equals $d(\pi)^{-1}\chi_\pi(h)$.*

Denote by χ_C the characteristic function of C in $G(F)/Z(F)$. Since ξ is a vector of length one, and so is $\pi(g)\xi$ since π unitarizable, we have $|(\pi(g)\xi, \xi)| \leq \chi_{C_B}(g)$.

Lemma 2.3 (Key Lemma). *Let C be a compact subset of $G(F)/Z(F)$. Then there is $c = c(C) > 0$ such that*

$$\int_{G(F)/Z(F)} \chi_C(\text{Int}(g)h) dg \leq c\Delta^{-1/2}(h)c(E)$$

for every $h \in G_{\text{ell}}^{\text{reg}}(F)$ where $E = F(h)$.

Assuming this, $|\varphi_r(h)| \leq c\Delta^{-1/2}(h)c(E)$ on $G_{\text{ell}}^{\text{reg}}(F)$. On the intersection of a compact subset in $G(F)$ with the complement $\text{Int}(G(F))(A^{\text{reg}}(F))$ of $G_{\text{ell}}^{\text{reg}}(F)$ in $G^{\text{reg}}(F)$, the sequence $\{\varphi_r\}$ is dominated by a multiple of $\Delta^{-1/2}$, and it converges uniformly in a compact subset of $\text{Int}(G(F))(A^{\text{reg}}(F))$, as shown in [4, §7, 2nd parag. after Prop. 7.5 to parag. before 7.6 (pp. 130 – 132)].

Lemma 2.4. *The function $\Delta^{-1/2}(h)c(E)$ is locally constant on the open set $G^{\text{reg}}(F)$.*

Proof. Note that $E = F(h)$ depends on h . We need to show it is bounded away from zero on any compact subset C of $G(F)$. We have to show that there is $c > 0$ with $\Delta^{-1/2}(h)c(E) \geq c$ on C . There is $z \in Z(F)$ such that the entries of each matrix in zC are integral. The function $\Delta^{-1/2}(h)c(E)$ takes the same value at zh , so we may assume that each matrix in C has integral entries, in \mathcal{O}_F . There is a number $c_1 > 0$ with $|\det h|_F^{1/2} \geq c_1$ on C , and $c_2 > 0$ with $|\text{tr}(h) - 4\det(h)|_F^{1/2} \leq c_2$ on C . Hence $\Delta^{-1/2}(h) \geq c_1/c_2$ on C . Define $\Delta^{-1/2}(h)c(E)$ to be ∞ on the singular set. Then this function is bounded away from 0 on the singular elements, and on the regular elements in C whose eigenvalues lie in F^\times .

If h is regular and lies in a nonsplit torus which splits over E , denote its eigenvalues by γ and $\bar{\gamma}$, in E . In fact, γ (and $\bar{\gamma}$) lie in $\mathcal{O}_E - \mathcal{O}_F$. Consider the \mathcal{O}_F -module generated by $1, \gamma$. Let $\{1, \delta\}$ be another basis for this module. Thus $\delta \in \mathcal{O}_E - \mathcal{O}_F$. Choose δ with smallest possible absolute value. Then $\gamma = a + b\delta$ for some $a, b \in \mathcal{O}_F$, and $\delta = c + d\gamma$ for some $c, d \in \mathcal{O}_F$. Then $\gamma = (a + bc) + bd\gamma$. Hence $a + bc = 0$ and $bd = 1$. Hence b is a unit, in \mathcal{O}_F^\times , and $|\gamma - \bar{\gamma}|_E = |\delta - \bar{\delta}|_E$.

If E/F is unramified then $c(E) = 1$ and $\delta = \varepsilon\pi_F^n$ for some unit ε in \mathcal{O}_F , $n \geq 0$. We claim that $\varepsilon - \bar{\varepsilon}$ is a unit. Indeed, by definition of δ , we have $\varepsilon \notin \mathcal{O}_F + \pi_F\mathcal{O}_E$. Hence $\{1, \varepsilon\}$ projects to a basis of $\mathcal{O}_E/\pi_F\mathcal{O}_E$. The Galois group $\text{Gal}(E/F)$ acts faithfully on $\mathcal{O}_E/\pi_F\mathcal{O}_E$, hence $\varepsilon - \bar{\varepsilon} \notin \pi_F\mathcal{O}_E$.

If E/F is ramified we may take δ to be of the form $\pi_F^n\pi_E, n \geq 0$. Since

$$|\pi_E - \bar{\pi}_E|_E = |(\pi_E - \bar{\pi}_E)^2|_F = c(E),$$

we have

$$|(\delta - \bar{\delta})^2|_F^{1/2} = |\delta - \bar{\delta}|_E^{1/2} = |\pi_F^n| \cdot c(E).$$

In both cases $\Delta^{-1/2}(\gamma)c(E) = |\det \gamma|_F^{1/2}|\pi_F|_F^{-n}$ is bounded from below since $n \geq 0$.

Recall that the discriminant can be defined to be $\det(\text{tr}_{E/F}(e_i e_j))$ where $\{e_i\}$ is a basis of \mathcal{O}_E over \mathcal{O}_F . Say it is $\{1, x\}$. Then $\det \begin{pmatrix} 1 & x+\bar{x} \\ x+\bar{x} & x^2+\bar{x}^2 \end{pmatrix} = -(x-\bar{x})^2$. In the unramified case take $x = \varepsilon$. In the ramified case $x = \pi_E$. ■

To show that the function $\Delta^{-1/2}c$ is locally integrable on $G(F)/Z(F)$ and $G(F)$, we need a lemma.

Suppose the torus $T(F)$ is the multiplicative group of the two dimensional F -algebra E . Choosing a measure dx on E we get a measure dt as the quotient of dx by $|\det x|$, and a measure dg/dt on $T(F)\backslash G(F)$. Let $n(E) > 0$ be the number for which $n(E)dx$ is self dual with respect to the character $x \mapsto \psi(\text{tr}(x))$ on E , where ψ is a nontrivial additive character on E .

Lemma 2.5. *For all $r \geq 0$ there is a number $d_r > 0$ such that for each torus $T(F)$ and each $t \in T(F)$ we have $\int_{T(F)\backslash G(F)} \chi_r(g^{-1}tg) \frac{dg}{dt} \leq d_r n(E) \Delta(t)^{-1/2}$, $\chi_r = \chi_{C_r}$.*

Proof. We may assume that t is regular. Suppose $T(F)$ is anisotropic. The integral is then

$$\text{vol}(T(F)/Z(F))^{-1} \int_{Z(F)\backslash G(F)} \chi_r(g^{-1}tg) \frac{dg}{dt}.$$

Denote the splitting field of $T(F)$ by E .

If E/F is unramified, $E^\times/F^\times = \mathcal{O}_E^\times/\mathcal{O}_F^\times$. The measure $n(E)dx$ assigns volume 1 to \mathcal{O}_E , hence $1 - q_E^{-1}$ to \mathcal{O}_E^\times , and volume 1 to \mathcal{O}_F , hence volume $1 - q_F^{-1}$ to \mathcal{O}_F^\times . But $q_E = q_F^2$ when E/F is unramified, hence

$$\int_{\mathcal{O}_E^\times/\mathcal{O}_F^\times} n(E)dx = (1 - q^{-2})/(1 - q^{-1}) = 1 + q^{-1},$$

and the volume of $T(F)/Z(F)$ is $n(E)^{-1}(1 + q_F^{-1})$.

If E/F is ramified, $n(E)dx$ assigns \mathcal{O}_E volume $c(E)$ and \mathcal{O}_E^\times the volume $(1 - q_E^{-1}) \text{vol}(\mathcal{O}_E)$, and \mathcal{O}_F^\times the volume $(1 - q_F^{-1}) \text{vol}(\mathcal{O}_F) = (1 - q_F^{-1})$. Here $q_E = q_F$, and E^\times/F^\times is the product of $\pi_E^\mathbb{Z}/\pi_F^\mathbb{Z}$ (of order 2) and $\mathcal{O}_E^\times/\mathcal{O}_F^\times$ (of volume $c(E)$). Hence the volume of $T(F)/Z(F)$ is $\frac{2}{n(E)}c(E)$. The lemma then follows from the Key Lemma.

The case of a split torus $T(F) = A(F)$ is discussed in [4, §7, p. 125]. ■

We can now prove the local integrability of $\Delta^{-1/2}c$, stated in Theorem 1.1, assuming the Key Lemma.

Proof. The function $\Delta(g)^{-1/2}c(F(g))$ is measurable. It is locally integrable on $G(F)$ if and only if it is locally integrable on $G(F)/Z(F)$. We have to show that

$$\int_{G(F)/Z(F)} \chi_C(g) \Delta(g)^{-1/2} c(F(g)) dg$$

is finite for any compact subset C of $G(F)/Z(F)$.

It suffice to show this for $C = C'_r$ for every $r \geq 0$. By the Weyl integration formula this integral is the sum of

$$\frac{1}{2} \int_{A(F)/Z(F)} \Delta(a)^{-1/2} \cdot \Delta(a) \left[\int_{A(F) \setminus G(F)} \chi_r(g^{-1}ag) \frac{dg}{da} \right] da$$

and

$$\frac{1}{2} \sum_{E \in S'} \int_{T(F)/Z(F)} \Delta(t)^{-1/2} c(F(t)) \cdot \Delta(t) \left[\int_{T(F) \setminus G(F)} \chi_r(g^{-1}tg) \frac{dg}{dt} \right] dt.$$

There is a compact set C_0 in $A(F)/Z(F)$ such that $\chi_r(g^{-1}ag) = 0$ for all g unless the projection of a lies in C_0 . The first integral needs only be taken over C_0 . The inner integral is $\leq d_r n(A(F)) \Delta(a)^{-1/2}$. But $\Delta(a)^{-1/2} \cdot \Delta(a) \cdot \Delta(a)^{-1/2} = 1$ on $A(F)$, so the first integral is bounded.

The sum over S' is bounded by

$$\frac{1}{2} \sum_{E \in S'} d_r n(E) c(E) \text{vol}(T(F)/Z(F)),$$

by Lemma 2.5. If the characteristic is not two the sum is finite, hence bounded. In general if T corresponds to the field E and $\pi_F^{1+t(E)} \mathcal{O}_F$ is the discriminant of E over F , then $c(E) = |\pi_F|^{(t(E)+1)/2}$ and $n(E) \int_{T(F)/Z(F)} dt \leq 2|\pi_F|^{(t(E)+1)/2}$. To complete the proof we need to show that $\sum_E q^{-(t(E)+1)}$ converges if F has characteristic two. The sum is over all separable quadratic extensions of F .

Let $M(t)$ be the number of extensions E/F for which $t(E) = t$. Associated to any such E is a quadratic character of F^\times with conductor π_F^{t+1} . Indeed $N_{E/F} E^\times = F^{\times 2} (1 + \pi_F^{t+1} \mathcal{O}_F)$ since $\pi_E + \bar{\pi}_E = u_F \pi_F^{t+1}$, so that the product of $1 + u\pi_E + u_2\pi_E^2 + \dots$ with its conjugate is

$$1 + u(\pi_E + \bar{\pi}_E) + u^2\pi_E\bar{\pi}_E + u_2\bar{\pi}_E^2 + \bar{u}_2\pi_E^2 + \dots$$

Hence

$$M(t) \leq [F^\times : (F^\times)^2 (1 + \pi_F^{t+1} \mathcal{O}_F)] = 2[\mathcal{O}_F^\times : \mathcal{O}_F^{\times 2} (1 + \pi_F^{t+1})]$$

if $t \geq 0$, and $M(-1) = 1$. Any element of \mathcal{O}_F^\times is represented modulo $1 + \pi_F^{t+1} \mathcal{O}_F$ by

$$a_0 + a_1\pi_F + \dots + a_t\pi_F^t.$$

Such a number is a square if $a_i = 0$ for all odd i . Hence $M(t) = O(q^{(t+1)/2})$, and the series $\sum_E q^{-(t(E)+1)}$ is bounded by a multiple of $\sum_{t \geq 0} q^{-(t+1)/2}$ so it converges, and the character is locally integrable. ■

It remains to prove the Key Lemma.

3. Proof of Key Lemma

The Key Lemma 2.3 asserts that for an elliptic regular $h \in \mathrm{GL}(2, F)$, thus $h \in G_{\mathrm{ell}}^{\mathrm{reg}}(F)$, the volume of the g in $G(F)/Z(F)$ with $\mathrm{Int}(g)h \in C$ is bounded by a constant multiple of $\Delta(h)^{-1/2}c(F(h))$. We may assume the compact C in $G(F)$ is K -biinvariant, thus $KCK = C$ for $K = G(\mathcal{O})$, and then take g of the form $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$. We shall *compute this integral precisely for $C = K$, then bound it from above for a general C* . First we need to describe h , namely the quadratic separable extensions E of F , when F is a local field of characteristic two. The case of other characteristics is well known and much easier, as then there are just two ramified quadratic extensions and one unramified, all separable. It is the case of $\theta = u_{\theta}t^{-1}$ ($k = 1$) below.

A local field of characteristic two has the form $F = \mathbb{F}_q((t))$, power series in the variable t over the field \mathbb{F}_q of $q = 2^z$ elements, $z \geq 1$. Its ring of integers $\mathcal{O} = \mathbb{F}_q[[t]]$ has the maximal ideal $t\mathcal{O}$, and group of units $\mathcal{O}^{\times} = \mathcal{O} - t\mathcal{O}$.

The *ramified quadratic separable extensions of F* are $E = F(r)$ where r is a root of $x^2 - x + \theta$, $\theta = u_{\theta}t^{1-2k}$, $k \geq 1$, $u_{\theta} \in \mathcal{O}^{\times}$ (the other root is $r + 1$). The unramified extension is the splitting field of the quadratic with $\theta \in \mathcal{O}^{\times}$, provided the polynomial is irreducible. Two extensions E are isomorphic if the corresponding θ 's differ by $a^2 - a$ for some a in F . This $x^2 - x + \theta$ is the characteristic polynomial of $r = \begin{pmatrix} 0 & -1 \\ \theta & 1 \end{pmatrix}$. Its centralizer consists of $\gamma = u + vr$, $u, v \in F$. We have $\det \gamma = u^2 + uv + v^2\theta$. We need to integrate over $A(F)/Z(F)$ and $N(F)$ the expression $n^{-1}a^{-1}\gamma an \in K$. Thus we compute

$$\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \theta & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -xy\theta & -x^{-1}-y-y^2x\theta \\ x\theta & 1+xy\theta \end{pmatrix}$$

and ask – denoting the identity by I – when does

$$\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} (uI + v \begin{pmatrix} 0 & -1 \\ \theta & 1 \end{pmatrix}) \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u-yxv\theta & -vy-y^2xv\theta-v/x \\ xv\theta & u+v+vxv\theta \end{pmatrix}$$

lie in K . If this happens then (1) $u^2 + uv + v^2\theta = \det \gamma \in \mathcal{O}^{\times}$, and (2) $v = \mathrm{tr} \gamma \in \mathcal{O}$.

These imply (3) $|u| = 1$ and $|v^2\theta| < 1$. Indeed, if $|u| > 1$ then $|u^2| = |v^2\theta|$, contradiction, and if $|u| < 1$ then $|v^2\theta| = 1$, contradiction again. We may then assume that $u = 1$.

We may write $v = at^k$, $a = u_a t^{\alpha}$, $\alpha \geq 0$ and $u_a \in \mathcal{O}^{\times}$. Then $r_1 = t^k r$ solves $r_1^2 - t^k r_1 + u_{\theta} t = 0$ (an irreducible Eisenstein polynomial). Thus we need to compute

$$\int \chi_K(g^{-1}(u + vr)g)dg = \int_y \int_x \chi_K \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} (1 + ar_1) \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) d^{\times} x dy$$

(on the left g ranges over $Z(F) \backslash G(F)/K$, on the right $x \in F^{\times}$ and $y \in F$).

The matrix lies in K implies that each of its entries is ≤ 1 in absolute value.

Thus (4) $|xv\theta| \leq 1$, and we distinguish between two cases.

If (5) $|y| \leq 1$, we get $|v/x| \leq 1$, thus $|v| \leq |x| \leq |v\theta|^{-1}$, or: $|v| \leq |x| \leq |v\theta|^{-1}q^{-2}$, $|x| = |v\theta|^{-1}q^{-1}$ or $|x| = |v\theta|^{-1}$.

If (6) $|y| > 1$ then we claim that (7) $|v/x| \leq 1$. Indeed, if $|v/x| > 1$ then $|v/x| = |vy||1 + xy\theta|$, which implies $1 = |xy||1 + xy\theta|$. Now if $|xy\theta| \geq 1$ then $|x^2y\theta| = 1$, but the valuation of θ is odd. If $|xy\theta| = 1$ then $|1 + xy\theta| \leq 1$, hence $1 \leq |xy| = |\theta|^{-1} < 1$. If $|xy\theta| < 1$ then $|xy| = 1$, hence $|\theta| < 1$, so (7) follows.

Also we have (8) $|vy| \leq 1$. For this we use: $|vy||1 + xy\theta| \leq 1$. If $|vy| > 1$ then $|1 + xy\theta| < 1$, hence $|xy\theta| = 1$, so $1 > |\theta|^{-1} = |xy| \geq |vy| > 1$.

Note that (9) $|xv\theta| \leq |y|^{-2} \leq q^{-2}$ (under (6)), and $|y| \leq |xv\theta|^{1/2}$ (this is $< |v|^{-1}$ as $|v| < |x\theta|$ implies $|v|^2 < |xv\theta|$).

In summary we integrate $\int d^\times x \int dy$ over:

(1) $|y| \leq 1$ and $|x| = |v\theta|^{-1}q^{-1}$ and $= |v\theta|^{-1}$; we get 2.

(2) $|v| \leq |x| \leq |v\theta|^{-1}q^{-2}$, $|y| \leq |xv\theta|^{1/2}$. Change $x \mapsto vx$ to get:

$$1 \leq |x| \leq |v^2\theta|^{-1}q^{-2}; \quad |y| \leq |xv^2\theta|^{-1/2}.$$

Put $|x| = q^m$; then

$$|v^2\theta|^{-1} = |t^{1+2\alpha}|^{-1} = q^{1+2\alpha}, \quad |v^2\theta x|^{-1/2} = q^{\alpha+1/2-m/2}.$$

To integrate $\int d^\times x \int dy$ we write a sum depending on the parity of $2\alpha + 1 - m$, i.e. of m ; we get two sums:

(A) $\sum q^{\alpha-m/2}$, sum over $0 \leq m \leq 2\alpha - 1$ (corresponding to $1 \leq |x| \leq |v^2\theta|^{-1}q^{-2}$) with even m , thus $m = 2k$ with $0 \leq k \leq \alpha - 1$, and if we put $1 \leq j = \alpha - k \leq \alpha$ we get

$$\sum_{0 \leq k \leq \alpha-1} q^{\alpha-k} = q \sum_{0 \leq j \leq \alpha-1} q^j = q \frac{q^\alpha - 1}{q - 1}.$$

(B) $\sum q^{\alpha+1/2-m/2}$, sum over odd m , $0 \leq m \leq 2\alpha - 1$, thus $m = 2k + 1$, $0 \leq k \leq \alpha - 1$, and $\frac{1}{2} - \frac{m}{2} = -k$. So we get the same sum and result as in (A). Thus in case (2) we get $2q(q^\alpha - 1)/(q - 1)$; plus 2 from (1) gives

$$2 \left(\frac{q^{\alpha+1} - q}{q - 1} + 1 \right) = 2 \frac{q^{\alpha+1} - 1}{q - 1} = 2(q^\alpha + q^{\alpha-1} + \dots + q + 1).$$

This is the orbital integral $I(\gamma, \chi_K)$ of the characteristic function χ_K of K at a regular $\gamma = u + vr$ in a ramified torus.

To compare this result with the claimed bound, note that $|v^2\theta|^{-1/2} = q^{\alpha+1/2}$. Also, the eigenvalues of γ are $u + vr$ and $u + v + vr$, hence

$$\Delta(\gamma) = \left| \frac{((u + vr) - (u + v + vr))^2}{u^2 + uv + v^2\theta} \right| = |v|^2$$

(or $|\det \gamma| = 1$ and $\text{tr } \gamma = v$ and $\Delta(\gamma) = \left| \frac{(\text{tr } \gamma)^2}{\det \gamma} - 4 \right|$). The discriminant of E is $t^{2k}\mathcal{O}^\times$, hence

$$c(E) = |t|^k = q^{-k} = |\theta|^{-1/2}q^{-1/2}, \quad \text{so } q^{\alpha+1/2} = \Delta(\gamma)^{-1/2}c(E)q^{1/2},$$

and $q^\alpha = \Delta^{-1/2}(\gamma)c(E)$.

In the general case, instead of $C = K$ we take the compact C to be the K -double coset $K \begin{pmatrix} \pi^p & 0 \\ 0 & \pi^s \end{pmatrix} K$. Multiplying by a central element we may assume that $(p, s) = (p, -p)$ or $(1 + p, -p)$. We aim to bound

$$\int_{Z \backslash G} \chi(g^{-1}(u + vr)g) dg = \text{vol}(KZ/Z) \int_x \int_y \chi(*) d^x x dy$$

where $*$ is the 2×2 matrix displayed before (3) above, and χ is the characteristic function of our double coset.

Consider first the case where $p + s$ is even. Then

$$\det \gamma = \det(u + vr) = u^2 + uv + v^2\theta$$

is $q_F^{-(p+s)}$. Hence (1) $|u^2| > |v^2\theta|$ (and $|v| < |u|$), as $|u^2| \leq |v^2\theta|$ implies

$$|uv| \leq |v^2||\theta|^{1/2} < |v^2\theta|,$$

but then $|\det \gamma| = |v^2\theta|$ has odd valuation. We assume $s = -p$, and then $u = 1$, hence $|v^2\theta| < 1$, $|v| < |\theta|^{-1/2} < 1$. To repeat, we are trying to find the x, y with

$$\begin{pmatrix} u-y \cdot xv\theta & vy-vy \cdot xy\theta-v/x \\ xv\theta & u+v+y \cdot xv\theta \end{pmatrix} \in K \begin{pmatrix} \pi^p & 0 \\ 0 & \pi^{-p} \end{pmatrix} K,$$

and we put $u = 1$. Then (2) $|xv\theta| \leq |\pi|^{-p}$. If (3) $|y| \leq 1$, then $|\pi|^p|v| \leq |x| \leq |v\theta|^{-1}|\pi|^{-p}$.

If (4) $|y| > 1$, then (5) $|x| \geq |\pi|^p|v|$. Indeed,

$$|vy - vy \cdot xy\theta - v/x| \leq |\pi|^{-p}.$$

Hence if $|v/x| > |\pi|^{-p}$ then $|v/x| = |vy||1 - xy\theta|$, so $1 = |xy||1 - xy\theta|$. If $|xy\theta| > 1$ then $|x^2y^2\theta| = 1$, but $\text{val}(\theta) = 1 - 2k$ is odd. If $|xy\theta| = 1$ then $|1 - xy\theta| \leq 1$, so $1 \leq |xy| = |\theta|^{-1} < 1$. If $|xy\theta| < 1$ then $|xy| = 1$ implies $|\theta| < 1$.

Also (6) $|vy| \leq |\pi|^{-p}$. Indeed, $|vy||1 - xy\theta| \leq |\pi|^{-p}$. If $|vy| > |\pi|^{-p}$ then $|1 - xy\theta| \leq |vy|^{-1}|\pi|^{-p} < 1$, so $|xy\theta| = 1$, hence $|\theta|^{-1} = |xy| \geq |\pi|^p|vy| > 1$.

Next observe that $|xy^2v\theta| \leq |\pi|^{-p}$ implies (7) $|y| \leq |xv\theta|^{-1/2}|\pi|^{-p/2}$, and this bound is $< |v|^{-1}|\pi|^{-p}$, namely $|v|^2|\pi|^{2p} < |xv\theta||\pi|^p$, or $|v| < |x\theta||\pi|^{-p}$ by (5). In particular $|xv\theta| \leq |y|^{-2}|\pi|^{-p} \leq |\pi|^{-p+2}$.

In summary, we need to integrate over (1) $|y| \leq 1$, $|\pi|^p|v| \leq |x| \leq |v\theta|^{-1}|\pi|^{-p}$; and over (2) $|y| > 1$, $|\pi|^p|v| \leq |x| \leq |v\theta|^{-1}|\pi|^{-p+2}$, $1 < |y| \leq |xv\theta|^{-1/2}|\pi|^{-p/2}$. The integral $\int_y dy$ gives $|xv\theta|^{-1/2}|\pi|^{-p/2} (> 1 \text{ as } |xv\theta| \leq |\pi|^{-p+2})$. The leftover from (1), $\int_x d^x x$ over $|x| = |v\theta|^{-1}|\pi|^{-p+1}$ and $= |v\theta|^{-1}|\pi|^{-p+2}$, is just a constant, so we need to estimate

$$\int_{|\pi|^p|v| \leq |x| \leq |v\theta|^{-1}|\pi|^{-p}} |xv\theta|^{-1/2} d^x x.$$

But

$$|v\theta|^{-1/2} \int_{c_1|v| \leq |x| \leq |v\theta|^{-1}c_2} |x|^{-1/2} d^x x = |v\theta|^{-1/2} \frac{c_1^{-1/2}|v|^{-1/2} - |v\theta|^{-1/2}c_2^{-1/2}}{1 - q^{-1/2}}$$

$$= \frac{c_1^{-1/2}|v^2\theta|^{-1/2} - c_2^{-1/2}}{1 - q^{-1/2}}$$

is bounded by a scalar multiple of $|v|^{-1}|\theta|^{-1/2}$, which is > 1 . It is dominated by $\Delta(\gamma)^{-1/2}c(E)$ since discriminant (E/F) is $\pi^{2k}\mathcal{O}_F$ so $c(E) = |\theta|^{-1/2}q_F^{-1/2}$, and

$$\Delta = \left| \frac{\text{tr}^2}{\det} - 4 \right| = \left| \frac{(v + 2u)^2 - 4(u^2 + uv + v^2\theta)}{u^2 + uv + v^2\theta} \right| = |v^2 - 4v^2\theta| = |v|^2.$$

If $g^{-1}\gamma g \in C = K \begin{pmatrix} \pi^{1+p} & 0 \\ 0 & \pi^{-p} \end{pmatrix} K$, $p \geq 0$, then $\det \gamma = u^2 + uv + v^2\theta$ has valuation 1. We have $|v^2\theta| > |u|^2$ (as if $|u|^2 > |v^2\theta| > |v|^2$ then $|\det \gamma| = |u|^2$ has even valuation). Also $|u| < |v\theta|$ (if $|u| \geq |v\theta|$ then $|u|^2 \geq |v^2\theta||\theta| > |v^2\theta|$) so $|uv| < |v^2\theta|$. As $\theta = u\theta r^{1-2k}$ ($t = \pi_F$) we must have $|v| = |\pi_F|^k$ so that $|\det \gamma| = |\pi_F|$. Then $|u| < 1$. But (2) – (7) of the case $(p, s) = (p, -p)$ remain valid in our case, as is the estimate for the integral, by the same computations.

There is only one unramified separable quadratic extension E of the local field F . It is the splitting field of an irreducible quadratic $x^2 - x + \theta$, where θ is a unit, in \mathcal{O}^\times . We compute the orbital integral of χ_K , the characteristic function of the maximal compact subgroup $K = \text{GL}(2, \mathcal{O})$ – the general case of double cosets $K \begin{pmatrix} \pi_0^p & 0 \\ 0 & \pi_0^s \end{pmatrix} K$ is similar, in the case that the characteristic is two. The other cases are easier. As in the ramified case, a solution of the equation is given by $r = \begin{pmatrix} 0 & -1 \\ \theta & 1 \end{pmatrix}$, and E by $\gamma = u + vr$, $u, v \in F$. Put $u(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, $d(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$. We need to integrate $d^\times x$ over $x \in F^\times$ and dy over $y \in F$ limited by

$$\begin{pmatrix} u-yxv\theta & -vy-y^2xv\theta-v/x \\ xv\theta & u+v+vxv\theta \end{pmatrix} = u(y)^{-1}d(x)^{-1}(u + vr)d(x)u(y) \in K.$$

Then $v = \text{tr } \gamma \in \mathcal{O}$ and $u^2 + uv + v^2\theta = \det \gamma \in \mathcal{O}^\times$ imply $\max(|u|, |v|) = 1$. Changing $y \mapsto y/x$ the integral $\int_y dy \int_x d^\times x$ becomes $\int_y dy \int_x d^\times x/|x|$, taken over

$$\begin{pmatrix} u-yv\theta & -(1+y+y^2\theta)v/x \\ xv\theta & u+v+yv\theta \end{pmatrix} \in K.$$

Thus $|xv| \leq 1$ and $|yv| \leq 1$. If $|y| \leq 1$ then $|v/x| \leq 1$ (if $|y| = 1$ then $z^2 + z + \theta \in \mathcal{O}^\times$ for $z = 1/y \in \mathcal{O}^\times$ by the irreducibility of $x^2 - x + \theta$). If $|y| > 1$ then $|vy^2/x| \leq 1$.

Hence we need integrate over (1) $|y| \leq 1$ and $|v| \leq |x| \leq |v|^{-1}$; (2) $1 < |y| \leq |v|^{-1}$, $|y| \leq |x/v|^{1/2}$ (note that $|x/v|^{1/2} \leq |v|^{-1}$ as $|xv| \leq 1$). In this case $|v| < 1$, hence $|u| = 1$, $|x/v|^{1/2} \geq q$ so $|x/v| \geq q^2$, thus $q^2|v| \leq |x| \leq |v|^{-1}$. The change $y \mapsto y/x$ of variables replaces $\int dy \int d^\times x$ by $\int dy \int d^\times x/|x|$, which is $\int_{|v| \leq |x| \leq |v|^{-1}} d^\times x/|x| = 1$ when $|v| = 1$. If $|v| < 1$, the integral is

$$\begin{aligned} & \int_{|v| \leq |x| \leq |v|^{-1}} d^\times x/|x| + \int_{q^2|v| \leq |x| \leq |v|^{-1}} \frac{d^\times x}{|x|} \int_{1 < |y| \leq |x/v|^{1/2}} dy \\ &= \int_{|x|=|v|} \frac{d^\times x}{|x|} + \int_{|x|=q|v|} \frac{d^\times x}{|x|} + \int_{q^2|v| \leq |x| \leq |v|^{-1}} \frac{d^\times x}{|x|} \int_{|y| \leq |x/v|^{1/2}} dy \\ &= |v|^{-1} + q^{-1}|v|^{-1} + |v| \int_{q^2|v|^2 \leq |x| \leq 1} \frac{d^\times x}{|x|} \int_{|y| \leq |x|^{1/2}|v|^{-1}} dy. \end{aligned}$$

To compute the last (double) integral as a sum, we write $|x| = q^{-m}$, $0 \leq m \leq 2r - 2$, $|v| = q^{-r}$, $r \geq 1$.

If $2|m = 2k$ then $\int_{|y| \leq |x|^{1/2}|v|^{-1}} dy = |x|^{1/2}|v|^{-1} = q^{-k}|v|^{-1}$, $0 \leq k \leq r - 1$.

If $2 \nmid m = 2k - 1$, $|x|^{1/2} = q^{-k+1/2}$ implies the integral over y is $q^{-k}|v|^{-1}$, $1 \leq k \leq r - 1$.

The double integral is then

$$\sum_{0 \leq k \leq r-1} q^k + \sum_{0 \leq k-1 \leq r-2} q^{k-1} = \frac{q^r - 1}{q - 1} + \frac{q^{r-1} - 1}{q - 1}.$$

Adding $q^r + q^{r-1}$ to this we get $|v|^{-1} \frac{q+1}{q-1} - \frac{2}{q-1}$. Recall that

$$|D(\gamma)| = \left| \frac{(\text{tr } \gamma)^2}{\det \gamma} - 4 \right| = |v|^2,$$

thus $|D(\gamma)|^{-1/2} = |v|^{-1}$. Since $\Lambda(\chi_K) = \frac{\mu(K/K \cap Z)}{\mu(N \cap K)} \cdot \frac{q}{q-1}$, where μ indicates volume, χ_K the characteristic function of K in G , and Λ is the orbital integral at the nontrivial unipotent conjugacy class, we conclude that the orbital integral of χ_K at γ is

$$I(\gamma, \chi_K) = -\frac{2}{q-1} \mu(K/K \cap Z) \chi_K(e) + \mu(N \cap K) \frac{q+1}{q} \Lambda(\chi_K) \cdot |D(\gamma)|^{-1/2}.$$

4. Determination of coefficients in a germ expansion

Let f be a locally constant function on $G(F)$ such that fdg lies in H . Let γ be an elliptic regular element in $G^{\text{reg}}(F)$. Denote by T the centralizer of γ in G . It is an elliptic torus. The orbital integral

$$I_f(\gamma) = I_f^Z(\gamma) = \int_{G/Z} f(\text{Int}(g)\gamma) dg = |T/Z| I_f^T(\gamma), \quad I_f^T(\gamma) = \int_{G/T} f(\text{Int}(g)\gamma) dg$$

converges, as does the orbital integral

$$\Lambda(f) = \int_{G/Z(u)} f(\text{Int}(g)u) dg, \quad u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of f at the unipotent regular (its centralizer $Z(u)$ has maximal dimension) element u . Near the identity the orbital integral has a *germ expansion*: for each elliptic torus T in G there exist functions $A_T(\gamma)$, $B_T(\gamma)$ such that for all $f \in H$ and all $\gamma \in T^{\text{reg}}(F)$ sufficiently close (depending on f and T) to e , we have

$$I_f(\gamma) = -A_T(\gamma)f(e) + B_T(\gamma)\Lambda(f).$$

Proposition 4.1. (1) *Let T be a quadratic torus in G . Then A_T is the positive constant $\frac{2}{q-1}|K|$, and $B_T(\gamma) = B_T|D(\gamma)|^{-1/2}$, where $B_T = \kappa_T c(T)$ is a positive constant, $c(T) = c(E)$ is the square root of the absolute value of a generator of the discriminant of the splitting field E of T over F , and κ_T is 2 if E/F is ramified,*

and $\frac{q+1}{q}$ if E/F is unramified.

(2) Let $f_K = \chi_K/|K|$ be the quotient of the characteristic function χ_K of K by the volume $|K|$ of K . Then

$$\begin{aligned} I_{f_K}(\gamma) &= |K|^{-1} \int_{G/Z} \chi_K(g\gamma g^{-1}) dg = \int_{K \backslash G/Z} \chi_K(g\gamma g^{-1}) dg \\ &= -\frac{2}{q-1} |K| f_K(e) + \kappa_T c(T) |D(\gamma)|^{-1/2} \Lambda(f_K) \end{aligned}$$

uniformly, for all γ in $T^{\text{reg}}(F) \cap K$.

Proof. Assertion (2) was proven above in the course of the proof of the Key Lemma. We showed that $I_{f_K}(\gamma) = -\frac{2}{q-1} + \kappa_T \frac{q}{q-1} q^\alpha$. Recall that

$$r^2 - r + \theta = 0, \quad \theta = u_\theta t^{1-2k}, \quad k \geq 1$$

(in the ramified case),

$$\begin{aligned} v &= at^k, \quad a = u_a t^\alpha, \quad \gamma = u + vr = u + u_a t^\alpha \cdot t^k r; \\ \Delta^{-1/2} &= |D(\gamma)|^{-1/2} = |v|^{-1}; \quad q^{\alpha+1/2} = |v^2 \theta|^{-1/2}, \\ c(E) &= c(T) = q^{-k} = |\theta|^{-1/2} q^{-1/2} \end{aligned}$$

so

$$q^\alpha = \Delta^{-1/2} c(E) = |D(\gamma)|^{-1/2} c(T).$$

For every f in H the integral $\Lambda(f) = \int_{Z(u) \backslash G} f(g^{-1}ug) dg$ is $|K|$ times

$$\int_{F^\times} f \left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) |a|^{-1} d^\times a = \int_{F^\times} f \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right) |a| d^\times a$$

as

$$g = ank = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k = \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k,$$

so $dg = |a|^{-1} dn da dk$, and $dg/dn = |a|^{-1} d^\times adk$. At $f = f_K$ we have that $\Lambda(f_K)$ is

$$\int_{|a| \leq 1} |a| d^\times a = \sum_{n=0}^{\infty} \int_{|a|=q^{-n}} |a| d^\times a = \sum_{n \geq 0} q^{-n} = \frac{1}{1 - q^{-1}} = \frac{q}{q-1}.$$

This shows that $A_T = \frac{2}{q-1} |K|$ and $B_T = \kappa_T c(T)$ when the splitting field E of T is ramified, where $c(T) = c(E)$ is the square root of the absolute value of a generator of the discriminant of E/F . When E/F is unramified, $c(E)$ is 1.

To compute A_T alone, we produce a function for which Λ is zero. Recall ([1]) that a pseudo-coefficient of the unramified twists of the Steinberg representation St is given by $z = |I|^{-1} \chi_I - 2|K|^{-1} \chi_K$, where χ_K is the characteristic function of $K = G(\mathcal{O})$, and χ_I is that of the Iwahori subgroup I . Since $K = I \cup IsI$, we have $|K/I| = q + 1$, and

$$z = |K|^{-1} ((q+1)\chi_I - 2\chi_K).$$

Denote by π_η the representation of G induced (in the normalized way) from the character η of the upper triangular subgroup AN . The restriction of the functions in its space to K gives a model F_χ consisting of functions on K transforming on the left according to the restriction χ of η to $AN \cap K$. If η is unramified, namely χ is 1 on $A(\mathcal{O})N(\mathcal{O})$, there are two nonzero I -invariant functions in F_χ (up to scalar multiples): φ_I , which takes the value 1 on I and 0 on $K - I$, and φ_{IsI} which is 0 on I and 1 on IsI . Hence π_η^I is two dimensional if $\eta|A(\mathcal{O}) = \chi$, namely η is unramified. There is one K -invariant nonzero function in F_χ (up to scalar multiples): $\varphi_K = \varphi_I + \varphi_{IsI}$, the characteristic function of K . Hence π_η^K is one dimensional, and $\text{tr } \pi_\eta(z) = 0$ if $\chi = 1$ (thus η is unramified). The same holds when η is ramified, as then $\pi_\eta^I = \{0\}$. If $\pi(g) = \rho(\det g)$, ρ unramified, then $\text{tr } \pi(z) = -1$, so $\text{tr } \text{St} \otimes \rho(z) = 1$ for the unramified twists $\text{St} \otimes \rho$ of the Steinberg representation St . If π is irreducible and ramified ($\pi^K = \{0\}$) then $\text{tr } \pi(z) = 0$.

Since z is a pseudo-coefficient of $\text{St} + \text{St} \otimes \rho$, where ρ is the nontrivial character of $F^\times/N_{E/F}E^\times$, E the unramified quadratic extension of F , we have that

$$I_z(\gamma) = \chi_{\text{St}}(\gamma) + \chi_{\text{St} \otimes \rho}(\gamma) = -2$$

on the regular elliptic set of γ in G , and $I_z(\gamma) = 0$ on the regular split set. Hence $\Lambda(z) = 0$, and also $z(e) = (q - 1)/|K|$.

Evaluating the germ expansion at $f = z$ and an elliptic regular γ , we obtain $-2 = -A_T(q - 1)/|K|$, or $A_T = 2|K|/(q - 1)$, a constant in γ , as asserted. In particular A_T depends linearly on the measure dg used in the definition of I_f .

The value of B_T is now obtained from the computation of the orbital integral of f_K above. ■

Finally we express the germ expansion more canonically in terms of measures $\mu = fdg$, where dg is a Haar measure. Write $|K|_{dg}$ to emphasize that the volume of K is measured using dg . Write δ_{dg} for the distribution $\delta_{dg}(\mu) = f(e)$. Then $\delta_{cdg} = c^{-1}\delta_{dg}$, and $|K|_{dg}\delta_{dg}$ is independent of the choice of dg , and we may drop dg from the notation. Note that $[K : I] = q + 1$, where I is an Iwahori subgroup of K .

Corollary 4.2. *Let Ω be an elliptic regular conjugacy class in G which is sufficiently close to e . Write E for the splitting field of the quadratic torus T – determined uniquely up to conjugation in G – by (any γ in) Ω . Then*

$$\begin{aligned} I_\Omega(\mu) &= -\frac{2}{q - 1} |K|_{dg} \delta_{dg}(\mu) + \kappa_E c(E) |D(\Omega)|^{-1/2} \Lambda(\mu) \\ &= -\frac{2(q + 1)}{q - 1} |I| \delta(\mu) + \kappa_E \cdot c(E) \cdot |D(\Omega)|^{-1/2} \Lambda(\mu). \end{aligned}$$

Here $I_\Omega(\mu) = I_\mu(\gamma)$ and $D(\Omega) = D(\gamma)$ for any γ in Ω . Both $\kappa_E = \kappa_T$ and $c(E) = c(T)$ are defined in 4.1.

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